

**NORM VARIETIES AND THE CHAIN LEMMA  
(AFTER MARKUS ROST)**

*Notes by Christian Haesemeyer and Chuck Weibel.*

The goal of this paper is to present proofs of two results of Markus Rost, the Chain Lemma 0.1 and the Norm Principle 0.3. These are the steps needed to complete the published verification of the *Bloch-Kato conjecture*, that the norm residue maps are isomorphisms  $K_n^M(k)/p \xrightarrow{\cong} H_{\text{ét}}^n(k, \mathbb{Z}/p)$  for every prime  $p$ , every  $n$  and every field  $k$  containing  $1/p$ .

Throughout this paper,  $p$  is a fixed prime, and  $k$  is a field of characteristic 0, containing the  $p$ -th roots of unity. We fix an integer  $n \geq 2$  and an  $n$ -tuple  $(a_1, \dots, a_n)$  of units in  $k$ , such that the symbol  $\{a\} = \{a_1, \dots, a_n\}$  is nontrivial in the Milnor  $K$ -group  $K_n^M(k)/p$ .

Associated to this data are several notions. A field  $F$  over  $k$  is a *splitting field* for  $\{a\}$  if  $\{a\}_F = 0$  in  $K_n^M(F)/p$ . A variety  $X$  over  $k$  is called a *splitting variety* if its function field is a splitting field;  $X$  is  *$p$ -generic* if any splitting field  $F$  has a finite extension  $E/F$  of degree prime to  $p$  with  $X(E) \neq \emptyset$ . A *norm variety* for  $\{a\}$  is a smooth projective  $p$ -generic splitting variety for  $\{a\}$  of dimension  $p^{n-1} - 1$ .

The following sequence of theorems reduces the Bloch-Kato conjecture to the Chain Lemma 0.1 and the Norm Principle 0.3; the notion of a *Rost variety* is defined in 0.5 below; the definition of a *Rost motive* is given in [14] and [15], and will not be needed in this paper.

- (0) The Chain Lemma 0.1 and the Norm Principle 0.3 hold; this is proven here.
- (1) Given (0), Rost varieties exist; this is Theorem 0.7 below, and is proven in [11, p. 253].
- (2) If Rost varieties exist then Rost motives exist; this is proven in [15].
- (3) If Rost motives exist then Bloch-Kato is true; this is proven in [13] and [14].

Here is the statement of the Chain Lemma, which we quote from [11, 5.1] and prove in §5. A field is  *$p$ -special* if  $p$  divides the order of every finite field extension.

**Theorem 0.1** (Rost's Chain Lemma). *Let  $\{a\} \in K_n^M(k)/p$  be a nontrivial symbol, where  $k$  is a  $p$ -special field. Then there exists a smooth projective cellular variety  $S/k$  and a collection of invertible sheaves  $J = J_1, J'_1, \dots, J_{n-1}, J'_{n-1}$  equipped with nonzero  $p$ -forms  $\gamma = \gamma_1, \gamma'_1, \dots, \gamma_{n-1}, \gamma'_{n-1}$  satisfying the following conditions:*

- (1)  $\dim S = p(p^{n-1} - 1) = p^n - p$ ;
- (2)  $\{a_1, \dots, a_n\} = \{a_1, \dots, a_{n-2}, \gamma_{n-1}, \gamma'_{n-1}\} \in K_n^M(k(S))/p$ ,  
 $\{a_1, \dots, a_{i-1}, \gamma_i\} = \{a_1, \dots, a_{i-2}, \gamma_{i-1}, \gamma'_{i-1}\} \in K_i^M(k(S))/p$  for  $2 \leq i < n$ .  
*In particular,  $\{a_1, \dots, a_n\} = \{\gamma, \gamma'_1, \dots, \gamma'_{n-1}\} \in K_n^M(k(S))/p$ ;*
- (3)  $\gamma \notin \Gamma(S, J)^{\otimes(-p)}$ , as is evident from (2);
- (4) for any  $s \in V(\gamma_i) \cup V(\gamma'_i)$ , the field  $k(s)$  splits  $\{a_1, \dots, a_n\}$ ;
- (5)  $I(V(\gamma_i)) + I(V(\gamma'_i)) \subseteq p\mathbb{Z}$  for all  $i$ , as follows from (4);
- (6)  $\deg(c_1(J)^{\dim S})$  is relatively prime to  $p$ .

(We explain  $p$ -forms in section 1 and show in Definition 1.6 how they may be used to define elements of  $K_n^M(k(S))/p$ . For any  $V$  of finite type over  $k$ ,  $I(V)$  is defined to be the subgroup of  $\mathbb{Z}$  generated by the  $[k(v) : k]$  for all closed  $v \in V$ .)

Rost's Norm Principle concerns the group  $\overline{A}_0(X, \mathcal{K}_1)$ , which we now define.

**Definition 0.2.** (Rost, [6]) For any regular scheme  $X$ , the group  $A_0(X, \mathcal{K}_1)$  is defined to be the group generated by symbols  $[x, \alpha]$ , where  $x$  is a closed point of  $X$  and  $\alpha \in k(x)^\times$ , modulo the relations (i)  $[x, \alpha][x, \alpha'] = [x, \alpha\alpha']$  and (ii) for every point  $y$  of dimension 1 the image of the tame symbol  $K_2(k(y)) \rightarrow \oplus k(x)^\times$  is zero.

The functor  $A_0(X, \mathcal{K}_1)$  is covariant in  $X$  for proper maps, because it is isomorphic to the motivic homology group  $H_{-1, -1}(X) = \text{Hom}_{DM}(\mathbb{Z}, M(X)(1)[1])$  (see [11, 1.1]). It is also the  $K$ -cohomology group  $H^d(X, \mathcal{K}_{d+1})$ , where  $d = \dim(X)$ .

The reduced group  $\overline{A}_0(X, \mathcal{K}_1)$  is defined to be the quotient of  $A_0(X, \mathcal{K}_1)$  by the difference of the two projections from  $A_0(X \times X, \mathcal{K}_1)$ . As observed in [11, 1.2], there is a well defined map  $N : \overline{A}_0(X, \mathcal{K}_1) \rightarrow k^\times$  sending  $[x, \alpha]$  to the norm of  $\alpha$ .

**Theorem 0.3** (Norm Principle). *Suppose that  $k$  is a  $p$ -special field and that  $X$  is a norm variety for some nontrivial symbol  $\{a\}$ . Let  $[z, \beta] \in \overline{A}_0(X, \mathcal{K}_1)$  be such that  $[k(z) : k] = p^\nu$  for  $\nu > 1$ . Then there exists a point  $x \in X$  with  $[k(x) : k] = p$  and  $\alpha \in k(x)^\times$  such that  $[z, \beta] = [x, \alpha]$  in  $\overline{A}_0(X, \mathcal{K}_1)$ .*

We will prove the Norm Principle 0.3 in section 9 below.

Our proofs of these two results are based on Rost's 1998 preprint [8], his web site [7] and Rost's lectures [Rost] in 1999-2000 and 2005. The idea for writing up these notes in publishable form originated during his 2005 course, and was reinvigorated by conversations with Markus Rost at the Abel Symposium 2007 in Oslo. As usual, all mistakes in this paper are the responsibility of the authors.

**Rost varieties.** In the rest of this introduction, we explain how 0.1 and 0.3 imply the problematic Theorem 0.7, and hence complete the proof of the Bloch-Kato conjecture. We first recall the notions of a  $\nu_i$ -variety and a Rost variety.

Let  $X$  be a smooth projective variety of dimension  $d > 0$ . Recall from [5, §16] that there is a characteristic number  $s_d : K_0(X) \rightarrow \mathbb{Z}$  corresponding to the symmetric polynomial  $\sum t_j^d$  in the Chern roots  $t_j$  of a bundle; we write  $s_d(X)$  for  $s_d$  of the tangent bundle  $T_X$ . When  $d = p^\nu - 1$ , we know that  $s_d(X) \equiv 0 \pmod{p}$ ; see [5, 16.6 and 16-E], [10, pp. 128-9] or [1, II.7].

**Definition 0.4.** (see [11, 1.20]) A  $\nu_{n-1}$ -variety over a field  $k$  is a smooth projective variety  $X$  of dimension  $d = p^{n-1} - 1$ , with  $s_d(X) \not\equiv 0 \pmod{p^2}$ .

For example,  $s_d(\mathbb{P}^d) = d + 1$  by [5, 16.6]. Thus the projective space  $\mathbb{P}^{p-1}$  is a  $\nu_1$ -variety, and so is any Brauer-Severi variety of dimension  $p - 1$ . In Section 8, we will show that the bundle  $\mathbb{P}(\mathcal{A})$  over  $S$  is a  $\nu_n$ -variety.

**Definition 0.5.** A *Rost variety* for a sequence  $\underline{a} = (a_1, \dots, a_n)$  of units in  $k$  is a  $\nu_{n-1}$ -variety such that:  $\{a_1, \dots, a_n\}$  vanishes in  $K_n^M(k(X))/p$ ; for each  $i < n$  there is a  $\nu_i$ -variety mapping to  $X$ ; and the motivic homology sequence

$$(0.6) \quad H_{-1, -1}(X \times X) \xrightarrow{\pi_0^* - \pi_1^*} H_{-1, -1}(X) \rightarrow H_{-1, -1}(k) \quad (= k^\times).$$

is exact. Part of Theorem 0.7 states that Rost varieties exist for every  $\underline{a}$ .

*Remark 0.6.1.* Rost originally defined a *norm variety* for  $\{\underline{a}\}$  to be a projective splitting variety of dimension  $p^{n-1}$  which is a  $\nu_{n-1}$ -variety. (See [Rost, 10/20/99].) Theorem 0.7(2) says that our definition agrees with Rost's when  $k$  is  $p$ -special.

Here is the statement of Theorem 0.7, quoted from [11, 1.21]. It assumes that the Bloch-Kato conjecture holds for  $n - 1$ .

**Theorem 0.7.** *Let  $n \geq 2$  and  $0 \neq \{\underline{a}\} = \{a_1, \dots, a_n\} \in K_n^M(k)/p$ . Then:*

0) *There exists a geometrically irreducible norm variety for  $\{\underline{a}\}$ .*

*Assume further that  $k$  is  $p$ -special. If  $X$  is a norm variety for  $\{\underline{a}\}$ , then:*

1)  *$X$  is geometrically irreducible.*

2)  *$X$  is a  $\nu_{n-1}$ -variety.*

3) *each element of  $\overline{A}_0(X, \mathcal{K}_1)$  is of the form  $[x, \alpha]$ , where  $x \in X$  is a closed point of degree  $p$  and  $\alpha \in k(x)^\times$ . 0.2 below.)*

The construction of geometrically irreducible norm varieties was carried out in [11, pp. 254–256]; this proves part (0) of Theorem 0.7. Part (1) was proven in [11, 5.4]. Part (2) was proven in [11, 5.2], assuming Rost's Chain Lemma (see 0.1), and part (3) was proven in [11, p. 271], assuming not only the Chain Lemma but also the Norm Principle (see 0.3 above).

As stated in the introduction of [11], the construction of norm varieties and the proof of Theorem 0.7 are part of an inductive proof of the Bloch-Kato conjecture. We point out that in the present paper, the inductive assumption (that the Bloch-Kato conjecture for  $n - 1$  holds) is never used. It only appears in [11] to prove that the candidates for norm varieties constructed there are  $p$ -generic splitting varieties. (However, the Norm Principle 0.3 is itself a statement about norm varieties.) In particular, the Chain Lemma 0.1 holds in all degrees independently of the Bloch-Kato conjecture.

## 1. FORMS ON VECTOR BUNDLES

We begin with a presentation of some well known facts about  $p$ -forms.

If  $V$  is a vector space over a field  $k$ , a  $p$ -form on  $V$  is a symmetric  $p$ -linear function on  $V$ , *i.e.*, a linear map  $\phi : \text{Sym}^p(V) \rightarrow k$ . It determines a  $p$ -ary form, *i.e.*, a function  $\varphi : V \rightarrow k$  satisfying  $\varphi(\lambda v) = \lambda^p \varphi(v)$ , by  $\varphi(v) = \phi(v, v, \dots, v)$ . If  $p!$  is invertible in  $k$ ,  $p$ -linear forms are in 1–1 correspondence with  $p$ -ary forms.

If  $V = k$  then every  $p$ -form may be written as  $\varphi(\lambda) = a\lambda^p$  or  $\phi(\lambda_1, \dots) = a \prod \lambda_i$  for some  $a \in k$ . Up to isometry, non-zero 1-dimensional  $p$ -forms are in 1–1 correspondence with elements of  $k^\times/k^{\times p}$ . Therefore an  $n$ -tuple of forms  $\varphi_i$  determine a well-defined element of  $K_n^M(k)/p$  which we write as  $\{\varphi_1, \dots, \varphi_n\}$ .

Of course the notion of a  $p$ -form on a projective module over a commutative ring makes sense, but it is a special case of  $p$ -forms on locally free modules (algebraic vector bundles), which we now define.

**Definition 1.1.** If  $\mathcal{E}$  is a locally free  $\mathcal{O}_X$ -module over a scheme  $X$  then a  $p$ -form on  $\mathcal{E}$  is a symmetric  $p$ -linear function on  $\mathcal{E}$ , *i.e.*, a linear map  $\phi : \text{Sym}^p(\mathcal{E}) \rightarrow \mathcal{O}_X$ . If  $\mathcal{E}$  is invertible, we will sometimes identify the  $p$ -form with the diagonal  $p$ -ary form  $\varphi = \phi \circ \Delta : \mathcal{E} \rightarrow \mathcal{O}_X$ ; locally, if  $v$  is a section generating  $\mathcal{E}$  then the form is determined by  $a = \varphi(v)$ :  $\varphi(tv) = at^p$ .

*Remark 1.1.1.* The geometric vector bundle over a scheme  $X$  whose sheaf of sections is  $\mathcal{E}$  is  $\mathbb{V} = \mathbf{Spec}(S^*(\mathcal{E}^\vee))$ , where  $\mathcal{E}^\vee$  is the dual  $\mathcal{O}_X$ -module of  $\mathcal{E}$ . We will sometimes describe  $p$ -forms in terms of  $\mathbb{V}$ .

The projective space bundle associated to  $\mathcal{E}$  is  $\pi : \mathbb{P}(\mathcal{E}) = \mathbf{Proj}(S^*) \rightarrow X$ ,  $S^* = S^*(\mathcal{E}^\vee)$ . The tautological line bundle on  $\mathbb{P}(\mathcal{E})$  is  $\mathbb{L} = \mathbf{Spec}(\mathrm{Sym} \mathcal{O}(1))$ , and its sheaf of sections is  $\mathcal{O}(-1)$ . The multiplication  $S^* \otimes \mathcal{E}^\vee \rightarrow S^*(1)$  in the symmetric algebra induces a surjection of locally free sheaves  $\pi^*(\mathcal{E}^\vee) \rightarrow \mathcal{O}(1)$  and hence an injection  $\mathcal{O}(-1) \rightarrow \pi^*(\mathcal{E})$ ; this yields a canonical morphism  $\mathbb{L} \rightarrow \pi^*(\mathbb{V})$  of the associated geometric vector bundles.

**Definition 1.2.** Any  $p$ -form  $\psi : \mathrm{Sym}^p(\mathcal{E}) \rightarrow \mathcal{O}_X$  on  $\mathcal{E}$  induces a canonical  $p$ -form  $\epsilon$  on the tautological line bundle  $\mathbb{L}$ :

$$\epsilon : \mathcal{O}(-p) = \mathrm{Sym}^p(\mathcal{O}(-1)) \rightarrow \mathrm{Sym}^p(\pi^*\mathcal{E}) = \pi^*\mathrm{Sym}^p(\mathcal{E}) \xrightarrow{\psi} \pi^*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}(\mathcal{E})}.$$

We will use the following notational shorthand. For a scheme  $Z$ , a point  $q$  on some  $Z$ -scheme and a vector bundle  $V$  on  $Z$  we write  $V|_q$  for the fiber of  $V$  at  $q$ , i.e., the  $k(q)$  vector space  $q^*(V)$  for  $q \rightarrow Z$ . If  $\varphi$  is a  $p$ -form on a line bundle  $L$ ,  $0 \neq u \in L|_q$  and  $a = \varphi|_q(u^p)$ , then  $\varphi|_q : (L|_q)^p \rightarrow k(q)$  is the  $p$ -form  $\varphi|_q(tu^p) = at^p$ .

*Example 1.3.* Given an invertible sheaf  $L$  on  $X$ , and a  $p$ -form  $\varphi$  on  $L$ , the bundle  $V = \mathcal{O} \oplus L$  has the  $p$ -form  $\psi(t, u) = t^p - \varphi(u)$ . Then  $\mathbb{P}(V) \rightarrow X$  is a  $\mathbb{P}^1$ -bundle, and its tautological line bundle  $\mathbb{L}$  has the  $p$ -form  $\epsilon$  described in 1.2.

Over a point in  $\mathbb{P}(V)$  of the form  $\infty = (0 : u)$ , the  $p$ -form on  $\mathbb{L}|_\infty$  is  $\epsilon(0, \lambda u) = -\lambda^p \varphi(u)$ . If  $q = (1 : u)$  is any other point on  $\mathbb{P}(V)$  then the 1-dimensional subspace  $\mathbb{L}|_q$  of the vector space  $V|_q$  is generated by  $v = (1, u)$  and the  $p$ -form  $\epsilon|_q$  on  $\mathbb{L}|_q$  is determined by  $\epsilon(v) = \psi(1, u) = 1 - \varphi(u)$  in the sense that  $\epsilon(\lambda v) = \lambda^p(1 - \varphi(u))$ .

One application of these ideas is the formation of the sheaf of Kummer algebras associated to a  $p$ -form. Recall that if  $L$  is a line bundle then the  $(p-1)$ st symmetric power of  $\mathbb{P}(\mathcal{O} \oplus L)$  is  $\mathrm{Sym}^{p-1}\mathbb{P}(\mathcal{O} \oplus L) = \mathbb{P}(\mathcal{A}(L))$ , where  $\mathcal{A}(L) = \bigoplus_{i=0}^{p-1} L^{\otimes i}$ .

**Definition 1.4.** If  $L$  is a line bundle on  $X$ , equipped with a  $p$ -form  $\phi$ , the *Kummer algebra*  $\mathcal{A}_\phi(L)$  is the vector bundle  $\mathcal{A}(L) = \bigoplus_{i=0}^{p-1} L^{\otimes i}$  regarded as a bundle of algebras as in [11, 3.11]; locally, if  $u$  is a section generating  $L$  then  $\mathcal{A}(L) \cong \mathcal{O}[u]/(u^p - \phi(u))$ . If  $x \in X$  and  $a = \phi|_x(u)$  then the  $k(x)$ -algebra  $\mathcal{A}|_x$  is the Kummer algebra  $k(x)(\sqrt[p]{a})$ , which is a field if  $a \notin k(x)^p$  and  $\prod k(x)$  otherwise.

Since the norm on  $\mathcal{A}_\phi(L)$  is given by a homogeneous polynomial of degree  $p$ , we may regard the norm as a map from  $\mathrm{Sym}^p \mathcal{A}_\phi(L)$  to  $\mathcal{O}$ . The canonical  $p$ -form  $\epsilon$  on the tautological line bundle  $\mathbb{L}$  on the projective bundle  $\mathbb{P} = \mathbb{P}(\mathcal{A}(L))$ , given in 1.2, agrees with the natural  $p$ -form:

$$\mathbb{L}^{\otimes p} \rightarrow \mathrm{Sym}^p \pi^* \mathcal{A}(L) \xrightarrow{N} \mathcal{O}_{\mathbb{P}},$$

where  $\pi : \mathbb{P} \rightarrow X$  is the structure map and the canonical inclusion of  $\mathbb{L}$  into  $\pi^*(\mathcal{A}(L)) = \bigoplus_{i=0}^{p-1} \pi^* L^{\otimes i}$  induces the first map.

Recall from 1.2 and 1.4 that  $\phi$  is a  $p$ -form on  $L$ ,  $\psi = (1, -\phi)$  is a  $p$ -form on  $\mathcal{O} \oplus L$  and  $\epsilon$  is the canonical  $p$ -form on  $\mathbb{L}$  induced from  $\psi$ .

