Relative Chern characters for nilpotent ideals

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1 Introduction

When A is a unital ring, the *absolute Chern character* is a group homomorphism $ch_* : K_*(A) \to HN_*(A)$, going from algebraic K-theory to negative cyclic homology (see [7, 11.4]). There is also a relative version, defined for any ideal I of A:

$$ch_* : K_*(A, I) \to HN_*(A, I). \tag{1.1}$$

Now suppose that A is a $\mathbb{Q}$-algebra and that I is nilpotent. In this case, Goodwillie proved in [4] that (1.1) is an isomorphism. His proof uses another character

$$ch'_* : K_*(A, I) \to HN_*(A, I) \tag{1.2}$$

which is defined only when I is nilpotent and $\mathbb{Q} \subset A$. Goodwillie showed that (1.2) is an isomorphism whenever it is defined, and that it coincides with (1.1) when $I^2 = 0$. Using this and a 5-Lemma argument, he deduced that (1.1) is an isomorphism for any nilpotent I. The question of whether ch_* and ch'_* agree for general nilpotent ideals I was left open in [4], and announced without proof in [7, 11.4.11]. This paper answers the question, proving in Theorem 6.5.1 that

$$ch_* = ch'_* \text{ for all nilpotent ideals } I. \tag{1.3}$$

Here are some applications of (1.3). Let I a nilpotent ideal in a commutative $\mathbb{Q}$-algebra A. Cathelineau proved in [1] that (1.2) preserves the direct sum decomposition coming from the eigenspaces of λ-operations and/or Adams operations. By (1.3), the relative Chern character (1.1) preserves the direct sum decomposition for nilpotent ideals. Next, Cathelineau's result and (1.3) are used in [2] to prove that the absolute Chern character ch_* also preserves the direct sum decomposition. In addition, our result (1.3) can be used to strengthen Ginot's results in [3].
This paper is laid out according to the following plan. In section 2 we show that the bar complex $\mathrm{B}(\mathsf{H})$ of a cocommutative Hopf algebra H has a natural cyclic module structure. The case $\mathsf{H} = k[G]$ is well known, and the case of enveloping algebras is implicit in [6]. In section 3, we relate this construction to the usual cyclic module of the algebra underlying H. In section 4 we consider a Lie algebra $\mathfrak{g}$ and factor the Loday-Quillen map $\wedge^n \mathfrak{g} \to C^{\lambda}_{n-1}(U\mathfrak{g})$ through our construction. In section 5 we consider a nilpotent Lie algebra $\mathfrak{g}$ and its associated nilpotent group G and relate the constructions for $U\mathfrak{g}$ and $\mathbb{Q}[G]$ using the Mal'cev theory of [10, App. A]. In section 6, we review the definitions of ch_* and ch'_* and prove our main theorem, that (1.3) holds.

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Notation

If $M = (M_*, b, B)$ is a mixed complex ([7, 2.5.13]), we will write $HH(M)$ for the chain complex (M_*, b), and $HN(M)$ for the total complex of Connes' left half-plane (b, B)-complex (written as $\mathcal{B}M^-$ in [7, 5.1.7]). By definition, the homology of $HH(M)$ is the Hochschild homology $HH_*(M)$ of M, the homology of $HN(M)$ is the negative cyclic homology $HN_*(M)$ of M, and the projection $\pi : HN(M) \to HH(M)$ induces the canonical map $HN_*(M) \to HH_*(M)$.
We refer the reader to [7, 2.5.1] for the notion of a cyclic module. There is a canonical cyclic k-module $C(A)$ associated to any algebra A, with $C_n(A) = A^{\otimes n+1}$, whose underlying simplicial module (C, b) is the Hochschild complex (see [7, 2.5.4]). We will write $HH(A)$ and $HN(A)$ in place of the more awkward expressions $HH(C(A))$ and $HN(C(A))$.
We write $C_{\geq n}$ for the good truncation $\tau_{\geq n}C$ of a chain complex C [13, 1.2.7]. If $n \geq 0$, then $C_{\geq n}$ can and will be regarded as a simplicial module via the Dold-Kan correspondence [13, 8.4].

2 Cyclic homology of cocommutative Hopf algebras

If A is the group algebra of a group G, then the bar resolution $\mathrm{E}(A) = k[\mathrm{E}G]$ admits a cyclic G-module structure and the bar complex $\mathrm{B}(A) = k[\mathrm{B}G]$ also admits a cyclic k-module structure [7]. In this section, we show that the cyclic modules $\mathrm{E}(A)$ and $\mathrm{B}(A)$ can be defined for any cocommutative Hopf algebra A.

2.1 Bar resolution and bar complex of an augmented algebra

Let k be a commutative ring, and A an augmented unital k-algebra, with augmentation $\epsilon : A \to k$. We write $\mathrm{E}(A)$ for the *bar resolution* of k as a left A-module ([13, 8.6.12]); this is the simplicial A-module $\mathrm{E}_n(A) = A^{\otimes n+1}$, whose face and degeneracy operators are given by

$$\begin{aligned} \mu_i : \mathrm{E}_n(A) &\to \mathrm{E}_{n-1}(A) \qquad (i = 0, \dots, n) \\ \mu_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \qquad (i < n) \\ \mu_n(a_0 \otimes \cdots \otimes a_n) &= \varepsilon(a_n) a_0 \otimes \cdots \otimes a_{n-1} \\ s_j : \mathrm{E}_n(A) &\to \mathrm{E}_{n+1}(A) \qquad (j = 0, \dots, n) \\ s_j(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_j \otimes 1 \otimes a_{j+1} \otimes \cdots \otimes a_n \end{aligned} \tag{2.1.1}$$

We write ∂' for the usual boundary map $\sum_{i=0}^{n} (-1)^i \mu_i : \mathrm{E}_n(A) \to \mathrm{E}_{n-1}(A)$. The augmentation induces a quasi-isomorphism $\epsilon : \mathrm{E}(A) \to k$. The unit $\eta : k \to A$ is a k-linear homotopy inverse of ϵ; we have $\epsilon\eta = 1$ and the extra degeneracy $s : \mathrm{E}(A)_n \to \mathrm{E}(A)_{n+1}$,

$$s(x) = 1 \otimes x \tag{2.1.2}$$

satisfies $1 - \eta\epsilon = [\partial', s]$. The *bar complex* of A is $\mathrm{B}(A) = k \otimes_A \mathrm{E}(A)$; $\mathrm{B}_n(A) = A^{\otimes n}$. We write $\partial = 1 \otimes \partial' : \mathrm{B}(A)_n \to \mathrm{B}(A)_{n-1}$ for the induced boundary map, and $\mathrm{E}(A)_{\mathrm{norm}}$ and $\mathrm{B}(A)_{\mathrm{norm}}$ for the normalized complexes.

