

MODULE STRUCTURE OF THE K -THEORY OF POLYNOMIAL-LIKE RINGS

CHRISTIAN HAESEMEYER AND CHARLES A. WEIBEL

ABSTRACT. Suppose Γ is a submonoid of a lattice, not containing a line. In this note, we use the natural Γ -grading on the monoid algebra $R[\Gamma]$ to prove structural results about the relative K -theory $K(R[\Gamma], R)$. When R contains a field, we prove a decomposition indexed by the rays in Γ , and a compatible action by the Witt vectors of R for each \mathbb{N} -grading of Γ . In characteristic zero, there is additionally an action by Witt vectors for the truncation set Γ . Finally, we apply this to get a ray-like description of $K_*(R[x_1, \dots, x_n])$ proposed by J. Davis.

If Γ is a commutative monoid, and R is a commutative ring, the monoid algebra $R[\Gamma]$ is naturally Γ -graded. In this note, we make use of this simple observation to introduce a natural decomposition of the relative K -theory groups $K_*(R[\Gamma], R)$ when Γ is a normal submonoid of \mathbb{Z}^n having no invertible elements except 0 and R contains a field. The decomposition is indexed by the set of rays of Γ , where a *ray* is the intersection of Γ with a line in $\mathbb{R}^n = \mathbb{Z}^n \otimes \mathbb{R}$. For each ray, the summand is a continuous module for the ring $W(R)$ of big Witt vectors of R . Of course, all this generalizes the well known $W(R)$ -module structure on $K_*(R[x], R)$ when $\Gamma = \mathbb{N} = \{0, 1, \dots\}$ [26]; in this case there is only one ray. These structures are transported from cyclic (or topological cyclic) homology via the Chern character (cyclotomic trace) maps.

We are motivated by the classical case where $\Gamma = \mathbb{N}^n$; the cartesian decomposition in [5] of the relative K -theory of $R[\mathbb{N}^n] = R[x_1, \dots, x_n]$ is compatible with the ray-like decomposition that Jim Davis obtained in [10], using the Farrell-Jones conjecture; this is discussed in section 7, which is independent of the rest of the paper.

Our results also apply to quotients $\bar{\Gamma} = \Gamma/I$ of Γ by an ideal I . For example, if $\Gamma = \mathbb{N}^n$ and I is generated by $\{(\dots, 0, a_i, 0, \dots) : i = 1, \dots, n\}$

Date: January 28, 2023.

Key words and phrases. Toric variety, algebraic K -theory, cyclic homology, big Witt vectors.

Haesemeyer was supported by ARC DP-170102328.

Weibel was supported by NSF grants, and the Simonyi Endowment at IAS.

then we obtain a $W(R)$ -module decomposition of the K -theory of

$$R[\Gamma/I] = R[x_1, \dots, x_n]/(x_1^{a_1}, \dots, x_n^{a_n})$$

which is related to the decomposition of the multi-relative groups

$$K_*(R[\Gamma/I], (x_1, \dots), \dots, (x_n)) \cong W_{\Gamma/I}/J \otimes_{\mathbb{Z}} E(x_1, \dots, x_{n-1})$$

found in [2, Thm. 1.3]. See Propositions 2.1 and 4.2 below.

We first discuss the special case where $\mathbb{Q} \subseteq R$ in sections 1 and 2; the results are slightly stronger in this case, because the ghost map is an isomorphism.

We treat the case where $\mathbb{F}_p \subseteq R$ in sections 3 through 6. In section 3, we relate the relative K -theory to TC using the cyclotomic trace. In section 4, we employ a result of Hesselholt and Nikolaus [16] to produce the ray decomposition on the relative K -theory. Sections 5 and 6 use Schlichtkrull's description of the transfer in topological Hochschild homology [24] to prove that the ray decomposition is preserved by Witt vector actions arising from \mathbb{N} -gradings of the monoid Γ .

In section 7, we extend Davis' formula for $K_*(R[x_1, \dots, x_n])$. In the Appendix, we develop some background material on the generalized Witt vectors $W_{\Gamma}(R)$.

Acknowledgements. We are grateful to Lars Hesselholt for his guidance on TC , especially in Section 4.

sec:HH

1. HOCHSCHILD AND CYCLIC HOMOLOGY

In this section, we consider the Hochschild and cyclic homology of A , relative to a commutative base ring k_0 , where A is a (possibly non-commutative) Γ -graded k_0 -algebra. Here Γ is a normal submonoid of \mathbb{Z}^n having no invertible elements except 0. The prototype Γ -graded algebra is the monoid ring $R[\Gamma]$ with $A_0 = R$; if I is an ideal of the monoid Γ , then $R[I]$ and $R[\Gamma/I]$ are also Γ -graded algebras.

The Hochschild complex $C_*(A)$ and the kernel $C_*(A, A_0)$ of $C_*(A) \rightarrow C_*(A_0)$ are Γ -graded chain complexes of A_0 -modules. Since Γ has no invertible elements, the Γ -degree 0 summand of $C_*(A)$ is the chain complex $C_*(A_0)$. Hence their homology $HH_n(A)$ and $HH_n(A, A_0)$ are Γ -graded R -modules, R being the center of A_0 , and $C_*(A, A_0)$ is a mixed complex of Γ -graded k_0 -modules with no Γ -degree 0 term.

Let R be a commutative k_0 -algebra. Recall from Corollary A.5(b) that $W_{\Gamma}(R)$ is the product over all rays ρ in Γ of the classical Witt ring $W(R)$. The ghost map gh sends $W_{\Gamma}(R)$ to the ring $\prod_{\Gamma} R$ of pointed functions $\Gamma \rightarrow R$; see Definition A.3. Thus any Γ -graded R -module M

with $M_0 = 0$ is a continuous $W_\Gamma(R)$ -module, by restriction of scalars along the ghost map; see Example A.7.

For example, when A is commutative, and $A_0 = R$, the Kähler differentials $\Omega_A^n = \Omega_{A/k_0}^n$ are Γ -graded with Ω_R^n in degree 0, so the Ω_A^n/Ω_R^n are continuous $W_\Gamma(R)$ -modules. They also form a trivial mixed complex of Γ -graded k_0 -modules; see [25, 9.8.9].

ctn-module

Lemma 1.1. *Let R be the center of A_0 . For each n , restriction of scalars along the ghost map makes each Hochschild homology group $HH_n(A, A_0)$ into a continuous $W_\Gamma(R)$ -module, and each cyclic homology group $HC_n(A, A_0)$ into a continuous $W_\Gamma(k_0)$ -module.*

Proof. This follows simply from Example A.7, combined with the observation that the differential b in the mixed complex $C_*(A, A_0)$ is R -linear, and the cyclic differential B is k_0 -linear. \square

Remark 1.1.1. If A_0 doesn't contain \mathbb{Q} , the ghost map is not an isomorphism, and there are other module structures on $HH_n(A, A_0)$; see [11] for details. Since our focus is on K -theory, we do not discuss them here.

By Example A.7, for every homogeneous element α in $HH_n(A, A_0)$, $HC_n(A, A_0)$ (or $\Omega_A^n/\Omega_{A_0}^n$ if A is commutative) of Γ -degree $\eta \neq 0$, $\gamma \in \Gamma$ has content $c(\gamma)$ and $r \in k_0$, we have

$$(1.2) \quad r[\gamma] * \alpha = \begin{cases} c(\gamma) r^e \alpha, & \text{if } \eta = e\gamma \text{ in } \Gamma, \\ 0, & \text{else.} \end{cases}$$

The de Rham differentials $d: A/R \rightarrow \Omega_A^1/\Omega_R^1$ and $\Omega_A^n/\Omega_R^n \rightarrow \Omega_A^{n+1}/\Omega_R^{n+1}$ are homomorphisms of $W_\Gamma(k_0)$ -modules, but they are not homomorphisms of $W_\Gamma(R)$ modules when $\Omega_{R/k_0}^1 \neq 0$. For example, if $t \in R$, and $\eta = e\gamma \neq 0$ with γ primitive, then for $a \in A_\eta$:

$$d(t[\gamma] * a) = d(t^e a) = t^e da + a d(t^e).$$

HH and HC of schemes. We can promote the above to schemes, using the fact that $HH_n(X)$ is the hypercohomology $\mathbb{H}_{\text{zar}}^{-n}(X; HH)$ of X with coefficients in HH , the sheafification of the presheaf C_* on affines; see [28].