**Lemma 1.5.** *Suppose that  $x \in X$  has  $\phi|_x \neq 0$  and that  $0 \neq u \in L|_x$ . Then  $\epsilon|_{(0:u)} \neq 0$ . Moreover,  $\phi(u) \in k(x)^{\times p}$  iff there is a point  $\ell \in \mathbb{P}(\mathcal{O} \oplus L)$  over  $x$  so that  $\epsilon|_\ell = 0$ .*

*Proof.* Let  $w = (t, su)$  be a point of  $\mathbb{L}|_x$  over  $\ell = (t : su) \in \mathbb{P}(\mathcal{O} \oplus L)|_x$ . If  $t = 0$  then  $\ell = (0 : u)$  and  $\epsilon(w) = -s^p \phi(u)$ , which is nonzero for  $s \neq 0$ . If  $t \neq 0$  then  $\epsilon|_\ell$  is determined by the scalar  $\epsilon(w) = \psi(t, su) = t^p - s^p \phi(u)$ . Thus  $\epsilon|_\ell = 0$  iff  $\phi(u) = (t/s)^p$ .  $\square$

*Remark 1.5.1.* Here is an alternative proof, using the Kummer algebra  $K = k(x)(a)$ ,  $a = \sqrt[p]{\phi(u)}$ . Since  $\epsilon(w) = \psi(t, su)$  is the norm of the nonzero element  $t - sa$  in  $K$ , the norm  $\epsilon(w)$  is zero iff the Kummer algebra is split, i.e.,  $\phi(u) = a^p \in k(x)^{\times p}$ .

Finally, the notation  $\{\gamma, \dots, \gamma'_{n-1}\}$  in the Chain Lemma 0.1 is a special case of the notation in the following definition.

**Definition 1.6.** Given line bundles  $H_1, \dots, H_n$  on  $X$ ,  $p$ -forms  $\alpha_i$  on  $H_i$ , and a point  $x \in X$  at which each form  $\alpha_i|_x$  is nonzero, we write  $\{\alpha_1, \dots, \alpha_n\}_x$  for the element  $\{\alpha_1|_x, \dots, \alpha_n|_x\}$  of  $K_n^M(k(x))/p$  described before 1.1: if  $u_i$  is a generator of  $H_i|_x$  and  $\alpha_i|_x(u_i) = a_i$  then  $\{\alpha_1, \dots, \alpha_n\}_x = \{a_1, \dots, a_n\}$ .

We record the following useful consequence of this construction. Recall that if  $(R, \mathfrak{m})$  is a regular local ring with quotient field  $F$ , then any regular sequence  $r_1, \dots$  generating  $\mathfrak{m}$  determines a specialization map  $K_*^M(F) \rightarrow K_*^M(R/\mathfrak{m})$ ; if  $a_1, \dots$  are units of  $R$ , any of these specializations sends  $\{a_1, \dots\}$  to  $\{\bar{a}_1, \dots\}$ .

**Lemma 1.7.** *Suppose that the  $p$ -forms  $\alpha_i$  are all nonzero at the generic point  $\eta$  of a smooth  $X$ . On the open subset  $U$  of  $X$  of points  $x$  on which each  $\alpha_i|_x \neq 0$ , the symbol  $\{\alpha_1|_x, \dots, \alpha_n|_x\}$  in  $K_n^M(k(x))/p$  is obtained by specialization from the symbol in  $K_n^M(k(X))/p$ .*

## 2. THE CHAIN LEMMA WHEN $n = 2$ .

The goal of this section is to construct certain iterated projective bundles together with line bundles and  $p$ -forms on them as needed in the case  $n = 2$  of the Chain Lemma 0.1. Our presentation is based upon Rost's lectures [Rost].

We begin with a generic construction, which starts with a pair  $K_0, K_{-1}$  of line bundles on a variety  $X_0 = X_{-1}$  and produces a tower of varieties  $X_r$ , equipped with distinguished line bundles  $K_r$ . Each  $X_r$  is a product of  $p - 1$  projective line bundles over  $X_{r-1}$ , so  $X_r$  has relative dimension  $r(p - 1)$  over  $X_0$ .

**Definition 2.1.** Given a morphism  $f_{r-1} : X_{r-1} \rightarrow X_{r-2}$  and line bundles  $K_{r-1}$  on  $X_{r-1}$ ,  $K_{r-2}$  on  $X_{r-2}$ , we form the projective line bundle  $\mathbb{P}(\mathcal{O} \oplus K_{r-1})$  over  $X_{r-1}$  and its tautological line bundle  $\mathbb{L}$ . By definition,  $X_r$  is the product  $\prod_1^{p-1} \mathbb{P}(\mathcal{O} \oplus K_{r-1})$  over  $X_{r-1}$ . Writing  $f_r$  for the projection  $X_r \rightarrow X_{r-1}$ , and  $\mathbb{L}_r$  for the external product  $\mathbb{L} \boxtimes \dots \boxtimes \mathbb{L}$  on  $X_r$ , we define the line bundle  $K_r$  on  $X_r$  to be  $K_r = (f_r \circ f_{r-1})^*(K_{r-2}) \otimes \mathbb{L}_r$ .

$$X_r \xrightarrow{f_r} X_{r-1} \xrightarrow{f_{r-1}} X_{r-2} \cdots X_1 \xrightarrow{f_1} X_0 = X_{-1}.$$

*Example 2.2 ( $k$ -tower).* The  $k$ -tower is the tower obtained when we start with  $X_0 = \text{Spec}(k)$ , using the trivial line bundles  $K_{-1}, K_0$ . Note that  $X_1 = \prod \mathbb{P}^1$  and  $K_1 = \mathbb{L}_1$ , while  $X_2$  is a product of projective line bundles over  $\prod \mathbb{P}^1$ , and  $K_2 = \mathbb{L}_2$ .

In the Chain Lemma (Theorem 0.1) for  $n = 2$  we have  $S = X_p$  in the  $k$ -tower, and the line bundles are  $J = J_1 = K_p$ ,  $J'_1 = f_p^*(K_{p-1})$ . Before defining the  $p$ -forms  $\gamma_1$  and  $\gamma'_1$  in 2.7, we quickly establish 2.6; this verifies part (6) of Theorem 0.1, that the degree of  $c_1(K_p)^{p^2-p}$  is prime to  $p$ .

If  $L$  is a line bundle over  $X$ , and  $\lambda = c_1(L)$ , the Chow ring of  $\mathbb{P} = \mathbb{P}(\mathcal{O} \oplus L)$  is  $CH(\mathbb{P}) = CH(X)[z]/(z^2 - \lambda z)$ , where  $z = c_1(\mathbb{L})$ . If  $\pi : \mathbb{P} \rightarrow X$  then  $\pi_*(z) = -1$

in  $CH(X)$ . Applying this observation to the construction of  $X_r$  out of  $X = X_{r-1}$  with  $\lambda_{r-1} = c_1(K_{r-1})$ , we have

$$CH(X_r) = CH(X_{r-1})[z_{r,1}, \dots, z_{r,p-1}] / (\{z_{r,j}^2 - \lambda_{r-1}z_{r,j} \mid j = 1, \dots, p-1\}),$$

where  $z_{r,j}$  is the first Chern class of the  $j$ th tautological line bundle  $\mathbb{L}$ . (Formally,  $CH(X_{r-1})$  is identified with a subring of  $CH(X_r)$  via the pullback of cycles.) By induction on  $r$ , this yields the following result:

**Lemma 2.3.**  *$CH^*(X_r)$  is a free  $CH^*(X_0)$ -module. A basis consists of the monomials  $\prod z_{i,j}^{e_{i,j}}$  for  $e_{i,j} \in \{0, 1\}$ ,  $0 < i \leq r$  and  $0 < j < p$ . As a graded algebra,  $CH^*(X_r)/p \cong CH^*(X_0)/p \otimes_{R_0} R_r$ , where  $R_0 = \mathbb{F}_p[\lambda_0, \lambda_{-1}]$  and*

$$R_r = \mathbb{F}_p[\lambda_{-1}, \lambda_0, \dots, \lambda_r, z_{1,1}, \dots, z_{r,p-1}] / I_r,$$

$$I_r = (\{z_{i,j}^2 - \lambda_{i-1}z_{i,j} \mid 1 \leq i \leq r, 0 < j < p\}, \{\lambda_i - \lambda_{i-2} - \sum_{j=1}^{p-1} z_{i,j} \mid 1 \leq i \leq r\}).$$

**Definition 2.4.** For  $r = 1, \dots, p$ , set  $z_r = \sum_{j=1}^{p-1} z_{r,j}$  and  $\zeta_r = \prod z_{r,j}$ . It follows from Lemma 2.3 that  $\lambda_i = \lambda_{i-2} + z_i$  and  $z_i^p = \sum z_{r,j}^p = \sum z_{r,j} \lambda_{r-1}^{p-1} = z_i \lambda_{i-1}^{p-1}$  in  $R_r$  and hence in  $CH(X_r)/p$ .

By Lemma 2.3, if  $1 \leq r \leq p$  then multiplication by  $\prod \zeta_i \in CH^{r(p-1)}(X_r)$  is an isomorphism  $CH_0(X_0)/p \xrightarrow{\sim} CH_0(X_r)/p$ . If  $X_0 = \text{Spec}(k)$  then  $CH_0(X_r)/p \cong \mathbb{F}_p$ , and is generated by  $\prod \zeta_i$ .

**Lemma 2.5.** *If  $y \in CH_0(X_0)$ , the degree of  $y \cdot \zeta_1 \cdots \zeta_r$  is  $(-1)^{r(p-1)} \deg(y)$ .*

*Proof.* The degree on  $X_r$  is the composition of the  $(f_i)_*$ . The projection formula implies that  $(f_r)_*(\zeta_r) = (-1)^{p-1}$ , and

$$(f_r)_*(y \cdot \zeta_1 \cdots \zeta_r) = (y \cdot \zeta_1 \cdots \zeta_{r-1}) \cdot (f_r)_*(\zeta_r) = (-1)^{p-1} y \cdot \zeta_1 \cdots \zeta_{r-1}.$$

Hence the result follows by induction on  $r$ .  $\square$

**Proposition 2.6.** *For every 0-cycle  $y$  on  $X_0$  and  $1 \leq r \leq p$ ,  $\lambda_r = c_1(K_r)$  satisfies  $y \lambda_r^{r(p-1)} \equiv y \zeta_1 \cdots \zeta_r$  in  $CH_0(X_r)/p$ , and  $\deg(y \lambda_r^{r(p-1)}) \equiv \deg(y) \pmod{p}$ .*

*For the  $k$ -tower 2.2 (with  $y = 1$ ), we have  $\deg(\lambda_p^{p^2-p}) \equiv 1 \pmod{p}$ .*

*Proof.* If  $r = 1$  this follows from  $y \lambda_{-1} = y \lambda_0 = 0$  in  $CH(X_0)$ :  $\lambda_1 = z_1 + \lambda_{-1}$  and  $y \cdot \zeta_1 \equiv y \lambda_1^{p-1}$ . For  $r \geq 2$ , we have  $\lambda_r = z_r + \lambda_{r-2}$  and  $z_r^p = z_r \lambda_{r-1}^{p-1}$  by 2.4. Because  $p - r \geq 0$ , we have

$$\begin{aligned} \lambda_r^{r(p-1)} &= (z_r + \lambda_{r-2})^{p(r-1) + (p-r)} \equiv (z_r^p + \lambda_{r-2}^p)^{r-1} \cdot (z_r + \lambda_{r-2})^{p-r} \pmod{p} \\ &= (z_r \lambda_{r-1}^{p-1} + \lambda_{r-2}^p)^{r-1} (z_r + \lambda_{r-2})^{p-r} \equiv \zeta_r \lambda_{r-1}^{(r-1)(p-1)} + T \pmod{p}, \end{aligned}$$

where  $T \in CH(X_{r-1})[z_r]$  is a homogeneous polynomial of total degree  $< p-1$  in  $z_r$ .

By 2.3, the coefficients of  $yT$  are elements of  $CH(X_{r-1})$  of degree  $> \dim(X_{r-1})$ , so  $yT$  must be zero. Then by the inductive hypothesis,

$$y \lambda_r^{r(p-1)} \equiv y \zeta_r \lambda_{r-1}^{(r-1)(p-1)} \equiv y \zeta_r \cdot (\zeta_1 \cdots \zeta_{r-1})$$

in  $CH^*(X_r)/p$ , as claimed. Now the degree assertion follows from Lemma 2.5.  $\square$

**The  $p$ -forms.** We now turn to the  $p$ -forms in the Chain Lemma 0.1, using the  $k$ -tower 2.2. We will inductively equip the line bundles  $\mathbb{L}_r$  and  $K_r$  of 2.2 with  $p$ -forms  $\Psi_r$  and  $\varphi_r$ ; the  $\gamma_1$  and  $\gamma'_1$  of the Chain Lemma 0.1 will be  $\varphi_p$  and  $\varphi_{p-1}$ .

When  $r = 0$ , we equip the trivial line bundles  $K_{-1}, K_0$  on  $X_0 = \text{Spec}(k)$  with the  $p$ -forms  $\varphi_{-1}(t) = a_2 t^p$  and  $\varphi_0(t) = a_1 t^p$ . The  $p$ -form  $\varphi_{r-1}$  on  $K_{r-1}$  induces a  $p$ -form  $\psi(t, u) = t^p - \varphi_{r-1}(u)$  on  $\mathcal{O} \oplus K_{r-1}$  and a  $p$ -form  $\epsilon$  on the tautological line bundle  $\mathbb{L}$ , as in Example 1.3. As observed in Example 1.3, at the point  $q = (1 : x)$  of  $\mathbb{P}(\mathcal{O} \oplus K_{r-1})$  we have  $\epsilon(y) = \psi(1, x) = 1 - \varphi_{r-1}(x)$ .

**Definition 2.7.** The  $p$ -form  $\Psi_r$  on  $\mathbb{L}_r$  is the product form  $\prod \epsilon$ :

$$\Psi_r(y_1 \boxtimes \cdots \boxtimes y_{p-1}) = \prod \epsilon(y_i).$$

The  $p$ -form  $\varphi_r$  on  $K_r = (f_{r-1} \circ f_r)^*(K_{r-2}) \otimes \mathbb{L}_r$  is defined to be

$$\varphi_r = (f_{r-1} \circ f_r)^*(\varphi_{r-2}) \otimes \Psi_r.$$

**Proposition 2.8.** *Let  $x = (x_1, \dots, x_{p-1}) \in X_r$  be a point with residue field  $E = k(x)$ . For  $-1 \leq i \leq r$ , choose generators  $u_i$  and  $v_i$  for the one-dimensional  $E$  vector spaces  $K_i|_x$  and  $\mathbb{L}_i|_x$  respectively, in such a way that  $u_i = u_{i-2} \otimes v_i$ .*

- (1) *If  $\varphi_i|_x = 0$  for some  $1 \leq i \leq r$  then  $\{a_1, a_2\}_E = 0 \in K_2(E)/p$ .*
- (2) *If  $\varphi_i|_x \neq 0$  for all  $i$ ,  $1 \leq i \leq r$ , then*

$$\{a_1, a_2\}_E = (-1)^r \{\varphi_{r-1}(u_{r-1}), \varphi_r(u_r)\} \in K_2(E)/p.$$

*Proof.* By induction on  $r$ . Both parts are obvious if  $r = 0$ . To prove the first part, we may assume that  $\varphi_i|_x \neq 0$  for  $1 \leq i \leq r-1$ , but  $\varphi_r|_x = 0$ . We have  $u_r = u_{r-2} \otimes v_r$  and by the definition of  $\varphi_r$ , we conclude that

$$0 = \varphi_r(u_r) = \varphi_{r-2}(u_{r-2})\Psi_r(v_r),$$

whence  $\Psi_r(v_r) = 0$ . Now the element  $v_r \neq 0$  is a tensor product of sections  $w_j$  and  $\Psi_r(v_r) = \prod \epsilon(w_j)$  so  $\epsilon(w_j) = 0$  for a nonzero section  $w_j$  of  $\mathbb{L}|_{x_j}$ . By Lemma 1.5,  $\varphi_{r-1}(u_{r-1})$  is a  $p$ th power in  $E$ . Consequently,  $\{\varphi_{r-2}(u_{r-2}), \varphi_{r-1}(u_{r-1})\}_E = 0$  in  $K_2(E)/p$ . This symbol equals  $\pm\{a_1, a_2\}_E$  in  $K_2(E)/p$ , by (2) and induction. This finishes the proof of the first assertion.

For the second claim, we can assume by induction that

$$\{a_1, a_2\}_E = \pm\{\varphi_{r-2}(u_{r-2}), \varphi_{r-1}(u_{r-1})\}_E.$$

Now  $\varphi_r(u_r) = \varphi_{r-2}(u_{r-2})\Psi_r(v_r)$ . Letting  $K = k(x)(\alpha)$ ,  $\alpha = \sqrt[p]{\varphi_{r-1}(u_{r-1})}$ , we have  $N_{K/k(x)}(t - s\alpha) = t^p - s\varphi_{r-1}u_{r-1} = \epsilon(t, s u_{r-1})$ , and hence  $\Psi_r(v_r) = N_{K/k(x)}(v')$  for some  $v' \in K$ . But  $\{\varphi_{r-1}(u_{r-1}), N_{K/k(x)}(v')\} = 0$  by Lemma 2.9 below. We conclude that

$$\{\varphi_{r-2}(u_{r-2}), \varphi_{r-1}(u_{r-1})\}_E \equiv -\{\varphi_{r-1}(u_{r-1}), \varphi_r(u_r)\}_E \pmod{p};$$

this concludes the proof of the second assertion.  $\square$

**Lemma 2.9.** *For any field  $k$  any  $a \in k^\times$  and any  $b$  in  $K_a = k[\sqrt[p]{a}]$ , the symbol  $\{a, N_{K_a/k}(b)\}$  is trivial in  $K_2(k)/p$ .*

*Proof.* Because  $\{a, b\} = p\{\sqrt[p]{a}, b\}$  vanishes in  $K_2(K_a)/p$ , we have  $\{a, N(b)\} = N\{a, b\} = pN(\{\sqrt[p]{a}, b\}) = 0$ .  $\square$

**Proof of the Chain Lemma 0.1 for  $n = 2$ .** We verify the conditions for the variety  $S = X_p$  in the  $k$ -tower 2.2; the line bundles  $J = J_1 = K_p$ ,  $J'_1 = f_p^*(K_{p-1})$ ; the  $p$ -forms  $\gamma_1$  and  $\gamma'_1$  in 0.1 are the forms  $\varphi_p$  and  $\varphi_{p-1}$  of 2.7. Part (1) of Theorem 0.1 is immediate from the construction of  $S = X_p$ ; parts (2) and (4) were proven in Proposition 2.8; parts (3) and (5) follow from (2) and (4); and part (6) is Proposition 2.6 with  $y = 1$ .  $\square$

### NORM PRINCIPLE FOR $n = 2$

The Norm Principle for  $n = 2$  was implicit in the Merkurjev-Suslin paper [4, 4.3]. We reproduce their short proof, which uses the Severi-Brauer variety  $X$  of the cyclic division algebra  $D = A_\zeta(a, b)$  attached to a nontrivial symbol  $\{a, b\}$  in  $K_2(k)/p$  and a  $p$ th root of unity  $\zeta$ ;  $X$  is a norm variety for the symbol  $\{a, b\}$ .

**Theorem 2.10** (Norm Principle for  $n = 2$ ). *If  $x \in X$  and  $[k(x) : k] = p^m$  for  $m > 1$  then for all  $\lambda \in k(x)$  there exists  $x' \in X$  and  $\lambda' \in k(x')$  so that  $[k(x') : k] \leq p$  and  $[x, \lambda] = [x', \lambda']$  in  $\overline{A}_0(X, \mathcal{K}_1)$ .*

*Proof.* By Merkurjev-Suslin [4, 8.7.2],  $N : \overline{A}_0(X, \mathcal{K}_1) \rightarrow k^\times$  is an injection with image  $\text{Nrd}(D) \subseteq k^\times$ . Therefore the unit  $N([x, \lambda])$  of  $k$  can be written as the reduced norm of an element  $\lambda' \in D$ . The subfield  $E = k(\lambda')$  of  $D$  has degree  $\leq p$ , and corresponds to a point  $x' \in X$ . Since  $N([x', \lambda']) = \text{Nrd}(\lambda') = N([x, \lambda])$ , we have  $[x, \lambda] = [x', \lambda']$  in  $\overline{A}_0(X, \mathcal{K}_1)$ .  $\square$

### 3. THE SYMBOL CHAIN

Here is the pattern of the chain lemma in all weights.