2.2 The cyclic module of a cocommutative coalgebra

If C is a k-coalgebra, with counit $\epsilon : C \to k$ and coproduct Δ, we have a simplicial k-module $R(C)$, with $R_n(C) = C^{\otimes n+1}$, and face and degeneracy operators given by

$$\begin{aligned} \varepsilon_i : R_n(C) &\to R_{n-1}(C) \qquad (i = 0, \dots, n) \\ \varepsilon_i(c_0 \otimes \cdots \otimes c_n) &= \epsilon(c_i) c_0 \otimes \cdots \otimes c_{i-1} \otimes c_{i+1} \otimes \dots c_n \\ \Delta_i : R_n(C) &\to R_{n+1}(C) \qquad (i = 0, \dots, n) \\ \Delta_i(c_0 \otimes \cdots \otimes c_n) &= c_0 \otimes \dots c_{i-1} \otimes c_i^{(0)} \otimes c_i^{(1)} \otimes \cdots \otimes c_n \end{aligned}$$

Here and elsewhere we use the (summationless) Sweedler notation $\Delta(c) = c^{(0)} \otimes c^{(1)}$ of [11]. We remark that each $R_n(C)$ has a coalgebra structure, and that the face maps ε_i are coalgebra homomorphisms. If in addition C is cocommutative, then the degeneracies Δ_j are also coalgebra homomorphisms, and $R(C)$ is a simplicial coalgebra. In fact $R_n(C)$ is the product of $n+1$ copies of C in the category of cocommutative coalgebras, and $R(C)$ is a particular case of the usual product simplicial resolution of an object in a category with finite products, which is a functor not only on the simplicial category of finite ordinals and monotone maps, but also on the larger category $\mathfrak{Fin}$ with the same objects, but where a homomorphism is just any set theoretic map, not necessarily order preserving. By [7, 6.4.5] we have:

Lemma 2.2.1. *For a cocommutative coalgebra C, the simplicial module $R(C)$ has the structure of a cyclic k-module, with cyclic operator*

$$\lambda(c_0 \otimes \cdots \otimes c_n) = (-1)^n c_n \otimes c_0 \otimes \cdots \otimes c_{n-1}.$$

Remark 2.2.2. Let G be a group; write $k[G]$ for the group algebra. Note $k[G]$ is a Hopf algebra, and in particular, a coalgebra, with coproduct determined by $\Delta(g) = g \otimes g$. The cyclic module $R(k[G])$ thus defined is precisely the cyclic module whose associated cyclic bicomplex was considered by Karoubi in [5, 2.21], where it is written $\tilde{C}_{**}(G)$.

2.3 The case of Hopf algebras

Let H be a Hopf algebra with unit η, counit ϵ and antipode S. We shall assume that $S^2 = 1$, which is the case for all cocommutative Hopf algebras. We may view $R(\mathsf{H})$ as a simplicial left H-module via the diagonal action:

$$a \cdot (h_0 \otimes \cdots \otimes h_n) = a^{(0)} h_0 \otimes \cdots \otimes a^{(n)} h_n.$$

Consider the maps (defined using summationless Sweedler notation):

$$\alpha : \mathrm{E}_n(\mathsf{H}) \to R_n(\mathsf{H}),$$

$$\alpha(h_0 \otimes \cdots \otimes h_n) = h_0^{(0)} \otimes h_0^{(1)} h_1^{(0)} \otimes \cdots \otimes h_0^{(n)} h_1^{(n-1)} \cdots h_{n-1}^{(1)} h_n \tag{2.3.1}$$

$$\beta : R_n(\mathsf{H}) \to \mathrm{E}_n(\mathsf{H}),$$

$$\beta(h_0 \otimes \cdots \otimes h_n) = h_0^{(0)} \otimes (S h_0^{(1)}) h_1^{(0)} \otimes \cdots \otimes (S h_{n-1}^{(1)}) h_n \tag{2.3.2}$$

A straightforward computation reveals:

Lemma 2.3.3. *The maps* (2.3.1) *and* (2.3.2) *are inverse isomorphisms of simplicial* H*-modules:* $\mathrm{E}_n(\mathsf{H}) \cong R_n(\mathsf{H})$.

2.4 Cyclic complexes of cocommutative Hopf algebras

From now on, we shall assume that H is a cocommutative Hopf algebra. In this case the cyclic operator $\lambda : R(\mathsf{H}) \to R(\mathsf{H})$ of 2.2.1 is a homomorphism of H-modules. Thus $R(\mathsf{H})$ is a cyclic H-module, and we can use the isomorphisms α and β of Lemma 2.3.3 to translate this structure to the bar resolution $\mathrm{E}(\mathsf{H})$. We record this as a corollary:

Corollary 2.4.1. *When* H *is a cocommutative Hopf algebra,* $\mathrm{E}(\mathsf{H})$ *is a cyclic* H*-module, and* $\mathrm{B}(\mathsf{H}) = k \otimes_{\mathsf{H}} \mathrm{E}(\mathsf{H})$ *is a cyclic k-module. The cyclic operator $t := \beta\lambda\alpha$ on* $\mathrm{E}(\mathsf{H})$ *is given by the formulas $t(h) = h$, $t(h_0 \otimes h_1) = -h_0 h_1^{(0)} \otimes S h_1^{(1)}$ and:*

$$t(h_0 \otimes \cdots \otimes h_n) = (-1)^n h_0 h_1^{(0)} \ldots h_n^{(0)} \otimes S(h_1^{(1)} \ldots h_n^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes h_{n-1}^{(2)}.$$

Remark 2.4.2. If $g_0, \dots, g_n \in \mathsf{H}$ are grouplike elements then

$$t(g_0 \otimes \cdots \otimes g_n) = (-1)^n g_0 \dots g_n \otimes (g_1 \dots g_n)^{-1} \otimes g_1 \otimes \cdots \otimes g_{n-1}.$$

In particular, for $\mathsf{H} = k[G]$, $\mathrm{B}(k[G])$ with the cyclic structure of 2.4.1 is the cyclic module associated to the cyclic set $\mathrm{B}(G,1)$ of [7, 7.3.3].

For the extra degeneracy of [7, 2.5.7],

$$s' = (-1)^{n+1} t s_n : \mathrm{E}_n(\mathsf{H}) \to \mathrm{E}_{n+1}(\mathsf{H}), \tag{2.4.3}$$

it is immediate from (2.1.1) and 2.4.1 that s' is signfree:

$$s'(h_0 \otimes \cdots \otimes h_n) = h_0 h_1^{(0)} \dots h_n^{(0)} \otimes S(h_1^{(1)} \dots h_n^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes h_n^{(2)}.$$

By 2.4.1, the Connes' operator B' is the H-module homomorphism:

$$B' = (1-t)s' \sum_{i=0}^{n} t^i : \mathrm{E}_n(\mathsf{H}) \to \mathrm{E}_{n+1}(\mathsf{H}). \tag{2.4.4}$$

(See [8, p. 569].) We write $B : \mathrm{B}(\mathsf{H})_n \to \mathrm{B}(\mathsf{H})_{n+1}$ for the induced k-module map.

