

# SLICES OF CO-OPERATIONS FOR $KGL$

PABLO PELAEZ AND CHARLES WEIBEL

ABSTRACT. We verify a conjecture of Voevodsky, concerning the slices of co-operations in motivic  $K$ -theory.

## INTRODUCTION

Fix a finite-dimensional Noetherian separated base scheme  $S$ , and consider the motivic stable homotopy category  $\mathcal{SH}(S)$  as defined in [19]. We write  $KGL$  for the motivic spectrum representing homotopy invariant  $K$ -theory in  $\mathcal{SH}(S)$ . In this paper, we study Voevodsky’s conjecture on slices of co-operations for  $KGL$  (Conjecture 8 in [20]), which describes the motivic slices of the motivic spectra  $KGL \wedge \cdots \wedge KGL$ . We verify the conjecture when  $S$  is smooth over a perfect field (see Theorem 0.1 below).

To describe the conjecture, recall that for any motivic spectrum  $E$ , Voevodsky introduced a natural “slice” tower  $\cdots \rightarrow f_{q+1}E \rightarrow f_qE \rightarrow \cdots$  of motivic spectra, and defined triangulated “slice” functors  $s_q$  fitting into cofibration sequences  $f_{q+1}E \rightarrow f_qE \rightarrow s_qE$ . Roughly,  $\{f_qE\}$  is the analogue of the Postnikov tower in topology;  $s_q$  is the analogue of the functor  $X \mapsto K(\pi_q X, q)$ .

If  $E$  is a topological ring spectrum, the ring  $E_*E = \pi_*(E \wedge E)$  is called the *ring of co-operations* for  $E$ ; the name comes from [3], where  $\pi_*(KU \wedge KU)$  is worked out (see Section 1 below). The title of this paper comes from viewing the motivic spectrum  $KGL \wedge KGL$  as giving rise to co-operations for  $K$ -theory. The easy part of Voevodsky’s Conjecture 8 says that there is an isomorphism

$$s_q(KGL \wedge KGL) \cong (T^q \wedge H\mathbb{Z}) \otimes \pi_{2q}(KU \wedge KU).$$

Here  $H\mathbb{Z}$  is the motivic Eilenberg-Mac Lane spectrum in  $\mathcal{SH}(S)$ ,  $T$  is the motivic space represented by the pointed projective line, and the tensor product of a spectrum with an abelian group has its usual meaning (see 0.5).

Since a ring spectrum  $E$  is a monoid object in spectra, co-operations fit into a cosimplicial spectrum  $N^\bullet E = E^{\wedge \bullet + 1}$ . This is a general construction: in any category with product  $\wedge$ , a monoid  $E$  determines a triple  $\top X = X \wedge E$  and an augmented cosimplicial object  $n \mapsto X \wedge E^{\wedge n+1}$  for any object  $X$ ; the cofaces are given by the unit  $\mathbf{1} \rightarrow E$  and the codegeneracies are given by the product  $E \wedge E \rightarrow E$ ; see [22, 8.6.4]. We write  $N^\bullet E$  when  $X = \mathbf{1}$ , so  $N^n E = E^{\wedge n+1}$ .

**Voevodsky’s Conjecture 8** (Slices of co-operations for  $KGL$ ). *As cosimplicial motivic spectra, the  $q^{\text{th}}$  slice satisfies*

$$s_q N^\bullet(KGL) \cong (T^q \wedge H\mathbb{Z}) \otimes \pi_{2q} N^\bullet(KU).$$

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One reformulation of this conjecture is to take the direct sum over  $q$  and use the fact that  $s_*E = \bigoplus s_q(E)$  is a graded motivic ring spectrum (see [10, 3.6.13]). It is convenient to adopt the notation that  $E \otimes A_*$  denotes  $\bigoplus_q (T^q \wedge E) \otimes A_q$  for a motivic spectrum  $E$  and a graded ring  $A_*$ . In this notation, we prove:

**Theorem 0.1.** *Assume that  $S$  is smooth over a perfect field. Then there is an isomorphism of cosimplicial ring spectra*

$$s_*N^\bullet(KGL) \cong H\mathbb{Z} \otimes \pi_{2*}N^\bullet(KU).$$

*Remark 0.1.1.* If  $S$  is over a field of characteristic 0, it seems likely that Theorem 0.1 would follow from the work of Spitzweck [16] on Landweber exact spectra. Such an approach would depend heavily on the Hopkins-Morel-Hoyois theorem [7].

Conjecture 8 is intertwined with Voevodsky's conjectures 1, 7 and 10 in [20], that  $H\mathbb{Z} \leftarrow \mathbf{1} \rightarrow KGL$  induces isomorphisms

$$H\mathbb{Z} \cong s_0(H\mathbb{Z}) \xleftarrow{\cong} s_0(\mathbf{1}) \xrightarrow{\cong} s_0(KGL),$$

and thus that  $\mathrm{Hom}_{\mathcal{SH}(S)}(s_0(\mathbf{1}), s_0(\mathbf{1})) \cong H^0(S, \mathbb{Z})$ . These are known to hold when the base  $S$  is smooth over a perfect field, or singular over a field of characteristic 0, by the work of Voevodsky and Levine (see [9, 10.5.1] and [9, 11.3.6]).

Here is our main result, which evidently implies Theorem 0.1.

**Theorem 0.2.** *Let  $S$  be a finite-dimensional separated Noetherian scheme. Then (a) there are isomorphisms for all  $n \geq 0$ :*

$$s_0(KGL) \otimes \pi_{2*}(KU^{\wedge n}) \xrightarrow{\cong} s_*(KGL^{\wedge n}).$$

*These isomorphisms commute with all of the coface and codegeneracy operators except possibly  $\partial^0$  and  $\sigma^0$ .*

*(b) Assume in addition that  $s_0(\mathbf{1}) \rightarrow s_0(KGL)$  is an isomorphism in  $\mathcal{SH}(S)$ , and that  $\mathrm{Hom}_{\mathcal{SH}(S)}(s_0(\mathbf{1}), s_0(\mathbf{1}))$  is torsionfree. Then the maps in (a) are the components of an isomorphism of graded cosimplicial motivic ring spectra:*

$$s_0(KGL) \otimes \pi_{2*}N^\bullet(KU) \xrightarrow{\cong} s_*N^\bullet(KGL).$$

The case  $n = 0$  of Theorem 0.2(a), that  $s_0(KGL) \otimes \pi_{2*}KU \cong s_*KGL$ , is immediate from the periodicity isomorphism  $T \wedge KGL \cong KGL$  defining the motivic spectrum  $KGL$  and the formula  $\pi_{2*}KU \cong \mathbb{Z}[u, u^{-1}]$ . The need to pass to slices is clear at this stage, because  $\pi_{2n,n}KGL \cong K_n(S)$  for  $S$  smooth over a perfect field.

The left side of Theorem 0.2 is algebraic in nature, involving only the cosimplicial ring  $\pi_{2*}(KU^{\wedge n+1})$  and  $H = KU_0(KU)$ . In fact,  $\pi_{2*}(KU^{\wedge n+1})$  is the cobar construction  $C_\Gamma^\bullet(R, R) \cong KU_* \otimes H^{\otimes n}$  for the Hopf algebroid  $(R, \Gamma) = (KU_*, KU_*KU)$ . We devote the first four sections to an analysis of this algebra, focussing on the rings  $F = KU_0(\mathbb{C}\mathbb{P}^\infty)$  and  $H$ . Much of this material is well known, and due to Frank Adams.

If  $E$  is an oriented motivic spectrum, the projective bundle theorem says that  $E \wedge \mathbb{P}^\infty \cong E \otimes F$  and hence that  $E \wedge \mathbb{P}^\infty \wedge^n \cong E \otimes F^{\otimes n}$ . Using this, we establish a toy version of Theorem 0.2 in Propositions 5.8 and 6.4 that, as cosimplicial spectra,  $KGL \wedge \mathbb{P}^\infty \wedge^{\bullet+1}$  is  $KGL \otimes F^{\otimes \bullet+1}$  (the cobar construction on  $F$ ) and

$$(0.3) \quad s_*(KGL \wedge \mathbb{P}^\infty \wedge^{\bullet+1}) \cong s_*(KGL) \otimes F^{\otimes \bullet+1} \cong s_0(KGL) \otimes KU_*(\mathbb{C}\mathbb{P}^\infty \wedge^n).$$

Using a theorem of Snaith ([5] and [17]), we use the toy model to show that  $KGL \wedge KGL \cong KGL \otimes H$  and more generally (in 7.5) that

$$(0.4) \quad KGL^{\wedge \bullet+2} = KGL \otimes H^{\otimes \bullet+1}$$

as cosimplicial spectra. Taking slices in (0.4) gives the isomorphisms in Theorem 0.2(a), and (with a little decoding) also proves compatibility with every coface and codegeneracy map except for  $\partial^0$  and  $\sigma^0$ . This proves Theorem 0.2(a).

Compatibility with the coface maps  $\partial^0$  is established in Proposition 8.5, using the extra hypothesis that  $s_0(\mathbf{1}) \cong s_0(KGL)$ , and compatibility with the codegeneracy map  $\sigma^0$  is established in Lemma 8.7, using the torsionfree hypothesis. This proves Theorem 0.2(b).

The paper is organized as follows. In Sections 1 and 2 we introduce the binomial rings  $F$  and  $H$ . As noted in Remark 1.6,  $KU_*(\mathbb{C}\mathbb{P}^\infty) = KU_* \otimes F$  and  $KU_*(KU) = KU_* \otimes H$ . In Section 3 we quickly review Hopf algebroids and the algebroid structure on  $(KU_*, KU_*KU)$ . In Section 4, we recall how  $\pi_*N^\bullet KU$  is the cobar complex for this algebroid, by showing that  $KU \wedge \mathbb{P}^\infty^{\wedge \bullet+1}$  is  $KU \otimes C^\bullet(F, F)$ , where  $C^\bullet(F, F)$  is the cobar complex of the Hopf algebra  $F$ . Most of this material is due (at least in spirit) to Adams.

In Section 5 we show that  $KGL \wedge \mathbb{P}^\infty^{\wedge \bullet+1}$  is  $KGL \otimes C^\bullet(F, F)$ , by mimicking the development of Section 4. In Section 6 we show that the slice functors commute with direct sums, and deduce (0.3). Section 7 establishes (0.4), and Theorem 0.2 is established in Section 8.

*Notation 0.5.* For any graded abelian group  $A$  and motivic spectrum  $E$ , we can form a motivic spectrum  $E \otimes A$ , as follows. If  $A$  is free with a basis of elements  $a_i$  in degrees  $d_i$ , then  $E \otimes A$  is the wedge of the  $E \wedge T^{d_i}$ , and we may regard  $a_i$  as a map from  $E \wedge T^{d_i}$  to  $E \otimes A$ . For general  $A$ , choose a free graded resolution  $0 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$  and define  $E \otimes A$  to be the cofiber of  $E \otimes P_1 \rightarrow E \otimes P_0$ ; Shanuel's Lemma implies that this is independent of the choice of resolution, up to isomorphism. The choice of a lift of a homomorphism  $A \rightarrow B$  is unique up to chain homotopy, so it yields a map  $E \otimes A \rightarrow E \otimes B$  unique up to homotopy. Given homomorphisms  $A \rightarrow B \rightarrow C$ , this yields a homotopy between the composition  $E \otimes A \rightarrow E \otimes B \rightarrow E \otimes C$  and  $E \otimes A \rightarrow E \otimes C$ . That is, this construction gives a lax functor from abelian groups to strict motivic spectra over  $E$ , and a functor to motivic spectra over  $E$ .