We start with a sequence  $a_1, a_2, \dots$  of units of  $k$ , and the function  $\Phi_0(t) = t^p$ . For  $r \geq 1$ , we inductively define functions  $\Phi_r$  in  $p^r$  variables and  $\Psi_r$  in  $p^r - p^{r-1}$  variables, taking values in  $k$ , and prove (in 3.4) that  $\{a_1, \dots, a_r, \Phi_r(\mathbf{x})\} \equiv 0 \pmod{p}$ . Note that  $\Phi_r$  and  $\Psi_r$  depend only upon the units  $a_1, \dots, a_r$ . We write  $\mathbf{x}_i$  for a sequence of  $p^r$  variables  $x_{ij}$  (where  $j = (j_1, \dots, j_r)$  and  $0 \leq j_i < p$ ), and we inductively define

$$(3.1) \quad \Psi_{r+1}(\mathbf{x}_1, \dots, \mathbf{x}_{p-1}) = \prod_{i=1}^{p-1} [1 - a_{r+1} \Phi_r(\mathbf{x}_i)],$$

$$(3.2) \quad \Phi_{r+1}(\mathbf{x}_0, \dots, \mathbf{x}_{p-1}) = \Phi_r(\mathbf{x}_0) \Psi_{r+1}(\mathbf{x}_1, \dots, \mathbf{x}_{p-1}).$$

We say that two rational functions are *birationally equivalent* if they can be transformed into one another by an automorphism (over the base field  $k$ ) of the field of rational functions.

*Example 3.3.*  $\Psi_1(x_1, \dots, x_{p-1}) = \prod(1 - a_1 x_i^p)$  and  $\Phi_1(x_0, \dots, x_{p-1})$  is  $x_0^p \prod(1 - a_1 x_i^p)$ , the norm of the element  $x_0 \prod(1 - x_i \alpha_1)$  in the Kummer extension  $k(\mathbf{x})(\alpha_1)$ ,  $\alpha_1 = \sqrt[p]{a_1}$ . Thus  $\Phi_1$  is birationally equivalent to symmetrizing in the  $x_i$ , followed by the norm from  $k[\sqrt[p]{a_1}]$  to  $k$ . More generally,  $\Psi_r(\mathbf{x}_1, \dots, \mathbf{x}_{p-1})$  is the product of norms of elements in Kummer extensions  $k(\mathbf{x}_1, \dots, \mathbf{x}_{p-1})(\sqrt[p]{b_i})$  of  $k(\mathbf{x}_1, \dots, \mathbf{x}_{p-1})$ .

*Example 3.3.1.* It is useful to interpret the map  $\Phi_1$  geometrically. Let  $R_{k(\alpha)/k} \mathbb{A}^1$  denote the variety, isomorphic to  $\mathbb{A}^p$ , which is the Weil restriction ([16]) of the affine line over  $k(\alpha)$ , so that there is a morphism  $N : R_{k(\alpha)/k} \mathbb{A}^1 \rightarrow \mathbb{A}^1$  corresponding to the norm map. The function  $k^p \rightarrow k(\alpha)$  defined by

$$(x_0, s_1, \dots, s_{p-1}) \mapsto x_0(1 - s_1 \alpha + s_2 \alpha^2 - \dots \pm s_{p-1} \alpha^{p-1})$$

induces a birational map  $\mathbb{A}^p \xrightarrow{m} R_{k(\alpha)/k}\mathbb{A}^1$ . Finally, let  $q : \mathbb{A}^{p-1} \rightarrow \mathbb{A}^{p-1}/\Sigma_{p-1} \cong \mathbb{A}^{p-1}$  be the symmetrizing map sending  $(x_1, \dots)$  to the elementary symmetric functions  $(s_1, \dots)$ . Then the following diagram commutes:

$$\begin{array}{ccc} \mathbb{A}^p = \mathbb{A}^1 \times \mathbb{A}^{p-1} & \xrightarrow{1 \times q} & \mathbb{A}^1 \times \mathbb{A}^{p-1} \xrightarrow{\text{birat.}} R_{k(\alpha)/k}\mathbb{A}^1 = \mathbb{A}^p \\ & \searrow \Phi_1 & \downarrow N \\ & & \mathbb{A}^1. \end{array}$$

*Remark 3.3.2.* If  $p = 2$ ,  $\Phi_1(x_0, x_1) = x_0^2(1 - a_1x_1^2)$  is birationally equivalent to the norm form  $u^2 - a_1v^2$  for  $k(\sqrt{a_1})/k$ , and  $\Phi_2 = \Phi_1(\mathbf{x}_0)[1 - a_2\Phi_1(\mathbf{x}_1)]$  is birationally equivalent to the norm form  $\langle\langle a_1, a_2 \rangle\rangle = (u^2 - a_1v^2)[1 - a_2(w^2 - a_1t^2)]$  for the quaternionic algebra  $A_{-1}(a_1, a_2)$ .

More generally,  $\Phi_n$  is birationally equivalent to the Pfister form

$$\langle\langle a_1, \dots, a_r \rangle\rangle = \langle\langle a_1, \dots, a_{r-1} \rangle\rangle \perp a_r \langle\langle a_1, \dots, a_{r-1} \rangle\rangle$$

and  $\Psi_r$  is equivalent to the restriction of the Pfister form to the subspace defined by the equations  $\mathbf{x}_0 = (1, \dots, 1)$ .

*Remark 3.3.3* (Rost [9]). Suppose that  $p = 3$ . Then  $\Phi_2$  is birationally equivalent to (symmetrizing, followed by) the reduced norm of the algebra  $A_\zeta(a_1, a_2)$  and  $\Phi_3$  is equivalent to the norm form of the exceptional Jordan algebra  $J(a_1, a_2, a_3)$ . When  $r = 4$ , Rost showed that the set of nonzero values of  $\Phi_4$  is a subgroup of  $k^\times$ .

For the next lemma, it is useful to introduce the function field  $F_r$  over  $k$  in the  $p^r$  variables  $x_{j_1, \dots, j_r}$ ,  $0 \leq j_t < p$ . Note that  $F_r$  is isomorphic to the tensor product of  $p$  copies of  $F_{r-1}$ .

**Lemma 3.4.**  $\{a_1, \dots, a_r, \Phi_r(\mathbf{x})\} = \{a_1, \dots, a_r, \Psi_r(\mathbf{x})\} = 0 \in K_{r+1}^M(F_r)/p$ .

If  $b \in k$  is a nonzero value of  $\Phi_r$ , then  $\{a_1, \dots, a_r, b\} = 0 \in K_{r+1}^M(k)/p$ .

*Proof.* By Lemma 2.9,  $\{a_r, \Psi_r(\mathbf{x})\} = 0$  because  $\Psi_r(\mathbf{x})$  is a product of norms of elements of  $k(\mathbf{x})(\sqrt[p]{b_i})$  by 3.3. If  $r = 1$  then  $\{a_1, \Phi_1(\mathbf{x})\} = \{a_1, x_0^p\} \equiv 0$  as well. The result for  $F_r$  follows by induction:

$$\{a_1, \dots, a_{r+1}, \Phi_{r+1}(\mathbf{x})\} = \{a_1, \dots, a_{r+1}, \Phi_r(\mathbf{x}_0)\}\{a_1, \dots, a_{r+1}, \Psi_{r+1}(\mathbf{x})\} = 0.$$

The result for  $b$  follows from the first assertion, and specialization from  $F_r$  to  $k$ , using the regular local ring at the point  $\mathbf{c}$  where  $\Phi_r(\mathbf{c}) = b$ .  $\square$

*Remark 3.5.* For any value  $b \in k^\times$  of  $\Phi_n$ , any desingularization  $X$  of the projective closure of the affine hypersurface  $X_b = \{\mathbf{x} : \Phi_n(\mathbf{x}) = b\}$  is a norm variety for the symbol  $\{a_1, \dots, a_n, b\}$  in  $K_{n+1}^M(k)/p$ . Note that  $\dim(X_b) = p^n - 1$ .

Indeed, we see from Lemma 3.4 that every affine point of  $X_b$  splits the symbol. In particular, the generic point of  $X_b$  is a splitting field for this symbol. By specialization, every point of  $X_b$  and  $X$  splits the symbol.

The symmetric group  $\Sigma_{p-1}$  acts on  $\{\mathbf{x}_1, \dots, \mathbf{x}_{p-1}\}$  and fixes  $\Phi_n$ , so it acts on  $X_b$ . It is easy to see that  $X_b/\Sigma_{p-1}$  is birationally isomorphic to the norm variety constructed in [11, §2] using the hypersurface  $W$  defined by  $N = b$  in the vector bundle of *loc. cit.* By [11, 1.19],  $X$  is also a norm variety.

The Chain Lemma is based upon the observation that certain manipulations (or “moves”) of Milnor symbols do not change the class in  $K_n^M(k)/p$ . Here is the class

of moves we will model geometrically in section 4; strings of these moves will be used in section 5 to prove the Chain Lemma.

**Definition 3.6.** A move of type  $C_n$  on a sequence  $a_1, \dots, a_n$  in  $k^\times$  is a transformation of the kind:

$$\text{Type } C_n : \quad (a_1, \dots, a_n) \mapsto (a_1, \dots, a_{n-2}, a_n \Psi_{n-1}(\mathbf{x}), a_{n-1}).$$

Here  $\Psi_{n-1}$  is a function of  $p^{n-1} - p^{n-2}$  new variables  $\mathbf{x}_i = \{\mathbf{x}_{1,1}, \dots, \mathbf{x}_{1,p-1}\}$ .

By Lemma 3.4,  $\{a_1, \dots, a_n\} = -\{a_1, \dots, a_{n-2}, a_n \Psi_{n-1}(\mathbf{x}), a_{n-1}\}$ , so the move does not change the symbol in  $K_n^M(k)/p$ . If we do this move  $p$  times, always with a new set of variables  $\mathbf{x}_i$ , we obtain a move  $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_{n-2}, \gamma_{n-1}, \gamma'_{n-1})$  in which  $\gamma_{n-1}, \gamma'_{n-1}$  are functions of  $p^n - p^{n-1}$  variables  $\mathbf{x}_{i,j}$ ,  $1 \leq i \leq p$ ,  $1 \leq j < p$ .

Since these moves do not change the symbol, we have

$$(3.7) \quad \{a_1, \dots, a_n\} = \{a_1, \dots, a_{n-2}, \gamma_{n-1}, \gamma'_{n-1}\}$$

in  $K_n^M(k)/p$ . The functions  $\gamma_{n-1}$  and  $\gamma'_{n-1}$  in (3.7) are the ones appearing in the Chain Lemma 0.1.

Formally, if  $k(\mathbf{x}_1)$  is the function field of the move of type  $C_n$ , then the function field  $F'_n$  of the move (3.7) is the tensor product  $k(\mathbf{x}_1) \otimes \dots \otimes k(\mathbf{x}_p)$ . We will define a variety  $S_{n-1}$  with function field  $F'_n$ .

Using  $p^{n-1} - p^{n-2}$  more variables  $\mathbf{x}'_{i,j}$  ( $1 \leq i < p$ ,  $1 \leq j \leq p$ ) we do  $p$  moves of type  $C_{n-1}$  on  $(a_1, \dots, a_{n-2}, \gamma_{n-1})$  to get the sequence  $(a_1, \dots, a_{n-3}, \gamma_{n-2}, \gamma'_{n-2}, \gamma'_{n-1})$ . The function field of this move is  $F'_{n-1} \otimes F'_n$ , and we will define a variety  $S_{n-2}$  with this function field, together with a morphism  $S_{n-2} \rightarrow S_{n-1}$ .

Next, apply  $p$  moves of type  $C_{n-2}$ , then  $p$  moves of type  $C_{n-3}$ , and so on, ending with  $p$  moves of type  $C_2$ . We have the sequence  $(\gamma_1, \gamma'_1, \gamma'_2, \dots, \gamma'_{n-1})$  in  $p^n - p$  variables  $\mathbf{x}_1, \dots, \mathbf{x}_{p-1}$ . Moreover, we see from Lemma 3.4 that

$$(3.8) \quad \{a_1, \dots, a_n\} = \{\gamma_1, \gamma'_1, \gamma'_2, \dots, \gamma'_{n-1}\} \quad \text{in } K_n^M(k).$$

The net effect will be to construct a tower

$$(3.9) \quad S = S_1 \xrightarrow{f_r} S_2 \longrightarrow \dots \longrightarrow S_{n-2} \longrightarrow S_{n-1} \longrightarrow S_n = \text{Spec}(k).$$

Let  $S$  be any variety containing  $U = \mathbb{A}^{p^n - p}$  as an affine open, so that  $k(S) = k(\mathbf{x}_1, \dots, \mathbf{x}_{p-1})$ , each  $\mathbf{x}_i$  is  $p^{n-1}$  variables  $x_{i,j}$  and all line bundles on  $U$  are trivial. Then parts (1) and (2) of the Chain Lemma 0.1 are immediate from (3.7) and (3.8).

Now the only thing left to do is to construct  $S = S_1$ , extend the line bundles (and forms) from  $U$  to  $S$ , and prove parts (4) and (6) of 0.1.

#### 4. MODEL $P_{n-1}$ FOR MOVES OF TYPE $C_n$

In this section, we construct a tower of varieties  $P_r$  and  $Q_r$  over a fixed base scheme  $S'$ , with  $p$ -forms on lines bundles over them, which will produce a model of the forms  $\Psi_r$  and  $\Phi_r$  in (3.1) and (3.2). This tower, depicted in (4.0), is defined in 4.2 below.

$$(4.0) \quad P_{n-1} \rightarrow \dots \rightarrow P_r \longrightarrow Q_{r-1} \rightarrow P_{r-1} \longrightarrow \dots \rightarrow Q_1 \rightarrow P_1 \longrightarrow Q_0 = S'$$

The passage from  $S'$  to the variety  $P_{n-1}$  is a model for the moves of type  $C_n$  defined in 3.6.

**Definition 4.1.** Let  $X$  be a variety over some fixed base  $S'$ . Given line bundles  $K, L$  on  $X$ , we can form the vector bundle  $V = \mathcal{O} \oplus L$ , the  $\mathbb{P}^1$ -bundle  $\mathbb{P}(V)$  over  $X$ , and  $\mathbb{L}$ . Taking products over  $S'$ , set

$$P = \prod_{\mathbb{1}}^{p-1} \mathbb{P}(\mathcal{O} \oplus L); \quad Q = X \times_{S'} P$$

On  $P$  and  $Q$ , we have the external products of the tautological line bundles:

$$\mathbb{L}(1, \dots, 1) = \mathbb{L} \boxtimes \mathbb{L} \boxtimes \dots \boxtimes \mathbb{L} \text{ on } P, \quad K \boxtimes \mathbb{L}(1, \dots, 1) \text{ on } Q.$$

Given  $p$ -forms  $\varphi$  and  $\phi$  on  $K$  and  $L$ , respectively, the line bundle  $\mathbb{L}$  has the  $p$ -form  $\epsilon$ , as in Example 1.3, and the line bundles  $\mathbb{L}(1, \dots, 1)$  and  $K \boxtimes \mathbb{L}(1, \dots, 1)$  are equipped with the product  $p$ -forms  $\Psi = \prod \epsilon$  and  $\Phi = \varphi \otimes \Psi$ .

*Remark 4.1.1.* Let  $x = (x_1, \dots, x_{p-1})$  denote the generic point of  $X^{p-1}$ . The function fields of  $P$  and  $Q$  are  $k(P) = k(x)(y_1, \dots, y_{p-1})$  and  $k(Q) = k(x_0) \otimes k(P)$ . We may represent the generic point of  $P$  in coordinate form as a  $(p-1)$ -tuple  $\{(1 : y_i)\}$ , where the  $y_i$  generate  $L$  over  $x_i$ . Then  $y = \{(1, y_i)\}$  is a generator of  $\mathbb{L}(1, \dots, 1)$  at the generic point, and  $\Psi(y) = \prod(1 - \phi(y_i))$  by 1.3. If  $v_0$  is a generator of  $K$  at the generic point  $x_0$  of  $X$ , then  $\Phi(y) = \varphi(v_0)\Psi(y)$ .

*Example 4.1.2.* An important special case arises when we begin with two line bundles  $H$  on  $S'$ ,  $K$  on  $X$ , with  $p$ -forms  $\alpha$  and  $\varphi$ . In this case, we set  $L = H \otimes K$  and equip it with the product form  $\phi(u \otimes v) = \alpha(u)\varphi(v)$ . At the generic point  $q$  of  $Q$  we can pick a generator  $u \in H|_q$  and set  $y_i = u \otimes v_i$ ; the forms resemble the forms of (3.1) and (3.2):

$$\Psi(y) = \prod(1 - \alpha(u)\varphi(v_i)), \quad \Phi(y) = \varphi(v_0)\Psi(y).$$

*Remark 4.1.3.* Suppose a group  $G$  acts on  $S', X, K$  and  $L$ , and  $K_0, L_0$  are nontrivial 1-dimensional representations so that at every fixed point  $x$  of  $X$  (a)  $k(x) = k$ , (b)  $L_x \cong L_0$ . Then  $G$  acts on  $P$  (resp.,  $Q$ ) with  $2^{p-1}$  fixed points  $y$  over each fixed point of  $X^{p-1}$  (resp., of  $X^p$ ), each with  $k(y) = k$ , and each fiber of  $\mathbb{L} = \mathbb{L}(1, \dots, 1)$  (resp.,  $K \boxtimes \mathbb{L}$ ) is the representation  $L_0^j$  (resp.,  $K_0 \otimes L_0^j$ ) for some  $j$  ( $0 \leq j < p$ ). Indeed,  $G$  acts nontrivially on each term  $\mathbb{P}^1$  of the fiber  $\prod \mathbb{P}^1$ , so that the fixed points in the fiber are the points  $(y_1, \dots, y_{p-1})$  with each  $y_i$  either  $(0 : 1)$  or  $(1 : 0)$ .

We now define the tower (4.0) of  $P_r$  and  $Q_r$  over a fixed base  $S'$ , by induction on  $r$ . We start with line bundles  $H_1, \dots, H_r$ , and  $K_0 = \mathcal{O}_{S'}$  on  $S'$ , and set  $Q_0 = S'$ .

**Definition 4.2.** Given a variety  $Q_{r-1}$  and a line bundle  $K_{r-1}$  on  $Q_{r-1}$ , we form the varieties  $P_r = P$  and  $Q_r = Q$  using the construction in Definition 4.1, with  $X = Q_{r-1}$ ,  $K = K_{r-1}$  and  $L = H_r \otimes K_{r-1}$  as in 4.1.2. To emphasize that  $P_r$  only depends upon  $S'$  and  $H_1, \dots, H_r$ , we will sometimes write  $P_r(S'; H_1, \dots, H_r)$ . As in 4.1,  $P_r$  has the line bundle  $\mathbb{L}(1, \dots, 1)$ , and  $Q_r$  has the line bundle  $K_r = K_{r-1} \boxtimes \mathbb{L}(1, \dots, 1)$ .

Suppose that we are given  $p$ -forms  $\alpha_i \neq 0$  on  $H_i$ , and we set  $\Phi_0(t) = t^p$  on  $K_0$ . Inductively, the line bundle  $K_{r-1}$  on  $Q_{r-1}$  is equipped with a  $p$ -form  $\Phi_{r-1}$ . As described in 4.1 and 4.1.2, the line bundle  $\mathbb{L}(1, \dots, 1)$  on  $P_r$  obtains a  $p$ -form  $\Psi_r$  from the  $p$ -form  $\alpha_r \otimes \Phi_{r-1}$  on  $L = H_r \otimes K_{r-1}$ , and  $K_r$  obtains a  $p$ -form  $\Phi_r = \Phi_{r-1} \otimes \Psi_r$ .

*Example 4.2.1.*  $Q_1 = P_1$  is  $\prod_1^{p-1} \mathbb{P}^1(\mathcal{O} \oplus H_1)$  over  $S'$ , equipped with the line bundle  $K_1 = \mathbb{L}(1, \dots, 1)$ . If  $H_1$  is a trivial bundle with  $p$ -form  $\alpha_1(t) = a_1 t^p$  then  $\Phi_1$  is the  $p$ -form  $\Phi_1$  of Example 3.3.

$P_2$  is  $\prod_1^{p-1} \mathbb{P}^1(\mathcal{O} \oplus H_2 \otimes K_1)$  over  $Q_1^{p-1}$ , and  $K_2 = K_1 \boxtimes \mathbb{L}(1, \dots, 1)$ .

**Lemma 4.3.** *If  $r > 0$  then  $\dim(P_r/S') = (p^r - p^{r-1})$  and  $\dim(Q_r/S') = p^r - 1$ .*

*Proof.* Set  $d_r = \dim(Q_r/S')$ . This follows easily by induction from the formulas  $\dim(P_{r+1}/S') = (p-1)(d_r+1)$ ,  $\dim(Q_{r+1}/S') = p(d_r+1) - 1$ .  $\square$

Choosing generators  $u_i$  for  $H_i$  at the generic point of  $S'$ , we get units  $a_i = \alpha_i(u_i)$ .