Definition 2.4.5. *We define the mixed* H*-module complex* $\mathrm{M}' = \mathrm{M}'(\mathsf{H})$*, and the mixed* k*-module complex* $\mathrm{M} = \mathrm{M}(\mathsf{H})$ *to be the normalized mixed complexes associated to the cyclic modules* $\mathrm{E}_*(\mathsf{H})$ *and* $\mathrm{B}_*(\mathsf{H})$ *of 2.4.1:*

$$\mathrm{M}'(\mathsf{H}) := (\mathrm{E}_*(\mathsf{H})_{\mathrm{norm}}, \partial', B')$$
$$\mathrm{M}(\mathsf{H}) := k \otimes_{\mathsf{H}} \mathrm{M}'(\mathsf{H}) = (\mathrm{B}_*(\mathsf{H})_{\mathrm{norm}}, \partial, B).$$

Remark 2.4.6. Consider the map $s'' = (-1)^n s_n : \mathrm{E}_n(\mathsf{H}) \to \mathrm{E}_{n+1}(\mathsf{H})$, and set $B'' = -ts''N$, where as usual $N = \sum t^i$ is the norm map. Then $B' = B''$ on $\mathrm{E}(\mathsf{H})_{\mathrm{norm}}$ because the relations $s_0 t_n = (-1)^n t_{n+1}^2 s_n$ and $tN = N$ yield for all $x \in \mathrm{E}_n(\mathsf{H})$:

$$\begin{aligned}(B'' - B')(x) &= ((-1)^{n+1} t s_n N + (-1)^n (1-t) t s_n N)(x) \\ &= (-1)^n (-t s_n N(x) + t s_n N(x) - t^2 s_n N(x)) \\ &= -s_0 t N(x) = -s_0 N(x) \equiv 0 \quad \text{in } \mathrm{E}(\mathsf{H})_{\mathrm{norm}}.\end{aligned}$$

Lemma 2.4.7. *The map* $B' : \mathrm{E}(\mathsf{H})_{\mathrm{norm}} \to \mathrm{E}(\mathsf{H})_{\mathrm{norm}}[1]$ *induced by* (2.4.4) *is given by the explicit formula:*

$$B'(h_0 \otimes \cdots \otimes h_n) =$$
$$\sum_{i=0}^{n} (-1)^{ni} h_0 h_1^{(0)} \cdots h_{n-i}^{(0)} \otimes h_{n-i+1}^{(0)} \otimes \cdots \otimes h_n^{(0)} \otimes S(h_1^{(1)} \cdots h_n^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes h_{n-i}^{(2)}.$$

Proof. For convenience, let us write $Sh^{(1)}$ for $S(h_1^{(1)} \cdots h_n^{(1)})$. It follows from 2.4.1, cocomutativity and induction on i that $t^i(h_0) = h_0$, and if $i \le n$ then:

$$t^i(h_0 \otimes \cdots \otimes h_n) =$$
$$(-1)^{ni} h_0 h_1^{(0)} \dots h_{n-i+1}^{(0)} \otimes h_{n-i+2}^{(0)} \otimes \cdots \otimes h_n^{(0)} \otimes Sh^{(1)} \otimes h_1^{(2)} \otimes \cdots \otimes h_{n-i}^{(2)}.$$

Let $s'' = (-1)^n s_n$ be as in Remark 2.4.6 and set $m = n(i+1)+1$. We have

$$\begin{aligned} -ts''t^i(h_0 \otimes \cdots \otimes h_n) &= (-1)^{n+1} t s_n t^i (h_0 \otimes \cdots \otimes h_n) \\ = (-1)^m t(h_0 h_1^0 \dots h_{n-i+1}^{(0)} &\otimes h_{n-i+2}^{(0)} \otimes \cdots \otimes h_n^{(0)} \otimes Sh^{(1)} \otimes h_1^{(2)} \otimes \cdots \otimes h_{n-i}^{(2)} \otimes 1) \\ = (-1)^{ni} h_0 h_1^{(0)} \dots h_{n-i}^{(0)} &\otimes h_{n-i+1}^{(0)} \otimes \cdots \otimes h_n^{(0)} \otimes Sh^{(1)} \otimes h_1^{(2)} \otimes \cdots \otimes h_{n-i}^{(2)}.\end{aligned}$$

Now sum up over i to get B'' and use Remark 2.4.6. □

Corollary 2.4.8. *Suppose that* $x_1, \ldots, x_n$ *are primitive elements of* H*, and* $h \in \mathsf{H}$*. Then* $B'(h \otimes x_1 \otimes \cdots \otimes x_n) = 0$ *in* $\mathrm{E}(\mathsf{H})_{\mathrm{norm}}$.

Proof. When x is primitive, $x^{(0)} \otimes x^{(1)} \otimes x^{(2)}$ is $1 \otimes 1 \otimes x + 2 \otimes x \otimes 1 + x \otimes 1 \otimes 1$. By Lemma 2.4.7, $B'(h \otimes x_1 \otimes \cdots \otimes x_n)$ is a sum of terms of the form

$$\pm h' \otimes x_{j+1}^{(0)} \otimes \cdots \otimes x_n^{(0)} \otimes S(x_1^{(1)} \cdots x_n^{(1)}) \otimes x_1^{(2)} \otimes \cdots \otimes x_j^{(2)}.$$

By inspection, each such term is degenerate, and vanishes in $\mathrm{E}(\mathsf{H})_{\mathrm{norm}}$. □

2.5 Adic filtrations and completion

As usual, we can use an adic topology on a Hopf algebra to define complete Hopf algebras, and topological versions of the above complexes.

First we recall some generalities about filtrations and completions of k-modules, following [10]. There is a category of filtered k-modules and filtration-preserving maps; a filtered module V is a module equipped with a decreasing filtration $V = \mathcal{F}_0(V) \supseteq \mathcal{F}_1(V) \supseteq \cdots$. The completion of V is $\hat{V} = \lim_n V/\mathcal{F}_n V$; it is a filtered module in the evident way. If W is another filtered k-module, then the tensor product $V \otimes W$ is a filtered module with filtration

$$\mathcal{F}_n(V \otimes W) = \sum_{p+q=n} \mathrm{image}(\mathcal{F}_p V \otimes \mathcal{F}_q W \to V \otimes W). \tag{2.5.1}$$

We define $\hat{V} \hat{\otimes} \hat{W}$ to be $\widehat{\hat{V} \otimes \hat{W}}$. Note that

$$\mathcal{F}_n(V \otimes W) \supseteq \mathrm{image}\Big(\mathcal{F}_n V \otimes W + V \otimes \mathcal{F}_n W \to V \otimes W\Big) \supseteq \mathcal{F}_{2n}(V \otimes W). \tag{2.5.2}$$

Hence the topology defined by $\{\mathcal{F}_n(V \otimes W)\}$ is the same as that defined by $\{\ker(V \otimes W \to V/\mathcal{F}_n V \otimes W/\mathcal{F}_n W)\}$. It follows that $\hat{V} \hat{\otimes} \hat{W}$ satisfies

$$\hat{V} \hat{\otimes} \hat{W} = \widehat{V \otimes W}. \tag{2.5.3}$$

All this has an obvious extension to tensor products of finitely many factors.