There is a natural associative map  $(\mathbf{1} \otimes A) \wedge (\mathbf{1} \otimes B) \rightarrow \mathbf{1} \otimes A \otimes B$ , at least if  $A$  and  $B$  have no summands  $\mathbb{Z}/2$ ,  $\mathbb{Z}/3$  or  $\mathbb{Z}/4$  [23, IV.2.8]; it is an isomorphism if  $A$  and  $B$  are free abelian groups. Thus if  $E$  is a ring spectrum and  $A$  is a ring, the composition  $(E \otimes A) \wedge (E \otimes A) \rightarrow E \otimes A$  makes  $E \otimes A$  into a ring spectrum.

## 1. UNIVERSAL BINOMIAL RINGS

Recall that a *binomial ring* is a subring  $R$  of a  $\mathbb{Q}$ -algebra which is closed under the operations  $r \mapsto \binom{r}{n}$ . It is a  $\lambda$ -ring with operations  $\lambda^n(r) = \binom{r}{n}$ .

For example, consider the subring  $F$  of  $\mathbb{Q}[t]$  consisting of *numerical polynomials* – polynomials  $f(t)$  with  $f(n) \in \mathbb{Z}$  for all integers  $n \gg 0$ . It is well known that  $F$  is free as an abelian group, and that the  $\alpha_n = \binom{t}{n}$  form a basis. It is not hard to verify the formula that

$$t \binom{t}{n} = n \binom{t}{n} + (n+1) \binom{t}{n+1}.$$

The general ring structure of  $F$  is determined by the combinatorial identity:

$$(1.1) \quad \alpha_i * \alpha_j = \sum_{k \leq i+j} \binom{k}{k-i, k-j, i+j-k} \alpha_k.$$

Here  $\binom{k}{a,b,c}$  denotes  $k!/a!b!c!$ . (To derive (1.1), note that the left side counts pairs of subsets of a set with  $t$  elements. If the union of an  $i$ -element set and a  $j$ -element set has  $k$  elements, the sets intersect in  $i+j-k$  elements.)

The universal polynomials for  $\lambda^n(\lambda^m(r))$  show that the numerical polynomials form a binomial ring. In fact,  $F$  is the free binomial ring on one generator  $t$ : if  $R$  is binomial and  $r \in R$  the canonical extension of the universal ring map  $\mathbb{Z}[t] \rightarrow R$  to  $F \rightarrow R \otimes \mathbb{Q}$  factors uniquely through a map  $F \rightarrow R$ .

**Definition 1.2.** Let  $H$  denote the localization  $F[1/t]$  of the ring of numerical polynomials; it is a subring of  $\mathbb{Q}[t, 1/t]$ .

Here is a useful criterion for membership in  $H$ .

**Lemma 1.3.**  $H = F[1/t]$  is the ring of all  $f(t) \in \mathbb{Q}[t, 1/t]$  such that for any positive integer  $a$  we have  $f(a) \in \mathbb{Z}[1/a]$ .

*Proof.* (Cf. [3, 5.3]) Multiplying  $f$  by a suitable power of  $t$ , we may assume that  $f(t) \in t^\nu \mathbb{Q}[t]$ , where  $\nu$  is the highest exponent of any prime occurring in the denominators of the coefficients of  $f$ . It suffices to show that  $f$  is a numerical polynomial. Fix  $a > 0$  and let  $p$  be a prime. If  $p|a$  then  $p$  does not appear in the denominator by construction; if  $p \nmid a$  then  $p$  does not appear in the denominator of  $f(a)$  by hypothesis. Hence  $f(a) \in \mathbb{Z}$ , as desired.  $\square$

Recall that  $\mathbb{Q}[t, 1/t]$  is a Hopf algebra with  $\Delta(t) = t_1 t_2$ . The usual proof [6, I.7.3] that the functions  $\binom{t}{n}$  form a basis of the ring  $F$  of numerical polynomials  $f(t)$  is easily modified to show that the functions  $\binom{t_1}{m} \binom{t_2}{n}$  form a  $\mathbb{Z}$ -basis of the ring of numerical polynomials  $f(t_1, t_2)$  in  $\mathbb{Q}[t_1, t_2]$ . Identifying  $\mathbb{Q}[t_1, t_2]$  with  $\mathbb{Q}[t] \otimes \mathbb{Q}[t]$ , we obtain a canonical isomorphism between the subring  $F \otimes F$  of  $\mathbb{Q}[t] \otimes \mathbb{Q}[t]$  and the ring of numerical polynomials in  $\mathbb{Q}[t_1, t_2]$ .

**Theorem 1.4.** (a)  $H$  is a Hopf subalgebra of  $\mathbb{Q}[t, 1/t]$ .  
(b)  $H$  is a binomial ring; it is the free binomial ring on a unit.

*Proof.* For (a), it suffices to show that  $\Delta : \mathbb{Q}[t] \rightarrow \mathbb{Q}[t_1, t_2]$  sends  $F[1/t] \subset \mathbb{Q}[1/t]$  into the subring  $F[1/t] \otimes F[1/t]$ . But  $\Delta$  sends  $\binom{t}{n}$  to  $f(t_1, t_2) = \binom{t_1 t_2}{n}$ , which is a numerical polynomial and hence belongs to  $F \otimes F$ . Thus  $\Delta$  sends  $t^{-k} \binom{t}{n}$  to a  $\mathbb{Z}$ -linear combination of the functions  $(t_1 t_2)^{-k} \binom{t_1}{m} \binom{t_2}{n}$ , which is in  $H \otimes H$ .

For (b), fix  $f(t)/t^m$  in  $F[1/t]$ , and set  $g_k(t) = \lambda^k(f/t^m)$ . For each nonzero  $a \in \mathbb{Z}$ ,  $\mathbb{Z}[1/a]$  is a binomial ring, so  $g_k(a) = \lambda^k(f(a)/a^m)$  is in  $\mathbb{Z}[1/a]$ .  $\square$

**Theorem 1.5.** *As an abelian group,  $H$  is free.*

*Proof.* For  $m \leq n$ , let  $F(m, n)$  denote the intersection of  $H$  with the  $\mathbb{Q}$ -span of  $t^m, \dots, t^n$  in  $\mathbb{Q}[t, 1/t]$ . Then the proof of [2, 2.2] goes through to prove that  $F(m, n) \cong R^{1+n-m}$  and that  $H$  is free abelian, given the following remark: For each  $k, m, n$ ,  $\binom{kt}{n}$  is a numerical polynomial, so  $(kt)^{-m} \binom{kt}{n}$  is certainly in  $F[1/t][1/k]$ .  $\square$

*Remark 1.6.* The ring  $KU_* \otimes F$  is  $KU_*(\mathbb{C}\mathbb{P}^\infty)$ , and  $KU_* \otimes H$  is isomorphic to the ring  $KU_0(KU)$ . These observations follow from [1, II.3] and [3, 2.3, 4.1, 5.3].

A  $\mathbb{Z}$ -basis of  $KU_*(KU)$  was given in [8, Cor. 13].

Since we will be interested in the algebras  $R \otimes F[1/t]$  over different base rings  $R$ , we now give a slightly different presentation of  $F_R = R \otimes F$  and  $F_R[1/t]$ . As an  $R$ -module,  $F_R$  is free with countable basis  $\{\alpha_0, \alpha_1, \dots\}$ , and we are given an  $R$ -module map  $T : F_R \rightarrow F_R$  (multiplication by  $t$ ), defined by

$$(1.7) \quad T(\alpha_n) = n\alpha_n + (n+1)\alpha_{n+1}.$$

Note that  $T(\alpha_0) = \alpha_1$ . The localization  $F_R[1/t]$  is the colimit of the system

$$(1.8) \quad F_R \xrightarrow{T} F_R \xrightarrow{T} F_R \xrightarrow{T} \dots$$

To describe it, we introduce the bookkeeping index  $t^{-j}$  to indicate the  $j^{\text{th}}$  term in this sequence. By [22, 2.6.8], there is a short exact sequence of  $R$ -modules:

$$0 \rightarrow \bigoplus_{j=0}^{\infty} F_R t^{-j} \xrightarrow{\Phi} \bigoplus_{j=0}^{\infty} F_R t^{-j} \rightarrow F_R[1/t] \rightarrow 0,$$

where  $\Phi(\alpha_n t^{-j}) = T(\alpha_n) t^{-j-1} - \alpha_n t^{-j}$ . Since  $\Phi$  is  $t^{-1}$ -linear, we may regard  $\Phi$  as an endomorphism of the free  $R[t^{-1}]$ -module  $F_R[t^{-1}] = \bigoplus F_R t^{-n}$  with basis  $\{\alpha_m\}$ , with

$$\Phi(\alpha_n) = T(\alpha_n) t^{-1} - \alpha_n = (n+1)t^{-1}\alpha_{n+1} + (nt^{-1} - 1)\alpha_n.$$

By abuse of notation, we write  $\alpha_n$  for the image in  $F_R[1/t]$  of the basis element  $\alpha_n$  of  $F_R$ . Thus  $F_R[1/t]$  may be presented as the  $R[1/t]$ -module with generators  $\alpha_n$  and relations

$$(1.9) \quad (1 - nt^{-1})\alpha_n = (n+1)t^{-1}\alpha_{n+1}, \quad n \geq 0.$$

Notice that  $\alpha_0 = t^{-1}\alpha_1$ . It is not hard to verify directly, beginning with (1.8), that multiplication by  $t^{-1}$  is an isomorphism on  $F_R[1/t]$ . It also follows directly from the ring structure on  $F$  and the identification of the colimit with  $F_R[1/t]$ .

If  $R$  contains no  $\mathbb{Z}$ -torsion, so that  $R \subseteq R \otimes \mathbb{Q}$ , it is easy to see that  $F_R[1/t]$  embeds in  $F_{R \otimes \mathbb{Q}}[1/t] = R \otimes \mathbb{Q}[t, t^{-1}]$ , as we saw at the beginning of this section. The presentation of  $R \otimes H$  as an  $R$ -module looks different when  $R$  has  $\mathbb{Z}$ -torsion, as we shall see in Section 2.

**Example 1.10.** (See [3, 2.2]) If  $R$  contains  $\mathbb{Q}$ , the presentation (1.9) shows that  $F_R[1/t]$  is the  $R[t^{-1}]$ -module with generators  $\{\alpha_0, \alpha_1, \dots\}$  modulo the relations:

$$t^{-n}\alpha_{n+1} = \frac{1}{(n+1)!} (1 - t^{-1})(1 - 2t^{-1}) \dots (1 - nt^{-1})\alpha_1.$$

That is,  $F_R[1/t] \cong R[t, t^{-1}]$  on generator  $\alpha_0$  with the relations

$$\alpha_n = \binom{t}{n} \alpha_0.$$

**Variante 1.11.** Adams uses a variant of the above construction. Fix a unit  $u$  of  $R$  and set  $v = ut$ , so  $t = v/u$ . We consider  $B = uT$  to be multiplication by  $v$  with (1.7) replaced by  $B(\beta_n) = nu\beta_n + (n+1)\beta_{n+1}$ , where  $\beta_n = u^n\alpha_n$ . Replacing the bookkeeping index  $t^{-1}$  by  $v^{-1}$ , (1.9) becomes  $(1 - nu/v)\beta_n = (n+1)v^{-1}\beta_{n+1}$ , or  $(n+1)\beta_{n+1} = (t-n)u\beta_n$ , and we recover

$$\beta_n = u^n \binom{v/u}{n} \beta_0 = u^n \binom{t}{n} \beta_0.$$

These  $\beta_n$  are the elements described by Adams in [1, II.13.7] as the generators of  $KU_*(KU)$ , regarded as a submodule of  $KU_*(KU) \otimes \mathbb{Q} = \mathbb{Q}[u, 1/u, v, 1/v]$ .