**Lemma 4.4.** *At the generic points of  $P_r$  and  $Q_r$ , the  $p$ -forms  $\Psi_n$  and  $\Phi_n$  of 4.2 agree with the forms defined in (3.1) and (3.2).*

*Proof.* This follows by induction on  $r$ , using the analysis of 4.1.2. Given a point  $q = (q_1, \dots, q_p)$  of  $Q_{r-1}^{p-1}$  and a point  $\{(1 : y_i)\}$  on  $P_r$  over it,  $y = \{(1, y_i)\}$  is a nonzero point on  $\mathbb{L}(1, \dots, 1)$  and  $y_i = 1 \otimes v_i$  for a section  $v_i$  of  $K_{r-1}$ . Since  $\epsilon(1, y_i) = 1 - a_r \Phi_{r-1}(v_i)$  and  $\Psi_r(y) = \prod \epsilon(1, y_i)$ , the forms  $\Psi_r$  agree. Similarly, if  $v_0$  is the generator of  $K_{r-1}$  over the generic point  $q_0$  then  $y' = v_0 \otimes y$  is a generator of  $K_r$  and

$$\Phi_r(y') = \Phi_{r-1}(v_0) \Psi_r(y),$$

which is also in agreement with the formula in (3.2).  $\square$

Recall that  $K_0$  is the trivial line bundle, and that  $\Phi_0$  is the standard  $p$ -form  $\Phi_0(v) = v^p$  on  $K_0$ . Every point of  $P_r = \prod \mathbb{P}(\mathcal{O} \oplus L)$  has the form  $w = (w_1, \dots, w_{p-1})$ , and the projection  $P_r \rightarrow \prod Q_{r-1}$  sends  $w \in P_r$  to a point  $x = (x_1, \dots, x_{p-1})$ .

**Proposition 4.5.** *Let  $s \in S'$  be a point such that  $a_1|_s, \dots, a_r|_s \neq 0$ .*

1. *If  $\Psi_r|_w = 0$  for some  $w \in P_r$ , then  $\{a_1, \dots, a_r\}$  vanishes in  $K_r^M(k(w))/p$ .*
2. *If  $\Phi_r|_q = 0$  for some  $q = (x_0, w) \in Q_r$ ,  $\{a_1, \dots, a_r\}$  vanishes in  $K_r^M(k(q))/p$ .*

*Proof.* Since  $\Phi_r = \Phi_{r-1} \otimes \Psi_r$ , the assumption that  $\Psi_r|_w = 0$  implies that  $\Phi_r|_q = 0$  for any  $x_0 \in Q_{r-1}$  over  $s$ . Conversely, if  $\Phi_r|_q = 0$  then either  $\Psi_r|_w = 0$  or  $\Phi_{r-1}|_{x_0} = 0$ . Since  $\Phi_0 \neq 0$ , we may proceed by induction on  $r$  and assume that  $\Phi_{r-1}|_{x_j} \neq 0$  for each  $j$ , so that  $\Phi_r|_q = 0$  is equivalent to  $\Psi_r|_w = 0$ .

By construction, the  $p$ -form on  $L = H_r \otimes K_{r-1}$  is  $\phi(u_r \otimes v) = a_r \Phi_{r-1}(v)$ , where  $u_r$  generates the vector space  $H_r|_s$  and  $v$  is a section of  $K_{r-1}$ . Since  $\Psi_r|_w$  is the product of the forms  $\epsilon|_{w_j}$ , some  $\epsilon|_{w_j} = 0$ . Lemma 1.5 implies that  $a_r \Phi_{r-1}(v)$  is a  $p$ th power in  $k(x_j)$ , and hence in  $k(w)$ , for any generator  $v$  of  $K_{r-1}|_{x_j}$ . By Lemma 3.4,  $\{a_1, \dots, a_{r-1}, \Phi_{r-1}\} = 0$  and hence

$$\{a_1, \dots, a_r\} = \{a_1, \dots, a_{r-1}, a_r \Phi_{r-1}\} = 0$$

in  $K_r^M(k(w))/p$ , as claimed.  $\square$

We conclude this section with some identities in  $CH(P_n)/pCH(P_n)$ , given in 4.8. To simplify the statements and proofs below, we write  $\text{ch}(X)$  for  $CH(X)/pCH(X)$ , and adopt the following notation.

**Definition 4.6.** Set  $\eta = c_1(H_n) \in \text{ch}^1(S')$ , and  $\gamma = c_1(\mathbb{L}(1, \dots, 1)) \in \text{ch}^1(P_n)$ . Writing  $\mathbb{P}$  for the bundle  $\mathbb{P}(\mathcal{O} \oplus H_n \otimes K_{n-1})$  over  $Q_{n-1}$ , let  $c \in \text{ch}(\mathbb{P})$  denote  $c_1(\mathbb{L})$  and let  $\kappa \in \text{ch}(Q_{n-1})$  denote  $c_1(K_{n-1})$ . We write  $c_j, \kappa_j \in \text{ch}(P_n)$  for the images of  $c$  and  $\kappa$  under the  $j$ th coordinate pullbacks  $\text{ch}(Q_{n-1}) \rightarrow \text{ch}(\mathbb{P}) \rightarrow \text{ch}(P_n)$ .

**Lemma 4.7.** *Suppose that  $H_1, \dots, H_{n-1}$  are trivial. Then*

- (a)  $\gamma^{p^n} = \gamma^{p^{n-1}} \eta^d$  in  $\text{ch}(P_n)$ , where  $d = p^n - p^{n-1}$ ;
- (b) If in addition  $H_n$  is trivial, then  $\gamma^d = -\prod c_j \kappa_j^e$ , where  $e = p^{n-1} - 1$ .
- (c) If  $S' = \text{Spec } k$  then the zero-cycles  $\kappa^e \in \text{ch}_0(Q_{n-1})$  and  $\gamma^d \in \text{ch}_0(P_n)$  have

$$\deg(\kappa^e) \equiv (-1)^{n-1} \quad \text{and} \quad \deg(\gamma^d) \equiv -1 \quad \text{modulo } p.$$

*Proof.* First note that because  $K_{n-1}$  is defined over the  $e$ -dimensional variety  $Q_{n-1}(\text{Spec } k; H_1, \dots, H_{n-1})$ , the element  $\kappa = c_1(K_{n-1})$  satisfies  $\kappa^{p^{n-1}} = 0$ . Thus  $(\eta + \kappa)^{p^{n-1}} = \eta^{p^{n-1}}$  and hence  $(\eta + \kappa)^d = \eta^d$ . Now the element  $c = c_1(\mathbb{L})$  satisfies the relation  $c^2 = c(\eta + \kappa)$  in  $\text{ch}(\mathbb{P})$  and hence

$$c^{p^n} = c^{p^{n-1}}(\eta + \kappa)^d = c^{p^{n-1}} \eta^d$$

in  $\text{ch}^{p^n}(\mathbb{P})$ . Now recall that  $P_n = \prod \mathbb{P}$ . Then  $\gamma = \sum c_j$  and

$$\gamma^{p^n} = \sum c_j^{p^n} = \sum c_j^{p^{n-1}} \eta^d = \gamma^{p^{n-1}} \eta^d.$$

When  $H_n$  is trivial we have  $\eta = 0$  and hence  $c^2 = c\kappa$ . Setting  $b_j = c_j^{p^{n-1}} = c_j \kappa_j^e$ , we have  $\gamma^d = \gamma^{p^{n-1}(p-1)} = (\sum b_j)^{p-1}$ . To evaluate this, we use the algebra trick that since  $b_j^2 = 0$  for all  $j$  and  $p = 0$  we have  $(\sum b_j)^{p-1} = (p-1)! \prod b_j = -\prod b_j$ .

For (c), note that if  $S' = \text{Spec } k$  then  $\eta = 0$  and  $\gamma^d$  is a zero-cycle on  $P_n$ . By the projection formula for  $\pi : P_n \rightarrow \prod Q_{n-1}$ , part (b) yields  $\pi_* \gamma^d = (-1)^p \prod \kappa_j^e$ . Since each  $Q_{n-1}$  is an iterated projective space bundle,  $CH(\prod Q_{n-1}) = \otimes_1^{p-1} CH(Q_{n-1})$ , and the degree of  $\prod \kappa_j^e$  is the product of the degrees of the  $\kappa_j^e$ . By induction on  $n$ , these degrees are all the same, and nonzero, so  $\deg(\prod \kappa_j^e) \equiv 1 \pmod{p}$ .

It remains to establish the inductive formula for  $\deg(\kappa^e)$ . Since it is clear for  $n = 0$ , and the  $Q_i$  are projective space bundles, it suffices to compute that  $c_1(K_n)^{p^{n-1}} = \kappa^e \gamma^d$  in  $\text{ch}(Q_n) = \text{ch}(Q_{n-1}) \otimes \text{ch}(P_n)$ . Since  $\kappa^{e+1} = 0$  and  $c_1(K_n) = \kappa + \gamma$  we have

$$c_1(K_n)^{p^{n-1}} = \kappa^{e+1} + \gamma^{p^{n-1}} = \gamma^{p^{n-1}},$$

and hence  $c_1(K_n)^d = \gamma^d$ . Since  $\gamma^{d+1} = 0$ , this yields the desired calculation:

$$c_1(K_n)^{p^{n-1}} = c_1(K_n)^e c_1(K_n)^d = (\kappa + \gamma)^e \gamma^d = \kappa^e \gamma^d. \quad \square$$

**Corollary 4.8.** *There is a ring homomorphism  $\mathbb{F}_p[\lambda, z]/(z^p - \lambda^{p-1}z) \rightarrow \text{ch}(P_n)$ , sending  $\lambda$  to  $\eta^{p^{n-1}}$  and  $z$  to  $\gamma^{p^{n-1}}$ .*

## 5. MODEL FOR $p$ MOVES

In this section we construct maps  $S_{n-1} \rightarrow S_n$  which model the  $p$  moves of type  $C_n$  defined in 3.6. Each such move introduces  $p^{n-1} - p^{n-2}$  new variables, and will be modelled by a map  $Y_r \rightarrow Y_{r-1}$  of relative dimension  $p^{n-1} - p^{n-2}$ , using the  $P_{n-1}$  construction in 4.2. The result (Definition 5.1) will be a tower of the form:

$$\begin{array}{ccccccc} J_{n-1} = L_p & & L_{p-1} & & L_2 & & L_1 & & L_0 = J_n \\ S_{n-1} = Y_p & \xrightarrow{f_p} & Y_{p-1} & \longrightarrow \cdots \longrightarrow & Y_2 & \xrightarrow{f_2} & Y_1 & \xrightarrow{f_1} & Y_0 = S_n. \end{array}$$

Fix  $n \geq 2$ , a variety  $S_n$ , and line bundles  $H_1, \dots, H_{n-2}, H_n$  and  $J_n = H_{n-1}$  on  $S_n$ . The first step in the tower is to form  $Y_0 = S_n$  and  $Y_1 = P_{n-1}(S_n; H_1, \dots, H_{n-2}, J_n)$ , with line bundles  $L_0 = J_n$  and  $L_1 = H_n \otimes \mathbb{L}(1, \dots, 1)$  as in 4.2. In forming the

other  $Y_r$ , the base in the  $P_{n-1}$  construction 4.2 will become  $Y_{r-1}$  and only the final line bundle will change (from  $J_n$  to  $L_{r-1}$ ). Here is the formal definition.

**Definition 5.1.** For  $r > 1$ , we define morphisms  $f_r : Y_r \rightarrow Y_{r-1}$  and line bundles  $\mathbb{L}_r^{\boxtimes}$  and  $L_r$  on  $Y_r$  as follows. Inductively, we are given a morphism  $f_{r-1} : Y_{r-1} \rightarrow Y_{r-2}$  and line bundles  $L_{r-1}$  on  $Y_{r-1}$ ,  $L_{r-2}$  on  $Y_{r-2}$ . Set  $\mathbb{L}_r^{\boxtimes} = \mathbb{L}(1, \dots, 1)$ ,

$$Y_r = P_{n-1}(Y_{r-1}; H_1, \dots, H_{n-2}, L_{r-1}) \xrightarrow{f_r} Y_{r-1}, \quad L_r = f_r^* f_{r-1}^*(L_{r-2}) \otimes \mathbb{L}_r^{\boxtimes}.$$

Finally, we write  $S_{n-1}$  for  $Y_p$  and set  $J_{n-1} = L_p$ ,  $J'_{n-1} = f_p^*(L_{p-1})$ . By Lemma 4.3,  $\dim(Y_r/Y_{r-1}) = p^{n-1} - p^{n-2}$  and hence  $\dim(S_{n-1}/S_n) = p^n - p^{n-1}$ .

For example, when  $n = 2$  and  $H_1$  is trivial, this tower is exactly the tower of 2.1: we have  $Y_r = P_1(Y_{r-1}; L_{r-1}) = \prod \mathbb{P}^1(\mathcal{O} \oplus L_{r-1})$ .

*Remark 5.1.1.* The line bundles  $J_{n-1}$  and  $J'_{n-1}$  will be the line bundles of the Chain Lemma 0.1. The rest of tower (3.9) will be obtained in Definition 5.8 by repeating this construction and setting  $S = S_1$ .

The rest of this section, culminating in Theorem 5.9, is devoted to proving part (6) of the Chain Lemma, that the degree of the zero-cycle  $c_1(J_1)^{\dim S}$  is relatively prime to  $p$ . In preparation, we need to compare the degrees of the zero-cycles  $c_1(J_{n-1})^{\dim S_{n-1}}$  on  $S_{n-1}$  and  $c_1(J_n)^{\dim S_n}$  on  $S_n$ . In order to do so, we introduce the following algebra.

**Definition 5.2.** We define the graded  $\mathbb{F}_p$ -algebra  $A_r$  and  $\bar{A}_r$  by  $\bar{A}_r = A_r/\lambda_{-1}A$  and:

$$A_r = \mathbb{F}_p[\lambda_{-1}, \lambda_0, \dots, \lambda_r, z_1, \dots, z_r]/(\{z_i^p - \lambda_{i-1}^{p-1}z_i, \lambda_i - \lambda_{i-2} - z_i \mid i = 1, \dots, r\}).$$

*Remark 5.2.1.* By Corollary 4.8, there is a homomorphism  $A_p \xrightarrow{\rho} \text{ch}(Y_p)$ , sending  $\lambda_r$  to  $c_1(L_r)^{p^{n-2}}$  and  $z_r$  to  $c_1(\mathbb{L}_r^{\boxtimes})^{p^{n-2}}$ . When  $H_{n-1}$  is trivial,  $\rho$  factors through  $\bar{A}_p$ .

**Lemma 5.3.** In  $\bar{A}_r$ , every element  $u$  of degree 1 satisfies  $u^{p^2} = u^p \lambda_0^{p^2-p}$ .

*Proof.* We will show that  $\bar{A}_r$  embeds into a product of graded rings of the form  $\Lambda_k = \mathbb{F}_p[\lambda_0][v_1, \dots, v_k]/(v_1^p, \dots, v_k^p)$ . In each entry,  $u = a\lambda_0 + v$  with  $v^p = 0$  and  $a \in \mathbb{F}_p$ , so  $u^p = a\lambda_0^p$  and  $u^{p^2} = a\lambda_0^{p^2}$ , whence the result.

Since  $\bar{A}_{r+1} = \bar{A}_r[z]/(z^p - \lambda_r^{p-1}z)$  is flat over  $\bar{A}_r$ , it embeds by induction into a product of graded rings of the form  $\Lambda' = \Lambda_k[z]/(z^p - u^{p-1}z)$ ,  $u \in \Lambda_k$ . If  $u \neq 0$ , there is an embedding of  $\Lambda'$  into  $\prod_{i=0}^{p-1} \Lambda_k$  whose  $i$ th component sends  $z$  to  $iu$ . If  $u = 0$ , then  $\Lambda' \cong \Lambda_{k+1}$ .  $\square$

*Remark 5.3.1.* It follows that if  $m > 0$  and  $(p^2 - p) \mid m$  then  $u^{kp+m} = \lambda_0^m u^{kp}$ .

**Proposition 5.4.** In  $\bar{A}_r$ ,  $\lambda_p^{p^N-p} = \lambda_0^{p^N-p^2} (\prod z_i^{p-1} + T\lambda_0)$ , where  $\deg(T) = p^2 - p - 1$ .

*Proof.* By Definition 5.2,  $\bar{A}_p$  is free over  $\mathbb{F}_p[\lambda_0]$ , with the elements  $\prod z_i^{m_i}$  ( $0 \leq m_i < p$ ) forming a basis. Thus any term of degree  $p^N - p$  is a linear combination of  $F = \lambda_0^{p^N-p^2} \prod z_i^{p-1}$  and terms of the form  $\lambda_0^{m_0} \prod z_i^{m_i}$  where  $\sum m_i = p^N - p^2$  and  $m_0 > p^N - p^2$ . It suffices to determine the coefficient of  $F$  in  $\lambda_p^{p^N-p}$ . Since  $\lambda_p^{p^N-p} = \lambda_0^{p^N-p^2} \lambda_p^{p^2-p}$  by Remark 5.3.1, it suffices to consider  $N = 2$ , when  $F = \prod z_i^{p-1}$ .

As in the proof of Proposition 2.6, if  $p \geq r \geq 2$  we compute in the ring  $\bar{A}_r$  that

$$\begin{aligned} \lambda_r^{r(p-1)} &= (z_r + \lambda_{r-2})^{p(r-1)+(p-r)} = (z_r^p + \lambda_{r-2}^p)^{r-1} \cdot (z_r + \lambda_{r-2})^{p-r} \\ &= (z_r \lambda_{r-1}^{p-1} + \lambda_{r-2}^p)^{r-1} (z_r + \lambda_{r-2})^{p-r} = z_r^{p-1} \lambda_{r-1}^{(r-1)(p-1)} + T, \end{aligned}$$

where  $T \in \bar{A}_{r-1}[z_r]$  is a homogeneous polynomial of total degree  $< p-1$  in  $z_r$ . By induction on  $r$ , the coefficient of  $(z_1 \cdots z_r)^{p-1}$  in  $\lambda_r^{r(p-1)}$  is 1 for all  $r$ .  $\square$

**Lemma 5.5.** *If  $S_n = \text{Spec}(k)$  and  $c = c_1(J_{n-1}) \in CH^1(S_{n-1})$ , then*

$$\deg(c^{\dim S_{n-1}}) \equiv 1 \pmod{p}.$$

*Proof.* Set  $d = \dim(S_{n-1}) = p^n - p^{n-1}$ ; under the map  $A_p \xrightarrow{\rho} \text{ch}(S_{n-1})$  of 5.2.1, the degree  $p^2 - p$  part of  $A_p$  maps to  $CH^d(S_{n-1})$ . In particular, the zero-cycle  $c^d = \rho(\lambda_p)^{p^2-p}$  equals the product of the  $\rho(z_i)^{p-1} = c_1(\mathbb{L}_i^{\boxtimes})^{d/p}$  by Proposition 5.4 (the  $T\lambda_0^*$  term maps to zero for dimensional reasons). Because  $S_{n-1} = Y_p$  is a product of iterated projective space bundles,  $CH_0(Y_p)$  is the tensor product of their  $CH_0$  groups, and the degree of  $c^d$  is the product of the degrees of the  $c_1(\mathbb{L}_i^{\boxtimes})^{d/p}$ , each of which is  $-1$  by Lemma 4.7. It follows that  $\deg(c^d) \equiv 1 \pmod{p}$ .  $\square$

**Theorem 5.6.** *If  $S_n$  has dimension  $p^M - p^n$  and  $H_1, \dots, H_{n-1}$  are trivial then the zero-cycles  $c_1(J_{n-1})^{\dim S_{n-1}} \in CH_0(S_{n-1})$  and  $c_1(J_n)^{\dim S_n} \in CH_0(S_n)$  have the same degree modulo  $p$ :*

$$\deg(c_1(J_{n-1})^{\dim S_{n-1}}) \equiv \deg(c_1(J_n)^{\dim S_n}) \pmod{p}.$$

*Proof.* By 5.2.1, there is a homomorphism  $A_p \xrightarrow{\rho} \text{ch}(S_{n-1})$ , sending  $\lambda_r$  to  $c_1(L_r)^{p^{n-2}}$  and  $z_r$  to  $c_1(\mathbb{L}_r^{\boxtimes})^{p^{n-2}}$ . Because  $H_{n-1}$  is trivial,  $\rho$  factors through  $\bar{A}_p$ .