If A is a filtered algebra (an algebra which is filtered as a k-module by ideals), then $\hat{A}$ is an algebra. If I is an ideal then $\{I^n\}$ is called the I-adic filtration on A.

Now suppose that H is a cocommutative Hopf algebra, equipped with the I-adic filtration, where I is both a (2-sided) ideal and a (2-sided) coideal of H, closed under the antipode S and satisfying $\epsilon(I) = 0$. The coideal condition on I means that

$$\Delta(I) \subset \mathsf{H} \otimes I + I \otimes \mathsf{H}.$$

This implies that $\Delta : \mathsf{H} \to \mathsf{H} \otimes \mathsf{H}$ is filtration-preserving; by (2.5.3) it induces a map $\hat{\Delta} : \hat{\mathsf{H}} \to \hat{\mathsf{H}} \hat{\otimes} \hat{\mathsf{H}}$, making $\hat{\mathsf{H}}$ into a *complete Hopf algebra* in the sense of [10].

Consider the induced filtrations (2.5.1) in $\mathrm{E}_n(\mathsf{H})$ and $\mathrm{B}_n(\mathsf{H})$ ($n \geq 0$). It is clear, from the formulas (2.1.1) and our assumption that $\epsilon(I) = 0$, that the simplicial structures of $\mathrm{E}(\mathsf{H})$ and $\mathrm{B}(\mathsf{H})$ are compatible with the filtration; i.e., $\mathrm{E}(\mathsf{H})$ and $\mathrm{B}(\mathsf{H})$ are simplicial filtered objects. Since I is a coideal closed under S, the formula in 2.4.1 implies that t preserves the filtration. Thus $\mathrm{E}(\mathsf{H})$ and $\mathrm{B}(\mathsf{H})$ are cyclic filtered modules. It follows that $\mathrm{M}'(\mathsf{H})$ and $\mathrm{M}(\mathsf{H})$ are filtered mixed complexes.

The identities (2.5.3) show that the completed objects $\hat{\mathrm{E}}(\mathsf{H})$, $\hat{\mathrm{B}}(\mathsf{H})$, etc. depend only on the topological Hopf algebra $\hat{\mathsf{H}}$, and can be regarded as its topological bar resolution, bar complex etc. which are defined similarly to their algebraic counterparts, but substituting $\hat{\otimes}$ for $\otimes$ everywhere. In this spirit, we shall write $\mathrm{E}^{\mathrm{top}}(\hat{\mathsf{H}})$, $\mathrm{B}^{\mathrm{top}}(\hat{\mathsf{H}})$, etc., for $\hat{\mathrm{E}}(\mathsf{H})$, $\hat{\mathrm{B}}(\mathsf{H})$, etc.

3 Comparison with the cyclic module of the algebra H

Let H be a cocommutative Hopf algebra. In this section, we construct an injective cyclic module map $\tau : \mathrm{B}(\mathsf{H}) \to C(\mathsf{H})$, from the cyclic bar complex $\mathrm{B}(\mathsf{H})$ of 2.4.1 to the canonical cyclic k-module $C(\mathsf{H})$ of the algebra underlying H ([7, 2.5.4]), and a lift c of τ to the negative cyclic complex $HN(\mathsf{H})$ of H.

In the group algebra case, these constructions are well-understood. The cyclic module inclusion $\tau : \mathrm{B}(k[G]) \subset C(k[G])$ is given in [7, 7.4]; see Example 3.2.2 below. Goodwillie proved in [4] that τ admits a natural lifting to a chain map $c : \mathrm{B}(k[G]) \to HN(k[G])$ to the negative cyclic complex, and that c is unique up to natural homotopy. An explicit formula for such a lifting was given by Ginot [3], in the normalized, mixed complex setting.

3.1 A natural section of the projection $\mathrm{H}N(\mathrm{M}'(\mathsf{H})) \to HH(\mathrm{M}'(\mathsf{H}))$.

Recall from Definition 2.4.5 that $\mathrm{M}' = \mathrm{M}'(\mathsf{H})$ is a mixed complex whose underlying chain complex is $HH(\mathrm{M}') = (\mathrm{E}(\mathsf{H})_{\mathrm{norm}}, \partial')$, and write π' for the projection from the negative cyclic complex $HN(\mathrm{M}')$ to $HH(\mathrm{M}')$. Following the method of Ginot [3], we shall define a natural H-linear chain homomorphism $\Upsilon' : HH(\mathrm{M}') \to HN(\mathrm{M}')$ such that $\pi' \Upsilon' = 1$.

We shall use a technical lemma about maps between chain complexes of modules over a k-algebra A. Assume given all of the following:

(i) A homomorphism of chain A-modules $f : C \to D$, with $C_n = 0$ for $n < n_0$.
(ii) A decomposition $C_n \cong A \otimes V_n$ for each n, where V_n is a k-module.
(iii) A k-linear chain contraction s for D.

Lemma 3.1.1. *Given (i)–(iii), there is an A-linear chain contraction κ^f of f, defined by $\kappa^f_{n_0}(av) = asf(v)$ and the inductive formula:*

$$\kappa^f_n : C_n = A \otimes V_n \to D_{n+1}, \quad \kappa^f_n(av) = a\, s\left(f - \kappa^f_{n-1} d\right)(v).$$

Proof. We have to verify the formula $f(av) = \kappa^f d(av) + d\kappa^f(av)$ for $a \in A$ and $v \in V_n$. When $n = n_0$, this is easy as $d(av) = 0$ and $f(v) = dsf(v)$:

$$d\kappa^f_{n_0}(av) = adsf(v) = af(v) = f(av).$$

Inductively, suppose that the formula holds for $n-1$. Since $dv \in C_{n-1}$, we have

$$d(f - \kappa^f d)(v) = f(dv) - d\kappa^f(dv) = (\kappa^f d)(dv) = 0.$$

Using this, and the definition of κ^f_n, we compute:

$$(d\kappa^f_n)(v) = ds(f - \kappa^f d)(v) = (1 - sd)(f - \kappa^f d)(v) = (f - \kappa^f d)(v).$$

Since $\kappa^f d(av) = \kappa^f(a\, dv) = a\kappa^f(dv)$ by construction,

$$d\kappa^f(av) + \kappa^f d(av) = ad\kappa^f(v) + a\kappa^f(dv) = af(v) = f(av). \quad \square$$

Lemma 3.1.2. *There is a sequence of* H*-linear maps* $\Upsilon'^n : \mathrm{E}(\mathsf{H}) \to \mathrm{E}(\mathsf{H})[2n]$*, starting with* $\Upsilon'^0 = 1$*, such that* $B'(\Upsilon'^n \partial' - \partial' \Upsilon'^n) = 0$.
They induce maps on the normalized complexes $\Upsilon'^n : HH(\mathrm{M}') \to HH(\mathrm{M}')[2n]$.