We now restrict to the case where A is commutative, and flat over k_0 , with $A_0 = R$ (such as $A = R[\Gamma]$ or $R[\Gamma/I]$), X is defined over R , and R contains a field, writing $X \otimes A$ for the scheme $X \times_R \text{Spec}(A)$. By the Künneth formula [25, Prop. 9.4.1], the shuffle product

$$HH(X) \otimes HH(A, R) \xrightarrow{\simeq} HH(X \otimes A, X)$$

is a quasi-isomorphism. Taking homology yields a decomposition:

$$HH_n(X \otimes A, X) \cong \bigoplus_i HH_i(X) \otimes HH_{n-i}(A, R).$$

(There are no Tor-terms because A is flat over k_0 .) Writing $\widetilde{HH} \otimes A$ and $HH\Gamma$ for the Γ -graded sheaves of mixed complexes associated to $HH(X \otimes A, X)$ and $HH(X \otimes R[\Gamma], X)$ on X , this proves:

sheaf=HHA

Lemma 1.3. *If X is a scheme over R , each $HH_n(X \otimes A, X)$ has a continuous $W_\Gamma(R)$ -module structure, natural in X . The same is true for the cdh-hypercohomology, and each*

$$HH_n(X \otimes A, X) \rightarrow \mathbb{H}_{\text{cdh}}^{-n}(X; \widetilde{HH} \otimes A)$$

is a continuous $W_\Gamma(R)$ -module map.

In particular, $\mathbb{H}_{\text{cdh}}^{-n}(X; HH\Gamma)$ is a continuous $W_\Gamma(R)$ -module. If I is an ideal of Γ , $\mathbb{H}_{\text{cdh}}^{-n}(X; \widetilde{HH} \otimes R[\Gamma/I])$ and $\mathbb{H}_{\text{cdh}}^{-n}(X; \widetilde{HH} \otimes R[I])$ are also continuous $W_\Gamma(R)$ -modules.

Recall that if $K \rightarrow C \xrightarrow{f} D \rightarrow TK$ is a distinguished triangle of chain complexes, the fiber of f is K and the mapping cone is TK .

F-version

Corollary 1.4. *Let $\mathcal{F}_{HH}(X)$ denote the fiber of the canonical map $HH(X) \rightarrow \mathbb{H}_{\text{cdh}}(X; HH)$. Then*

$$\mathcal{F}_{HH}(X \otimes A/R) \cong \mathcal{F}_{HH}(X) \otimes HH(A, R).$$

Thus, its homotopy groups are also continuous $W_\Gamma(R)$ -modules.

Remark 1.5. If A or X is not flat over k_0 , we may consider the derived version of the Hochschild complexes $HH(A)$ and $HH(X)$. Lemma 1.3 and Corollary 1.4 go through in the derived setting. We leave the details to the reader.

Similarly, the cyclic homology $HC_n(X)$ is the hypercohomology module $\mathbb{H}_{\text{zar}}^{-n}(X, HC)$ of X with coefficients in the sheafification HC of the complex giving cyclic homology; see [27]. We write $X[\Gamma]$ for $X \otimes_R R[\Gamma]$ and $\mathcal{F}_{HC}(X)$ for the fiber of the canonical map $HC(X) \rightarrow \mathbb{H}_{\text{cdh}}(X; HC)$.

scheme-HC

Lemma 1.6. *If $\mathbb{Q} \subseteq R$ and X is a scheme over R , each $HC_n(X[\Gamma], X)$ has a continuous $W_\Gamma(R)$ -module structure, natural in X . The same is true for the cdh-hypercohomology, and each*

$$HC_n(X[\Gamma], X) \rightarrow \mathbb{H}_{\text{cdh}}^{-n}(X; HC[\Gamma])$$

is a continuous $W_\Gamma(R)$ -module map. In addition, the homotopy groups of $\mathcal{F}_{HC}(X[\Gamma], X)$ are also continuous $W_\Gamma(R)$ -modules.

If I is an ideal of Γ then the homotopy groups of $\mathcal{F}_{HC}(X[I], X)$ and $\mathcal{F}_{HC}(X[\Gamma/I], X)$ are also continuous $W_\Gamma(R)$ -modules.

Proof. Because $\mathbb{Q}[\Gamma]$ has at least one \mathbb{N} -grading with \mathbb{Q} in degree 0, the S operator acts as zero on $HC_*(A, \mathbb{Q})$; see [25, 9.9.1]. This means that $HC_*(A, \mathbb{Q})$ is a trivial $HC_*(\mathbb{Q})$ -comodule. By [17, (3.2)], this implies that $HC_*(R[\Gamma], R) \cong HH_*(R) \otimes HC_*(\mathbb{Q}[\Gamma], \mathbb{Q})$, and similarly with R replaced by X . Since a continuous $W(R)$ -module is a colimit of $W(R)/V_m W(R)$ -modules, the tensor product of an R -module with a continuous $W(\mathbb{Q})$ -module is a continuous $W(R)$ -module. Hence $HC_*(R[\Gamma], R)$ has the structure of a continuous $W_\Gamma(R)$ -module. \square

Remark 1.7. Suppose that k is a field containing \mathbb{Q} , and that X is a toric variety over k . By [4, 2.4], the Zariski sheaf HH is the product of complexes $HH^{(i)}$, $i \geq 0$, and $\mathbb{H}_{\text{cdh}}(X; HH^{(i)}) \simeq \mathbb{H}_{\text{cdh}}(X; \Omega^i)[i]$. As pointed out in [4, 2.1–2], $\mathbb{H}_{\text{cdh}}^n(X; HH)$ is the direct sum of the groups $H_{\text{cdh}}^{n+i}(X; \Omega^i)$. Similarly, HC is the product of complexes $HC^{(i)}$, $i \geq 0$, and $\mathbb{H}_{\text{cdh}}(X; HC)$ is the product of the complexes $\mathbb{H}_{\text{cdh}}(X; HC^{(i)})$.

2. K -GROUPS IN CHARACTERISTIC 0

sec:cdh

Let Sch/k be the category of schemes of finite type over a field k of characteristic 0, and write $X[\bar{\Gamma}]$ for $X \times \text{Spec}(\mathbb{Z}[\bar{\Gamma}])$, where $\bar{\Gamma}$ is Γ , I or Γ/I . The purpose of this section is to show that the relative K -groups

$$K_n(X[\bar{\Gamma}], X) = K_n(X[\bar{\Gamma}])/K_n(X)$$

are continuous $W_\Gamma(k)$ -modules, by relating them to cyclic homology relative to $k_0 = \mathbb{Q}$.

Let $\mathcal{F}_K(X)$ denote the fiber of the canonical map $K(X) \rightarrow \mathbb{H}_{\text{cdh}}(X, K)$. By [4, Thm. 1.6], the Chern character $K \rightarrow HN$ induces a weak equivalences $\mathcal{F}_K(X) \xrightarrow{\sim} |\mathcal{F}_{HC}(X)[1]|$ for each X in Sch/k , where $|C|$ denotes the associated spectrum of a chain complex C . Given Lemma 1.6, the following result is a rephrasing of [6, Thm. 5.6]; in that theorem, $\bar{\Gamma}$ is assumed to be finitely generated as a monoid but, since K -theory commutes with direct limits, this is not a problem.