## 2. THE $\ell$ -PRIMARY DECOMPOSITION

In this section, we suppose that  $R$  is an algebra over  $\mathbb{Z}/\ell^\nu$  and give a basis for the colimit  $H_R = F_R[1/t]$  of the sequence  $F_R \xrightarrow{T} F_R \xrightarrow{T} F_R \xrightarrow{T} \cdots$  of (1.8).

Recall from Section 1 that the elements of  $F$  are polynomial functions  $\mathbb{N} \rightarrow \mathbb{Z}$ , and that the  $\alpha_n = \binom{t}{n}$  form a basis of  $F$ .

**Lemma 2.1.** *For each  $q \in \mathbb{Z}$ ,  $F \otimes \mathbb{Z}/q$  embeds in the ring of all functions  $\mathbb{N} \rightarrow \mathbb{Z}/q$ .*

*Proof.* If  $f \in F$  satisfies  $f(a) \equiv 0 \pmod{q}$  for every  $a > 0$  then  $h(t) = f(t)/q$  is a numerical polynomial, and  $f(t) = qh(t)$  is in  $qF$ .  $\square$

**Example 2.2.** Suppose that  $\mathbb{Z}/2 \subseteq R$ . Then the relations  $\alpha_{2k} = t^{-1}\alpha_{2k+1}$  and  $(1 - t^{-1})\alpha_{2k+1} = 0$  imply that  $H_R = F_R[1/t]$  is the free  $R$ -module with basis  $\{\alpha_{2k+1}, k \geq 0\}$ , with  $\alpha_{2k} = \alpha_{2k+1}$ .

In fact,  $F/2F$  and  $H/2H$  are Boolean rings by Lemma 2.1. In particular, each  $\alpha_n$  is idempotent in the ring  $F/2F$ , including  $t = \alpha_1$ , and  $H/2H = tF/2F$  is a factor ring of  $F/2F$ . In addition, if  $m \leq 2^r - 1$  then (1.1) implies that  $\alpha_m\alpha_{2^r-1} = \alpha_{2^r-1}$ .

When  $R$  is a  $\mathbb{Z}/\ell^\nu$ -algebra,  $F_R$  has a similar block decomposition. To prepare for it, let  $L'$  denote the free  $R$ -module on basis  $\alpha_0, \alpha_1, \dots, \alpha_{\ell-1}$ , and let  $L$  denote the submodule of  $L'$  on  $\alpha_1, \dots, \alpha_{\ell-1}$ . Following (1.7), we define maps  $b_k : L' \rightarrow L'$  by

$$(2.3) \quad b_k(\alpha_i) = (k\ell + i)\alpha_i + (k\ell + i + 1)\alpha_{i+1}, \quad i = 0, \dots, \ell - 2$$

and  $b_k(\alpha_{\ell-1}) = (k\ell + \ell - 1)\alpha_{\ell-1}$ . Note that  $b_k(L) \subseteq L$ .

**Lemma 2.4.** *The maps  $L \rightarrow L[1/b_k] \rightarrow L'[1/b_k]$  are isomorphisms for all  $k$ .*

*Proof.* The restriction of  $b_k$  to  $L$  is represented by a lower triangular matrix, whose determinant  $\prod_{i=1}^{\ell-1} (k\ell + i)$  is a unit in  $R$ . Thus each  $b_k$  restricts to an automorphism of  $L$ . Since  $\ell^\nu = 0$  in  $R$ ,  $(b_k)^\nu$  maps  $\alpha_0$  into  $L$ . The result is now straightforward.  $\square$

We can now describe the  $R$ -module  $H_R = R \otimes F_R[1/t]$  when  $R = \mathbb{Z}/\ell$ ; the case  $\ell = 2$  was given in Example 2.2. The maps  $\phi_k : L' \rightarrow F_R$ ,  $\phi_k(\alpha_i) = \alpha_{k\ell+i}$ , induce an isomorphism  $\bigoplus \phi_k : \bigoplus_{k=0}^{\infty} L' \rightarrow F_R$  under which the map  $\bigoplus b_k$  is identified with the map  $T$  of (1.7).

**Corollary 2.5.** *If  $\mathbb{Z}/\ell \subseteq R$  with  $\ell$  prime, the map  $\bigoplus \phi_k : \bigoplus_{k=0}^{\infty} L \rightarrow H_R$  is an isomorphism. Thus the elements  $\alpha_n$ ,  $n \not\equiv 0 \pmod{\ell}$  form a basis of  $H_R$ .*

*Proof.* The maps  $\phi_k : L' \rightarrow F$  satisfy  $\phi_k \circ \bigoplus b_k = T \circ \phi$ . Hence the map  $\bigoplus \phi_k$  induces an isomorphism between  $\bigoplus L \cong \bigoplus L'[1/b_k]$  and  $H_R = F_R[1/t]$ .  $\square$

**Theorem 2.6.** *If  $R$  is a  $\mathbb{Z}/\ell^\nu$ -algebra, the  $R$ -module map  $\bigoplus \phi_k : \bigoplus L \xrightarrow{\cong} H_R$  is an isomorphism. In particular, the elements  $\alpha_n$ ,  $n \not\equiv 0 \pmod{\ell}$ ,  $n > 0$ , form a basis.*

*Proof.* Let  $F_n$  denote the free  $R$ -submodule of  $F_R$  on basis  $\{\alpha_{nq+i} : 0 \leq i < q\}$ , where  $q = \ell^\nu$ . Then  $F_R$  is the direct sum of the  $F_n$ . Since  $T(\alpha_{nq-1}) = (nq-1)\alpha_{nq-1}$ ,  $T$  sends  $F_n$  into itself, and there are isomorphisms  $F_0 \rightarrow F_n$ ,  $\alpha_i \mapsto \alpha_{nq+i}$  commuting with  $T$ . Therefore  $H_R = F_R[1/t]$  is isomorphic to  $\bigoplus F_n[1/t]$ , and it suffices to show that  $F_0[1/t]$  is free on the  $\alpha_n$  with  $0 < n < q$  and  $n \not\equiv 0 \pmod{\ell}$ .

For  $k = 0, \dots, \ell^\nu - 1$ , let  $F_0^{\geq k\ell}$  denote the  $R$ -submodule of  $F_0$  generated by the  $\alpha_i$  with  $k\ell \leq i < q$ . These form a filtration of  $F_0$ , and the maps  $\phi_k : L' \rightarrow F_0^{\geq k\ell}$  induce  $R$ -module isomorphisms with the filtration quotients

$$\bar{\phi}_k : L' \xrightarrow{\cong} F_0^{\geq k\ell} / F_0^{\geq (k+1)\ell}$$

such that  $T \circ \bar{\phi}_k = b_k \circ \phi_k$ . By Lemma 2.4, it follows that  $\phi_k$  induces an isomorphism

$$L \cong L'[1/b_k] \xrightarrow{\cong} F_0^{\geq k\ell} / F_0^{\geq k\ell + \ell}[1/t].$$

By induction on  $k$ , it follows that  $\bigoplus \phi_k$  induces an isomorphism  $\bigoplus L \cong F_0[1/t]$ .  $\square$

### 3. HOPF ALGEBROIDS

Recall from [14, A1.1.1] that a Hopf algebraoid  $(R, \Gamma)$  is a pair of commutative rings, with maps  $\eta_L, \eta_R : R \rightarrow \Gamma$ ,  $\varepsilon : \Gamma \rightarrow R$ ,  $c : \Gamma \rightarrow \Gamma$  and  $\Delta : \Gamma \rightarrow \Gamma \otimes_R \Gamma$  satisfying certain axioms, listed in *loc. cit.*

Let  $M$  be a  $\Gamma$ -comodule with structure map  $M \xrightarrow{\psi} M \otimes_R \Gamma$ . Recall [14, A1.2.11] that the *cobar complex*  $C_\Gamma^\bullet(M, R)$  is the cosimplicial comodule with  $C^0 = M$  and  $C^n = M \otimes_R \Gamma^{\otimes n}$ , with cofaces given by  $\psi$ ,  $\eta_L$  and  $\Delta : \Gamma \rightarrow \Gamma \otimes_R \Gamma$ . In particular, when  $M = R$  (with  $\psi = \eta_R$ ), the cobar complex is a cosimplicial ring.

**Example 3.1.** For any commutative algebra  $R$ ,  $(R, R \otimes R)$  is a Hopf algebraoid with  $\eta_L(r) = r \otimes 1$ ,  $\eta_R(r) = 1 \otimes r$ ,  $c(r \otimes s) = s \otimes r$  and  $\Delta(r \otimes s) = r \otimes 1 \otimes s$ . The cobar complex  $C_{R \otimes R}^\bullet(R, R)$  is the standard cosimplicial module  $n \mapsto R^{\otimes n+1}$ .

If  $\Gamma$  is a Hopf algebra over  $R$ , then  $(R, \Gamma)$  is a Hopf algebraoid with  $\eta_L = \eta_R$  the unit,  $\varepsilon$  the counit,  $c$  the antipode and  $\Delta$  the coproduct. In this case, the cobar complex is classical.

**Example 3.2.** The pair  $(R_{\mathbb{Q}}, \Gamma_{\mathbb{Q}}) = (\mathbb{Q}[u, 1/u], \mathbb{Q}[u, 1/u, t, 1/t])$  is a Hopf algebraoid with  $\eta_L(u) = u \otimes 1$ ,  $\eta_R(u) = c(u) = tu$ ,  $c(t) = 1/t$ ,  $\varepsilon(t) = 1$  and  $\Delta(t) = t \otimes t$ .

Recall from Definition 1.2 that  $H$  is the subalgebra of  $\mathbb{Q}[t, 1/t]$  generated by the  $\binom{t}{n}$ . If  $R = \mathbb{Z}[u, 1/u]$  and  $\Gamma = R \otimes H$ , then  $(R, \Gamma)$  is a sub-Hopf algebraoid. If we set  $v = tu$  then we have  $\eta_R(u) = v$ ,  $c(u) = v$  and

$$\Delta(u) = u \otimes 1, \quad \Delta(t) = t \otimes t, \quad \Delta(v) = ut \otimes t = 1 \otimes v.$$

Of course,  $KU_* = R$  and  $\pi_*(KU \wedge KU) = \Gamma$ , and the formulas given by Adams in [1, II.13.4] show that  $(R, \Gamma)$  is the original Hopf algebraoid  $(KU_*, KU_*KU)$ .