Set  $N = M - n + 2$  and  $y = \lambda_0^{p^N - p^2}$ , so  $\rho(y) = c_1(J_n)^{\dim S_n} \in \text{ch}_0(S_n)$ . From Proposition 5.4 we have  $\lambda_p^{p^N - p} \equiv y \prod z_i^{p-1}$  modulo  $\ker(\rho)$ . From Lemma 2.5, the degree of this element equals the degree of  $y$  modulo  $p$ .  $\square$

**The  $p$ -forms.** We now define the  $p$ -forms on the line bundles  $J_{n-1}$  and  $J'_{n-1}$  using the tower (5.1). Suppose that the line bundles  $L_{-1} = H_n$  and  $L_0 = J_n$  on  $S_n$  are equipped with the  $p$ -forms  $\beta_{-1}$  and  $\beta_0$ . We endow the line bundle  $L_1$  in Definition 5.1 with the  $p$ -form  $\beta_1 = f^*(\beta_{-1}) \otimes \Psi_{n-1}(\beta_0)$ ; inductively, we endow the line bundle  $L_r$  with the  $p$ -form

$$\beta_r = f^*(\beta_{r-2}) \otimes \Psi_{n-1}(\beta_{r-1}).$$

*Example.* When  $n = 2$  and  $H_1$  is trivial, we saw that the tower 5.1 is exactly the tower of 2.1. In addition, the  $p$ -form  $\beta_r = \Psi_1(\beta_{r-1})$  agrees with the  $p$ -form  $\varphi_r = f^*(\varphi_{r-2}) \otimes \Psi_r$  of 2.7.

**Lemma 5.7.** *If  $\beta_0 = \alpha_{n-1}$  and  $\beta_{-1} = \alpha_n$ , then (at the generic point of  $Y_1$ ) the  $p$ -form  $\beta_p$  agrees with the form  $\alpha_n \Psi_{n-1}$  in (3.6).*

*Proof.* By Lemma 4.4, the form agrees with the form of (3.1).  $\square$

**Definition 5.8.** The tower (3.9) of varieties  $S_i$  is obtained by downward induction, starting with  $S_n = \text{Spec}(k)$  and  $J_n = H_{n-1}$ . Construction 5.1 yields  $S_{n-1}$ ,  $J_{n-1}$  and  $J'_{n-1}$ . Inductively, we repeat construction 5.1 for  $i$ , starting with the output  $S_{i+1}$  and  $J_{i+1}$  of the previous step, to produce  $S_i$ ,  $J_i$  and  $J'_i$ .

By downward induction in the tower (3.9), each  $J_i$  and  $J'_i$  carries a  $p$ -form, which we call  $\gamma_i$  and  $\gamma'_i$ , respectively. By 5.7, these forms agree with the forms  $\gamma_i$  and  $\gamma'_i$  of (3.7) and (3.8).

Since  $\dim(S_i/S_{i-1}) = p^{i+1} - p^i$  we have  $\dim(S_i/S_n) = p^n - p^i$ . Thus if we combine Lemma 5.5 and Theorem 5.6, we obtain the following result.

**Theorem 5.9.** *For each  $i < n$ ,  $\deg(c_1(J_i)^{\dim S_i}) \equiv -1 \pmod{p}$ .*

Theorem 5.9 establishes part (6) of the Chain Lemma 0.1, that  $\deg(c_1(J_1)^{\dim S_1})$  is relatively prime to  $p$ .

*Proof of the Chain Lemma 0.1.* We verify the conditions for the variety  $S = S_1$  in the tower (3.9); the line bundles  $J_i$  and  $J'_i$  and their  $p$ -forms are obtained by pulling back from the bundles and forms defined in 5.8. Part (1) of Theorem 0.1 is immediate from the construction of  $S$ ; part (6) is Theorem 5.9, combined with Lemma 5.5. Part (2) was just established, and part (4) was proven in Proposition 4.5; parts (3) and (5) follow from (2) and (4). This completes the proof of the Chain Lemma.  $\square$

## 6. NICE $G$ -ACTIONS

We will extend the Chain Lemma to include an action by  $G = \mu_p^n$  on  $S$ ,  $J_i$ , and  $J'_i$  leaving  $\gamma_i$  and  $\gamma'_i$  invariant, such that the action is admissible in the following sense. Here  $\mu_p$  is the cyclic group of  $p^{\text{th}}$  roots of unity.

**Definition 6.1.** (Rost, cf. [8, p.2]) Let  $G$  be a group acting on a  $k$ -variety  $X$ . We say that the action is *nice* if  $\text{Fix}_G(X)$  is 0-dimensional, and consists of  $k$ -points.

When  $G$  also acts on a line bundle  $L$  over  $X$ , the action on the geometric bundle  $L$  is *nice* exactly when  $G$  acts nontrivially on  $L|_x$  for every fixed point  $x \in X$ , and in this case  $\text{Fix}_G(L)$  is the zero-section over  $\text{Fix}_G(X)$ .

Suppose that  $G$  acts nicely on each of several line bundles  $L_i$  over  $X$ . We say that  $G$  *acts nicely* on  $\{L_1, \dots, L_r\}$  if for each fixed point  $x \in X$  the image of the canonical representation  $G \rightarrow \prod \text{Aut}(L_i|_x) = \prod k(x)^\times$  is  $\prod G_i$ , with each  $G_i$  nontrivial.

*Remark 6.1.1.* If  $X_i \rightarrow S$  are equivariant maps and the  $X_i$  are nice, then  $G$  also acts nicely on  $X_1 \times_S X_2$ . However, even if  $G$  acts nicely on line bundles  $L_i$  it may not act nicely on  $L_1 \boxtimes L_2$ , because the representation over  $(x_1, x_2)$  is the product representation  $L_1|_{x_1} \otimes L_2|_{x_2}$ .

*Example 6.2.* Suppose that  $G$  acts nicely on a line bundle  $L$  over  $X$ . Then the induced  $G$ -action on  $\mathbb{P} = \mathbb{P}(\mathcal{O} \oplus L)$  and its canonical line bundle  $\mathbb{L}$  is nice. Indeed, if  $x \in X$  is a fixed point then the fixed points of  $\mathbb{P}|_x$  consist of the two  $k$ -points  $\{[\mathcal{O}], [L]\}$ , and if  $L|_x$  is the representation  $\rho$  then  $G$  acts on  $\mathbb{L}$  at these fixed points as  $\rho$  and  $\rho^{-1}$ , respectively.

By 6.1.1,  $G$  also acts nicely on the products  $P = \prod \mathbb{P}(\mathcal{O} \oplus L)$  and  $Q = X \times_{S'} P$  of Definition 4.1, but it may not act nicely on  $\mathbb{L}(1, \dots, 1)$ .

*Example 6.3.* If a  $p$ -group  $G$  acts nicely on  $L$ , it also acts nicely on the projective space  $\mathbb{P}(\mathcal{A})$  of the Kummer algebra bundle  $\mathcal{A} = \mathcal{A}(L)$  of 1.4. Indeed, an elementary calculation shows that  $\text{Fix}_G \mathbb{P}(\mathcal{A})$  consists of the  $p$  sections  $[L^i]$ ,  $0 \leq i < p$  over  $\text{Fix}_G(X)$ . In each fiber, the (vertical) tangent space at each fixed point is the representation  $\rho \oplus \dots \oplus \rho^{p-1}$ . If  $G = \mu_p$ , this is the reduced regular representation.

Over any fixed point  $x \in X$ ,  $L|_x$  is trivial, and the cyclic group of order  $p$  acts on the bundle  $\mathcal{A}|_x$  by  $L^i \mapsto L^{i+1}$ , rotating the fixed points. This induces  $G$ -isomorphisms between the tangent spaces at these points.

*Example 6.3.1.* The action of  $G$  on  $Y = \mathbb{P}(\mathcal{O} \oplus \mathcal{A})$  is not nice. In this case, an elementary calculation shows that  $\text{Fix}_G(Y)$  consists of the points  $[L^i]$  of  $\mathbb{P}(\mathcal{A})$ ,  $0 < i < p$ , together with the projective line  $\mathbb{P}(\mathcal{O} \oplus \mathcal{O})$  over every fixed point  $x$  of  $X$ . For each  $x$ , the (vertical) tangent space at  $[L^i]$  is  $1 \oplus \rho \oplus \cdots \oplus \rho^{p-1}$ ; if  $G = \mu_p$ , this is the regular representation.

When  $G = \mu_p^n$ , the following lemma allows us to assume that the action on  $L|_x$  is induced by the standard representation  $\mu_p \subset k^\times$ , via a projection  $G \rightarrow \mu_p$ .

**Lemma 6.4.** *Any nontrivial 1-dimensional representation  $\rho$  of  $G = \mu_p^n$  factors as the composition of a projection  $G \rightarrow \mu_p$  with the standard representation of  $\mu_p$ .*

*Proof.* The representation  $\rho$  is a nonzero element of  $(\mathbb{Z}/p)^n = G^* = \text{Hom}(\mu_p^n, \mathbb{G}_m)$ , and  $\pi$  is the Pontryagin dual of the induced map  $\mathbb{Z}/p \rightarrow G^*$  sending 1 to  $\rho$ .  $\square$

The construction of the  $P_r$  and  $Q_r$  in 4.2 is natural in the given line bundles  $H_1, \dots, H_n$  over  $S'$ , and so is the construction of the  $Y_r$ ,  $S_r$  and  $S$  in 5.1 and 5.8. Since  $\prod_{i=1}^n \text{Aut}(H_i)$  acts on the  $H_i$ , this group (and any subgroup) will act on the variety  $S$  of the Chain Lemma. We will show that it acts nicely on  $S$ .

Recall from Definition 4.2 that  $P_r$  and  $Q_r$  are defined by the construction 4.1 using the line bundle  $L_r = H_r \otimes K_{r-1}$  over  $Q_{r-1}$ .

**Lemma 6.5.** *If  $S' = \text{Spec}(k)$ , then  $G = \mu_p^r$  acts nicely on  $L_r$ ,  $P_r$  and  $Q_r$ .*

This implies that any subgroup of  $\prod_{i=1}^r \text{Aut}(H_i)$  containing  $\mu_p^r$  also acts nicely.

*Proof.* We proceed by induction on  $r$ , the case  $r = 1$  being 6.2, so we may assume that  $\mu_p^{r-1}$  acts nicely on  $Q_{r-1}$ . By 6.1.1, it suffices to show that  $G = \mu_p^r$  acts nicely on  $\mathbb{P}(\mathcal{O} \oplus L_r)$ , where  $L_r = H_r \otimes K_{r-1}$ . Since the final component  $\mu_p$  of  $G$  acts trivially on  $K_{r-1}$  and  $Q_{r-1}$  and nontrivially on  $H_r$ ,  $G = \mu_p^{r-1} \times \mu_p$  acts nicely on  $L_r$ . By Example 6.2,  $G$  acts nicely on  $\mathbb{P}(\mathcal{O} \oplus L_r)$ .  $\square$

The proof of Lemma 6.5 goes through in slightly greater generality.

**Corollary 6.6.** *Suppose that  $G = \mu_p^n$  acts nicely on  $S'$  and on the line bundles  $\{H_1, \dots, H_r\}$  over it. Then  $G$  acts nicely on  $L_r$ ,  $P_r$  and  $Q_r$ .*

*Proof.* Without loss of generality, we may replace  $S'$  by a fixed point  $s \in S'$ , in which case  $G$  acts nicely on  $\{H_1, \dots, H_r\}$  through the surjection  $\mu_p^n \rightarrow \mu_p^r$ . Now we are in the situation of Lemma 6.5.  $\square$

*Example 6.6.1.* Since  $\mu_p^{n-1}$  acts nicely on  $Y = P_{n-1}(S'; H_1, \dots, H_{n-1})$  and on the bundle  $K_{n-1}$ , while  $\mu_p$  of  $G = \mu_p^n$  acts solely on  $H_n$ , it follows that the group  $\mu_p^n = \mu_p^{n-1} \times \mu_p$  acts nicely on  $\{H_1, \dots, H_{n-1}, H_n \otimes \mathbb{L}(1, \dots, 1)\}$  over  $Y$ .

We can now process the tower of varieties  $Y_r$  defined in 5.1. For notational convenience, we write  $H_{n-1}$  for  $J_n$ . The case  $r = 0$  of the following assertion uses the convention that  $L_0 = H_{n-1}$  and  $L_{-1} = H_n$ .

**Proposition 6.7.** *Suppose that  $G = G_0 \times \mu_p^n$  acts nicely on  $S_n$  and (via  $G \rightarrow \mu_p^n$ ) on  $\{H_1, \dots, H_n\}$ . Then  $G$  acts nicely on each  $Y_r$ , and on its line bundles  $\{H_1, \dots, H_{n-2}, L_r, L_{r-1}\}$ .*

*Proof.* The question being local, we may replace  $S'$  by a fixed point  $s \in S'$ , and  $G$  by  $\mu_p^n$ . We proceed by induction on  $r$ , the case  $r = 1$  being Example 6.6.1, since  $L_1 = H_n \otimes \mathbb{L}(1, \dots, 1)$ . Inductively, suppose that  $G$  acts nicely on  $Y_r$  and on  $\{H_1, \dots, H_{n-2}, L_r, L_{r-1}\}$ . Thus there is a factor of  $G$  isomorphic to  $\mu_p$  which acts nontrivially on  $L_r$  but acts trivially on  $\{H_1, \dots, H_{n-2}, L_r\}$ . Hence this factor acts trivially on  $Y_{r+1} = P_{n-1}(Y_r; H_1, \dots, H_{n-2}, L_r)$  and its line bundle  $\mathbb{L}^{\boxtimes}$ , and nontrivially on  $L_{r+1} = L_{r-1} \otimes \mathbb{L}^{\boxtimes}$ . The assertion follows.  $\square$

**Corollary 6.8.**  $G = \mu_p^n$  acts nicely on  $(S, J)$ .

*Proof.* By Definition 5.1,  $S_{n-1} = Y_p$ ,  $J_{n-1} = L_p$  and  $J'_{n-1} = L_{p-1}$ . By 6.7 with  $r = p$ ,  $G$  acts nicely on  $S_{n-1}$  and on  $\{H_1, \dots, H_{n-2}, J_{n-1}, J'_{n-1}\}$ . By downward induction,  $G = \mu_p^{n-i} \times \mu_p^i$  acts nicely on  $S_i$  and  $\{H_1, \dots, H_{i-1}, J_i, J'_i\}$  for all  $i \leq n$ . The case  $i = 1$  is the conclusion, since  $(S, J) = (S_1, J_1)$ .  $\square$

*Remark 6.8.1.* If  $G = \mu_p^n$  acts nicely on  $S'$ , Rost [8, p.2] would say that a fixed point  $s \in S'$  is *twisting* for  $\{H_1, \dots, H_r\}$  if the map  $G \rightarrow \mu_p^r \subset \prod k(s)^\times = \prod \text{Aut}(H_i|_s)$  is a surjection.

## 7. $G$ -FIXED POINT EQUIVALENCES

Let  $\mathcal{A} = \mathcal{A}(J)$  be the Kummer algebra over the variety  $S$  of the Chain Lemma 0.1, as in 1.4. The group  $G = \mu_p^n$  acts nicely on  $S$  and  $J$  by 6.8, and on  $\mathcal{A}$  and  $\mathbb{P}(\mathcal{A})$  by 6.3. In this section, we introduce two  $G$ -varieties  $\bar{Y}$  and  $Q$ , parametrized by norm conditions, and show that they are  $G$ -fixed point equivalent to  $\mathbb{P}(\mathcal{A})$  and  $\mathbb{P}(\mathcal{A})^p$ , respectively. This will be used in the next section to show that  $\bar{Y}$  is  $G$ -fixed point equivalent to the Weil restriction of  $Q_E$  for any Kummer extension  $E$  of  $k$ .

We begin by defining fixed point equivalence and the variety  $Q$ .

**Definition 7.1.** Let  $G$  be an algebraic group. We say that two  $G$ -varieties  $X$  and  $Y$  are  *$G$ -fixed point equivalent* if  $\text{Fix}_G X$  and  $\text{Fix}_G Y$  are 0-dimensional, lie in the smooth locus of  $X$  and  $Y$ , and there is a separable extension  $K$  of  $k$  and a bijection  $\text{Fix}_G(X_K) \rightarrow \text{Fix}_G(Y_K)$  under which the families of tangent spaces at the fixed points are isomorphic as  $G$ -representations over  $K$ .

**Definition 7.2.** Recall from 1.4 that the norm  $\mathcal{A} \xrightarrow{N} \mathcal{O}_S$  is equivariant, and homogeneous of degree  $p$ . We define the  $G$ -variety  $Q$  over  $S \times \mathbb{A}^1$ , and its fiber  $Q_w$  over  $w \in k$ , by the equation  $N(\beta) = t^p w$ :

$$Q = \{([\beta, t], w) \in \mathbb{P}(\mathcal{A} \oplus \mathcal{O}) \times \mathbb{A}^1 : N(\beta) = t^p w\},$$

$$Q_w = \{[\beta, t] \in \mathbb{P}(\mathcal{A} \oplus \mathcal{O}) : N(\beta) = t^p w\}, \quad \text{for } w \in k.$$

Since  $\dim(S) = p^n - p$  we have  $\dim(Q_w) = p^n - 1$ . If  $w \neq 0$ , then it is proved in [11, §2] that  $Q_w$  is geometrically irreducible and that the open subscheme where  $t \neq 0$  is smooth.

If  $w \neq 0$ ,  $Q_w$  is disjoint from the section  $\sigma : S \cong \mathbb{P}(\mathcal{O}) \rightarrow \mathbb{P}(\mathcal{A} \oplus \mathcal{O})$ ; over each point of  $S$ , the point  $(0 : 1)$  is disjoint from  $Q_w$ . Hence the projection  $\mathbb{P}(\mathcal{A} \oplus \mathcal{O}) - \sigma(S) \rightarrow \mathbb{P}(\mathcal{A})$  from these points induces an equivariant morphism  $\pi : Q_w \rightarrow Y = \mathbb{P}(\mathcal{A})$ ,  $\pi(\beta, t) = \beta$ . This is a cover of degree  $p$  over its image, since  $\pi(\beta, t) = \pi(\beta, \zeta t)$  for all  $\zeta \in \mu_p$ .

**Theorem 7.3.** *If  $w \neq 0$ ,  $G$  acts nicely on  $Q_w$  and  $\text{Fix}_G Q_w \cap (Q_w)_{\text{sing}} = \emptyset$ . Moreover,  $Q_w$  and  $Y = \mathbb{P}(\mathcal{A})$  are  $G$ -fixed point equivalent over the field  $\ell = k(\sqrt[p]{b})$ .*

*Proof.* Since the maps  $Q_w \xrightarrow{\pi} Y \rightarrow S$  are equivariant,  $\pi$  maps  $\text{Fix}_G Q_w$  to  $\text{Fix}_G Y$ , and both lie over the finite set  $\text{Fix}_G S$  of  $k$ -rational points. Since the tangent space  $T_y$  is the product of  $T_s S$  and the tangent space of the fiber  $Y_s$ , and similarly for  $Q_w$ , it suffices to consider a  $G$ -fixed point  $s \in S$ .

By 6.7 and Lemma 6.4,  $G$  acts nontrivially on  $L = J|_s$  via a projection  $G \rightarrow \mu_p$ . By Example 6.3,  $G$  acts nicely on  $\mathbb{P}(\mathcal{A})$ . Thus there is no harm in assuming that  $G = \mu_p$  and that  $L$  is the standard 1-dimensional representation.

Let  $y \in Y$  be a  $G$ -fixed point lying over  $s$ . By 6.2, the tangent space of  $Y|_s$  at  $y$  is the reduced regular representation, and  $y$  is one of  $[1], [L], \dots, [L^{p-1}]$ .

We saw in Example 6.3.1 that a fixed point  $[a_0 : a_1 : \dots : a_{p-1} : t]$  of  $G$  in  $\mathbb{P}(\mathcal{A} \oplus \mathcal{O})|_s$  is either one of the points  $e_i = [\dots : 0 : a_i : 0 \dots : 0]$ , which do not lie on  $Q_w$ , or a point on the projective line  $\{[a_0 : 0 : t]\}$ . By inspection,  $Q_w \otimes_k \ell$  meets the projective line in the  $\ell$ -points  $[\zeta \sqrt[p]{b} : 0 : \dots : 0 : 1]$ ,  $\zeta \in \mu_p$ . Each of these  $p$  points is smooth on  $Q_w$ , and the tangent space (over  $s$ ) is the reduced regular representation of  $G$ .  $\square$

*Remark 7.3.1.* Since  $\pi([\zeta \sqrt[p]{b} : 0 : \dots : 0 : 1]) = [1 : 0 : \dots : 0]$  for all  $\zeta \in \mu_p$ ,  $\text{Fix}_G(Q_w) \xrightarrow{\pi} \text{Fix}_G(Y)$  is *not* a scheme isomorphism over  $\ell$ .