K-W module

Proposition 2.1. *For all n , $K_n(X[\bar{\Gamma}], X) \cong \mathbb{H}_{\text{cdh}}^{1-n}(X, \mathcal{F}_{HC\bar{\Gamma}})$. Hence $K_n(X[\bar{\Gamma}], X)$ is a continuous $W_{\bar{\Gamma}}(k)$ -module.*

In particular, $K_n(X[\bar{\Gamma}], X)$ decomposes as a direct sum of $W(k)$ -modules $K_n(X[\bar{\Gamma}], X)_\rho$, indexed by the rays ρ of Γ .

Proof. By [14], $\mathbb{H}_{\text{cdh}}(X; K) \simeq KH(X)$, which is homotopy invariant. Because $k[\bar{\Gamma}]$ has an \mathbb{N} -grading with k in degree 0, $\mathbb{H}_{\text{cdh}}(X; K) \simeq \mathbb{H}_{\text{cdh}}(X[\bar{\Gamma}]; K)$, i.e., $\mathbb{H}_{\text{cdh}}(X[\bar{\Gamma}], X; K) \simeq 0$. Hence

$$|\mathcal{F}_{HC}(X[\bar{\Gamma}], X)[1]| \xleftarrow{\sim} \mathcal{F}_K(X[\bar{\Gamma}], X) \xrightarrow{\sim} K(X[\bar{\Gamma}], X).$$

The final assertion follows from Lemma A.6. \square

Example 2.2. If $\bar{\Gamma} = \mathbb{N}^n$, this proves that $K(k[x_1, \dots, x_n])/K_n(k)$ is a continuous $W_\Gamma(k)$ -module. We give more detail in Corollary 7.4 below.

ex:xy=zz

Example 2.3. Let Γ be the submonoid of \mathbb{N}^2 generated by $x = (1, 0)$, $y = (1, 2)$ and $z = (1, 1)$ so that $k[\Gamma]$ is $k[x, y, z]/(z^2 = xy)$. By [8, 4.3], $K_1(k[\Gamma], k) \cong k$ with the k in Γ -degree $(2, 2)$; $a \in k$ corresponds to the class of the matrix $\begin{pmatrix} 1+az & ax \\ ay & 1-az \end{pmatrix}$. More generally, if $i > 0$ then $K_{2i}(\mathbb{Q}[\Gamma], \mathbb{Q}) = 0$ and $K_{2i-1}(\mathbb{Q}[\Gamma], \mathbb{Q}) \cong \mathbb{Q}$.

Since k is a field, it follows that $K_q(k[\Gamma], k) \cong \bigoplus_{r \geq 0} \Omega_k^{q-1-2r}$. Replacing k by $k[\mathbb{N}^p] = k[x_1, \dots, x_p]$, it follows that

$$K_1(k[\Gamma \wedge \mathbb{N}^p], k[\mathbb{N}^p]) \cong k[\mathbb{N}^p] \cong K_1(k[\Gamma], k) \oplus \bigoplus_{\rho \subset \mathbb{N}^p} tk[t]$$

(the rays ρ in \mathbb{N}^p correspond to maximal inclusions of $k[t]$ into $k[\mathbb{N}^p]$), and more generally that $K_q(k[\Gamma \wedge \mathbb{N}^p], k)$ is a sum $\bigoplus_{r \geq 0} \Omega_{k[x_1, \dots, x_p]}^{q-1-2r}$, where

$$\Omega_{k[x_1, \dots, x_p]}^{q-1-2r} \cong \bigoplus_{i+j=q-1-2r} \Omega_k^i \otimes \Omega_{\mathbb{Q}[x_1, \dots, t_p]}^j.$$

sec:TC

3. TOPOLOGICAL CYCLIC HOMOLOGY

The purpose of this section is to relate the K -groups $K_n(X[\bar{\Gamma}], X)$ to topological cyclic homology, in preparation for Sections 4 and 6.

We begin by recalling the definition of $\mathrm{TC}(R)$, following Nikolaus–Scholze [21, II.1]. Let T denote the circle group. A *cyclotomic spectrum* is a spectrum E together with T -equivariant maps $E \rightarrow E^{tC_p}$ for all primes p .

For example, if R is a ring then $\mathrm{THH}(R)$ is a cyclotomic spectrum, and $\mathrm{TC}(R) = \mathrm{TC}(\mathrm{THH}(R))$ is defined in [21, II.1.8] to be the mapping spectrum

$$\mathrm{TC}(R) = \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}, \mathrm{THH}(R)).$$

We may also define $\mathrm{TC}^-(R) = \mathrm{THH}(R)^{hT}$; defining $TP(R)$ as the product over all primes of $(\mathrm{THH}(R)^{tC_p})^{hT}$, there is a functorial fiber sequence (see [21, II.1.9]):

SBI (3.1)
$$\mathrm{TC}(R) \rightarrow \mathrm{TC}^-(R) \rightarrow TP(R).$$

Let X be a finite dimensional noetherian scheme. There is no difficulty in defining $\mathrm{THH}(X)$ by Zariski descent, and we define $\mathrm{TC}(X)$ to be $\mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}, \mathrm{THH}(X))$.

The cyclotomic trace map $K \rightarrow \mathrm{TC}$ of [21, 1.2] induces a homotopy cartesian square:

$$\boxed{\text{square:K-TC}} \quad (3.2) \quad \begin{array}{ccc} K(X) & \longrightarrow & \mathbb{H}_{\mathrm{cdh}}(X, K) \\ \downarrow & & \downarrow \\ \mathrm{TC}(X) & \longrightarrow & \mathbb{H}_{\mathrm{cdh}}(X, \mathrm{TC}). \end{array}$$

This follows from Land–Tamme [20, B.3], given the fact [20, 3.6] that the fiber K^{inf} of the cyclotomic map $K \rightarrow \mathrm{TC}$ is “truncating”; a spectrum-valued functor E on perfect ∞ -categories is called *truncating* if for each connective A_∞ -spectrum A , $E(A) \rightarrow E(\pi_0 A)$ is an equivalence, and E sends exact sequences to fiber sequences. Thus the homotopy fibers \mathcal{F}_K and $\mathcal{F}_{\mathrm{TC}}$ of the horizontal maps in (3.2) are equivalent. (This was proven under restrictive hypotheses by Geisser and Hesselholt in [13, Thm. C].)