**Lemma 3.3.** *The Hopf algebraoid  $(R, R \otimes H)$  of Example 3.2 is split.*

*Proof.* The natural inclusion of the Hopf algebra  $(\mathbb{Z}, H)$  into  $(R, R \otimes H)$  is a map of algebroids [14, A1.1.9], and the identity map on  $R \otimes H$  is an algebra map.  $\square$

Fix  $R$  and  $\Gamma \cong R \otimes H$  as in Example 3.2. Because the algebroid  $(R, \Gamma)$  is split, we can say more:  $C_\Gamma^\bullet(M, R) = C_H^\bullet(M, \mathbb{Z})$  (see [14, A1.2.17]), and the latter is the usual cobar complex of  $M$  as a comodule over the Hopf algebra  $H$ . For  $M = R$  we have the following description.

**Corollary 3.4.** *The cobar complex  $C_\Gamma^\bullet(R, R)$  has  $C^n = R \otimes H^{\otimes n}$ . The coface maps  $C^n \rightarrow C^{n+1}$  are given by the units  $\eta_L, \eta_R : R \rightarrow R \otimes H = \Gamma$  of  $\Gamma$  and  $\Delta_H$ .*

#### 4. ORIENTED TOPOLOGICAL SPECTRA

This section is intended to serve as a template for the motivic constructions which follow, and is restricted to topological spectra. No originality is claimed for these results. To emphasize our analogy, we will write  $\mathbb{P}^\infty$  for  $\mathbb{C}\mathbb{P}_+^\infty$  in this section.

If  $E$  is a topological ring spectrum, then combining the ring structure of  $E$  with the  $H$ -space structure of  $\mathbb{P}^\infty$  we obtain a natural ring spectrum  $E \wedge \mathbb{P}^\infty$ . Following Adams [1] and Snaith [15], the motivation for studying  $F[1/t]$  comes from the computation of  $E \wedge KU$ , where  $E$  is an oriented spectrum.

Recall that a commutative topological ring spectrum  $E$  with unit  $\mathbf{1} \xrightarrow{\eta_E} E$  is said to be *oriented* if there is a map  $\mathbb{P}^\infty \wedge \mathbf{1} \xrightarrow{x} S^2 \wedge E$  whose restriction to  $\mathbb{C}\mathbb{P}_+^1 \wedge \mathbf{1}$  is  $1 \wedge \eta_E$  (using  $S^2 = \mathbb{C}\mathbb{P}^1$ ). As in [1, II.2], this data yields a formal group law on  $E_*[[x]] \cong E^*(\mathbb{P}^\infty)$ , and dual elements  $\beta_n$  in  $E_{2n}(\mathbb{P}^\infty) = \pi_{2n}(E \wedge \mathbb{P}^\infty)$ .

When  $E$  is orientable, the  $\beta_n$  induce maps

$$E \wedge S^{2n} \xrightarrow{1 \wedge \beta_n} E \wedge E \wedge \mathbb{P}^\infty \rightarrow E \wedge \mathbb{P}^\infty,$$

and these maps induce an isomorphism from  $\bigoplus_{n=0}^\infty E \wedge S^{2n}$  to  $E \wedge \mathbb{P}^\infty$  (by [1, p. 42]). Consequently, we have

$$E_*(\mathbb{P}^\infty) = \pi_*(E \wedge \mathbb{P}^\infty) \cong \bigoplus_{n=0}^\infty \pi_*(E \wedge S^{2n}).$$

Thus  $E_*(\mathbb{P}^\infty)$  is a free graded  $E_*$ -module on generators  $\beta_n$  in degree  $2n$ ,  $n \geq 0$ .

Let  $\xi : S_+^2 \cong \mathbb{C}\mathbb{P}_+^1 \rightarrow \mathbb{P}^\infty$  be the map that classifies the tautological line bundle on  $\mathbb{C}\mathbb{P}^1$ , and let  $b : \mathbb{P}^\infty \wedge \mathbf{1} \rightarrow S^{-2} \wedge \mathbb{P}^\infty \wedge \mathbf{1}$  denote the adjoint of the stable map

$$S^2 \wedge \mathbb{P}^\infty \wedge \mathbf{1} \xrightarrow{\text{Hopf}} S_+^2 \wedge \mathbb{P}^\infty \wedge \mathbf{1} \xrightarrow{\xi \wedge 1} \mathbb{P}^\infty \wedge \mathbb{P}^\infty \wedge \mathbf{1} \xrightarrow{m} \mathbb{P}^\infty \wedge \mathbf{1}.$$

Let  $j = \xi - 1$  denote the map  $\mathbb{P}^\infty \rightarrow BU_+ \rightarrow KU$  classifying the virtual tautological line bundle of rank 0. By Snaith [15, 2.12], the map  $j$  induces an equivalence from the homotopy colimit of the sequence

$$\mathbb{P}^\infty \wedge \mathbf{1} \xrightarrow{b} S^{-2}\mathbb{P}^\infty \wedge \mathbf{1} \xrightarrow{S^{-2}b} S^{-4}\mathbb{P}^\infty \wedge \mathbf{1} \rightarrow \dots$$

to the spectrum  $KU$ . We write  $u = u_{KU}$  for the map  $S^2 \rightarrow \mathbb{P}^\infty \xrightarrow{j} KU$ . The map

$$S^2 \wedge KU \xrightarrow{u \wedge 1} KU \wedge KU \xrightarrow{m} KU$$

is the periodicity isomorphism for  $KU$  (cf. (7.1)). Composing  $j$  with the inverse of the periodicity isomorphism, we obtain an orientation for  $KU$  whose formal group law is  $x + y + uxy$ .

Smashing with  $E$  yields maps  $b_E : E \wedge \mathbb{P}^\infty \rightarrow S^{-2} \wedge E \wedge \mathbb{P}^\infty$  and an equivalence between  $E \wedge KU$  and the homotopy colimit of the sequence of spectra:

$$(4.1) \quad E \wedge \mathbb{P}^\infty \xrightarrow{b_E} S^{-2} \wedge E \wedge \mathbb{P}^\infty \xrightarrow{S^{-2}b_E} S^{-4} \wedge E \wedge \mathbb{P}^\infty \longrightarrow \dots$$

We now make the connection with the algebraic considerations of Section 1.

**Lemma 4.2.** *The coefficient of  $x^i y^j$  in  $(x + y + uxy)^k$  is  $\binom{k}{k-i, k-j, i+j-k} u^{i+j-k}$ .*

*Proof.* The product has  $3^k$  terms. The only terms producing the monomial  $x^i y^j$  are those with exactly  $k - j$  of the factors must be  $x$ , exactly  $k - i$  must be  $y$ , and the remaining  $i + j - k$  must be  $uxy$ .  $\square$

**Corollary 4.3.** *Suppose that  $E$  is an oriented spectrum with a multiplicative group law  $\mu(x, y) = x + y + uxy$  such that  $u$  is a unit of  $E_*$ . Then the homotopy groups of the cosimplicial spectrum  $E \wedge \mathbb{P}^\infty \wedge^{\bullet+1}$  form a cosimplicial group  $E_* \otimes F^{\otimes \bullet+1}$ , isomorphic to the cobar complex  $E_* \otimes C_{F \otimes F}^\bullet(F, F)$  of Example 3.1.*

*Proof.* By the above remarks,  $\pi_*(E \wedge \mathbb{P}^\infty \wedge^{n+1})$  is a free  $E_*$ -module with basis  $\beta_{i_0} \wedge \dots \wedge \beta_{i_n}$ , which we identify with  $E_* \otimes F^{\otimes n+1}$  by sending  $\beta_i$  to  $u^i \alpha_i$ , as in 1.11. The coface maps are easily seen to correspond, and for the codegeneracies it suffices to consider  $\sigma^0 : \pi_*(E \wedge \mathbb{P}^\infty \wedge \mathbb{P}^\infty) \rightarrow \pi_*(E \wedge \mathbb{P}^\infty)$ . Adams showed in [1, II(3.4–5)] that the coefficient  $a_{ij}^k$  of  $\sigma^0(\beta_i \otimes \beta_j)$  is the coefficient of  $x^i y^j$  in  $(x + y + uxy)^k$ , which is given by Lemma 4.2, and agrees with  $(u^i \alpha_i) * (u^j \alpha_j)$  by (1.1).  $\square$

**Lemma 4.4.** *Suppose that  $E$  is an oriented spectrum with a multiplicative group law  $\mu(x, y) = x + y + uxy$  such that  $u$  is a unit of  $E_*$ . Then*

$$\pi_*(E \wedge KU) \cong \varinjlim \pi_*(S^{-2n} \wedge E \wedge \mathbb{P}^\infty) \cong E_* \otimes \varinjlim F \cong E_* \otimes H.$$

*Proof.* Identifying  $R = E_*$  and  $F_R = E_*(\mathbb{P}^\infty) = \pi_*(E \wedge \mathbb{P}^\infty)$ , Adams shows in [1, II(3.6)] that multiplication by  $\beta_1$  is given by the formula (1.7), with  $t$  replaced by  $v = ut$  as in Variant 1.11. Comparing with (1.8), we see that the ring  $E_*(\mathbb{P}^\infty)[1/\beta_1]$  is isomorphic to  $E_* \otimes H$ . From (4.1), we obtain the result.  $\square$

*Remark 4.4.1.* Lemma 4.4 fails dramatically if  $u = 0$ , for example when  $E = H\mathbb{Z}$ .

Replacing  $E$  by  $E \wedge KU$ , which we orient by  $x \wedge \eta_{KU}$ , Lemma 4.4 yields  $\pi_*(E \wedge KU \wedge KU) \cong E_* \otimes H \otimes H$ . The two maps  $E_* \otimes H \rightarrow E_* \otimes H \otimes H$ , induced by  $E \wedge KU \rightarrow E \wedge KU \wedge^2$ , are given by  $\eta_E \otimes 1$  and  $1 \otimes \Delta_H$ , and  $E_* \otimes H \otimes H \xrightarrow{\sigma^0} E_* \otimes H$  is given by Corollary 4.3. An inductive argument establishes:

**Corollary 4.5.** *The homotopy groups of the cosimplicial spectrum form the cosimplicial group:*

$$\pi_*(E \wedge KU \wedge^{n+1}) = E_* \otimes H^{\otimes n+1}.$$

*This is the cobar complex  $C_\Gamma^\bullet(M, E_*)$  of Example 3.2 with  $M = E_* \otimes H$ .*

When  $E = KU$ , we can also form the cosimplicial spectrum  $N^\bullet(KU)$ , as in the introduction, and we have:

**Proposition 4.6.** *The cosimplicial ring  $\pi_*(N^\bullet KU)$  is  $KU_* \otimes H^{\otimes \bullet}$ , the cobar complex  $C_\Gamma^\bullet(KU_*, KU_*)$  of 3.4. In particular, for every  $n$  we have an isomorphism*

$$\pi_*(KU \wedge^{n+1}) \cong KU_* \otimes H^{\otimes n}.$$

$KU^{\wedge \bullet}$  with finite coefficients

Fix a prime  $\ell$  and a power  $q = \ell^\nu$ , and let  $P$  denote the Moore spectrum for  $\mathbb{Z}/q$ . We will assume that  $q$  is chosen so that  $P$  is a ring spectrum with unit. For example, if  $\ell \neq 2, 3$  then this is the case for any  $\nu \geq 1$ ; see [23, IV.2.8]. Smashing  $P$  with  $KU$  yields the standard cosimplicial spectrum  $P \wedge KU^{\wedge n+1}$  associated to the triple  $- \wedge KU$ . Set  $R = \pi_*(P \wedge KU)$ ; the Universal Coefficient Theorem yields  $R = \mathbb{Z}/q[u, 1/u]$ . Since  $P \wedge KU$  is oriented by the orientation on  $KU$ , we see from Lemma 4.4 and Theorem 2.6 that  $\pi_*(P \wedge KU \wedge KU) = R \otimes H$  is a free  $R$ -module with basis  $\{\beta_n : n \not\equiv 0 \pmod{\ell}\}$ , and from (4.5) that  $\pi_*(P \wedge KU^{\wedge n+1}) = R \otimes H^{\otimes n}$ . In fact,  $(R, R \otimes H)$  is the mod  $q$  reduction of the Hopf algebroid of Lemma 3.3.