Proof. Inductively, we suppose we have constructed Υ'^n satisfying $B'[\Upsilon'^n, \partial'] = 0$. Now any chain map from $C = \mathrm{E}(\mathsf{H})$ to $\mathrm{E}(\mathsf{H})[2n+1]$ must land in the good truncation $D = \mathrm{E}(\mathsf{H})[2n+1]_{\geq 0}$, and the k-linear chain contraction $-s$ of (2.1.2) is also a contraction of D. We claim that the H-linear map $f = -B'\Upsilon'^n : C \to D$ is a chain map. Since the differential on D is $-\partial'$, the claim follows from:

$$f(\partial') - (-\partial')f = -B'\Upsilon'^n\partial' - \partial' B'\Upsilon'^n = B'(\partial'\Upsilon'^n - \Upsilon'^n\partial') = 0.$$

We define Υ'^{n+1} to be the H-linear chain contraction of $f = -B'\Upsilon'^n$ given by the formulas in Lemma 3.1.1. That is,

$$\Upsilon'^{n+1} := \kappa^f = \kappa^{-B'\Upsilon'^n} \qquad (n \geq 0). \tag{3.1.3}$$

The chain contraction condition $[\Upsilon'^{n+1}, \partial'] = f$ for (3.1.3) implies that the inductive hypothesis $B'[\Upsilon'^{n+1}, \partial'] = B'f = 0$ holds.
Finally, note that the normalized mixed complex $HH(M')$ is a quotient of $\mathrm{E}(\mathsf{H})$, and its terms have the form $HH(M')_n = \mathsf{H} \otimes W_n$ for a quotient module W_n of V_n. By naturality of κ^f in f, the above construction also goes through with $\mathrm{E}(\mathsf{H})$ replaced by $M'(\mathsf{H})$, and the maps Υ' on $\mathrm{E}(\mathsf{H})$ and $HH(M')$ are compatible. □

We define maps $\Upsilon' : HH(\mathrm{E}(\mathsf{H})) \to HN(\mathrm{E}(\mathsf{H}))$ and $\Upsilon' : HH(\mathrm{M}') \to HN(\mathrm{M}')$ by

$$\Upsilon' = \sum_{n=0}^{\infty} \Upsilon'^n. \tag{3.1.4}$$

That is, $\Upsilon'(x)$ is $(\cdots, \Upsilon'^n(x), \cdots, \Upsilon'^1(x), x)$.

Lemma 3.1.5. *The maps* Υ' *in* (3.1.4) *are morphisms of chain* H*-modules, and* $\pi'\Upsilon' = 1$*. Here* π' *is the appropriate canonical projection, either* $HN(\mathrm{E}(\mathsf{H})) \to HH(\mathrm{E}(\mathsf{H}))$ *or* $HN(\mathrm{M}') \to HH(\mathrm{M}')$.

Proof. It is clear that Υ' is H-linear and that $\pi'\Upsilon' = \Upsilon'^0 = 1$. To see that it is a chain map, we observe that the nth coordinate of $(B' + \partial')\Upsilon' - \Upsilon'\partial'$ is $B'\Upsilon'^{n-1} + \partial'\Upsilon'^n - \Upsilon'^n\partial'$. This is zero by the chain contraction condition for (3.1.3). □

Remark 3.1.6. Write $[1]^n$ for the element $1 \otimes \cdots \otimes 1$ of $k^{\otimes n}$. By induction, we may check that $\Upsilon'^n(1) = (-1)^n (2n)!/n!\,[1]^{2n+1}$. Thus $\Upsilon'(1) = (0, \ldots, 0, 1)$ in $HN(M')$.

Recall from Definition 2.4.5 that $M = k \otimes_{\mathsf{H}} M'$, and that $\mathrm{B}(A) = k \otimes_A \mathrm{E}(A)$.

Corollary 3.1.7. *There are morphisms of chain* k*-modules,* $\Upsilon : HH(\mathrm{B}(\mathsf{H})) \to HN(\mathrm{B}(\mathsf{H}))$ *and* $\Upsilon\colon HH(\mathrm{M}) \to HN(\mathrm{M})$*, defined by*

$$\Upsilon = \sum_{n=0}^{\infty} 1_k \otimes_{\mathsf{H}} \Upsilon'^n,$$

and $\pi\Upsilon = 1$*. Here* π *is the appropriate projection* $\pi\colon HN \to HH$.

3.2 The lift $HH(\mathrm{B}(\mathsf{H})) \xrightarrow{c} HN(\mathsf{H})$

Recall that $C(\mathsf{H})$ denotes the canonical cyclic complex of the algebra underlying H ([7, 2.5.4]). We set $\tau_0 = \eta : k \to \mathsf{H}$.

Lemma 3.2.1. *Let* H *be a cocommutative Hopf algebra. Then the* k*-linear map*

$$\tau : \mathrm{B}(\mathsf{H}) \to C(\mathsf{H})$$

$$\tau(h_1 \otimes \cdots \otimes h_n) = S(h_1^{(0)} \ldots h_n^{(0)}) \otimes h_1^{(1)} \otimes \cdots \otimes h_n^{(1)}, \quad n > 0,$$

is an injective homomorphism of cyclic k*-modules. It induces an injection of the associated mixed complexes,* $M(\mathsf{H}) \hookrightarrow C(\mathsf{H})_{\mathrm{norm}}$.

Proof. One has to check that τ commutes with the face, degeneracy and cyclic operators; these are all straightforward, short calculations. The fact that the maps are injective follows from the antipode identity $(Sh^{(0)})h^{(1)} = \eta\epsilon(h)$. $\square$

Remark 3.2.2. If $g_1, \dots, g_n \in \mathsf{H}$ are grouplike, then

$$\tau(g_1 \otimes \cdots \otimes g_n) = (g_1 \dots g_n)^{-1} \otimes g_1 \otimes \cdots \otimes g_n.$$

Thus for $\mathsf{H} = k[G]$, the τ of 3.2.1 is the map $k[\mathrm{B}(G,1)] \hookrightarrow HH(k[G])$ of [7, 7.4.5].

We define $c : \mathrm{B}(\mathsf{H}) \to HN(\mathsf{H})$ to be the natural chain map

$$c : \mathrm{B}(\mathsf{H}) \xrightarrow{\Upsilon} HN(\mathrm{B}(\mathsf{H})) \xrightarrow{\tau} HN(\mathsf{H}). \tag{3.2.3}$$

We will also write c for the normalized version $HH(M) \to HN(\mathsf{H})_{\mathrm{norm}}$ of this map.