By [18, 6.3] or [20, A.5], $\mathbb{H}_{\mathrm{cdh}}(X, K) \cong KH(X)$ for any finite dimensional noetherian scheme X . By homotopy invariance, if $\bar{\Gamma}$ is either Γ , I or Γ/I then $KH(X[\bar{\Gamma}], X) = 0$. This proves:

$\boxed{\text{K-TC}}$ **Proposition 3.3.** *For any finite-dimensional noetherian scheme X ,*

$$K(X[\bar{\Gamma}], X) \xrightarrow{\simeq} \mathcal{F}_{\mathrm{TC}}(X[\bar{\Gamma}], X).$$

4. RAY DECOMPOSITION OF K -GROUPS IN CHARACTERISTIC p .

$\boxed{\text{sec:TC-K}}$

Let \mathbb{Z} denote the spectrum $H\mathbb{Z}$, regarded as a cyclotomic spectrum with trivial action by the circle group T . As pointed out in [16], when $k = \mathbb{Z}/p$, there is a T -equivariant map

$$\mathbb{Z} \rightarrow \mathbb{Z}/p \simeq \tau_{\geq 0} \mathrm{TC}(k) \rightarrow \mathrm{THH}(k).$$

(This fails for $k = \mathbb{Z}$.) By the argument of [16, Sec. 1], this gives an equivariant decomposition of cyclotomic spectra for each scheme X over \mathbb{Z}/p :

$$\mathrm{THH}(X[\bar{\Gamma}]) \cong \mathrm{THH}(X) \otimes_{\mathbb{S}} B^{\mathrm{cy}}\bar{\Gamma}_+ \cong \mathrm{THH}(X) \otimes_{\mathbb{Z}} (\mathbb{Z} \otimes_{\mathbb{S}} B^{\mathrm{cy}}\bar{\Gamma}_+).$$

Here $B^{\mathrm{cy}}\bar{\Gamma}$ is the cyclic nerve of $\bar{\Gamma}$; it is also called $N^{\mathrm{cy}}(\bar{\Gamma})$. With respect to this decomposition, each Frobenius $\varphi : \mathrm{THH} \rightarrow \mathrm{THH}^{tC_p}$ factors as a composition $\varphi \otimes \tilde{\varphi}$, where $\varphi(X) : \mathrm{THH}(X) \rightarrow \mathrm{THH}(X)^{tC_p}$, and $\tilde{\varphi} : B^{\mathrm{cy}}\bar{\Gamma} \rightarrow (B^{\mathrm{cy}}\bar{\Gamma})^{hC_p}$ is the unstable Frobenius.

By [16, Sec. 1], there is an equivalence between the ∞ -category of chain complexes with T -action, $D(\mathbb{Z})^{BT}$, and the ∞ -category of mixed complexes over \mathbb{Z} . Under this equivalence, the object $\mathbb{Z} \otimes B^{\mathrm{cy}}\bar{\Gamma}_+$ of

$D(\mathbb{Z})^{BT}$ corresponds to the mixed complex $C_*(\mathbb{Z}[\bar{\Gamma}])$ of $\mathbb{Z}[\bar{\Gamma}]$. The $\bar{\Gamma}$ -grading gives a decomposition of $\mathrm{THH}(X[\bar{\Gamma}], X)$ into a sum of terms $\mathrm{THH}(X) \otimes C_*(X[\bar{\Gamma}], X)_\rho$ indexed by the rays ρ of Γ .

TC-decomp

Proposition 4.1. *For every scheme X over \mathbb{Z}/p , each of the spectra $TC(X[\bar{\Gamma}], X)$, $TC^-(X[\bar{\Gamma}], X)$ and $TP(X[\bar{\Gamma}], X)$ decomposes as the sum of factors $TC(X[\bar{\Gamma}], X)_\rho$, etc., indexed by the rays ρ of Γ . There are similar decompositions for TC^- and TP .*

Proof. The complex $C_*(\mathbb{Z}[\bar{\Gamma}])$ is $\bar{\Gamma}$ -graded, and each unstable Frobenius $\tilde{\varphi}$ preserves the rays ρ of $\bar{\Gamma}$; for each $\eta \in \bar{\Gamma}$, $\tilde{\varphi}$ sends the degree η summand to the degree η^p summand. Hence $TC^-(X[\bar{\Gamma}]) \xrightarrow{\varphi^{hT}} TP(X[\bar{\Gamma}])$ decomposes as a sum of maps $TC^-_\rho \rightarrow TP_\rho$. It follows that $TC(\mathbb{Z}[\bar{\Gamma}], \mathbb{Z})$ is the sum over all rays ρ of $TC(\mathbb{Z}[\bar{\Gamma}])_\rho$. \square

The ray-indexed decompositions in Proposition 4.1 are compatible with the cdh fibrant replacement functor $TC \rightarrow \mathbb{H}_{\mathrm{cdh}}(-, TC)$. Thus they give a decomposition of the fiber $\mathcal{F}_{TC}(X[\bar{\Gamma}], X)$ as the sum over all rays ρ in Γ of $\mathcal{F}_{TC}(X[\bar{\Gamma}], X)_\rho$.

Combining this with Proposition 3.3, this proves

cor:decomp

Corollary 4.2. *For any finite-dimensional noetherian scheme X over \mathbb{Z}/p , the spectrum $K(X[\bar{\Gamma}], X)$ decomposes as a sum of spectra $K(X[\bar{\Gamma}], X)_\rho$, indexed by the rays ρ of Γ . Hence there are natural decompositions $K_n(X[\bar{\Gamma}], X) \cong \bigoplus_\rho K_n(X[\bar{\Gamma}], X)_\rho$.*

Schlichtkrull

5. THE TRANSFER IN THH

Let G be a pointed, abelian monoid, and K a submonoid such that, as a K -set, G is free of rank m . Schlichtkrull [24] constructed a transfer map $THH(RG) \rightarrow THH(RH)$ in the special case when G and K are groups (with a disjoint basepoint); the purpose of this section is to show that much of his construction goes through in the monoid setting.

def:EGG

Definition 5.1. Let $\mathcal{E}G$ denote the simplicial set $EG \times_G G^{\mathrm{ad}}$, where G^{ad} denotes G as a trivial G -set. It is the sum $\bigvee_{g \in G} EG \times \{g\}$.

In [24, Sec. 6], Schlichtkrull constructs maps

$$\mathcal{E}G \xrightarrow{\mathrm{trf}} \Gamma^+(EG \times_K G^{\mathrm{ad}}), \quad EG \times_K G^{\mathrm{ad}} \xrightarrow{\kappa} \Gamma^+ \mathcal{E}K.$$

The first is the topological transfer map of the m -sheeted covering space $EG \times_K G^{\mathrm{ad}} \rightarrow \mathcal{E}G$, and the second is the K -set projection $G \cong \bigvee_i K\gamma_i \rightarrow K\gamma_1 \cong K$, where $\{\gamma_1, \dots, \gamma_m\}$ is a K -basis of the K -set G . He writes Res for the composition

$$\Gamma^+(\kappa) \circ \mathrm{trf} : \Sigma^\infty(\mathcal{E}G) \rightarrow \Gamma^+ \Sigma^\infty(\mathcal{E}K).$$

We will apply his results to $G = \bar{\Gamma} \times \langle t \rangle$ and $K = \bar{\Gamma} \times \langle t^m \rangle$.

The following result is based on Schlichtkrull's calculations in [24]. For simplicity, we write $\bar{\Gamma}[t]$ for the monoid $\bar{\Gamma} \times \langle t \rangle$ and fix γt^n in $\bar{\Gamma}[t]$. Recall from [7, 2.2] that $THH(R[\bar{\Gamma}])$ is S^1 -equivariantly $THH(R) \wedge B^{\text{cy}}\bar{\Gamma}$, and that the $\bar{\Gamma}$ -grading comes from $B^{\text{cy}}\bar{\Gamma} = \bigvee_{g \in \bar{\Gamma}} B^{\text{cy}}(\bar{\Gamma}; g)$. There is a similar equivalence for $B^{\text{cy}}(\bar{\Gamma}[t])$.

prop:trf

Proposition 5.2. *For $\text{trf} : THH(R) \wedge B^{\text{cy}}\bar{\Gamma}[t] \rightarrow THH(R) \wedge B^{\text{cy}}\bar{\Gamma}[t]$:*
 (a) *if $m \nmid n$ then $\text{trf} = 0$ on the γt^n component of $THH(R) \wedge B^{\text{cy}}\bar{\Gamma}[t]$;*
 (b) *if $m|n$, trf sends the γt^n -component of $THH(R) \wedge B^{\text{cy}}\bar{\Gamma}[t]$ to itself.*

Proof. If $n = 0$, we get $B^{\text{cy}}\bar{\Gamma}[t]_{\gamma} \simeq B^{\text{cy}}\bar{\Gamma}_{\gamma} \wedge B^{\text{cy}}\langle t \rangle_1$ and $B^{\text{cy}}\langle t \rangle_1 \simeq S^0$ by [7, 2.9.2] and [24]. The same is true for $B^{\text{cy}}\langle t^m \rangle$, and the transfer is simply the identity.