Set  $E = P \wedge KU$ . Then each  $\beta_n$  determines a map  $S^{2n} \wedge E \rightarrow E \wedge \mathbb{P}^\infty \rightarrow E \wedge KU$ . By Lemma 4.4, the direct sum of these maps is a homotopy equivalence of spectra

$$\bigoplus E \wedge S^{2n} \xrightarrow{\simeq} E \wedge \mathbb{P}^\infty$$

There is a sequence (4.1) for  $E \wedge \mathbb{P}^\infty$ , with colimit  $E \wedge KU$ . By Example 3.2,  $\pi_*(E \wedge KU) \cong E_* \otimes H$ . Since  $R = \pi_*(E)$  is a  $\mathbb{Z}/q$ -module, Theorem 2.6 applies: the  $\beta_n$  with  $n \not\equiv 0 \pmod{\ell}$  form a basis of  $E_* \otimes H$ .

**Proposition 4.7.** *For  $E = P \wedge KU$ , there is a homotopy equivalence of spectra*

$$\bigoplus_{n \not\equiv 0 \pmod{\ell}} E \wedge S^{2n} \xrightarrow{\simeq} E \wedge KU.$$

*Proof.* The argument we gave in Corollary 2.5 shows that the left side is the colimit of (4.1). i.e.,  $E \wedge KU$ . In effect, we just notice that the components  $E \wedge S^{2n} \rightarrow E \wedge S^{2n-2}$  of  $b$  are multiplication by  $n$  and are thus null-homotopic when  $\ell|n$ , while the maps  $E \wedge S^{2n} \rightarrow E \wedge S^{2n}$  of  $b$  are multiplication by  $(n+1)$  and are thus null-homotopic when  $\ell|n+1$ .  $\square$

5. THE MOTIVIC  $E \wedge N^\bullet(\mathbb{P}^\infty)$ 

We now turn to motivic spectra. The goal of this section is to describe the cosimplicial spectrum  $E \wedge N^\bullet(\mathbb{P}^\infty) = E \wedge \mathbb{P}^\infty^{\wedge \bullet+1}$  associated to an oriented motivic spectrum  $E$ . Here the motivic space  $\mathbb{P}^\infty$  is defined as the union of the spaces represented by the schemes  $\mathbb{P}^n$ .

The Segre maps  $\mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^\infty$  induce a map  $m : \mathbb{P}^\infty \times \mathbb{P}^\infty \rightarrow \mathbb{P}^\infty$  making  $\mathbb{P}^\infty \wedge \mathbf{1}$  into a ring  $T$ -spectrum. Then  $n \mapsto E \wedge \mathbb{P}^\infty^{\wedge n+1}$  is the standard cosimplicial spectrum associated to a spectrum  $E$  and the triple  $- \wedge \mathbb{P}^\infty$  [22, 8.6.4]. If  $E$  is a ring  $T$ -spectrum, then this is a cosimplicial ring spectrum.

A commutative ring  $T$ -spectrum  $E$  with unit  $\mathbf{1} \xrightarrow{\eta} E$  is said to be *oriented* if there is a map  $\mathbb{P}^\infty \wedge \mathbf{1} \xrightarrow{t} T \wedge E$  whose restriction to  $\mathbb{P}^1 \wedge \mathbf{1}$  is  $T \wedge \eta$ . (Compare with Section 4.) As pointed out in Example 2.6.4 of [13],  $KGL$  is an oriented ring spectrum in  $\mathcal{SH}$  with a multiplicative formal group law [13, 1.3.1 and 1.3.3].

When  $E$  is oriented, the projective bundle theorem in [4, 3.2] yields an isomorphism with inverse  $\oplus c^n$ :

$$(5.1) \quad \oplus \beta_n : \bigoplus_{n=0}^{\infty} E \wedge T^n \xrightarrow{\cong} E \wedge \mathbb{P}^\infty.$$

If we let  $F_{\text{gr}}$  denote the free graded abelian group on generators  $\beta_n$  in degree  $n \geq 0$ , then  $E \wedge \mathbb{P}^\infty$  is the free graded  $E$ -module spectrum  $E \otimes F_{\text{gr}}$ . By (5.1), the terms  $E \wedge \mathbb{P}^\infty^{\wedge n+1}$  are isomorphic to the direct sum of terms  $E \wedge T^{i_0} \wedge \cdots \wedge T^{i_n}$ ,  $i_j \geq 0$ .

**Lemma 5.2.** *The coface map  $\partial^j : E \wedge \mathbb{P}^\infty^{\wedge n} \rightarrow E \wedge \mathbb{P}^\infty^{\wedge n+1}$  sends the summand  $E \wedge T^{i_1} \wedge \cdots \wedge T^{i_n}$  to  $E \wedge T^{i_1} \wedge \cdots \wedge T^{i_j} \wedge T^0 \wedge T^{i_{j+1}} \wedge \cdots \wedge T^{i_n}$ .*

*Proof.* Since the coface maps are determined by the unit map  $\mathbf{1} \rightarrow \mathbb{P}^\infty$ , it suffices to observe that the map

$$E = E \wedge T^0 \rightarrow E \wedge \mathbb{P}^\infty \cong \oplus E \wedge T^n$$

is just the inclusion of  $T^0$  into  $\oplus T^n$ , by the projective bundle theorem.  $\square$

At this point, we may interpret the coface maps in  $E \wedge N^\bullet(\mathbb{P}^\infty)$  in terms of  $F_{\text{gr}}$ . Recall (from [22, 8.1.9]) that a *semi-cosimplicial* object in a category is a sequence of objects  $K^n$  with coface operators  $\partial^i : K^{n-1} \rightarrow K^n$  ( $0 \leq i \leq n$ ) satisfying the cosimplicial identities  $\partial^j \partial^i = \partial^i \partial^{j-1}$  if  $i < j$ .

For example,  $n \mapsto F_{\text{gr}}^{\otimes n+1}$  is a semi-cosimplicial graded abelian group, where the codegeneracies insert  $1 = \beta_0$ . (The generators  $\beta_n$  of  $F_{\text{gr}}$  lie in degree  $n$ .) We may also form the semi-cosimplicial spectrum  $E \otimes F_{\text{gr}}^{\otimes n+1}$ . Note that the codegeneracy maps  $F_{\text{gr}}^{\otimes n+1} \rightarrow F_{\text{gr}}^{\otimes n}$  associated to the product  $F \otimes F \rightarrow F$  are not graded, as (1.1) shows, so we do not get a cosimplicial spectrum in this way.

**Corollary 5.3.** *Let  $E$  be an oriented spectrum. Then there is an isomorphism of semi-cosimplicial ring spectra,  $E \otimes F_{\text{gr}}^{\otimes \bullet+1} \xrightarrow{\cong} E \wedge \mathbb{P}^\infty^{\wedge \bullet+1}$*

**Lemma 5.4.** *For any oriented  $E$ , the component maps  $E \wedge T^n \rightarrow E \wedge T^i \wedge T^j$  of the diagonal  $E \wedge \mathbb{P}^\infty \rightarrow E \wedge \mathbb{P}^\infty \wedge \mathbb{P}^\infty$  are the canonical associativity isomorphisms when  $i + j = n$ , and zero otherwise.*

*Proof.* The maps  $\beta_n : E \wedge T^n \rightarrow E \wedge \mathbb{P}^\infty$  in (5.1) satisfy  $c^k \circ \beta_n = \delta_{kn}$ , where  $c^n$  is the projection  $E \wedge \mathbb{P}^\infty \rightarrow E \wedge T^n$ . We have a commutative diagram:

$$\begin{array}{ccccccc} E \wedge T^n & \xrightarrow{\beta_n} & E \wedge \mathbb{P}^\infty & \xrightarrow{\Delta} & E \wedge (\mathbb{P}^\infty)^{\wedge i+j} & \xrightarrow{c, c, \dots, c} & E \wedge T^{\wedge i+j} \\ & & \downarrow \Delta & & \cong \downarrow & & \downarrow \cong \\ & & E \wedge \mathbb{P}^\infty \wedge \mathbb{P}^\infty & \xrightarrow{\Delta} & E \wedge (\mathbb{P}^\infty)^{\wedge i} \wedge (\mathbb{P}^\infty)^{\wedge j} & \xrightarrow{c, \dots, c} & E \wedge T^{\wedge i} \wedge T^{\wedge j}. \end{array}$$

The top horizontal composite is  $c^{i+j} \circ \beta_n = \delta_{n, i+j}$ , and the entire composition is the component map in question. The result follows.  $\square$

We now turn to the codegeneracies  $\sigma^j : E \wedge \mathbb{P}^\infty^{\wedge n+1} \rightarrow E \wedge \mathbb{P}^\infty^{\wedge n}$ . These are all induced from the product  $\mathbb{P}^\infty \wedge \mathbb{P}^\infty \rightarrow \mathbb{P}^\infty$ . After taking the smash product with  $E$ , and using (5.1), we may rewrite the product as an  $E$ -module map

$$\bigoplus_{i, j \geq 0} E \wedge T^i \wedge T^j = (E \wedge \mathbb{P}^\infty) \wedge (E \wedge \mathbb{P}^\infty) \rightarrow E \wedge \mathbb{P}^\infty = \bigoplus_{k \geq 0} E \wedge T^k.$$

Since each term  $E \wedge T^i \wedge T^j$  is a compact  $E$ -module, its image lies in a finite sum. Thus the product is given by a matrix whose entries  $a_{ij}^k$  are the component maps  $E \wedge T^i \wedge T^j \rightarrow E \wedge T^k$ .

In terms of motivic homotopy groups,  $E_{*,*}(\mathbb{P}^\infty) \cong \bigoplus_{n=0}^{\infty} E_{*,*}(T^n)$  is a free graded  $E_{*,*}$ -module with generators  $\beta_n \in E_{2n,n}(\mathbb{P}^\infty)$ ,  $n \geq 0$ , corresponding to the unit of  $E_{2n,n}(T^n) \cong E_{0,0}$ . Dually,  $E^{*,*}(\mathbb{P}^\infty) \cong E^{*,*}[[x]]$  and (as in topology) the orientation yields a formal group law of the form  $\mu = x + y + \sum_{i,j \geq 1} a_{i,j} x^i y^j$  in  $E_*[[x, y]] \cong E^*(\mathbb{P}^\infty \times \mathbb{P}^\infty)$ , where

$$(5.5) \quad a_{i,j} \in E_{2i+2j-2, i+j-1} = \text{Hom}_{S\mathcal{H}}(T^{i+j}, E \wedge T).$$

We can now extend a result of Adams to the motivic setting.