Theorem 3.2.4. *The following diagram commutes*

$$\begin{array}{ccc} & & HN(\mathsf{H}) \\ & \overset{c}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \mathrm{B}(\mathsf{H}) & \xrightarrow[\tau]{} & HH(\mathsf{H}). \end{array}$$

Proof. By (3.2.3), Lemma 3.2.1 and Corollary 3.1.7, $\pi c = \pi\tau\Upsilon = \tau\pi\Upsilon = \tau$. $\square$

Remark 3.2.5. Goodwillie proved in [4, II.3.2] that, up to chain homotopy, there is a unique chain map $\mathrm{B}(k[G]) \to HN(k[G])$ lifting τ, natural in the group G. Ginot [3] has given explicit formulas for one such map; it follows that Ginot's map is naturally chain homotopic to the map c constructed in (3.2.3) for $\mathsf{H} = k[G]$.

3.3 Passage to completion

If A is a filtered algebra, the induced filtration (2.5.1) on the canonical cyclic module $C(A)$ makes it a cyclic filtered module. Passing to completion we obtain a cyclic module $C^{\mathrm{top}}(\hat{A})$ with $C^{\mathrm{top}}_n(\hat{A}) = \hat{A}^{\hat{\otimes} n+1}$. In the spirit of subsection 2.5, we write $HH^{\mathrm{top}}(\hat{A})$, $HN^{\mathrm{top}}(\hat{A})$, etc. for the Hochschild and cyclic complexes etc. of the mixed complex associated to $C^{\mathrm{top}}(\hat{A})$.
In particular this applies if $A = \mathsf{H}$ is a cocommutative Hopf algebra, equipped with an I-adic filtration, where I is an ideal and coideal of H with $\epsilon(I) = 0$, closed under the antipode S. Write $\hat{\mathsf{H}}$ for the associated complete Hopf algebra.
It is clear from the formula in Lemma 3.2.1 that τ is a morphism of cyclic filtered modules. Hence it induces continuous maps $\hat{\tau}$ between the corresponding complexes for HH, HN, etc.

Propostion 3.3.1. *The map c of* (3.2.3) *induces a continuous map $\hat{c}$ which fits into a commutative diagram*

$$\begin{array}{ccc} & & HN^{\mathrm{top}}(\hat{\mathsf{H}}) \\ & \overset{\hat{c}}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \mathrm{B}^{\mathrm{top}}(\hat{\mathsf{H}}) & \xrightarrow[\hat{\tau}]{} & HH^{\mathrm{top}}(\hat{\mathsf{H}}). \end{array}$$

Proof. It suffices to show that Υ (and hence c) is a filtered morphism. We observed in 2.5 that $\mathrm{E}(\mathsf{H})$ is a cyclic filtered module. Similarly, it is clear that s, s' and B' are filtered morphisms from their definitions in (2.1.2), (2.4.3) and (2.4.4). The recursion formulas in Lemma 3.1.1 and (3.1.3) show that each Υ'^n is filtered, whence so are Υ' and Υ, as required. $\square$

4 The case of universal enveloping algebras of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over a commutative ring k. Then the enveloping algebra $U\mathfrak{g}$ is a cocommutative Hopf algebra, so the constructions of the previous sections apply to $U\mathfrak{g}$. In particular a natural map $c : \mathrm{B}(U\mathfrak{g}) \to HN(U\mathfrak{g})$ was constructed in (3.2.3). In this section, we show that the Loday-Quillen map

$$\wedge\mathfrak{g} \xrightarrow{\theta} C^{\lambda}(U\mathfrak{g})[-1] \xrightarrow{B} HN(U\mathfrak{g})$$

factors through c up to chain homotopy. (See Theorem 4.2.2.)

4.1 Chevalley-Eilenberg complex

The Chevalley-Eilenberg resolution of k as a $U\mathfrak{g}$-module has the form $(U\mathfrak{g} \otimes \wedge\mathfrak{g}, d')$, and is given in [13, 7.7]. Tensoring it over $U\mathfrak{g}$ with k, we obtain a complex $(\wedge\mathfrak{g}, d)$. Kassel showed in [6, 8.1] that the anti-symmetrization map

$$e : \wedge^n\mathfrak{g} \to (U\mathfrak{g})^{\otimes n} \tag{4.1.1}$$
$$e(x_1 \wedge \cdots \wedge x_n) = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sg}(\sigma)x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$$

induces chain maps $e : \wedge\mathfrak{g} \to \mathrm{B}(U\mathfrak{g})$ and $1 \otimes e : U\mathfrak{g} \otimes \wedge\mathfrak{g} \to \mathrm{E}(U\mathfrak{g})$, because $ed = \partial e$ and $(1 \otimes e)d' = \partial'(1 \otimes e)$. Moreover, e and $1 \otimes e$ are quasi-isomorphisms; see [6, 8.2].

Lemma 4.1.2. *The map $\psi' : U\mathfrak{g} \otimes \wedge\mathfrak{g} \to HN(\mathrm{M}'(U\mathfrak{g}))$ defined by*

$$\psi'(x) = (\dots, 0, 0, 0, 1 \otimes e(x))$$

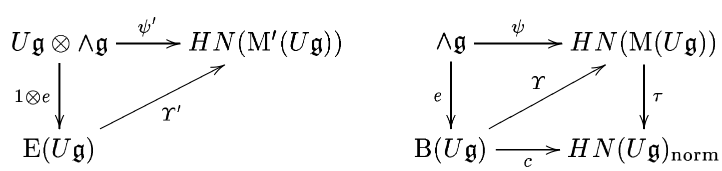
is a morphism of chain $U\mathfrak{g}$-modules, and the map $\psi : \wedge\mathfrak{g} \to HN(\mathrm{M}(U\mathfrak{g}))$ defined by

$$\psi(x) = (\dots, 0, 0, 0, e(x))$$

is a morphism of chain k-modules.

Proof. Consider $U\mathfrak{g} \otimes \wedge\mathfrak{g}$ and $\wedge\mathfrak{g}$ as mixed complexes with trivial Connes' operator. By Corollary 2.4.8, $B'(1 \otimes e) = 0$ in $M'(U\mathfrak{g})$. Thus both $1 \otimes e$ and e induce morphisms of mixed complexes $U\mathfrak{g} \otimes \wedge\mathfrak{g} \to \mathrm{M}'(U\mathfrak{g})$ and $\wedge\mathfrak{g} \to \mathrm{M}'(U\mathfrak{g})$. □

Lemma 4.1.3. *The diagrams*



commute up to natural $U\mathfrak{g}$-linear (resp., natural k-linear) chain homotopy.

Proof. By 3.1.7, 4.1.2 and (3.2.3), it suffices to consider the left diagram.
Consider the mixed subcomplex $N \subset \mathrm{M}'(U\mathfrak{g})$ given by $N_0 = 0$, $N_1 = \ker \partial'$ and $N_n = \mathrm{E}_n(U)_{\mathrm{norm}}$. Because $\psi'(1) = \Upsilon'(1)$ by Example 3.1.6, the difference $f = \Upsilon'(1 \otimes e) - \psi'$ factors through $HN(N)$. Put $\phi^n = (-1)^n(sB')^n s : N \to N[2n+1]$. One checks that $\phi := \sum_{i=0}^{\infty} \phi^n$ is a natural, k-linear contracting homotopy for $HN(N)$. Now apply Lemma 3.1.1. □

There are simple formulas for τ and $\tau\psi$ in the normalized complexes.