If $n > 0$, we get $B^{\text{cy}}\bar{\Gamma}[t]_{\gamma t^n} = B^{\text{cy}}\bar{\Gamma}_{\gamma} \wedge B^{\text{cy}}\langle t \rangle_{t^n}$; by [23, 3.20–21] this is $B^{\text{cy}}\bar{\Gamma}_{\gamma} \wedge B^{\text{cy}}\langle t, 1/t \rangle_{t^n}$. In addition, the following diagram commutes.

$$\begin{array}{ccc}
 THH(R) \wedge B^{\text{cy}}\bar{\Gamma}_{\gamma} \wedge B^{\text{cy}}\langle t \rangle_{t^n} & \xrightarrow{\simeq} & THH(R) \wedge B^{\text{cy}}\bar{\Gamma}_{\gamma} \wedge B^{\text{cy}}\langle t, 1/t \rangle_{t^n} \\
 \downarrow \simeq & & \downarrow \simeq \\
 THH(R[t]_{t^n}) \wedge B^{\text{cy}}\bar{\Gamma}_{\gamma} & \xrightarrow{\simeq} & THH(R[t, 1/t]_{t^n}) \wedge B^{\text{cy}}\bar{\Gamma}_{\gamma} \\
 \downarrow \text{trf} \wedge 1 & & \downarrow \text{trf} \wedge 1 \\
 THH(R[t^m]) \wedge B^{\text{cy}}\bar{\Gamma} & \xrightarrow{\simeq} & THH(R[t^m, 1/t^m]) \wedge B^{\text{cy}}\bar{\Gamma}
 \end{array}$$

The vertical compositions are the transfer maps from $THH(R\bar{\Gamma}[t]_{\gamma t^n})$ to $THH(R\bar{\Gamma}[t^m])$ and $THH(R\bar{\Gamma}[t^m, 1/t^m])$, respectively. Now apply [24, Thm. A] to see that the lower right vertical map is 0 unless $m|n$, and maps to the γt^n -component otherwise. \square

Remark 5.2.1. In [24, Prop. 6.1], Schlichtkrull assumes that G and K are groups (with disjoint basepoints), so that $\mathcal{E}G \simeq BG_+$, and compares Res to the restriction map $N^{\text{cy}}G \rightarrow N^{\text{cy}}K$ using a map $\phi : N^{\text{cy}}G \rightarrow \mathcal{E}G$. This approach does not carry over to monoids.

THH-W

Corollary 5.3. *Let ϕ and ψ be graded $R\bar{\Gamma}$ -algebra endomorphisms of $R\bar{\Gamma}[t]$. The $\bar{\Gamma}$ -grading on $THH(R\bar{\Gamma})$ is preserved by the composition*

$$THH(R\bar{\Gamma}) \xrightarrow{i} THH(R\bar{\Gamma}[t]) \xrightarrow{\psi \circ \text{trf} \circ \phi} THH(R\bar{\Gamma}[t]) \xrightarrow{t=1} THH(R\bar{\Gamma}).$$

TC-W

Corollary 5.4. *Let ϕ and ψ be graded $R\bar{\Gamma}$ -algebra endomorphisms of $R\bar{\Gamma}[t]$. The composition*

$$TC(R\bar{\Gamma}) \xrightarrow{i} TC(R\bar{\Gamma}[t]) \xrightarrow{\psi \circ \text{trf} \circ \phi} TC(R\bar{\Gamma}[t]) \xrightarrow{t=1} TC(R\bar{\Gamma})$$

preserves the ray decomposition on $TC(R\bar{\Gamma})$.

Proof. This follows because TC is constructed from the cyclotomic structure on THH in a way that preserves the ray decomposition (but *not* the $\bar{\Gamma}$ -grading). Indeed, the transfer maps the γ -component of $TC(R[\bar{\Gamma}], R)$ to the γ^p component; see the discussion on page 144 of [16]. \square

K-W

Corollary 5.5. *Let ϕ and ψ be graded $R\bar{\Gamma}$ -algebra endomorphisms of $R\bar{\Gamma}[t]$. The ray decomposition of $K(R\bar{\Gamma})$ is preserved by the composition*

$$K(R\bar{\Gamma}) \xrightarrow{i} K(R\bar{\Gamma}[t]) \xrightarrow{\psi \circ \text{trf} \circ \phi} K(R\bar{\Gamma}[t]) \xrightarrow{t=1} K(R\bar{\Gamma}).$$

Proof. Since everything is natural in R -algebras, the transfer trf and composition $\psi \circ \text{trf} \circ \phi$ act on the $\bar{\Gamma}$ -graded spectrum $\mathbb{H}_{\text{cdh}}(\text{Spec } R, TC(\mathcal{O}[\bar{\Gamma}], \mathcal{O}))$, and the action is compatible with the ray decomposition. Hence the same is true for the homotopy fiber $\mathcal{F}(R[\bar{\Gamma}], R)$ of the map

$$TC(R\bar{\Gamma}, R) \rightarrow \mathbb{H}(\text{Spec } R, TC(\mathcal{O}[\bar{\Gamma}], \mathcal{O})).$$

Finally, we observe that the isomorphism $K_*(R\bar{\Gamma}, R) \cong \pi_* \mathcal{F}_{TC}(R\bar{\Gamma}, R)$ is compatible with the $\bar{\Gamma}$ -decomposition and with the actions of trf and $\psi \circ \text{trf} \circ \phi$. \square

6. RAY COMPONENTS ARE $W(k)$ -MODULES

sec:main

In [26], a functorial formula was given for a continuous action of the Witt vectors $W(k)$ on the relative K -theory $K_*(A, A_0)$ of any \mathbb{N} -graded k -algebra A . This applies to Γ -graded A once we choose a homomorphism $p : \Gamma \rightarrow \mathbb{N}$ such that $p(\gamma) \neq 0$ for all γ with $A_\gamma \neq 0$; these exist by our assumption that Γ contains no line. Each grading $\Gamma \rightarrow \mathbb{N}$ also defines a $W(k)$ -action on $HH_*(A, A_0)$, etc.; see [11]. Let $i : A \rightarrow A[t]$ denote the injection sending $a \in A_n$ to at^n . Recall from [26] that multiplication by $\omega \in W(k)$ on $K(A, A_0)$ is the composition

eq:K(A)

$$(6.1) \quad K(A, A_0) \xrightarrow{i} K(A[t], A) \xrightarrow{\omega} K(A[t], A) \xrightarrow{t=1} K(A, A_0).$$

Here is one way to choose a grading $\Gamma \rightarrow \mathbb{N}$. Recall that Γ is the monoid of lattice points in the “dual cone” of a rational polyhedral cone $\sigma \subset \mathbb{R}^n$. Set $\Gamma^\vee = \{w \in \mathbb{Z}^n : w \cdot \Gamma \geq 0\}$, and choose $w \in \Gamma^\vee$ with $w \cdot v > 0$ for all $0 \neq v \in \Gamma$. This gives an \mathbb{N} -grading on any Γ -graded algebra A such that A_0 is the ring of elements of (\mathbb{N} -) degree 0, and hence a $W(k)$ -module structure on any $K_*(A, A_0)$, and in particular on $K_*(k[\Gamma], k)$. However, as pointed out in [26, 5.3], the $W(k)$ -module structure can depend on the choice of w .