**Theorem 5.6.** *If  $E$  is an oriented motivic ring spectrum, then  $a_{1j}^k = k a_{1,1+j-k}$ , with  $a_{1,1+j-k}$  as in (5.5).*

*Proof.* This follows directly from Lemma 5.4, together with Adams' calculation in [1, II(3.6), p. 46].  $\square$

*Remark 5.6.1.* For fixed  $i, j$  the maps  $a_{ij}^k$  are zero unless  $k \leq i + j$ . Therefore the image of  $E \wedge T^i \wedge T^j$  under the product lands in the finitely many terms  $E \wedge T^k$  with  $k \leq i + j$ .

Our next task is to replace  $E \otimes F_{\text{gr}}^{\otimes \bullet+1}$  by  $E \otimes F^{\otimes \bullet+1}$  to get a cosimplicial spectrum. This is possible when  $E$  is periodic in the following sense. If  $E_*$  has a unit in  $E_{2,1}$ , represented by a map  $T \xrightarrow{u} E$ , then  $u \wedge E$  induces an isomorphism  $T \wedge E \rightarrow E$  with inverse  $u^{-1} \wedge E$ . We call it a *periodicity map* for  $E$ , and say that  $E$  is *periodic*. If  $E$  is oriented and periodic, we can use the periodicity map for  $E$  to define maps  $\alpha_n : E \rightarrow E \wedge \mathbb{P}^\infty$  such that  $\beta_n = u^n \alpha_n$ , resulting in a rewriting of the projective bundle formula (5.1) as

$$(5.7) \quad \bigoplus \alpha_n : \bigoplus_{n=0}^{\infty} E \xrightarrow{\cong} E \wedge \mathbb{P}^\infty.$$

Using this new basis, each  $E \wedge \mathbb{P}^\infty^{\wedge n}$  is isomorphic to  $E \otimes F^{\otimes n}$ .

Now suppose that  $E$  is an oriented spectrum with a multiplicative formal group law  $\mu = x + y + uxy$ , with  $u$  a unit in  $E_{2,1}$ . In this case, it is convenient to change

our orientation to eliminate  $u$ , as suggested in [1, II(2.1)]. This produces the new formal group law  $x + y + xy$ . To see this, let  $t'$  denote the element  $ut$  of  $E^0(\mathbb{P}^\infty)$ ; then the formal group law implies that

$$\mu(t') = (ux) + (uy) + (ux)(uy) = x' + y' + x'y'.$$

**Proposition 5.8.** *Let  $E$  be an oriented ring spectrum  $E$  with multiplicative group law  $x + y + xy$ ,  $u$  a unit. Then the cosimplicial ring spectrum  $E \wedge \mathbb{P}^{\infty \wedge \bullet+1}$  is  $E \otimes C_{F \otimes F}^\bullet(F, F)$ .*

*In particular,  $KGL \wedge \mathbb{P}^{\infty \wedge \bullet+1}$  is isomorphic to  $KGL \otimes C_{F \otimes F}^\bullet(F, F)$ .*

*Proof.* After the change in orientation indicated above, the projection bundle formula yields isomorphisms  $E \otimes F^{\otimes n} \xrightarrow{\cong} E \wedge \mathbb{P}^{\infty \wedge n}$ . The respective coface maps are insertion of 1 and  $T^0$  (by Lemma 5.2), so they agree. The codegeneracies of the left side are given by the product  $F \otimes F \rightarrow F$ , whose coefficients are given by (1.1). These agree with the matrix entries  $a_{ij}^k$  on the right by Lemma 4.2 and Corollary 4.3, as the proof of 4.3 shows.  $\square$

## 6. THE SLICE FILTRATION

Recall [20, 2.2] that  $s_q(E)$  denotes the  $q^{\text{th}}$  slice of a motivic spectrum  $E$ , and that  $s_q(E \wedge T^i) = s_{q-i}(E) \wedge T^i$ . The following result is well known to experts.

**Proposition 6.1.** *The slice functors  $s_q$  commute with direct sums.*

*Proof.* It suffices to prove that  $f_q$  commutes with direct sums, since there is a distinguished triangle of the form  $f_{q+1} \rightarrow f_q \rightarrow s_q$ . For simplicity, we will restrict ourselves to the case  $q = 0$ ; the argument for arbitrary  $q$  is exactly the same.

Consider a direct sum  $E = \bigoplus_\alpha E_\alpha$  in  $\mathcal{SH}$ . Since  $\mathcal{SH}^{\text{eff}}$  is closed under direct sums,  $\bigoplus_\alpha f_0(E_\alpha)$  is effective. By [10, 3.1.14], there is a family of compact objects  $K$  such that, for every  $X \rightarrow Y$ ,  $f_0(X) \rightarrow f_0(Y)$  is an isomorphism if and only if each  $[K, X] \rightarrow [K, Y]$  is an isomorphism. In the case at hand, for every such  $K$  we have

$$[K, \bigoplus_\alpha f_0(E_\alpha)] \cong \bigoplus_\alpha [K, f_0(E_\alpha)] \cong \bigoplus_\alpha [K, E_\alpha] \cong [K, \bigoplus_\alpha E_\alpha].$$

Therefore  $\bigoplus_\alpha f_0(E_\alpha) \rightarrow f_0(\bigoplus_\alpha E_\alpha)$  is an isomorphism.  $\square$

**Example 6.2.** Suppose that  $E$  is oriented. Applying  $s_q$  to (5.1) yields the formula  $\bigoplus_{s_{q-n}}(E) \wedge T^n \xrightarrow{\cong} s_q(E \wedge \mathbb{P}^\infty)$  with component maps  $s_q(\beta_n)$ . More generally, the slice  $s_q(E \wedge \mathbb{P}^{\infty \wedge n})$  is isomorphic to

$$\bigoplus s_r(E) \wedge T^{i_1} \wedge \cdots \wedge T^{i_n},$$

where  $q = r + i_1 + \cdots + i_n$ , and the coface maps in  $s_q(E \wedge \mathbb{P}^{\infty \wedge \bullet})$  are given by Lemma 5.2, with  $E$  replaced by  $s_q(E)$ . Thus, except for the codegeneracy maps, the cosimplicial spectrum  $\bigoplus_q s_q(E \wedge \mathbb{P}^{\infty \wedge \bullet+1})$  is isomorphic to  $\bigoplus_q s_q(E) \otimes F_{\text{gr}}^{\otimes \bullet+1}$ .

We will need the following observation: for any motivic ring spectrum  $E$ ,  $s_*E = \bigoplus s_q(E)$  is a graded motivic ring spectrum; see [10, 3.6.13]). Since  $\mathbb{Z}[u, u^{-1}]$  is a graded ring (with  $u$  in degree 1), we can form the graded motivic ring spectrum  $s_0(E) \otimes \mathbb{Z}[u, u^{-1}] = \bigoplus T^q \wedge s_0(E)$ .

**Lemma 6.3.** *As ring spectra,  $s_0(KGL) \otimes \mathbb{Z}[u, u^{-1}] \xrightarrow{\cong} \bigoplus s_q(KGL)$ .*

*More generally,  $s_0E \otimes \mathbb{Z}[u, u^{-1}] \xrightarrow{\cong} s_*E$  for any oriented ring spectrum  $E$  with a unit  $u$  in  $E_{2,1}$ .*

*Proof.* The periodicity isomorphisms  $u^q : KGL \wedge T^q \rightarrow KGL$ , and more generally  $E \wedge T^q \rightarrow E$ , induce isomorphisms  $s_0 E \wedge T^q \cong s_q E$  compatible with multiplication. These assemble to give the result. (Cf. [16, 6.2].)  $\square$

**Proposition 6.4.** *Suppose that  $E$  is oriented, and has a multiplicative group law  $x + y + xy$ . If  $u$  is a unit, we have isomorphisms of cosimplicial ring spectra:*

$$s_*(E \wedge \mathbb{P}^{\infty \wedge \bullet+1}) \cong s_*(E) \otimes F^{\otimes \bullet+1} \cong s_0(E) \otimes \mathbb{Z}[u, u^{-1}] \otimes C_{F \otimes F}^\bullet(F, F).$$

*In particular,  $s_*(KGL \wedge \mathbb{P}^{\infty \wedge \bullet+1}) \cong s_0(KGL) \otimes \mathbb{Z}[u, u^{-1}] \otimes C_{F \otimes F}^\bullet(F, F)$ .*

*Proof.* Applying  $s_q$  to 5.7 yields the simple formula, with component maps  $s_q(\alpha_n)$ :

$$s_q(E) \otimes F = \bigoplus_{n=0}^{\infty} s_q(E) \xrightarrow{\cong} s_q(E \wedge \mathbb{P}^{\infty}).$$

The rest is immediate from Proposition 5.8 and Example 6.2.  $\square$

When  $E = KGL$ , Propositions 5.8 and 6.4 give the formulas for  $KGL \wedge \mathbb{P}^{\infty \wedge \bullet+1}$  and its slices which were mentioned in the Introduction.

## 7. $E \wedge KGL$

In this section, we describe the augmented cosimplicial spectrum  $E \wedge KGL^{\wedge \bullet+1}$  associated to an oriented spectrum  $E$  and the triple  $- \wedge KGL$ .

$$E \xrightarrow{\eta_L} E \wedge KGL \rightrightarrows E \wedge KGL \wedge KGL \rightrightarrows E \wedge KGL^{\wedge 3} \dots$$

Replacing  $E$  by  $E \wedge KGL^{\wedge n}$ , we are largely reduced to the description of  $E \wedge KGL$ ,  $\eta_L$  and the map  $\sigma^0 : E \wedge KGL \wedge KGL \rightarrow E \wedge KGL$ .

We begin with a few generalities. The  $T$ -spectrum  $KGL$  comes with a periodicity isomorphism  $T \wedge KGL \rightarrow KGL$ ; see [19, 6.8] and [20, 3.3]; our description of it is taken from [13, 1.3]. Let  $\xi : T \rightarrow \mathbb{P}^{\infty}$  be the map that classifies the tautological line bundle on  $\mathbb{P}^1$ , and let  $b : \mathbf{1} \wedge \mathbb{P}^{\infty} \rightarrow T^{-1} \wedge \mathbf{1} \wedge \mathbb{P}^{\infty}$  denote the adjoint of the map

$$T \wedge \mathbb{P}^{\infty} \xrightarrow{\xi \wedge 1} \mathbb{P}^{\infty} \wedge \mathbb{P}^{\infty} \xrightarrow{m} \mathbb{P}^{\infty}.$$

As observed by Gepner-Snaith [5, 4.17] and Spitzweck-Østvær [17],  $KGL$  is the homotopy colimit of the resulting sequence,

$$\mathbf{1} \wedge \mathbb{P}^{\infty} \xrightarrow{b} T^{-1} \wedge \mathbf{1} \wedge \mathbb{P}^{\infty} \xrightarrow{T^{-1}b} T^{-2} \wedge \mathbf{1} \wedge \mathbb{P}^{\infty} \xrightarrow{T^{-2}b} \dots,$$

and the colimit map  $\mathbf{1} \wedge \mathbb{P}^{\infty} \rightarrow KGL$  is the map classifying the virtual tautological line bundle  $\xi - 1$  of degree 0. We write  $u_K$  for the map  $T \rightarrow \mathbf{1} \wedge \mathbb{P}^{\infty} \xrightarrow{\xi-1} KGL$ . Then multiplication by  $u_K$  is the periodicity isomorphism

$$(7.1) \quad T \wedge KGL \xrightarrow{u_K \wedge 1} KGL \wedge KGL \xrightarrow{m} KGL$$

for the  $T$ -spectrum  $KGL$ ; see [13, 1.3.3].