*Remark 7.4.* For any  $w \in k^\times$  of  $N$ , any desingularization  $Q'$  of  $Q_w$  is a smooth, geometrically irreducible splitting variety for the symbol  $\{a_1, \dots, a_n, w\}$  in  $K_{n+1}^M(k)/p$ .

Assuming the Bloch-Kato conjecture for  $n$ , Suslin and Joukhovitski show  $Q'$  is a norm variety in [11, §2]. Note that the variety  $X_w$  of 3.5 is birationally a cover of  $Q_w$ .

To construct  $\bar{Y}$ , we fix a Kummer extension  $E = k(\epsilon)$  of  $k$ . Let  $\mathcal{B}$  be the  $\mathcal{O}_S$ -subbundle  $(\mathcal{A} \otimes 1) \oplus (\mathcal{O}_S \otimes \epsilon)$  of  $\mathcal{A}_E = \mathcal{A} \otimes_k E$  and let  $N_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{O}_S \otimes_k E$  be the map induced by the norm on  $\mathcal{A}_E$ .

**Definition 7.5.** Let  $U$  be the variety  $\mathbb{P}(\mathcal{A}) \times_k \mathbb{P}(\mathcal{B})^{\times(p-1)}$  over  $S^{\times p}$ , and let  $L$  be the line bundle  $\mathbb{L}(\mathcal{A}) \boxtimes \mathbb{L}(\mathcal{B})^{\boxtimes(p-1)}$  over  $U$ , given as the external product of the tautological bundles. The product of the various norms defines an algebraic morphism  $N : L \rightarrow \mathcal{O}_S \otimes E$ .

**Lemma 7.6.** *Let  $u \in U$  be a point over  $(s_0, s_1, \dots, s_{p-1})$ , and write  $A_i$  for the  $k(s_i)$ -algebra  $\mathcal{A}|_{s_i}$ . Then the following hold.*

- (1) *If  $\{\underline{a}\}$  doesn't split at any of the points  $s_0, \dots, s_{p-1}$ , then the norm map  $N : L_u \rightarrow k(u) \otimes E$  is non-zero.*
- (2) *If  $\{\underline{a}\}|_{s_0} \neq 0$  in  $K_n^M(k(s_0))/p$ , then  $A_0$  is a field.*
- (3) *For  $i \geq 1$ , if  $\{\underline{a}\}|_{E(s_i)} \neq 0$  in  $K_n^M(E(s_i))/p$  then  $A_i \otimes E$  is a field.*

*Proof.* The first assertion follows from part (4) of the Chain Lemma 0.1, since by 1.4 the norm on  $L$  is induced from the  $p$ -form  $\gamma_1$  on  $J$ . Assertions (2–3) follow from part (2) of the Chain Lemma, since  $\{\underline{a}\} \neq 0$  implies that  $\gamma$  is nontrivial.  $\square$

**Definition 7.7.** Let  $\mathbb{A}^E$  denote the Weil restriction  $\text{Res}_{E/k} \mathbb{A}^1$ , characterized by  $\mathbb{A}^E(F) = F \otimes_k E$  ([16]). Let  $\bar{Y}$  denote the subvariety of  $\mathbb{P}(L \oplus \mathcal{O}) \times \mathbb{A}^E$  consisting of all points  $([\alpha : t], w)$  such that  $N(\alpha) = t^p w$  in  $E$ . We write  $\bar{Y}_w$  for the fiber over a point  $w \in \mathbb{A}^E$ . Note that  $\dim(\bar{Y}_w) = p^{n+1} - p = p \dim(Q_w)$ .

*Notation 7.8.* Let  $([\alpha : t], w)$  be a  $k$ -rational point on  $\bar{Y}$ , so that  $w \in \mathbb{A}^E(k) = E$ . We may regard  $[\alpha : t] \in \mathbb{P}(L \oplus \mathcal{O})(k)$  as being given by a point  $u \in U(k)$ , lying

over a point  $(s_0, \dots, s_{p-1}) \in S(k)^{\times p}$ , and a nonzero pair  $(\alpha, t) \in L_u \times k$  (up to scalars). From the definition of  $L$ , we see that (up to scalars)  $\alpha$  determines a  $p$ -tuple  $(b_0, b_1 + t_1\epsilon, \dots, b_{p-1} + t_{p-1}\epsilon)$ , where  $b_i \in \mathcal{A}|_{s_i}$  and  $t_i \in k$ . When  $\alpha \neq 0$ ,  $b_0 \neq 0$  and for all  $i > 0$ ,  $b_i \neq 0$  or  $t_i \neq 0$ . Finally, writing  $A_i$  for  $\mathcal{A}|_{s_i}$ , the norm condition says that in  $E$ :

$$N_{A_0/k}(b_0) \prod_{i=1}^{p-1} N_{A_i \otimes E/E}(b_i + t_i\epsilon) = t^p w.$$

If  $k \subseteq F$  is a field extension, then an  $F$ -point of  $\bar{Y}$  is described as above, replacing  $k$  by  $F$  and  $E$  by  $E \otimes_k F$  everywhere.

*Remark 7.8.1.* If  $w \neq 0$ , then  $\alpha \neq 0$ , because  $N(\alpha) = t^p w$  and  $(\alpha, t) \neq (0, 0)$ .

**Lemma 7.9.** *If  $\bar{Y}$  has a  $k$ -point with  $t = 0$  then  $\{\underline{a}\}|_E = 0$  in  $K_n^M(E)/p$ .*

*Proof.* We use the description of a  $k$ -point of  $\bar{Y}$  from 7.8. If  $t = 0$ , then  $\alpha \neq 0$ , therefore  $b_0 \neq 0 \in A_0$  and  $b_i + t_i\epsilon \neq 0 \in A_i \otimes E$ . By Lemma 7.6, if  $\{\underline{a}\}|_E \neq 0$  in  $K_n^M(E)/p$  then  $A_0$  and all the algebras  $A_i \otimes E$  are fields, so that  $N(\alpha) = N_{A_0/k}(b_0) \prod_{i=1}^{p-1} N_{A_i \otimes E/E}(b_i + t_i\epsilon) \neq 0$ , a contradiction to  $t^p w = 0$ .  $\square$

Consider the projection  $\bar{Y} \rightarrow \mathbb{A}^E$  onto the second factor, and write  $\bar{Y}_w$  for the (scheme-theoretic) fiber over  $w \in \mathbb{A}^E$ . Combining 7.6 with 7.9 we obtain the following consequence (in the notation of 7.8):

**Corollary 7.10.** *If  $\{\underline{a}\} \neq 0$  in  $K_n^M(E)/p$  and  $w \neq 0$  is such that  $\bar{Y}_w$  has a  $k$ -point, then  $A_0$  and the  $A_i \otimes E$  are fields and  $w$  is a product of norms of an element of  $A_0$  and elements in the subsets  $A_i + \epsilon$  of  $A_i \otimes_k E$ .*

*Remark 7.10.1.* In Theorem 7.13 we will see that if  $w$  is a generic element of  $E$  then such a  $k$ -point exists.

The group  $G = \mu_p^n$  acts nicely on  $S$  and  $J$  by 6.8, and on  $\mathcal{A}$  and  $\mathbb{P}(\mathcal{A})$  by 6.3. It acts trivially on  $\mathbb{A}^E$ , so  $G$  acts on  $B$ ,  $U$  and  $\bar{Y}$  (but not nicely; see 6.1.1).

In the notation of 7.8, if  $([\alpha : t], w)$  is a fixed point of the  $G$ -action on  $\bar{Y}$  then the points  $u_0 \in \mathbb{P}(\mathcal{A})$  and  $s_i \in S$  are fixed, and therefore are  $k$ -rational (see 6.1). If  $u$  is defined over  $F$ , each point  $(b_i : t_i)$  is fixed in  $\mathcal{B}|_{s_i}$ . Since  $S$  acts nicely on  $J$ , Example 6.3.1 shows that if  $t = 0$  then either  $t_i \neq 0$  (and  $b_i \in F \subset A_i \otimes F$ ) or else  $t_i = 0$  and  $0 \neq b_i \in J|_{s_i}^{\otimes r_i} \otimes F \subseteq A_i \otimes F$  is for some  $r_i$ ,  $0 \leq r_i < p$ .

**Lemma 7.11.** *For all  $w$ ,  $\text{Fix}_G \bar{Y}_w$  is disjoint from the locus where  $t = 0$ .*

*Proof.* Suppose  $([\alpha : 0], w)$  is a fixed point defined over a field  $F$  containing  $k$ . As explained above,  $b_0 \neq 0$  and (for each  $i > 0$ )  $b_i + t_i\epsilon \neq 0$  and either  $t_i \neq 0$  or there is an  $r_i$  so that  $b_i \in J|_{s_i}^{r_i} \otimes F$ . Let  $I$  be the set of indices such that  $t_i \neq 0$ .

By Example 6.3,  $b_0 \in J|_{s_0}^{\otimes r_0}$  for some  $r_0$ , and hence  $N_{A_0}(b_0)$  is a unit in  $k$ , because the  $p$ -form  $\gamma$  is nontrivial on  $J|_{s_0}$ . Likewise, if  $i \notin I$ , then  $N_{A_i \otimes F/F}(b_i)$  is a unit in  $F$ .

Now suppose  $i \in I$ , i.e.,  $t_i \neq 0$ , and recall that in this case  $b_i \in F \subset A_i \otimes F$ . If we write  $EF$  for the algebra  $E \otimes F \cong F[\epsilon]/(\epsilon^p - e)$ , then the norm from  $A_i \otimes EF$  to  $EF$  is simply the  $p$ -th power on elements in  $EF$ , so  $N_{A_i \otimes EF/EF}(b_i + t_i\epsilon) = (b_i + t_i\epsilon)^p$  as an element in the algebra  $EF$ . Taking the product, and keeping in mind  $t = 0$ , we get the equation

$$\prod_{i \in I} N_{A_i \otimes EF/EF}(b_i + t_i\epsilon) = \prod_{i \in I} (b_i + t_i\epsilon)^p = 0.$$

Because  $EF$  is a separable  $F$ -algebra, it has no nilpotent elements. We conclude that

$$\prod_{i \in I} (b_i + t_i \epsilon) = 0.$$

The left hand side of this equation is a polynomial of degree at most  $p - 1$  in  $\epsilon$ ; since  $\{1, \epsilon, \dots, \epsilon^{p-1}\}$  is a basis of  $F \otimes E$  over  $F$ , that polynomial must be zero. This implies that  $b_i = t_i = 0$  for some  $i$ , a contradiction.  $\square$

**Proposition 7.12.** *If  $w \in \mathbb{A}^E$  is generic then  $\text{Fix}_G \bar{Y}_w$  lies in the open subvariety where  $t \prod_{i=1}^p t_i \neq 0$ .*

*Remark 7.12.1.* The open subvariety in 7.12 is  $G$ -isomorphic (by setting  $t$  and all  $t_i$  to 1) to a closed subvariety of  $\mathbb{A}(\mathcal{A})^p$ , namely the fiber over  $w$  of the map  $N_{\mathcal{A} \otimes E/E} : \mathbb{A}(\mathcal{A})^p \rightarrow \mathbb{A}^E$  defined by

$$N(b_0, \dots, b_{p-1}) = N_{A_0/k}(b_0) \prod_{i=1}^{p-1} N_{A_i \otimes E/E}(b_i + \epsilon).$$

Indeed,  $\mathbb{A}(\mathcal{A})^p$  is  $G$ -isomorphic to an open subvariety of  $\bar{Y}$  and  $N_{A_i \otimes E/E}$  is the restriction of  $\alpha \mapsto N(\alpha)$ .

*Proof.* By Lemma 7.11,  $\text{Fix}_G \bar{Y}_w$  is disjoint from the locus where  $t = 0$ , so we may assume that  $t = 1$ . Since  $w$  is generic, we may also take  $w \neq 0$ . So let  $([\alpha : 1], w)$  be a fixed point defined over  $F \supseteq k$  for which  $t_j = 0$ . As in the proof of the previous lemma, we collect those indices  $i$  such that  $t_i \neq 0$  into a set  $I$ , and write  $EF$  for  $E \otimes_k F$ . Recall that for  $i \in I$ , we have  $b_i \in F$ . Since  $j \notin I$ , we have that  $|I| \leq p - 2$ . For  $i \notin I$ ,

$$N_{A_i \otimes EF/EF}(b_i + t_i \epsilon) = N_{A_i \otimes F/F}(b_i) \in F^\times$$

(the norm cannot be 0 as  $t^p w = w \neq 0$  by assumption). So we get that

$$\prod_{i \in I} (b_i + t_i \epsilon)^p = \xi w$$

for some  $\xi \in F^\times$ . If we view  $\xi w$  as a point in  $\mathbb{P}(E)(F) = (EF - \{0\})/F^\times$ , then we get an equation of the form

$$\left[ \prod_{i \in I} (b_i + t_i \epsilon)^p \right] = [w].$$

But the left-hand side lies in the image of the morphism  $\prod_{i \in I} \mathbb{P}^1 \rightarrow \mathbb{P}(E)$  which sends  $[b_i : t_i] \in \mathbb{P}^1(F)$  to  $[\prod (b_i + t_i \epsilon)^p] \in \mathbb{P}(E)(F)$ . Since  $|I| \leq p - 2$ , this image is a proper closed subvariety, proving the assertion for generic  $w$ .  $\square$

**Theorem 7.13.** *For a generic closed point  $w \in \mathbb{A}^E$ ,  $\bar{Y}_w$  is  $G$ -fixed point equivalent to the disjoint union of  $(p - 1)!$  copies of  $\mathbb{P}(\mathcal{A})^p$*

*Proof.* Since both lie over  $S$ , it suffices to consider a  $G$ -fixed point  $s = (s_0, \dots, s_{p-1})$  in  $S(k)^p$  and prove the assertion for the fixed points over  $s$ . Because  $G$  acts nicely on  $S$  and  $J$ ,  $k(s) = k$  and (by Lemma 6.4)  $G$  acts on  $J_s$  via a projection  $G \rightarrow \mu_p$  as the standard representation of  $\mu_p$ . Note that  $J_s = J_{s_i}$  for all  $i$ .

By Example 6.3, there are precisely  $p$  fixed points on  $\mathbb{P}(\mathcal{A})$  lying over a given fixed point  $s_i \in S(k)$ , and at each of these points the (vertical) tangent space is the reduced regular representation of  $\mu_p$ . Thus each fixed point in  $\mathbb{P}(\mathcal{A})^p$  is  $k$ -rational, the number of fixed points over  $s$  is  $p^p$ , and each of their tangent spaces is the sum of  $p$  copies of the reduced regular representation.

Since  $w$  is generic, we saw in 7.12 that all the fixed points of  $\bar{Y}_w$  satisfy  $t \neq 0$  and  $t_i \neq 0$  for  $1 \leq i \leq p - 1$ . By Remark 7.12.1, they lie in the affine open

$\mathbb{A}(\mathcal{A})^p$  of  $\mathbb{P}(L \oplus \mathcal{O})$ . Because  $\mu_p$  acts nicely on  $J_s$ , an  $F$ -point  $b = (b_0, \dots, b_{p-1})$  of  $\mathbb{A}(\mathcal{A})^p$  is fixed if and only if each  $b_i \in F$ . That is,  $\text{Fix}_G(\mathbb{A}(\mathcal{A})^p) = \mathbb{A}^p$ . Now the norm map restricted to the fixed-point set is just the map  $\mathbb{A}^p \rightarrow \mathbb{A}^E$  sending  $b$  to  $b_0^p \prod_{i=1}^{p-1} (b_i + \epsilon)^p$ . This map is finite of degree  $p^p(p-1)!$ , and étale for generic  $w$ , so  $\text{Fix}_G(\overline{Y}_w)$  has  $p^p(p-1)!$  geometric points for generic  $w$ . This is the same number as the fixed points in  $(p-1)!$  copies of  $\mathbb{P}(\mathcal{A})$  over  $s$ , so it suffices to check their tangent space representations.

At each fixed point  $b$ , the tangent space of  $\mathbb{A}(\mathcal{A})^p$  (or  $\overline{Y}$ ) is the sum of  $p$  copies of the regular representation of  $\mu_p$ . Since this tangent space is also the sum of the tangent space of  $\mathbb{A}^p$  (a trivial representation of  $G$ ) and the normal bundle of  $\mathbb{A}^p$  in  $\overline{Y}$ , the normal bundle must then be  $p$  copies of the reduced regular representation of  $\mu_p$ . Since the tangent space of  $\mathbb{A}^p$  maps isomorphically onto the tangent space of  $\mathbb{A}^E$  at  $w$ , the tangent space of  $\overline{Y}_w$  is the same as the normal bundle of  $\mathbb{A}^p$  in  $\overline{Y}$ , as required.  $\square$

*Remark 7.13.1.* The fixed points in  $\overline{Y}_w$  are not necessarily rational points, and we only know that the isomorphism of the tangent spaces at the fixed points holds on a separable extension of  $k$ . This is parallel to the situation with the fixed points in  $Q_w$  described in Theorem 7.3.

## 8. A $\nu_n$ -VARIETY

The following result will be needed in the proof of the norm principle.

**Theorem 8.1.** *Let  $S$  be the variety of the chain lemma for some symbol  $\{a\} \in K_n^M(k)/p$  and  $\mathcal{A} = \bigoplus_{i=0}^{p-1} J^{\otimes i}$  the sheaf of Kummer algebras over  $S$ . Then the projective bundle  $\mathbb{P}(\mathcal{A})$  has dimension  $d = p^n - 1$  and  $p^2 \nmid s_d(\mathbb{P}(\mathcal{A}))$ .*

*Proof.* Let  $\pi : \mathbb{P}(\mathcal{A}) \rightarrow S$  be the projection. The statement about the dimension is trivial. In the Grothendieck group  $K_0(\mathbb{P}(\mathcal{A}))$ , we have that

$$[T_{\mathbb{P}(\mathcal{A})}] = \pi^*([T_S]) + [T_{\mathbb{P}(\mathcal{A})/S}]$$

where  $T_{\mathbb{P}(\mathcal{A})/S}$  is the relative tangent bundle. The class  $s_d$  is additive, and the dimension of  $S$  is less than  $d$ , so we conclude that  $s_d(\mathbb{P}(\mathcal{A})) = s_d(T_{\mathbb{P}(\mathcal{A})/S})$ . Now  $[T_{\mathbb{P}(\mathcal{A})/S}] = [\pi^*(\mathcal{A}) \otimes \mathcal{O}(1)_{\mathbb{P}(\mathcal{A})/S}] - 1$ ; applying additivity again, together with the definition of  $s_d$  and the decomposition of  $\mathcal{A}$  and hence  $\pi^*(\mathcal{A})$  into line bundles, we obtain

$$s_d(\mathbb{P}(\mathcal{A})) = \deg \sum_{i=0}^{p-1} c_1(\pi^* J^{\otimes i} \otimes \mathcal{O}(1))^d.$$

The projective bundle formula presents the Chow ring  $CH^*(\mathbb{P}(\mathcal{A}))$  as:

$$CH^*(\mathbb{P}(\mathcal{A})) = CH^*(S)[y] / \left( \prod_{i=0}^{p-1} (y - ix) \right)$$

where  $x = -c_1(J) \in CH^1(S)$  and  $y = c_1(\mathcal{O}(1)) \in CH^1(\mathbb{P}(\mathcal{A}))$ . Then  $s_d(\mathbb{P}(\mathcal{A}))$  is the degree of the following element of the ring  $CH^*(\mathbb{P}(\mathcal{A}))$ :

$$s'_d(\mathbb{P}(\mathcal{A})) = \sum_{i=0}^{p-1} (y - ix)^d = \sum_{i=0}^{p-1} a_i y^i x^{d-i}$$

for some integer coefficients  $a_i$ . Since  $x \in CH^1(S)$ , we have  $x^r = 0$  for any  $r > \dim(S) = p^n - p$ . It follows that  $s'_d(\mathbb{P}(\mathcal{A})) = a_{p-1} y^{p-1} x^{\dim(S)}$ . By part (6) of the

Chain Lemma 0.1, the degree of  $x^{\dim(S)} = (-1)^{\dim(S)} c_1(J)^{\dim(S)}$  is prime to  $p$ . In addition,  $\pi_*(y^{p-1}) = \pi_*(c_1(\mathcal{O}(1))^{p-1}) = [S] \in CH^0(S)$ . By the projection formula  $s_d(\mathbb{P}(\mathcal{A})) = a_{p-1} \deg x^{\dim(S)}$ . Thus to prove the theorem, it suffices to show that  $a_{p-1} \equiv p \pmod{p^2}$ ; this algebraic calculation is achieved in Lemma 8.2 below.  $\square$

**Lemma 8.2.** *In the ring  $R = \mathbb{Z}/p^2[x, y]/(\prod_{i=0}^{p-1}(y - ix))$ , the coefficient of  $y^{p-1}$  in  $u_m = \sum_{i=0}^{p-1}(y - ix)^{p^m-1}$  is  $px^b$ , with  $b = p^m - p$ .*

*Proof.* Since  $u_m$  is homogeneous of degree  $p^m - 1$ , it suffices to determine the coefficient of  $y^{p-1}$  in  $u_m$  in the ring

$$R/(x-1) = \mathbb{Z}/p^2[y]/(\prod_{i=0}^{p-1}(y-i)) \cong \prod_{i=0}^{p-1} \mathbb{Z}/p^2.$$

If  $m = 1$ , then  $u_1 = \sum_{i=0}^{p-1}(y-i)^{p-1}$  is a polynomial of degree  $p-1$  with leading term  $py^{p-1}$ . Inductively, we use the fact that for all  $a \in \mathbb{Z}/p^2$ , we have

$$a^{p^2-p} = \begin{cases} 0, & \text{if } p \mid a \\ 1, & \text{otherwise.} \end{cases}$$

Thus for  $m \geq 2$ , if we set  $k = (p^{m-1} - 1)/(p-1)$ , then  $a^{p^m-1} = a^{(p-1)+k(p^2-p)} = a^{p-1} \in \mathbb{Z}/p^2$ , and therefore

$$u_m = \sum_{i=0}^{p-1}(y-i)^{p^m-1} = \sum_{i=0}^{p-1}(y-i)^{p-1} = u_1$$

holds in  $R/(x-1)$ ; the result follows.  $\square$

## 9. THE NORM PRINCIPLE

We now turn to the Norm Principle, which concerns the group  $A_0(X, \mathcal{K}_1)$  associated to a variety  $X$ . In the literature, this group is also known as  $H_{-1, -1}(X)$  and  $H^d(X, \mathcal{K}_{d+1})$ , where  $d = \dim(X)$ . We recall the definition from 0.2.