Let R be a commutative k -algebra, and X a finite-dimensional noetherian scheme with $R = H^0(X, \mathcal{O}_X)$. Note that $W(R)$ is a $W(k)$ -algebra. Fixing a choice of w , we obtain:

thm:main

Theorem 6.2. *Each $K_n(R[\bar{\Gamma}], R)$ and $K_n(X[\bar{\Gamma}], X)$ is a continuous $W(R)$ -module; the latter decomposes as a sum $\bigoplus_{\rho} K_n(X[\bar{\Gamma}], X)_{\rho}$ of continuous $W(R)$ -modules, indexed by the rays ρ of Γ .*

Proof. Since every $\omega \in W(R)$ is a product $\omega = \prod_m (1 - r_m t^m)$, it suffices to consider the action of each $\omega = (1 - r t^m)$ on $K_n(R[\bar{\Gamma}], R)$; the case $K_n(X[\bar{\Gamma}], X)$ is similar. Set $A = R[\bar{\Gamma}]$.

By [26], ω acts on $K(A[t], A)$ as $F_m[r]V_m$. Now $[r]$ and V_m correspond to the graded A -algebra homomorphisms $t \mapsto r t$ and $t \mapsto t^m$, and the operator F_m corresponds to the transfer map $K(A[t]) \rightarrow K(A[t^m])$ followed by the equivalence associated with $A[t^m] \cong A[t]$. Thus, the composition (6.1) with $\omega = (1 - r t^m)$ preserves the ray decomposition by Corollary 5.5. \square

7. POLYNOMIAL RINGS

sec:poly

The results in this section are independent of the rest of this paper. Let k be any unital ring. According to the Fundamental Theorem of K -theory [29, III.2.2, V.8.2], $K_q(k[t_1, \dots, t_n])/K_q(k)$ is the direct sum (for $1 \leq i \leq n$) of $\binom{n}{i}$ copies of $N^i K_q(k)$.

Recall that $k[\mathbb{Z}^n] = k[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ is the group ring of the free abelian group \mathbb{Z}^n , and the Farrell–Jones conjecture in K -theory is known for the free abelian group \mathbb{Z}^n ; see Quinn [22]. Using this, J. Davis proved in [10] (compare [19]) that $K_q(k[\mathbb{Z}^n])$ is the direct sum of $\binom{n}{r}$ copies of $K_{q-r}(k)$ ($r = 0, \dots, n$) and the direct sum over the rays $\rho \subset \mathbb{Z}^n$ of many copies of $NK_{q-r}(k)$ ($r = 0, \dots, n-1$):

$$\bigoplus_{r=0}^n \binom{n}{r} K_{q-r}(k) \quad \oplus \quad \bigoplus_{\rho \subset \mathbb{Z}^n} \bigoplus_{r=0}^{n-1} \binom{n-1}{r} NK_{q-r}(k),$$

(Every maximal cyclic subgroup of \mathbb{Z}^n is the union of two rays.) Using the notation that $L^r K_q = K_{q-r}$ and $L^r NK_q = NK_{q-r}$, this becomes

eq:Davis

$$(7.1) \quad K_q(k[\mathbb{Z}^n]) \cong (1 + L)^n K_q(k) \oplus \bigoplus_{\rho \subset \mathbb{Z}^n} (1 + L)^{n-1} NK_q(k).$$

When $n = 1$ there are only 2 rays, so we recover the usual formula for $K_q(k[x, 1/x])$. The formula for $n = 2$ implies the formula:

$$K_q(k[x, y], k) \cong \bigoplus_{\rho \subset \mathbb{N}^2} NK_q(k) \oplus NK_{q-1}(k).$$

Let us write \mathbb{N}_+^p for the sub-semigroup of elements (n_1, \dots, n_p) in \mathbb{N}^p with all $n_i > 0$. Since $\mathbb{N}^2 - \{0\} = (\mathbb{N}_+ \times 0) \cup (0 \times \mathbb{N}_+) \cup \mathbb{N}_+^2$, the above

formula implies that

$$N^2 K_q(k) \cong \bigoplus_{\rho \in \mathbb{N}_+^2} N K_q(k) \oplus N K_{q-1}(k).$$

thm:NnK

Theorem 7.2. *For every unital ring k ,*

$$N^n K_q(k) \cong \bigoplus_{\rho \in \mathbb{N}_+^n} (1 + L)^{n-1} N K_q(k).$$

This was conjectured by Davis in [10], who proved it in [10, Cor. 15] when the NK_i are countable torsion groups. We first give a quick argument to establish Theorem 7.2 when k is a \mathbb{Q} -algebra.

Proof of 7.2 for commutative \mathbb{Q} -algebras. Identifying $t\mathbb{Q}[t]$ and $\Omega_{\mathbb{Q}[t]/\mathbb{Q}}^1$, the formula of [5, 4.2] is that for $p > 0$:

$$N^{p+1} K_q(k) \cong x_1 \cdots x_p \mathbb{Q}[x_1, \dots, x_p] \otimes_{\mathbb{Q}} (1 + L)^p N K_q.$$

Plugging in $NK_q = TK_q \otimes t\mathbb{Q}[t]$ (from [5, 0.1 and 4.2]) yields Theorem 7.2 in this case. \square

For the proof of 7.2 for arbitrary k , recall that the wreath group $W_n = (\mathbb{Z}/2) \wr S_n$ of order $2^n n!$ acts as automorphisms on the monomials in $k[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$; the subgroup $(\mathbb{Z}/2)^n$ interchanges the x_i and x_i^{-1} pairwise, and the subgroup S_n permutes the x_i . It also acts on \mathbb{Z}^n (by signed permutation matrices), on the primitive vectors in \mathbb{Z}^n , and on the rays in \mathbb{Z}^n . For each $r < n$, let R_r denote the set of all rays in $(x_1, \dots, x_r, 0, \dots, 0)$ with all x_i positive. The orbit of R_r under W_n has $2^r \binom{n}{r}$ disjoint components, and the stabilizer of R_r is $S_r \times W_{n-r}$. For example, when $r = 1$ and $n = 2$, the orbit of $(1, 0) \in R_1$ consists of the 4 points $(\pm 1, 0)$ and $(0, \pm 1)$.

Similarly, the group S_n acts on \mathbb{N}^n , and the orbit of R_r has $\binom{n}{r}$ disjoint components, and the stabilizer of R_r is $S_r \times S_{n-r}$.

Proof of Theorem 7.2. We proceed by induction on n , the case $n = 1$ being trivial. Let Q denote the right hand side of Theorem 7.2.

Recall that the W_r -orbit of \mathbb{N}_+^r in \mathbb{Z}^r has order 2^r . By induction, for $r < n$ the images of the 2^r embeddings of $N^r K_q(k) \subset K_q(k[\mathbb{N}^r])$ into $K_q(k[\mathbb{Z}^r])$ give embeddings of $(1 + L)^{r-1} N K_q(k)$ into $K_q(k[\mathbb{Z}^r])$. The image is the sum of copies of $(1 + L)^{r-1} N K_q(k)$, indexed by the rays in the 2^r copies of R_r . As r varies, we get a subgroup of $K_q(k[\mathbb{Z}^n])$ whose quotient is 2^r copies of Q .

This induces a W_n -equivariant isomorphism from $\bigoplus_{2^n} N^n K_q(k)$ to $\bigoplus_{2^n} Q$. Taking invariants yields a map from $N^n K_q(k)$ to

$$(\bigoplus_{2^n} Q)^{W_n} = \bigoplus_{\rho \in \mathbb{N}_+^n} (1 + L)^{n-1} N K_q(k). \quad \square$$

Remark 7.3. The referee points out that the case of general n follows from the case $n = 2$ by induction. When $n = 2$, the rays on $(1, 0)$ and $(0, 1)$ correspond to the two canonical inclusions of $NK_q(k)$ in $K_q(k[x, y])$, given by the subrings $k[x]$ and $k[y]$ of $k[x, y]$.