Smashing  $b$  with any spectrum  $E$  yields maps  $b_E : E \wedge \mathbb{P}^{\infty} \rightarrow T^{-1} \wedge E \wedge \mathbb{P}^{\infty}$ , and yields a sequence of  $T$ -spectra

$$(7.2) \quad E \wedge \mathbb{P}^{\infty} \xrightarrow{b_E} T^{-1} \wedge E \wedge \mathbb{P}^{\infty} \xrightarrow{T^{-1}b_E} T^{-2} \wedge E \wedge \mathbb{P}^{\infty} \xrightarrow{T^{-2}b_E} \dots,$$

with homotopy colimit  $E \wedge KGL$ , parallel to the sequence (4.1) in topology.

When  $E$  is oriented, the adjoint  $E \wedge T \wedge \mathbb{P}^{\infty} \rightarrow E \wedge \mathbb{P}^{\infty}$  of  $b_E$  is the composition of the map  $\beta_1 : E \wedge T \rightarrow E \wedge \mathbb{P}^{\infty}$  of (5.1), smashed with  $\mathbb{P}^{\infty}$ , with the product on  $\mathbb{P}^{\infty}$ .

Therefore  $b_E$  corresponds to multiplication by  $\beta_1$ , and the following observation follows from Theorem 5.6.

**Lemma 7.3.** *If  $E$  is oriented, the components  $E \wedge T^j \rightarrow T^{-1} \wedge E \wedge T^k$  of the map  $b_E$  of (7.2) are multiplication by  $k a_{1,1+j-k}$ .*

We now assume that  $E$  has a multiplicative formal group law  $\mu = x + y + uxy$  with  $u \in E_{2,1}$  a unit of  $E_*$ . If  $\alpha_n : E \rightarrow E \otimes F \cong E \wedge \mathbb{P}^\infty$  is the map defined by (5.7), Lemma 7.3 says that

$$b_E(\beta_n) = n u \beta_n + (n+1)\beta_{n+1}, \quad \text{or} \quad b_E(\alpha_n) = n \alpha_n + (n+1)\alpha_{n+1}.$$

Comparing with (1.7), this shows that the map  $b_E : E \wedge \mathbb{P}^\infty \rightarrow T^{-1} \wedge E \wedge \mathbb{P}^\infty$  is induced from the homomorphism  $T : F \rightarrow F$ ,  $T(f) = tf$ . Recall from Definition 1.2 that the Hopf algebra  $H$  is defined by  $H = F[1/t]$  and that  $C_H^n(\mathbb{Z}, \mathbb{Z}) = H^{\otimes n}$  in the classical cobar complex (see Example 3.1).

**Theorem 7.4.** *If  $E$  has a multiplicative group law  $x + y + uxy$  with  $u$  a unit, we have  $E \wedge KGL \cong E \otimes H$ , and isomorphisms of cosimplicial spectra for all  $q \in \mathbb{Z}$ :*

$$\begin{aligned} E \wedge KGL^{\wedge \bullet+1} &\cong E \otimes H^{\otimes \bullet+1} \cong E \otimes C_H^\bullet(\mathbb{Z}, \mathbb{Z}); \\ s_*(E \wedge KGL^{\wedge \bullet+1}) &\cong s_*(E) \otimes H^{\otimes \bullet+1} \cong s_*(E) \otimes C_H^\bullet(\mathbb{Z}, \mathbb{Z}). \end{aligned}$$

In addition,  $E \rightarrow E \wedge KGL$  corresponds to  $E \otimes \mathbb{Z} \rightarrow E \otimes H$ .

*Proof.* By (1.7), multiplication by  $t$  on  $E \otimes F$  is given by the same formula as multiplication by  $b_E$  on  $E \wedge \mathbb{P}^\infty$ . It follows that the homotopy colimit  $E \wedge KGL$  of the sequence (7.2) is the same as  $E \otimes \text{colim}(F \xrightarrow{t} F \rightarrow \dots) = E \otimes H$ . Replacing  $E$  by  $E \wedge KGL^{\wedge n}$  shows that  $E \wedge KGL^{\wedge n+1} \cong E \otimes H^{\otimes n+1}$  and hence that  $s_*(E \wedge KGL^{\wedge n+1}) \cong s_*(E) \otimes H^{\otimes n+1}$ . The coface and codegeneracies of these cosimplicial spectra are identified by Propositions 5.8 and 6.4.  $\square$

When  $E = KGL$  we get the formula mentioned in the Introduction:

**Corollary 7.5.**  *$KGL \wedge KGL \cong KGL \otimes H$ , and  $KGL^{\wedge \bullet+2} \cong KGL \otimes H^{\otimes \bullet+1}$ . In addition,  $\eta_L : KGL \rightarrow KGL \wedge KGL$  corresponds to  $KGL \otimes \mathbb{Z} \rightarrow KGL \otimes H$ .*

**Example 7.6.** When  $E = E \otimes \mathbb{Q}$ , we have  $E \wedge KGL^{\wedge n} \cong E \otimes \mathbb{Q}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ , because  $\mathbb{Q} \otimes H = \mathbb{Q}[t, t^{-1}]$ .

Set  $q = \ell^\nu$  and write  $E/q$  for the cofiber  $E \otimes \mathbb{Z}/q$  of  $E \xrightarrow{q} E$ . An elementary calculation shows that  $(E/q) \otimes H \cong E \otimes H/qH$ .

**Corollary 7.7.** *Suppose that  $E$  has a multiplicative formal group law  $x + y + uxy$ , such that  $u$  is a unit of  $E_{*,*}$ . Then the family of maps  $\beta_n : E \wedge T^n \rightarrow E \wedge \mathbb{P}^\infty \rightarrow E \wedge KGL$  with  $n \not\equiv 0 \pmod{\ell}$  induces an isomorphism*

$$E \otimes H/\ell^\nu H \cong \bigoplus_{n \neq 0} (E/\ell^\nu) \wedge T^n \xrightarrow{\cong} (E/\ell^\nu) \wedge KGL.$$

Moreover, the slices  $s_m(E/\ell^\nu \wedge KGL)$  are isomorphic to  $\bigoplus_{n \neq 0} s_{m-n}(E/\ell^\nu) \wedge T^n$ .

*Proof.* Immediate from Theorems 2.6 and 7.4.  $\square$

8. THE COSIMPLICIAL  $KGL$  SPECTRUM

In this section, we complete the information in the previous section to determine the slices of the cosimplicial spectrum  $N^\bullet KGL$ , with  $N^n KGL = KGL^{\wedge n+1}$ .

$$KGL \rightrightarrows KGL \wedge KGL \rightrightarrows KGL^{\wedge 3} \rightrightarrows \dots$$

Its codegeneracies are given by the product on  $KGL$ , and its coface maps are given by insertion of the unit  $\eta_K : \mathbf{1} \rightarrow KGL$ ; in particular,  $\partial^0, \partial^1 : KGL \rightarrow KGL \wedge KGL$  are the canonical maps  $\eta_R = \eta_K \wedge 1$  and  $\eta_L = 1 \wedge \eta_K$ , respectively. There is also an involution  $c$  of  $KGL \wedge KGL$  swapping the two factors, and we have  $\eta_R = c \circ \eta_L$ .

By Corollary 7.5, there are isomorphisms  $KGL \otimes H^{\otimes n} \xrightarrow{\cong} KGL^{\wedge n+1}$  such that the diagram

$$\begin{array}{ccc} KGL \otimes \mathbb{Z} & \xrightarrow{1 \wedge \text{incl}} & KGL \otimes H \\ \downarrow \cong & & \downarrow \cong \\ KGL & \xrightarrow{\eta_L} & KGL \wedge KGL \end{array}$$

commutes. As  $\partial^1 = \eta_L$ , this establishes the initial case of the following lemma.

**Lemma 8.1.** *The isomorphisms  $KGL \otimes H^{\otimes n} \xrightarrow{\cong} KGL^{\wedge n+1}$  of Corollary 7.5 are compatible with all the coface and codegeneracy operators of  $N^\bullet KGL$ , except possibly for  $\partial^0$  and  $\sigma^0$ .*

*Proof.* Recall the standard construction (the dual path space of [22, 8.3.14]) which takes a cosimplicial object  $X^\bullet$  and produces a new cosimplicial object  $Y^\bullet$  with  $Y^n = X^{n+1}$ , coface maps  $\partial_Y^i = \partial_X^{i+1}$  and codegeneracy maps  $\sigma_Y^i = \sigma_X^{i+1}$ . Applying this construction to  $N^\bullet KGL$  yields the cosimplicial spectrum  $KGL^{\wedge \bullet+2}$  of Corollary 7.5. Since this is isomorphic to the cosimplicial spectrum  $KGL \otimes H^{\otimes \bullet}$  by Theorem 7.4, the result follows.  $\square$

*Proof of Theorem 0.2(a).* Taking slices in Lemma 8.1, we see that the isomorphisms  $s_*(KGL) \otimes H^{\otimes n} \xrightarrow{\cong} s_*(KGL^{\wedge n+1})$  are compatible with all of the coface and codegeneracy operators, except possibly for  $\partial^0$  and  $\sigma^0$ .  $\square$

*Remark 8.2.* It is tempting to consider the cosimplicial spectrum  $KGL \otimes C_H^\bullet(\mathbb{Z}, \mathbb{Z})$ , where  $C_H^\bullet(\mathbb{Z}, \mathbb{Z})$  is the cobar complex over  $H$ . However, its coface maps  $\partial^0$  and codegeneracy maps  $\sigma^0$  are not the same as in  $N^\bullet KGL$ .

To finish the proof of Theorem 0.2(b), we need to show that the isomorphisms in Proposition 8.1 are indeed compatible with  $\partial^0$  and  $\sigma^0$ , under the two additional assumptions. We will do so in Propositions 8.5 and 8.11

**Definition 8.3.** The map  $v_K : T \wedge \mathbf{1} \rightarrow KGL \wedge KGL$  is defined to be  $\eta_R(u_K)$ , where  $u_K : T \rightarrow KGL$  is the map of 7.1.

We define  $v : T \wedge KGL \wedge KGL \rightarrow KGL \wedge KGL$  to be multiplication by  $v_K$ .