Lemma 4.1.4. *Let $x_1,\dots,x_n \in \mathfrak{g}$. Then in $C(U\mathfrak{g})_{\mathrm{norm}}$ we have:*

$$\tau(x_1\otimes\cdots\otimes x_n) = 1\otimes x_1\otimes\cdots\otimes x_n$$

Proof. Let $\nabla^{(n)} : U\mathfrak{g}^{\otimes n}\to U\mathfrak{g}$ be the multiplication map and $\sigma\in\mathfrak{S}_{2n}$ the (bad) riffle shuffle $\sigma(2i-1)=i$, $\sigma(2i)=n+i$. By definition (see 3.2.1),

$$\tau = (S\otimes 1^{\otimes n})\circ(\nabla^{(n)}\otimes 1^{\otimes n})\circ\sigma\circ\Delta^{\otimes n} \tag{4.1.5}$$

in $C(U\mathfrak{g})$. Since the x_i are primitive,

$$\Delta^{\otimes n}(x_1\otimes\cdots\otimes x_n) = (x_1\otimes 1 + 1\otimes x_1)\otimes\cdots\otimes(x_n\otimes 1+1\otimes x_n)$$

Expanding this product gives a sum in which

$$x = 1\otimes x_1\otimes 1\otimes x_2\otimes\cdots\otimes 1\otimes x_n$$

is the only term not mapped to a degenerate element of $C(U\mathfrak{g})$ under the composition (4.1.5). Thus in $C(U\mathfrak{g})_{\mathrm{norm}}$ we have:

$$\begin{aligned}\tau(x_1\otimes\cdots\otimes x_n) &= (S\otimes 1^{\otimes n})(\nabla^{(n)}\otimes 1^{\otimes n})\sigma(x)\\ &= (S\otimes 1^{\otimes n})(\nabla^{(n)}\otimes 1^{\otimes n})(1\otimes\cdots\otimes 1\otimes x_1\otimes\cdots\otimes x_n)\\ &= (S\otimes 1^{\otimes n})(1\otimes x_1\otimes\cdots\otimes x_n) = 1\otimes x_1\otimes\cdots\otimes x_n. \quad\square\end{aligned}$$

Corollary 4.1.6. *We have:* $\tau\psi(x_1\wedge\cdots\wedge x_n) = (\dots,0,0,0,1\otimes e(x_1\wedge\cdots\wedge x_n))$.

Proof. Combine Lemmas 4.1.2 and 4.1.4. $\square$

4.2 The Loday-Quillen map

We can now show that $\tau\psi$ factors through the Connes' complex $C^\lambda(U\mathfrak{g}) = \operatorname{coker}(1-t : U^{\otimes *}\to HH(U\mathfrak{g}))$. We have a homomorphism θ which lifts the Loday-Quillen map of [7, 10.2.3.1, 11.3.12] to $U\mathfrak{g}$:

$$\begin{gathered}\theta : \wedge^{n+1}\mathfrak{g}\to C^\lambda_n(U\mathfrak{g})\\ \theta(x_0\wedge x_1\wedge\cdots\wedge x_n) = x_0\otimes e(x_1\wedge\cdots\wedge x_n).\end{gathered}$$

Because we are working modulo the image of $1-t$, θ is well defined. The following result is implicit in the proof of [7, 10.2.4] for $r=1$, $A=U\mathfrak{g}$.

Lemma 4.2.1. *θ is a chain homomorphism $\wedge\mathfrak{g}\to C^\lambda(U\mathfrak{g})[-1]$.*

Proof. To show that $b\theta=-\theta d$, we fix a monomial $x_1\wedge\cdots\wedge x_n$ and compute that $b\theta(x_0\wedge\cdots\wedge x_n) = \sum_{\sigma\in\mathfrak{S}_n} b(x_0\otimes x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}) = A+B$, where A equals:

$$\begin{aligned}&\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sg}(\sigma)\Big(x_0x_{\sigma(1)}\otimes x_{\sigma(2)}\otimes\cdots\otimes x_{\sigma(n)} + (-1)^n x_{\sigma(n)}x_0\otimes x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n-1)}\Big)\\ &= \sum_{\sigma\in\mathfrak{S}_n}\mathrm{sg}(\sigma)\Big(x_0x_{\sigma(1)}\otimes x_{\sigma(2)}\otimes\cdots\otimes x_{\sigma(n)} - x_{\sigma(1)}x_0\otimes x_{\sigma(2)}\otimes\cdots\otimes x_{\sigma(n)}\Big)\\ &= \sum_{i=1}^n\sum_{\substack{\sigma\in\mathfrak{S}\\ \sigma(i)=i}}(-1)^{i-1}\mathrm{sg}(\sigma)[x_0,x_i]\otimes x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(i-1)}\otimes x_{\sigma(i+1)}\otimes\cdots\otimes x_{\sigma(n)}\\ &= \sum_{i=1}^n(-1)^{i-1}[x_0,x_i]\otimes e(x_1\wedge\cdots\wedge x_{i-1}\wedge x_{i+1}\wedge\cdots\wedge x_n),\end{aligned}$$

and B equals

$$\sum_{\sigma\in\mathfrak{S}_n}\sum_{i=1}^{n-1}(-1)^i \mathrm{sg}(\sigma) x_0\otimes x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(i)}x_{\sigma(i+1)}\otimes\cdots\otimes x_{\sigma(n)}$$
$$= x_0\otimes e(d(x_1\wedge\cdots\wedge x_n)).$$

Similarly, $\theta\, d(x_0\wedge\cdots\wedge x_n)$ is the sum of $-A$ and

$$\sum_{0<i<j\le n}(-1)^{i+j+1}x_0\otimes e([x_i,x_j]\wedge x_1\wedge\cdots\wedge x_n) = -x_0\otimes e(d(x_1\wedge\cdots\wedge x_n))$$

which equals $-B$. Therefore $b\theta(x_0\wedge\cdots\wedge x_n) = -\theta d(x_0\wedge\cdots\wedge x_n)$. $\square$

Warning: the sign convention used for d in [7, 10.1.3.3] differs by -1 from the usual convention, used here and in [13, 7.7] and [6].

It is well known and easy to see that, because Connes' operator B vanishes on the image of $1-t$, it induces a chain map for every algebra A:

$$B : C^\lambda(A)[-1]\to HN(A)$$
$$B[x] = (\ldots,0,0,0,Bx).$$

Let $\wedge^+\mathfrak{g}$ denote the positive degree part of $\wedge\mathfrak{g}$.