`cor:NnK`

Corollary 7.4. *For every unital ring k ,*

$$K_q(k[x_1, \dots, x_n]) \cong K_q(k) \oplus \bigoplus_{r=1}^n \bigoplus_{\rho \subseteq \mathbb{N}_+^r} \binom{n}{r} (1+L)^{r-1} NK_q(k).$$

For example, the summand for $r = 1$ is n copies of $NK_q(k)$, as $\rho = \mathbb{N}_+^1$. When $n = 2$ we have the decomposition:

$$\begin{aligned} K_q(k[x, y]) &\cong K_q(k) \oplus 2NK_q(k) \oplus \bigoplus_{\rho \subseteq \mathbb{N}_+^2} (NK_q(k) \oplus NK_{q-1}(k)) \\ &\cong K_q(k) \oplus \bigoplus_{\rho \subseteq \mathbb{N}^2} NK_q(k) \oplus \bigoplus_{\rho \subseteq \mathbb{N}_+^2} NK_{q-1}(k). \end{aligned}$$

This is a strengthening of Bass' formula

$$K_q(k[x, y]) \cong (1+N)^2 K_q(k) = K_q(k) \oplus 2NK_q(k) \oplus N^2 K_q(k).$$

APPENDIX A. WITT VECTORS ON AFFINE MONOIDS

`sec:Gamma-Witt`

Let Γ be a submonoid of \mathbb{Z}^n having no invertible elements except 0. We assume that Γ is *normal* in the sense that if $v \in \mathbb{Z}^n$ and $dv \in \Gamma$ for some positive integer d then $v \in \Gamma$. The following definition is generalized from [3], where $\Gamma = \mathbb{N}^r$.

Definition A.1. A subset S of $\Gamma \setminus \{0\}$ is a Γ -*truncation set* if, for all natural numbers d and $s \in \Gamma$, $ds \in S$ implies $s \in S$.

Remark A.1.1. Let I be an ideal in Γ . Then the subset $\Gamma \setminus (I \cup \{0\})$ is a Γ -truncation set. Indeed, if $\gamma \in \Gamma$ and $d\gamma \notin I$ for some $d \in \mathbb{N}$, then $\gamma \notin I$.

In this section, we fix a commutative ring k and define the rings $W_\Gamma(k)$ and $W_{\Gamma/I}(k)$, so that when $\Gamma = \mathbb{N}$ and $I = \{n \in \mathbb{N} \mid n \geq d\}$ we recover the classical ring $W(k)$ of big Witt vectors and truncated Witt vectors on k . Much of this section is adopted from [2], [15] and [30].

`def:prim`

Definition A.2. A nonzero element v of Γ is called *primitive* if it is not a multiple of any other element of Γ . Each primitive element v generates a maximal cyclic submonoid ρ of Γ , called a *ray*. Because Γ has no invertible elements, each ray has the form $\mathbb{N}v$ for a unique primitive v . As a pointed set, Γ is the wedge of its rays.

Fix a commutative ring k . The ring of *big Witt vectors* $W_\Gamma(k)$ over k has for its underlying set k^Γ , the pointed functions $\Gamma \xrightarrow{a} k$ (functions which send 0 to 0). Addition and multiplication in $W_\Gamma(k)$ are determined by Lemma A.4 below, exactly as in the classical ring $W(k)$. We write a_γ for $a(\gamma)$, so a can be written as the vector $\{a_\gamma\}_{\gamma \in \Gamma}$. If $S \subset \Gamma \setminus \{0\}$ is a Γ -truncation set, we can similarly define the truncated Witt vectors $W_S(k)$; the underlying set is the set of functions $S \rightarrow k$. If $S = \Gamma \setminus (I \cup \{0\})$ for an ideal I , then we write $W_{\Gamma/I}(k)$ for the ring $W_S(k)$. In this case, $S \cong \Gamma/I \setminus \{0\}$.

Here are some special elements of $W_\Gamma(k)$. For each $\gamma \in \Gamma$, we write $[\gamma]$ for the Kronecker δ -function sending γ to 1 and η to 0 if $\eta \neq \gamma$. For $r \in k$, $r[\gamma]$ sends γ to r and $\eta \neq \gamma$ to 0. We write δ_{prim} for the Kronecker δ -function sending primitive elements to 1 and non-primitives to 0.

For any nonzero $\gamma \in \Gamma$, we define its *content* $c(\gamma)$ to be the least common multiple of $\{e \in \mathbb{N} : \gamma \in e \cdot \Gamma\}$. As an element of \mathbb{Z}^n , $\gamma = (e_1, \dots, e_n)$, and $c(\gamma) = \gcd\{e_i\}$. It is easy to see that γ is primitive if and only if $c(\gamma) = 1$. Because Γ is normal, $v = \gamma/c(\gamma)$ is a primitive element of Γ and the ray $\rho = \mathbb{R}\gamma \cap \Gamma$ is $\mathbb{N}v$.

Given a Γ -truncation set S , let $\prod_S k$ denote the set k^S of functions, made into a ring under the termwise sum $(a+b)_\gamma = a_\gamma + b_\gamma$ and product $(ab)_\gamma = a_\gamma b_\gamma$. This ring is just the product of copies of the ring k , indexed by S .

ghost

Definition A.3. The *ghost map* $\text{gh} : W_S(k) \rightarrow \prod_S k$ is the function taking $a = \{a_\gamma\}$ to the function $\text{gh}(a) : S \rightarrow k$ defined by

$$\text{gh}(a) : \eta \mapsto \sum_{e\gamma=\eta} c(\gamma)a_\gamma^e.$$

The η -component of $\text{gh}(r[\gamma])$ is $c(\gamma)r^e$ if $\eta = e\gamma$, and 0 otherwise.

W-ring

Lemma A.4. [2, 2.3] *There is a unique way, functorial in k , to put a ring structure on the set $W_S(k)$ so that $\text{gh} : W_S(k) \rightarrow \prod_S k$ is a ring homomorphism.*

The identity element of the ring $W_S(k)$ is δ_{prim} , and $\text{gh}(\delta_{\text{prim}})$ is the identity of $\prod_S k$.

Proof. $W_S(k)$ and k^S are the products over all rays $\rho = \mathbb{N}v$ in Γ of the sets $W_{\mathbb{N}v \cap S}(k)$ and $k^{\mathbb{N}v \cap S}$, respectively. Since the ghost map respects this decomposition, we may assume that S is a truncation set in \mathbb{N} . In this case, the result is classical. \square

idempotent

Corollary A.5. *a) If v is primitive, $[v]$ is an idempotent element of $W_\Gamma(k)$; the idempotent $\text{gh}([v])$ of $\prod_{\Gamma \setminus \{0\}} k$ is the projection onto $\prod_{\mathbb{N}v \setminus \{0\}} k$, and $[v] \cdot W_\Gamma(k) \cong W(k)$.*

b) $W_\Gamma(k)$ is the product over all rays $\rho = \mathbb{N}v$ in Γ of the classical ring $W(k)$ of big Witt vectors:

$$W_\Gamma(k) = \prod_{\mathbb{N}v} W_{\mathbb{N}v}(k) \cong \prod_{\text{rays } \rho} W(k).$$

In particular, when $\Gamma = \mathbb{N}$, the ring $W_{\mathbb{N}}(k)$ is the classical ring $W(k)$ of big Witt vectors over k .

When $\mathbb{Q} \subseteq k$, the ghost map is an isomorphism: $W_\Gamma(k) \cong \prod_{\Gamma \setminus \{0\}} k$. This is because it is an isomorphism in the classical setting.