**Lemma 8.4.** *We have commutative diagrams*

$$\begin{array}{ccc} T \wedge KGL & \xrightarrow{T \wedge \eta_R} & T \wedge KGL \wedge KGL \\ \downarrow u & & \downarrow v \\ KGL & \xrightarrow{\eta_R} & KGL \wedge KGL \end{array} \quad \begin{array}{ccc} T \wedge s_q(KGL) & \xrightarrow{T \wedge \eta_R} & T \wedge s_q(KGL \wedge KGL) \\ \downarrow u & & \downarrow v \\ s_{q+1}(KGL) & \xrightarrow{\eta_R} & s_{q+1}(KGL \wedge KGL). \end{array}$$

*Proof.* For reasons of space, we write  $K$  for  $KGL$ . In the diagram below, the top rectangle commutes by definition of  $v_K$ , and the bottom rectangle commutes because  $\eta_R$  is a morphism of ring spectra.

$$\begin{array}{ccccc}
T \wedge \mathbf{1} & \xrightarrow{T \wedge \eta_K} & T \wedge K & \xrightarrow{T \wedge \eta_R} & T \wedge (K \wedge K) & \xleftarrow{T \wedge \eta_{(K \wedge K)}} & T \wedge \mathbf{1} \\
& & \downarrow u_K \wedge K & & \downarrow v_K \wedge K \wedge K & & \\
& & K \wedge K & \xrightarrow{\eta_R \wedge \eta_R} & (K \wedge K)^{\wedge 2} & & \\
& & \downarrow m_K & & \downarrow m_{K \wedge K} & & \\
& & K & \xrightarrow{\eta_R} & K \wedge K & & 
\end{array}$$

The left and right verticals are the maps  $u$  and  $v$ , respectively. This establishes commutativity of the first square in Lemma 8.4. As the second square is the composition of the natural isomorphism  $T \wedge s_q(E) \xrightarrow{\cong} s_{q+1}(T \wedge E)$  and the  $q^{\text{th}}$  slice of the first square, it also commutes.  $\square$

Recall that  $s_0(\mathbf{1}) \rightarrow s_0(KGL)$  is an isomorphism if the base  $S$  is smooth over a perfect field  $k$ , or any scheme over a field of characteristic 0. This is so because it is true over any perfect field [9], and  $s_0$  commutes with the pullback  $\mathcal{SH}(k) \rightarrow \mathcal{SH}(S)$  in these cases; see [12, 2.16] and [12, 3.7].

**Proposition 8.5.** *Assume that  $s_0(\mathbf{1}) \rightarrow s_0(KGL)$  is an isomorphism. Then the isomorphisms  $s_q(KGL) \otimes H^{\otimes n} \xrightarrow{\cong} s_q(KGL^{\wedge n+1})$  are compatible with  $\partial^0$ .*

*Proof.* Since  $\mathbf{1}$  is an initial object, there is a unique map from  $\mathbf{1}$  to  $KGL \wedge KGL$ ; the assumption implies that  $\eta_L = \eta_R$  as maps  $s_0(KGL) \rightarrow s_0(KGL \wedge KGL)$ . Now for any  $q \in \mathbb{Z}$ , Lemma 6.3 says that  $u^q : T^q \wedge s_0(KGL) \xrightarrow{\cong} s_q(KGL)$ . By Lemma 8.4, we have a diagram

$$\begin{array}{ccc}
T^q \wedge s_0(KGL) & \xrightarrow{T^q \wedge \eta_R} & T^q \wedge s_0(KGL \wedge KGL) \\
\cong \downarrow u^q & & \downarrow v^q \\
s_q(KGL) & \xrightarrow{\eta_R} & s_q(KGL \wedge KGL).
\end{array}$$

This determines the maps  $\eta_R : s_q(KGL) \rightarrow s_q(KGL \wedge KGL)$ . Summing over  $q$ , and invoking Lemma 6.3 twice, we see that  $\eta_R$  is the map

$$s_0(KGL) \otimes \mathbb{Z}[u, u^{-1}] \rightarrow s_0(KGL) \otimes \mathbb{Z}[u, u^{-1}, v, v^{-1}]$$

sending the copy of  $s_0(KGL)$  indexed by  $u^q$  to the copy of  $s_0(KGL)$  indexed by  $v^q$ . Since we saw in Example 3.2 that this is the same as the map induced from  $\eta_R : \mathbb{Z}[u, u^{-1}] \rightarrow \mathbb{Z}[u, u^{-1}] \otimes H$ , this shows that we have a commutative diagram

$$\begin{array}{ccc}
\bigoplus s_q(KGL) & \xrightarrow{1 \otimes \eta_R} & \bigoplus s_q(KGL) \otimes H \\
\downarrow = & & \downarrow \cong \\
\bigoplus s_q(KGL) & \xrightarrow{\partial^0} & \bigoplus s_q(KGL \wedge KGL).
\end{array}$$

The result follows for  $n > 0$ , since  $\partial^0 : KGL^{\wedge n} \rightarrow KGL^{\wedge n+1}$  is  $\eta_R \wedge KGL^{\wedge n-1}$ .  $\square$

To conclude the proof of Theorem 0.2, we have to compare the slices of the codegeneracy  $\sigma^0 : KGL \wedge KGL \rightarrow KGL$  (the product) with the map of Example 3.2,

$$(8.6) \quad s_0(KGL) \otimes \mathbb{Z}[u, u^{-1}] \otimes H \xrightarrow{\varepsilon} s_0(KGL) \otimes \mathbb{Z}[u, u^{-1}], \quad \varepsilon(u) = \varepsilon(v) = u.$$

Recall from 1.2 that  $\mathbb{Z}[t, t^{-1}]$  is a subring of  $H$ , and that  $\varepsilon : H \rightarrow \mathbb{Z}$  sends  $t$  to 1.

**Lemma 8.7.** *The restriction of the product  $s_*(KGL \wedge KGL) \rightarrow s_*(KGL)$  to*

$$s_*(KGL) \otimes \mathbb{Z}[t, t^{-1}] \rightarrow s_*(KGL) \otimes H \cong s_*(KGL \wedge KGL)$$

*equals the composition*

$$s_*(KGL) \otimes \mathbb{Z}[t, t^{-1}] \xrightarrow{\varepsilon} s_*(KGL) \otimes \mathbb{Z} \cong s_*(KGL).$$

*Proof.* Since  $\sigma^0 : KGL \wedge KGL \rightarrow KGL$  is a left inverse to both  $\eta_L$  and  $\eta_R$ , and  $u = \eta_L(u)$ ,  $v = \eta_R(u)$ , we see that  $\sigma^0$  sends the copies of  $s_0(KGL)$  indexed by the monomials  $u^q$  and  $v^q$  (in  $\mathbb{Z}[u, u^{-1}] \otimes H$ ) to the copies indexed by  $u^q$ . Since  $\sigma^0$  is a map of ring spectra, the copies indexed by  $u^i \otimes t^j = u^{i-j} v^j$  map to the copies indexed by  $u^i$ .  $\square$

Recall from Example 7.6 that  $KGL^{\wedge n+1} \otimes \mathbb{Q} \cong KGL \otimes \mathbb{Q}[t_0, t_0^{-1}, \dots, t_n, t_n^{-1}]$ . This is  $KGL \otimes C_{\mathbb{Q}[t, t^{-1}]}^\bullet(\mathbb{Z}, \mathbb{Z})$ , where  $C_{\mathbb{Q}[t, t^{-1}]}^\bullet(\mathbb{Z}, \mathbb{Z})$  is the  $n^{\text{th}}$  term of the cobar complex of Example 3.2 for the Hopf algebra  $\mathbb{Q}[u, u^{-1}]$ .

Now consider  $(R, \Gamma)$  with  $R = \mathbb{Q}[u, u^{-1}]$  and  $\Gamma = R[t, t^{-1}]$ , as in Example 3.2. Combining Lemma 8.1, Proposition 8.5 and Lemma 8.7, we obtain the rational version of the KGL slice conjecture :

**Corollary 8.8.** *Let  $(R, \Gamma)$  be as above, and assume that  $s_0(\mathbf{1}) \rightarrow s_0(KGL)$  is an isomorphism. As cosimplicial motivic ring spectra,*

$$s_* N^*(KGL \otimes \mathbb{Q}) \cong s_0(KGL) \otimes C_\Gamma^\bullet(R, R).$$

As in [3, 6.1], it is easy to verify that the diagram

$$(8.9) \quad \begin{array}{ccc} KGL \wedge \mathbb{P}^\infty & \xrightarrow{\xi^{-1}} & KGL \wedge KGL \\ \downarrow b_K & & \downarrow v \\ T^{-1} \wedge KGL \wedge \mathbb{P}^\infty & \xrightarrow{\xi^{-1}} & T^{-1} \wedge KGL \wedge KGL \end{array}$$

commutes, where  $b_K$  is  $KGL \wedge b$ ,  $b$  is the map in (7.2) and the right side is multiplication by  $v_K$  (smashed with  $T^{-1}$ ).

**Lemma 8.10.** *If  $S$  is smooth over a perfect field, then  $s_0(\mathbf{1}) \rightarrow H[\mathbb{Z}]$  is an isomorphism in  $\mathcal{SH}(S)$ .*

*Proof.* This is true over the ground field  $F$ , by [9, 11.3.6]. Now the slice functors  $s_q$  commute with the pullback  $\pi^*$  over  $\pi : S \rightarrow \text{Spec}(F)$ ; see [12, 2.16]. Because  $S$  is smooth,  $H\mathbb{Z}$  also pulls back over  $\pi^*$  (see [19, 6.1], [21, 3.18]), and we have

$$s_0(\mathbf{1}_S) = \pi^* s_0(\mathbf{1}_F) = \pi^*(H\mathbb{Z}_F) = H\mathbb{Z}_S. \quad \square$$

Now the group  $\text{Hom}_{\mathcal{SH}(S)}(H\mathbb{Z}, H\mathbb{Z}) \cong H^0(S, \mathbb{Z})$  is torsionfree, and equal to  $\mathbb{Z}$  if  $S$  is connected; cf. [18, 3.7]. It follows that if  $S$  is smooth over a perfect field, so that  $s_0(\mathbf{1}) \cong H\mathbb{Z}$ , the hypothesis of the following Proposition is satisfied.

**Proposition 8.11.** *Assume that  $\text{Hom}_{\mathcal{SH}(S)}(s_0(\mathbf{1}), s_0(\mathbf{1}))$  is torsionfree. Then the isomorphisms  $s_q(KGL) \otimes H^{\otimes n} \xrightarrow{\cong} s_q(KGL^{\wedge n+1})$  are compatible with  $\sigma^0$ .*

*Proof.* Every element of  $H$  is a power of  $t$  times a numerical polynomial, so it suffices to check the test maps  $\alpha_n : s_0(K) \rightarrow s_0(KGL) \otimes H \cong s_0(KGL \wedge KGL)$  corresponding to the elements  $\binom{t}{n}$  of  $H$ . If  $n = 0, 1$  we are done by Lemma 8.7. For  $n \geq 2$  the composition

$$s_0(KGL) \xrightarrow{n!} s_0(KGL) \xrightarrow{\alpha_n} s_0(KGL \wedge KGL) \xrightarrow{\sigma^0} s_0(KGL)$$

is given by  $\sigma^0(t(t-1)\cdots(t-n+1)) = 0$  in the ring  $[s_0(KGL), s_0(KGL)]$ , which is  $\mathbb{Z}$  by Lemma 8.10. Since  $\sigma^0\alpha_n$  corresponds to an integer, and is killed by  $n!$ , we must have  $\sigma^0\alpha_n = 0$ .  $\square$

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ U.S.A.  
*E-mail address:* `pablo.pelaez@gmail.com`

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ U.S.A.  
*E-mail address:* `weibel@math.rutgers.edu`