**Definition 9.1.** If  $X$  is a regular scheme then  $A_0(X, \mathcal{K}_1)$  is the cokernel of the map  $\oplus_y K_2(k(y)) \xrightarrow{(\partial_{xy})} \oplus_x k(x)^\times$ . In this expression, the first sum is taken over all points  $y \in X$  of dimension 1, and the second sum is over all closed points  $x \in X$ . The map  $\partial_{xy} : K_2(k(y)) \rightarrow k(x)^\times$  is the tame symbol associated to the discrete valuation on  $k(y)$  associated to  $x$ ; if  $x$  is not a specialization of  $y$  then  $\partial_{xy} = 0$ . If  $x \in X$  is closed and  $\alpha \in k(x)^\times$  we write  $[x, \alpha]$  for the image of  $\alpha$  in  $A_0(X, \mathcal{K}_1)$ .

The group  $A_0(X, \mathcal{K}_1)$  is covariant for proper morphisms  $X \rightarrow Y$ , and clearly  $A_0(\text{Spec } k, \mathcal{K}_1) = k^\times$  for every field  $k$ . Thus if  $X \rightarrow \text{Spec}(k)$  is proper then there is a morphism  $N : A_0(X, \mathcal{K}_1) \rightarrow k^\times$ , whose restriction to the group of units of a closed point  $x$  is the norm map  $k(x)^\times \rightarrow k^\times$ . That is,  $N[x, \alpha] = N_{k(x)/k}(\alpha)$ .

**Definition 9.2.** When  $X$  is smooth and proper over  $k$ , we write  $\overline{A}_0(X, \mathcal{K}_1)$  for the quotient of  $A_0(X, \mathcal{K}_1)$  by the relation that  $[x_1, N_{x/x_1}(\alpha)] = [x_2, N_{x/x_2}(\alpha)]$  for every closed point  $x = (x_1, x_2)$  of  $X \times_k X$  and every  $\alpha \in k(x)^\times$ .

It is proven in [11, 1.5–1.7] that if  $X$  has a  $k$ -rational point then  $\overline{A}_0(X, \mathcal{K}_1) = k^\times$ ; if  $X(k) = \emptyset$ , then both the kernel and cokernel of  $N : \overline{A}_0(X, \mathcal{K}_1) \rightarrow k^\times$  have exponent  $n$ , where  $n$  is the gcd of the degrees  $[k(x) : k]$  for closed  $x \in X$ . In addition, if  $x, x'$  are two points of  $X$  then for any field map  $k(x') \rightarrow k(x)$  over  $k$  and any  $\alpha \in k(x)^\times$  we have  $[x, \alpha] = [x', N_{x/x'}\alpha]$  in  $\overline{A}_0(X, \mathcal{K}_1)$ .

To illustrate the advantage of passing to  $\overline{A}_0$ , consider a cyclic field extension  $E/k$ . Then  $A_0(\text{Spec } E, \mathcal{K}_1) = E^\times$  and by Hilbert 90, there is an exact sequence

$$0 \rightarrow \overline{A}_0(\text{Spec } E, \mathcal{K}_1) \rightarrow k^\times \rightarrow \text{Br}(K/k) \rightarrow 0.$$

We now suppose that  $k$  is a  $p$ -special field, so that the kernel and cokernel of  $N : \overline{A}_0(X, \mathcal{K}_1) \rightarrow k^\times$  are  $p$ -groups, and that  $X$  is a norm variety (a  $p$ -generic splitting variety of dimension  $p^n - 1$ ). The Norm Principle is concerned with reducing the degrees of the field extensions  $k(x)$  used to represent elements of  $\overline{A}_0(X, \mathcal{K}_1)$ . For this, the following definition is useful.

**Definition 9.3.** Let  $\tilde{A}_0(k)$  denote the subset of elements  $\theta$  of  $\overline{A}_0(X, \mathcal{K}_1)$  represented by  $[x, \alpha]$  where  $k(x) = k$  or  $[k(x) : k] = p$ . If  $E/k$  is a field extension,  $\tilde{A}_0(E)$  denotes the corresponding subset of  $\overline{A}_0(X_E, \mathcal{K}_1)$ .

**Lemma 9.4.** *If  $k$  is  $p$ -special and  $X$  is a norm variety, then  $\tilde{A}_0(k)$  is a subgroup of  $\overline{A}_0(X, \mathcal{K}_1)$ .*

*Proof.* By the Multiplication Principle [11, 5.7], which depends upon the Chain Lemma 0.1, we know that for each  $[x, \alpha], [x', \alpha']$  in  $\tilde{A}_0(k)$ , there is a  $[x'', \alpha''] \in \tilde{A}_0(k)$  so that  $[x, \alpha] + [x', \alpha'] = [x'', \alpha'']$  in  $\overline{A}_0(X, \mathcal{K}_1)$ . Hence  $\tilde{A}_0(k)$  is closed under addition. It is nonempty because  $E = k[\sqrt[p]{a_1}]$  splits the symbol and therefore  $X(E) \neq \emptyset$ . It is a subgroup because  $[x, \alpha] + [x, \alpha^{-1}] = [x, 1] = 0$ .  $\square$

**Lemma 9.5** ([11, 1.24]). *If  $k$  is  $p$ -special and  $X$  is a norm variety, the restriction of  $\overline{A}_0(X, \mathcal{K}_1) \xrightarrow{N} k^\times$  to  $\tilde{A}_0(k)$  is an injection.*

*Proof.* Let  $[x, \alpha]$  represent  $\theta \in \tilde{A}_0(k)$ . If  $N(\theta) = N_{k(x)/k}(\alpha) = 1$  then  $\alpha = \sigma(\beta)/\beta$  for some  $\beta$  by Hilbert's Theorem 90. But  $[x, \sigma(\beta)] = [x, \beta]$  in  $\tilde{A}_0(k)$ ; see [11, 1.5].  $\square$

*Example 9.5.1.* If  $X$  has a  $k$ -point  $z$ , then the norm map  $N$  of 0.2 is an isomorphism  $\tilde{A}_0(k) \cong \overline{A}_0(X, \mathcal{K}_1) \xrightarrow{\cong} k^\times$ , split by  $\alpha \mapsto [z, \alpha]$ . Indeed, for every closed point  $x$  of  $X$  we have  $[x, \alpha] = [z, N_{k(x)/k}\alpha]$  in  $\overline{A}_0(X, \mathcal{K}_1)$ , by [11, 1.5].

Our goal in the next section is to prove the following theorem. Let  $E/k$  be a field extension with  $[E : k] = p$ . Since  $k$  has  $p$ th roots of unity, we can write  $E = k(\epsilon)$  with  $\epsilon^p \in k$ .

**Theorem 9.6.** *Suppose that  $k$  is  $p$ -special,  $\{\underline{a}\}_E \neq 0$  and that  $X$  is a norm variety for  $\{\underline{a}\}$ . For  $[z, \alpha] \in \tilde{A}_0(E)$ , there exist points  $x_i \in X$  of degree  $p$  over  $k$ ,  $t_i \in k$  and  $b_i \in k(x_i)$  such that  $N_{E(z)/E}(\alpha) = \prod N_{E(x_i)/E}(b_i + t_i\epsilon)$ .*

The proof of Theorem 9.6 uses Theorem 10.4, which in turn depends upon the cobordism results in the appendix. Theorem 9.6 is the key ingredient in the proof of Theorem 9.7, and we will see that the Norm Principle follows easily from Theorem 9.7.

**Theorem 9.7.** *If  $k$  is  $p$ -special and  $[E : k] = p$  then  $\overline{A}_0(X_E, \mathcal{K}_1) \xrightarrow{N_{E/k}} \overline{A}_0(X, \mathcal{K}_1)$  sends  $\tilde{A}_0(E)$  to  $\tilde{A}_0(k)$ .*

*Proof.* If  $\{\underline{a}\}_E = 0$  then the generic splitting variety  $X$  has an  $E$ -point  $x$ , and Theorem 9.7 is immediate from Example 9.5.1. Indeed, in this case  $X_E$  has an  $E$ -point  $x'$  over  $x$ , every element of  $\tilde{A}_0(E) \cong E^\times$  has the form  $[x', \alpha]$ , and  $N_{E/k}[x', \alpha] =$

$[x, \alpha]$ . Hence we may assume that  $\{\underline{a}\}_E \neq 0$ . This has the advantage that  $E(x_i) = E \otimes_k k(x_i)$  is a field for every  $x_i \in X$ .

Choose  $\theta = [z, \alpha] \in \tilde{A}_0(E)$  and let  $x_i \in X$ ,  $t_i$  and  $b_i$  be the data given by Theorem 9.6. Each  $x_i$  lifts to an  $E(x_i)$ -point  $x_i \otimes E$  of  $X_E$  so we may consider the element

$$\theta' = \theta - \sum [x_i \otimes E, b_i + t_i \epsilon] \in \bar{A}_0(X_E, \mathcal{K}_1).$$

By 9.4 over  $E$ ,  $\theta'$  belongs to the subgroup  $\tilde{A}_0(E)$ . By Theorem 9.6, its norm is

$$N(\theta') = N_{E(z)/E}(\alpha) / \prod N_{E(x_i)/E}(b_i + t_i \epsilon) = 1.$$

By Lemma 9.5,  $\theta' = 0$ . Hence  $N_{E/k}(\theta) = \sum [x_i, N_{E(x_i)/k(x_i)}(b_i + t_i \epsilon)]$  in  $\bar{A}_0(X, \mathcal{K}_1)$ . Since  $\tilde{A}_0(k)$  is a group by 9.4, this is an element of  $\tilde{A}_0(k)$ .  $\square$

**Corollary 9.8** (Theorem 0.7(3)). *If  $k$  is  $p$ -special then  $\tilde{A}_0(k) = \bar{A}_0(X, \mathcal{K}_1)$ , and  $N : \bar{A}_0(X, \mathcal{K}_1) \rightarrow k^\times$  is an injection.*

*Proof.* We may suppose that  $X(k) = \emptyset$ . For every closed  $z \in X$  there is an intermediate subfield  $E$  with  $[k(z) : E] = p$  and a  $k(z)$ -point  $z'$  in  $X_E$  over  $z$ . Since  $[z', \alpha] \in \tilde{A}_0(E)$ , Theorem 9.7 implies that  $[z, \alpha] = N[z', \alpha]$  is in  $\tilde{A}_0(k)$ . This proves the first assertion. The second follows from this and Lemma 9.5.  $\square$

The Norm Principle of the Introduction follows from Theorem 9.7.

**Proof of the Norm Principle (Theorem 0.3).** We consider a generator  $[z, \alpha]$  of  $\bar{A}_0(X, \mathcal{K}_1)$ . Since  $[k(z) : k] = p^\nu$  for  $\nu > 0$ , there is a subfield  $E$  of  $k(z)$  with  $[k(z) : E] = p$ , and  $z$  lifts to a  $k(z)$ -point  $z'$  of  $X_E$ . By construction,  $[z', \alpha] \in \tilde{A}_0(E)$  and  $\bar{A}_0(X_E, \mathcal{K}_1) \rightarrow \bar{A}_0(X, \mathcal{K}_1)$  sends  $[z', \alpha]$  to  $[z, \alpha]$ . By Theorem 9.7,  $[z, \alpha]$  is in  $\tilde{A}_0(k)$ , i.e., is represented by an element  $[x, \alpha]$  with  $[k(x) : k] = p$ .  $\square$

## 10. EXPRESSING NORMS

Recall that  $E = k(\epsilon)$  is a fixed Kummer extension of a  $p$ -special field  $k$ , and  $X$  is a norm variety over  $k$  for the symbol  $\{\underline{a}\}$ . The purpose of this section is to prove Theorem 9.6, that if an element  $w \in E$  is a norm for a Kummer point of  $X_E$  then  $w$  is a product of norms of the form specified in Theorem 9.6.

Recall from 7.2 that  $Q \subseteq \mathbb{P}(\mathcal{A} \oplus \mathcal{O}) \times \mathbb{A}_k^1$  is the variety of all points  $([\beta, t], w)$  such that  $N(\beta) = t^p w$ , and let  $q : Q \rightarrow \mathbb{A}_k^1$  be the projection. Extending the base field to  $E$  and applying the Weil restriction functor, we obtain a morphism

$$Rq = \text{Res}_{E/k}(q_E) : RQ = \text{Res}_{E/k}(Q_E) \rightarrow \mathbb{A}^E.$$

Moreover, choose once and for all a resolution of singularities  $\tilde{Q} \rightarrow Q$ , which is an isomorphism where  $t \neq 0$ . This is possible since  $Q$  is smooth where  $t \neq 0$ , see 7.2.

*Remark 10.1.* Since  $k$  is  $p$ -special, so is  $E$ . As stated in Lemma 9.5, the norm map  $\tilde{A}_0(E) \rightarrow E^\times$  is injective; we identify  $\tilde{A}_0(E)$  with its image. Thus  $[z, \alpha] \in \tilde{A}_0(E)$  is identified with  $N_{E(z)/E}(\alpha) \in E^\times$ . By [11, Theorem 5.5], there is a point  $s \in S$  such that  $E(z) = \mathcal{A}_s \otimes E$ ; Under the correspondence  $E(z) \cong \mathbb{A}(\mathcal{A})_s(E)$ , we identify  $\alpha$  with a point of  $\mathbb{A}(\mathcal{A})(E)$ , lying over  $s \in S$ . Then  $N_{E(z)/E}(\alpha) = Rq([\alpha, 1], N(\alpha))$ . In other words,  $\tilde{A}_0(E) \subseteq E^\times$  is equal to  $q(Q(E)) - \{0\}$ .

To prove Theorem 9.6 it therefore suffices to show that  $\overline{Y}_w(k)$  is non-empty when  $w = Rq([\beta, 1], w)$ . To do this, we will produce a correspondence  $Z \rightarrow \overline{Y} \times_{\mathbb{A}^E} RQ$  that is dominant and of degree prime to  $p$  over  $RQ$ . We construct the correspondence  $Z$  using the Multiplication Principle of [11, 5.7] in the following form.

**Lemma 10.2** (Multiplication Principle). *Let  $k$  be a  $p$ -special field. Then the set of values of the map  $N : \mathbb{A}(\mathcal{A})(k) \rightarrow k$  is a multiplicative subset of  $k^\times$ .*

*Proof.* Given Remark 10.1, this is a consequence of Lemma 9.4.  $\square$

**Lemma 10.3.** *Let  $F = k(\overline{Y})$  be the function field. Then there exists a finite extension  $L/F$ , of degree prime to  $p$ , and a point  $\xi \in RQ(L)$  lying over the generic point of  $\mathbb{A}^E$ .*

*Proof.* Let  $F'$  be the maximal prime-to- $p$  extension of  $F$ ; then the field  $EF' = E \otimes_k F'$  is  $p$ -special. We may regard the generic point of  $\overline{Y}$  as an element in  $\overline{Y}(F)$ . Applying the inclusion  $F \subset F'$  to this element, followed by the projection  $\overline{Y} \rightarrow \mathbb{A}^E$ , we obtain an element  $\omega$  of  $\mathbb{A}^E(F') = EF'$ . By 7.8,  $\omega$  is a product of norms from  $\mathbb{A}(\mathcal{A})(EF')$ . By the Multiplication Principle 10.2, there exists  $\beta \in \mathbb{A}(\mathcal{A})(EF')$  such that  $N(\beta) = \omega$ . Now let  $\xi$  be the point  $([\beta, 1], \omega) \in RQ(F')$ . Then  $Rq(\xi) = \omega$  and  $\xi$  is defined over some finite intermediate extension  $F \subseteq L \subseteq F'$ , with  $[L : F]$  prime to  $p$ .  $\square$

Write  $\eta_L$  for the point of  $\overline{Y}(L)$  defined by the inclusion  $F \subseteq L$ . We can now define  $\overline{Y} \xleftarrow{f} Z \xrightarrow{g} RQ$  to be a (smooth, projective) model of  $(\eta_L, \xi) \in (\overline{Y} \times_{\mathbb{A}^E} RQ)(L)$ .

**Theorem 10.4.** *The morphism  $g : Z \rightarrow RQ$  is proper and dominant (hence onto) and of degree prime to  $p$ .*

*Proof.* Let  $\omega \in \mathbb{A}^E$  be the generic point,  $k(\omega)$  the function field and  $E(\omega) = E \otimes k(\omega)$ . As degree is a generic notion and invariant under extension of the base field, we may replace  $\overline{Y} \leftarrow Z \rightarrow RQ$  by its basechange along the morphism

$$\mathrm{Spec}(E(\omega)) \rightarrow \mathrm{Spec}(k(\omega)) \xrightarrow{\omega} \mathbb{A}^E,$$

to obtain morphisms  $f : Z_{E(\omega)} \rightarrow \overline{Y}_{E(\omega)}$  and  $g : Z_{E(\omega)} \rightarrow RQ_{E(\omega)}$ . Using the normal basis theorem, we can write  $E(\omega) = E(\omega_1, \dots, \omega_p)$  for transcendentals  $\omega_i$  that are permuted under the action of the cyclic group  $\mathrm{Gal}(E/k)$ .

We will apply the DN Theorem A.1 with base field  $k' = E(\omega)$ . In the notation of Theorem A.1, we let  $r = p$ ; we write  $Y$  for some desingularization of  $\overline{Y}_{E(\omega)}$ ; we let  $X$  be  $R\tilde{Q}_{E(\omega)}$ , and we let  $W$  be a model for  $Z_{E(\omega)}$  mapping to  $Y$  and  $X$ . Finally, we let  $u_i = \{a_1, \dots, a_n, \omega_i\} \in K_{n+1}^M(k')/p$ .

Observe that our base field contains  $E$ , so  $R\tilde{Q}_{E(\omega)} = \mathrm{Res}_{E/k}(\tilde{Q}_E) \times_{\mathbb{A}^E} E(\omega)$  splits as a product  $R\tilde{Q}_{E(\omega)} = \prod_{i=1}^p \tilde{Q}_{\omega_i}$ , where  $Q_{\omega_i}$  is the fiber of  $Q \rightarrow \mathbb{A}^1$  over the point  $\omega_i \in \mathbb{A}^1(E(\omega)) = E(\omega)$ . Therefore we have  $X = \prod_{i=1}^p X_i$  where  $X_i$  is  $\tilde{Q}_{\omega_i}$ , the resolution of singularities of  $Q_{\omega_i}$ . By Remark 7.4,  $X_i$  is a smooth, geometrically irreducible splitting variety for the symbol  $u_i$  of dimension  $p^n - 1$ . Thus, hypothesis (1) of the DN Theorem A.1 is satisfied.

By Theorem A.10,  $t_{d,1}(X_i) = t_{d,1}(\mathbb{P}(\mathcal{A}))$ ; by Lemma A.6, we conclude that  $s_d(X_i) \equiv vs_d(\mathbb{P}(\mathcal{A})) \pmod{p^2}$  for some unit  $v \in \mathbb{Z}/p$ . Since  $s_d(\mathbb{P}(\mathcal{A})) \not\equiv 0$  by Theorem 8.1, we conclude that hypothesis (3) of the DN Theorem A.1 is satisfied.