Remark A.5.1. By abuse of notation, for each primitive $v \in \Gamma$, we will write t_v for $[v]$ and t_v^n for $[nv]$, so that the abelian group structure on $W_{\mathbb{N}v}(k)$ may be interpreted as the multiplicative group $(1 + t_v k[[t_v]])^\times$. Since the abelian group structure on $W_\Gamma(k)$ corresponds to the group $\prod_v (1 + t_v k[[t_v]])^\times$, and not a subgroup of the units in $k[[\Gamma]]$, the power series interpretation of $W_\Gamma(k)$ is inconvenient in our setting.

Remark A.5.2. If $I \subset \Gamma$ is an ideal of Γ , let Γ/I denote the quotient monoid with underlying set $\Gamma \setminus I$. The quotient map $\Gamma \rightarrow \Gamma/I$ yields a ring homomorphism $W_\Gamma(k) \rightarrow W_{\Gamma/I}(k)$ sending the vector $\{r_\gamma\}_{\gamma \in \Gamma \setminus \{0\}}$ to $\{r_\gamma\}_{\gamma \in I \cup \{0\}}$.

Continuous $W_\Gamma(k)$ -modules. We say that a $W_\Gamma(k)$ -module A is *continuous* if for each element $a \in A$ there is a finite subset S_a of Γ such that the products $[\gamma] * a$ vanish for all $\gamma \notin S_a$.

Since $W_\Gamma(k)$ is a product ring, every $W_\Gamma(k)$ -module A is contained in a product $\prod A_\rho$ of $W(k)$ -modules, indexed by the rays $\rho = \mathbb{N}v$ in Γ ; $A_\rho = [v] \cdot A$. The following characterization is immediate.

oplus

Lemma A.6. *A $W_\Gamma(k)$ -module A is a continuous module if and only if $A = \bigoplus_\rho A_\rho$ and each A_ρ is a continuous $W(k)$ -module.*

gh-module

Example A.7. Any Γ -graded k -module $A = \bigoplus_{\gamma \neq 0} A_\gamma$ may be regarded as a continuous module over the product ring $\prod_\Gamma k$, with $(x_\gamma) \in \prod_\Gamma k$ acting via multiplication by x_γ on the component A_γ , $\gamma \neq 0$. The restriction along the ghost map $W_\Gamma(k) \rightarrow \prod_\Gamma k$ defines a $W_\Gamma(k)$ -module structure on A . By Definition A.3, if $a \in A_\eta$ then

$$r[\gamma] * a = \begin{cases} c(\gamma)r^e a, & \text{if } \eta = e\gamma \text{ in } \Gamma, \\ 0, & \text{else.} \end{cases}$$

Since Γ is a submonoid of \mathbb{Z}^n , this module structure is continuous.

If $\mathbb{Q} \subseteq k$, the category of continuous $W_\Gamma(k)$ -modules is equivalent to the category of Γ -graded k -modules $A = \bigoplus_{\gamma \neq 0} A_\gamma$, where $\{A_\gamma\}_{\gamma \in \Gamma}$ is a family of k -modules.

REFERENCES

- [1] V. Angeltveit, Witt vectors and truncation posets, *Theory Appl. Categ.* 32 (2017), paper 8, 258–285.
- [2] V. Angeltveit, T. Gerhardt, M. Hill and A. Lindenstrauss, On the algebraic K -theory of truncated polynomial algebras in several variables, *J. K-theory* 13 (2014), 57–81. arXiv:1206.0247
- [3] J. Borger, The basic geometry of Witt vectors, I: The affine case, *Algebra Number Theory* 5 (2011), 231–285.
- [4] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, K -regularity, cdh -fibrant Hochschild homology, and a conjecture of Vorst, *J. AMS* 21 (2008), 547–561.
- [5] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, Bass’ NK groups and cdh -fibrant Hochschild homology, *Invent. Math.* 181 (2010), 421–448.
- [6] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, The K -theory of toric varieties, *Trans. AMS* 361 (2009), 3325–3341.
- [7] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, The K -theory of toric varieties in positive characteristic p , *J. Topology* 7 (2014), 247–286.
- [8] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, The K -theory of cones of smooth varieties, *J. Alg. Geom.* 22 (2013), 13–34.
- [9] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, Toric Varieties, Monoid Schemes and cdh descent, *J. reine angew. Math.* 698 (2015), 1–54.
- [10] J.F. Davis, Some remarks on Nil groups in algebraic K -theory, available at arXiv:0803.1641 (2008).
- [11] B. Dayton and C. Weibel, Module structures on the Hochschild and Cyclic Homology of Graded Rings, pp. 63–90 in *Algebraic K-theory and Algebraic Topology*, NATO ASI Series C, 407, Kluwer, 1993.
- [12] W. Fulton, *Introduction to Toric Varieties*, Princeton Univ. Press, 1993.
- [13] T. Geisser and L. Hesselholt, On the vanishing of negative K -groups, *Math. Ann.* 348 (2010), 707–736.
- [14] C. Haesemeyer, Descent properties of homotopy K -theory, *Duke Math. J.* 125 (2004), 589–619.
- [15] L. Hesselholt, The big de Rham–Witt complex, *Acta Math.* 214 (2015), 135–207.
- [16] L. Hesselholt and Nikolaus, Algebraic K -theory of planar cuspidal curves, K -theory in algebra, analysis and topology, *AMS Contemp. Math.* 749 (2020), 139–148.
- [17] C. Kassel, Cyclic homology, comodules, and mixed complexes, *J. Alg.* 107 (1987), 195–216.
- [18] M. Kerz, F. Strunk and G. Tamme, Algebraic K -theory and descent for blow-ups, *Invent. Math.* 211 (2018), 523–577.
- [19] W. Lück and D. Rosenthal, On the K - and L -theory of hyperbolic and virtually finitely generated groups, *Firum Math.* 26 (2014), 1565–1609.
- [20] M. Land and G. Tamme, On the K -theory of pullbacks, *Annals of Math.* 190 (2019), 877–930.
- [21] T. Nikolaus and P. Scholze, On topological cyclic homology, *Acta Math.* 221 (2018), 203–409.
- [22] F. Quinn, Algebraic K -theory over virtually abelian groups, *J. Pure Appl. Algebra* 216 (2012), 170–183.

- Rognes [23] J. Rognes, Topological logarithmic structures, In *New topological contexts for Galois theory and algebraic geometry (BIRS 2008)*, volume 16 of *Geom. Topol. Monogr.*, pages 401–544, Geom. Topol. Publ, Coventry, 2009.
- Schlicht1 [24] C. Schlichtkrull, The transfer map in topological Hochschild homology, *J. Pure Appl. Algebra* 133 (1998), 289–316.
- WH [25] C. Weibel, *An introduction to homological algebra*, Cambridge U. Press, 1994.
- W87 [26] C. Weibel, Module structures on the K -theory of graded rings, *J. Algebra* 105 (1987), 465–483.
- W96 [27] C. Weibel, Cyclic homology for schemes, *Proc. AMS* 124 (1996), 1655–1662.
- W97 [28] C. Weibel, The Hodge filtration and cyclic homology, *K-theory* 12 (1997), 145–164.
- WK [29] C. Weibel, *The K-book*, Grad. Studies in Math. 145, AMS, 2013.
- Wick [30] K. Wickelgren, Cartier’s first theorem for Witt vectors on $\mathbb{Z}_{\geq 0}^n - 0$, pp. 321–328 in *Algebraic topology: applications and new directions*, AMS Contemp. Math. 620, 2014.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MELBOURNE,
VIC 3010, AUSTRALIA

MATH. DEPT., RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ 08901, USA
Email address: weibel@math.rutgers.edu
URL: <http://math.rutgers.edu/~weibel>