Furthermore,  $K = k'(X_1 \times \dots \times X_{i-1})$  is contained in a rational function field over  $E$ ; in fact, the field  $E(\omega_j)(Q_{\omega_j})$  becomes a rational function field once we

adjoin  $\sqrt[p]{\gamma}$ . Since  $E$  does not split  $\{\underline{a}\}$ ,  $K$  does not split  $\{\underline{a}\}$  either. It follows that  $K$  does not split  $u_i = \{\underline{a}\} \cup \{\omega_i\}$ , verifying hypothesis (2) of Theorem A.1.

We have now checked the hypotheses (1–3) of Theorem A.1. It remains to check that  $X$  and  $Y$  are  $G$ -fixed point equivalent up to a prime-to- $p$  factor. In fact, we proved in Theorem 7.13 that  $\overline{Y}_{E(\omega)}$  is  $G$ -fixed point equivalent to  $(p-1)!$  copies of  $\mathbb{P}(\mathcal{A})^p$ , hence so is  $Y$  (since the fixed points lie in the smooth locus), and in Theorem 7.3 that  $X_i$  is  $G$ -fixed point equivalent to  $\mathbb{P}(\mathcal{A})$ . That is,  $Y$  is  $G$ -fixed point equivalent to  $(p-1)!$  copies of  $X$ . Therefore the DN Theorem applies to show that  $g$  is dominant and of degree prime to  $p$ , as asserted.  $\square$

*Proof of Theorem 9.6.* We have proved that there is a diagram  $\overline{Y} \xleftarrow{f} Z \xrightarrow{g} RQ$  such that the degree of  $g$  is prime to  $p$ . By blowing up if necessary we may assume that  $g : Z \rightarrow RQ$  factors through  $\tilde{g} : Z \rightarrow R\tilde{Q}$ , with  $\deg(\tilde{g})$  prime to  $p$ .

Let  $[z, \alpha] \in \tilde{A}_0(E)$ , and set  $w = N_{E(z)/E}(\alpha)$ . By Remark 10.1, there exists a point  $([\beta, 1], w) \in RQ(k)$ . Lift this to a point in  $R\tilde{Q}(k)$  (recall that  $R\tilde{Q} \rightarrow RQ$  is an isomorphism where  $t \neq 0$ ). Since  $Z \rightarrow R\tilde{Q}$  is a morphism of smooth projective varieties of degree prime to  $p$  and  $k$  is  $p$ -special, we can lift  $([\beta, 1], w)$  to a  $k$ -point of  $Z$ , and then apply  $f : Z \rightarrow \overline{Y}$  to get a  $k$ -point in  $\overline{Y}_w$ . By the definition of  $\overline{Y}$  and Corollary 7.10, this means that we can find Kummer extensions  $k(x_i)/k$  (corresponding to points  $s_i \in S$ , and determining points  $x_i \in X$  because  $X$  is a  $p$ -generic splitting variety), elements  $b_i \in k(x_i)$  and  $t_i \in k$  such that  $w = \prod_i N_{E(x_i)/E}(b_i + t_i\epsilon)$ , as asserted.  $\square$

## A. APPENDIX: THE DN THEOREM

In this appendix, we give a proof of the following Degree Theorem, which is used in the proof of Theorem 9.7, which is the key step in establishing the Norm Principle. Throughout,  $k$  will be a fixed field of characteristic 0,  $p$  will be a prime,  $n \geq 1$  will be an integer and we fix  $d = p^n - 1$ .

Recall from Definition 7.1 that if  $X$  and  $Y$  are  $G$ -fixed point equivalent then  $\dim(X) = \dim(Y)$ , the fixed points are 0-dimensional and their tangent space representations are isomorphic (over  $\bar{k}$ ).

**Theorem A.1** (DN Theorem). *For  $r \geq 1$ , let  $u_1, \dots, u_r$  be symbols in  $K_{n+1}^M(k)/p$  and let  $X = \prod_1^r X_i$ , where the  $X_i$  are irreducible smooth projective  $G$ -varieties of dimension  $d = p^n - 1$  such that:*

- (1)  $k(X_i)$  splits  $u_i$ ;
- (2)  $u_i$  is non-zero over  $k(X_1 \times \dots \times X_{i-1})$ ; and
- (3)  $p^2 \nmid s_d(X_i)$

*Let  $Y$  be a smooth irreducible projective  $G$ -variety which is  $G$ -fixed point equivalent to the disjoint union of  $m$  copies of  $X$ , where  $p \nmid m$ . Let  $F$  be a finite extension of  $k(Y)$  of degree prime to  $p$ , and  $\text{Spec}(F) \rightarrow X$  a point, with model  $f : W \rightarrow X$ . Then  $f$  is dominant and of degree prime to  $p$ .*

$$\begin{array}{ccc}
 \text{Spec}(F) & \longrightarrow & W \\
 \text{finite} \downarrow & & \downarrow g \\
 \text{Spec}(k(Y)) & \longrightarrow & Y
 \end{array}
 \begin{array}{c}
 \\
 \\
 \searrow f \text{ (dominant)} \\
 X
 \end{array}$$

The proof will use two ingredients: the degree formulas A.2 and A.5 below, due to Levine and Morel; and a standard localization result A.10 in (complex) cobordism theory. The former concern the algebraic cobordism ring  $\Omega_*(k)$ , and the latter concern the complex bordism ring  $MU_*$ . These are related via the Lazard ring  $\mathbb{L}_*$ ; combining Quillen's theorem [1, II.8] and the Morel-Levine theorem [3, 4.3.7], we have graded ring isomorphisms:

$$\Omega_*(k) \cong \mathbb{L}_* \cong MU_{2*}.$$

Here is the Levine-Morel generalized degree formula for an irreducible projective variety  $X$ , taken from [3, Theorem 4.4.15]. It concerns the ideal  $M(X)$  of  $\Omega_*(k)$  generated by the classes  $[Z]$  of smooth projective varieties  $Z$  such that there is a  $k$ -morphism  $Z \rightarrow X$ , and  $\dim(Z) < \dim(X)$ .

**Theorem A.2** (Generalized Degree Formula). *Let  $f : Y \rightarrow X$  be a morphism of smooth projective  $k$ -varieties. If  $\dim(X) = \dim(Y)$  then  $[Y] - \deg(f)[X] \in M(X)$ .*

Trivially, if  $[Z] \in M(X)$  then  $M(Z) \subseteq M(X)$ . We also have:

**Lemma A.3.** *Let  $X$  be a smooth projective  $k$ -variety. If  $Z$  and  $Z'$  are birationally equivalent, then  $[Z] \in M(X)$  holds if and only if  $[Z'] \in M(X)$ .*

*Proof.* By [3, 4.4.17], the class of  $Z$  modulo  $M(Z)$  is a birational invariant. Thus  $[Z'] - [Z] \in M(Z)$ . Because  $M(Z) \subseteq M(X)$ , the result follows.  $\square$

We shall also need the Levine-Morel ‘‘higher degree formula’’ A.5, which is taken from [3, Theorem 4.4.24], and concerns the mod  $p$  characteristic numbers  $t_{d,r}(X)$  of Definition A.4, where  $p$  is prime,  $n \geq 1$  and  $d = p^n - 1$ .

Choose a graded ring homomorphism  $\psi : \mathbb{L}_* \rightarrow \mathbb{F}_p[v_n]$  corresponding to some height  $n$  formal group law, where  $v_n$  has degree  $d$ ; many such group laws exist, and the class  $t_{d,r}$  will depend on this choice, but only up to a unit. (See [3, 4.4].)

**Definition A.4.** For  $r > 0$ , the homomorphism  $t_{d,r} : \Omega_{rd}(k) \cong \mathbb{L}_{rd} \rightarrow \mathbb{F}_p$  sends  $x$  to the coefficient of  $v_n^r$  in  $\psi(x)$ . If  $X$  is a smooth projective variety over  $k$ , of dimension  $rd$ , then  $X$  determines a class  $[X]$  in  $\Omega_{rd}(k)$ , and  $t_{d,r}(X)$  is  $t_{d,r}([X])$ .

**Theorem A.5 (Higher Degree Formula).** *Let  $f : W \rightarrow X$  be a morphism of smooth projective varieties of dimension  $rd$  and suppose that  $X$  admits a sequence of surjective morphisms*

$$X = X^{(r)} \rightarrow X^{(r-1)} \rightarrow \dots \rightarrow X^{(0)} = \text{Spec}(k)$$

such that

- (1)  $\dim(X^{(i)}) = id$ .
- (2) If  $\eta$  is a zero-cycle on  $X^{(i)} \times_{X^{(i-1)}} k(X^{(i-1)})$ , then  $p$  divides the degree of  $\eta$ .

Then  $t_{d,r}(W) = \deg(f) t_{d,r}(X)$ .

Here are some properties of this characteristic number that we shall need. Recall that if  $\dim(X) = d$  then  $p$  divides  $s_d(X)$ , so that  $s_d(X)/p$  is an integer.

**Lemma A.6.** *Let  $X/k$  be a smooth projective variety, and  $k \subseteq \mathbb{C}$  an embedding.*

- (1) For  $r = 1$ , there is a unit  $u \in \mathbb{F}_p$  such that  $t_{d,1}(X) \equiv u s_d(X)/p$ .
- (2) If  $X = \prod_{i=1}^r X_i$  and  $\dim(X_i) = d$  for all  $i$ , then  $t_{d,r}(X) = \prod_{i=1}^r t_{d,1}(X_i)$ .
- (3)  $t_{d,r}(X)$  depends only on the class of  $(X \times_k \mathbb{C})^{an}$  in the complex cobordism ring.

*Proof.* Part (1) is [3, Proposition 4.4.22]. Part (2) is immediate from the definition of  $t_{d,r}$  and the graded multiplicative structure on  $\Omega_*(k)$ . Finally, part (3) is a consequence of the fact that the natural homomorphism  $\Omega_*(k) \rightarrow MU_{2*}$  is an isomorphism (since both rings are isomorphic to the Lazard ring).  $\square$

*Remark A.6.1.* The class called  $s_d$  in this article is the  $S_d$  in [3]; the class called  $s_d(X)$  in [3] is our class  $s_d(X)/p$ .

The next lemma is a variant of Theorem A.5. It uses the same hypotheses.

**Lemma A.7.** *Let  $X$  be as in Theorem A.5. Then  $\psi(M(X)) = 0$ .*

*Proof.* Consider  $Z$  with  $[Z] \in M(X)$ . If  $d$  does not divide  $\dim(Z)$ , then  $\psi([Z]) = 0$  for degree reasons. If  $\dim(Z) = 0$ , then the image of  $Z$  is a closed point of  $X$ ; since the degree of such a closed point is divisible by  $p$ , we have  $\psi([Z]) = 0$ . Hence we may assume that  $\dim(Z) = sd$  for some  $0 < s < r$ . The cases  $r = 1$  and  $s = 0$  are immediate, so we proceed by induction on  $r$  and  $s$ .

Let  $f : Z \rightarrow X$  be a  $k$ -morphism with  $\dim(Z) = sd$ , and let  $f_s : Z \rightarrow X^{(s)}$  be the obvious composition. As  $\dim(Z) = \dim(X^{(s)})$ , the generalized degree formula A.2 applies to show that  $[Z] - \deg(f_s)([X^{(s)}]) \in M(X^{(s)})$ . By induction on  $r$ ,  $\psi(M(X^{(s)})) = 0$ , so  $\psi([Z]) = \deg(f_s)\psi([X^{(s)}])$ . We claim that  $\deg(f_s) \equiv 0 \pmod{p}$ , which yields  $\psi([Z]) = 0$ , as desired.

If  $f_s$  is not dominant, then  $\deg(f_s) = 0$  by definition. On the other hand, if  $f_s$  is dominant, then the generic point of  $Z$  maps to a closed point  $\eta$  of  $X^{(s+1)} \times_{X^{(s)}} k(X^{(s)})$ . By condition (2) of Theorem A.5,  $p$  divides  $\deg(\eta) = \deg(f_s)$ .  $\square$

We will need to show that  $\psi(M(Y)) = 0$  for the  $Y$  appearing in Theorem A.1. This is accomplished in the next lemma.

**Lemma A.8.** *Suppose  $X, Y$  and  $W$  are smooth projective varieties of dimension  $rd$  over  $k$ , and  $f : W \rightarrow X$  and  $g : W \rightarrow Y$  are morphisms. Suppose further that  $\psi(M(X)) = 0$  and that  $p$  does not divide  $\deg(g)$ . Then  $\psi(M(Y)) = 0$ .*

*Proof.* Suppose  $[Z] \in M(Y)$ . As  $g : W \rightarrow Y$  is a proper morphism of smooth varieties, of degree prime to  $p$ , we can lift the generic point  $\text{Spec}(k(Z)) \rightarrow Y$  to a point  $q : \text{Spec}(F) \rightarrow W$  for some field extension  $F/k(Z)$  of degree  $e$  prime to  $p$ . Let  $\tilde{Z}$  be a smooth projective model of  $F$  possessing a morphism to  $Z$  and a morphism to  $X$  extending the  $k$ -morphism  $f \circ q : \text{Spec}(F) \rightarrow X$ . Hence  $[\tilde{Z}] \in M(X)$ . By the degree formula for the map  $\tilde{Z} \rightarrow Z$ ,  $e[Z] - [\tilde{Z}] \in M(Z)$ . If  $\dim(Z) = 0$ , then  $M(Z) = (0)$ . In general,  $M(Z)$  is generated by the classes of varieties of dimension less than  $\dim(Z)$  that map to  $Z$  (hence a fortiori also map to  $Y$ ) over  $k$ . By induction on the dimension of  $Z$ , we may assume that  $\psi(M(Z)) = 0$ . Moreover,  $\psi([\tilde{Z}]) = 0$  by assumption; since  $p$  does not divide  $e$ , we conclude that  $\psi([Z]) = 0$  as asserted.  $\square$

Finally, we will use the following standard bordism localization result.

**Lemma A.9.** *Suppose that the abelian  $p$ -group  $G = \mu_p^n$  acts without fixed points on an compact complex analytic manifold  $M$ , by holomorphic maps. Then  $\psi([M]) = 0$  in  $\mathbb{F}_p$ .*

*Proof.* By [12],  $[M]$  is in the ideal of  $MU_*$  generated by  $\{p, [M_1] \dots, [M_{n-1}]\}$ , where  $\dim_{\mathbb{C}}(M_i) = p^i - 1$ . Since  $p$  is the only generator of this ideal whose dimension is a multiple of  $d = p^n - 1$ ,  $\psi$  is zero on every generator and hence on the ideal.  $\square$

**Theorem A.10.** *Let  $G$  be  $\mu_p^n$  and let  $X$  and  $Y$  be compact complex  $G$ -manifolds which are  $G$ -fixed point equivalent. Then  $\psi([X]) = \psi([Y])$ .*

*Proof.* Remove equivariantly isomorphic small balls about the fixed points of  $X$  and  $Y$ , and let  $M = X \cup -Y$  denote the result of joining the rest of  $X$  and  $Y$ , with the opposite orientation on  $Y$ . Then  $X$  and  $Y$  determine canonical weakly complex structures on punctured neighborhoods of the fixed points. As these structures are completely determined by the normal bundles at the fixed points (see [2, 3.1]), they glue to give  $M$  a canonical weakly complex structure. Moreover,  $G$  acts on  $M$  with no fixed points, and  $[X] - [Y] = [M]$  in  $MU_*$ . By Lemma A.9,  $\psi([X]) - \psi([Y]) = \psi([M]) = 0$ .  $\square$

We can now prove Theorem A.1. Note that the inclusion  $k(Y) \subset F$  induces a dominant rational map  $W \rightarrow Y$ ; we may replace  $W$  by a blowup to eliminate the points of indeterminacy and obtain a morphism  $g : W \rightarrow Y$ , whose degree is prime to  $p$ , without affecting the statement of Theorem A.1.

*Proof of the DN Theorem A.1.* We will apply Theorem A.5 to  $X$  and the  $X^{(t)} = \prod_{i=1}^t X_i$ . We must first check that the hypotheses are satisfied. The first condition is obvious. For the second condition, it is convenient to fix  $t$  and set  $F = k(X_1 \times \dots \times X_{t-1})$ ,  $X' = X^{(t)} \times_{X^{(t-1)}} F$ . By hypotheses (1–2) of Theorem A.1, the symbol  $u_t$  is nonzero over  $F$  but splits over the generic point of  $X'$ ; by specialization, it splits over all closed points. A transfer argument implies that the degree of any

closed point  $\eta$  of  $X'$  is divisible by  $p$ ; this is the second condition. Hence Theorem A.5 applies and we have  $t_{d,r}(W) = \deg(f) t_{d,r}(X)$ .

By Lemmas A.8 and A.7, we have that  $\psi(M(Y)) = 0$ ; by the generalized degree formula A.2, we conclude that  $\psi([W]) = \deg(g) \psi([Y])$ , so that  $t_{d,r}(W) = \deg(g) t_{d,r}(Y) \neq 0$ . Hence

$$\deg(f) t_{d,r}(X) = \deg(g) t_{d,r}(Y).$$

By Theorem A.10 and Lemma A.6(3),  $mt_{d,r}(X) = t_{d,r}(Y)$ . Condition (3) of Theorem A.1 and Lemma A.6 imply that  $t_{d,1}(X_i) \neq 0$  for all  $i$  and hence that  $t_{d,r}(X) \neq 0$ . It follows that  $m \deg(g) \equiv \deg(f) \neq 0$  modulo  $p$ , as required.  $\square$

**Acknowledgements.** We would like to state the obvious: we are deeply indebted to Markus Rost for proving the results presented in this paper, and for lecturing on them during Spring 2000 and Spring 2005 terms at the Institute for Advanced Study. We are also grateful to the Institute for Advanced Study for providing the conditions for these lectures. We are also grateful to Marc Levine and Peter Landweber for their help with the cobordism theory used in this paper. Finally, we thank the anonymous referee for numerous corrections.

#### REFERENCES

- [1] J. F. Adams, *Stable homotopy and generalised homology*, Univ. Chicago Press, 1974.
- [2] P. Conner and E. Floyd, Periodic maps which preserve a complex structure, *Bull. Amer. Math. Soc.* 70 (1964), 574–579.
- [3] M. Levine and F. Morel, *Algebraic Cobordism*, Springer Monographs in Mathematics, 2007.
- [4] A. Merkurjev and A. Suslin,  $K$ -cohomology of Severi-Brauer varieties, *Izv. Akad. Nauk SSSR Ser. Mat.* 46 (1982), 1011–1046 (Russian). English translation *Math. USSR Izvestiya* 21 (1983), 307–340.
- [5] J. Milnor and J. Stasheff, *Characteristic Classes*, Princeton Univ. Press, 1974.
- [6] M. Rost, Chow groups with coefficients, *Doc. Math. J. DMV* 1 (1996), 319–393.
- [7] M. Rost, Chain lemma for splitting fields of symbols, *Preprint*, 1998. Available at <http://www.math.uni-bielefeld.de/~rost/chain-lemma.html>
- [8] M. Rost, Construction of Splitting Varieties, *Preprint*, 1998. Available at <http://www.math.uni-bielefeld.de/~rost/chain-lemma.html>
- [9] M. Rost, The chain lemma for Kummer elements of degree 3. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(3):185–190, 1999.
- [Rost] M. Rost, *Notes on Lectures given at IAS*, 1999–2000 and Spring Term 2005.
- [10] R. Stong, *Notes on Cobordism Theory*, Princeton Univ. Press, 1968.
- [11] A. Suslin and S. Joukhovitski, Norm Varieties, *J. Pure Appl. Alg.* 206 (2006), 245–276.
- [12] T. tom Dieck, Actions of finite abelian  $p$ -groups without stationary points, *Topology* 9 (1970), 359–366.
- [13] V. Voevodsky, On Motivic Cohomology with  $\mathbb{Z}/l$  coefficients, *Preprint*, 2003. Available at <http://www.math.uiuc.edu/K-theory/0639/>.
- [14] C. Weibel, Axioms for the Norm Residue Isomorphism, *Preprint*, 2006. To appear in *Proceedings Valladolid Conference*, available at <http://www.math.uiuc.edu/K-theory/0809/>.
- [15] C. Weibel, Patching the Norm Residue Isomorphism Theorem, *Preprint*, 2007. Available at <http://www.math.uiuc.edu/K-theory/0844/>.
- [16] A. Weil, *Adeles and Algebraic Groups*, Birkhäuser, 1982. Based on lectures given 1959–1960.