Theorem 4.2.2. *We have $\tau\,\psi = B\,\theta$ as chain maps $\wedge^+\mathfrak{g}\to HN(U\mathfrak{g})_{\mathrm{norm}}$.*
Hence the following diagram commutes up to natural chain homotopy.

$$\begin{array}{ccc} \wedge^+\mathfrak{g} & \xrightarrow{\;ce\;} & \\ \downarrow{\scriptstyle\theta} & & \searrow \\ C^\lambda(U\mathfrak{g})[-1] & \xrightarrow{\;B\;} & HN(U\mathfrak{g})_{\mathrm{norm}}. \end{array}$$

Theorem 4.2.2 fails for the degree 0 part k of $\wedge\mathfrak{g}$. Indeed, $\theta(1)=0$ for degree reasons, while from Example 3.1.6 we see that $c\,e(1)=\tau\,\psi(1)=(\ldots,0,1)$ is nonzero.

Proof. By Lemma 4.1.3, it suffices to check that $\tau\,\psi = B\,\theta$. By Corollary 4.1.6 and (4.1.1), we have

$$\tau\psi(x_1\wedge\cdots\wedge x_n) = (\ldots,0,0,\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sg}(\sigma)1\otimes x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}). \tag{4.2.3}$$

Note the expression above contains no products in $U\mathfrak{g}$; neither do the formulas for $\theta(x_0\wedge x_1\wedge\cdots\wedge x_n)$ and the Connes' operator in $C(U\mathfrak{g})$. Thus we may assume that $\mathfrak{g}$ is abelian, and that $U\mathfrak{g}=S\mathfrak{g}$ is a (commutative) symmetric algebra. We may interpret $\theta(x_1\wedge\cdots\wedge x_n)$ as the shuffle product $x_1\star B(x_2)\star\cdots\star B(x_n)$ (see [7, 4.2.6]), and the nonzero coordinate of (4.2.3) as the shuffle product

$$(1\otimes x_1)\star\cdots\star(1\otimes x_n) = B(x_1)\star\cdots\star B(x_n)$$
$$= B\big(x_1\star B(x_2)\star\cdots\star B(x_n)\big) \qquad \text{by [8, 3.1] or [7, 4.3.5]}$$
$$= B(\theta(x_1\wedge\cdots\wedge x_n)). \qquad \square$$

5 Nilpotent Lie algebras and nilpotent groups

In this and the remaining sections we shall fix the ground ring $k = \mathbb{Q}$. Let $\mathfrak{g}$ be a nilpotent Lie algebra; consider the completion $\hat{U}\mathfrak{g}$ of its enveloping algebra with respect to the augmentation ideal, and set

$$G = \exp \mathfrak{g} \subset \hat{U}\mathfrak{g}.$$

This is a nilpotent group, and $\mathsf{H} = \mathbb{Q}[G]$ is a Hopf algebra. The inclusion $G \subset \hat{U}\mathfrak{g}$ induces a homomorphism $\mathsf{H} = \mathbb{Q}[G] \hookrightarrow \hat{U}\mathfrak{g}$, and $\hat{\mathsf{H}} \cong \hat{U}\mathfrak{g}$ by [10, A.3].
On the other hand, Suslin and Wodzicki showed in [12, 5.10] that there is a natural quasi-isomorphism $sw : \mathrm{B}(\mathbb{Q}[G]) \to \wedge\mathfrak{g}$. Putting these maps together with those considered in the previous sections, we get a diagram

$$\begin{array}{ccccc}
\mathrm{B}(\mathbb{Q}[G]) & & \xrightarrow{\quad c \quad} & HN(\mathbb{Q}[G]) & \\
\downarrow{\scriptstyle sw} & \searrow & & & \searrow \\
\wedge\mathfrak{g} & & \mathrm{B}^{\mathrm{top}}(\hat{\mathsf{H}}) & \xrightarrow{\quad \hat{c} \quad} & HN^{\mathrm{top}}(\hat{\mathsf{H}}). \\
\downarrow{\scriptstyle e} & \nearrow & & & \nearrow \\
B(U\mathfrak{g}) & & \xrightarrow{\quad c \quad} & HN(U\mathfrak{g}) &
\end{array} \tag{5.1}$$

Propostion 5.2. *Diagram* (5.1) *commutes up to natural chain homotopy.*

Proof. The two parallelograms commute by naturality. The triangle on the left of (5.1) commutes up to natural homotopy by Lemma 5.3 below. □

Lemma 5.3. *The following diagram commutes up to natural chain homotopy.*

$$\begin{array}{ccc}
\mathrm{B}(\mathbb{Q}[G]) & \longrightarrow & \mathrm{B}^{\mathrm{top}}(\hat{\mathsf{H}}) \\
\downarrow{\scriptstyle sw} & & \uparrow{\scriptstyle \iota} \\
\wedge\mathfrak{g} & \xrightarrow{\quad e \quad} & \mathrm{B}(U\mathfrak{g})
\end{array}$$

Proof. By construction (see [12, 5.10]), the map sw is induced by a map $\mathrm{E}(\mathbb{Q}[G]) \to \hat{\mathsf{H}}\hat{\otimes} \wedge \mathfrak{g}$. Let ι be the upward vertical map; ιe is induced by $1 \otimes \iota e : \hat{\mathsf{H}}\hat{\otimes} \wedge \mathfrak{g} \to \mathrm{E}^{\mathrm{top}}(\hat{\mathsf{H}})$. Thus it suffices to show that the composite $\mathrm{E}(\mathbb{Q}[G]) \to \mathrm{E}^{\mathrm{top}}(\hat{\mathsf{H}})$ is naturally homotopic to the map induced by the homomorphism $\mathbb{Q}[G] \to \hat{\mathsf{H}}$. By definition, these maps agree on $\mathrm{E}_0(\mathbb{Q}[G])$; thus their difference goes to the subcomplex $\ker(\hat{\epsilon} : \mathrm{E}^{\mathrm{top}}(\hat{\mathsf{H}}) \to \mathbb{Q})$ which is contractible, with contracting homotopy induced by (2.1.2). Hence we may apply Lemma 3.1.1; this finishes the proof. □

Remark 5.4. Let $\mathfrak{g} = J_{\mathrm{Lie}}$ be the Lie algebra associated to an ideal J in a $\mathbb{Q}$-algebra A ($\mathfrak{g}$ is J with the commutator bracket); there is a canonical algebra map $U\mathfrak{g} \to A$ sending $\mathfrak{g}$ onto J. If J is a nilpotent ideal then $\mathfrak{g}$ is a nilpotent Lie algebra and the induced algebra map $\hat{U}\mathfrak{g} \to A$ restricts to an isomorphism $G = \exp(\mathfrak{g}) \overset{\cong}{\to} (1 + J)^{\times}$, as is proven in [12, 5.2].
Let $C(A, J)$ denote the kernel of $C(A) \to C(A/J)$, and let $C^{\lambda}(U\mathfrak{g}, \mathcal{I})$ denote the kernel of $C^{\lambda}(U\mathfrak{g}) \to C^{\lambda}(k)$. The composite of θ with the map induced by $U\mathfrak{g} \to A$ factors through $C^{\lambda}(A, J)$, giving rise to a commutative diagram

$$\begin{array}{ccc}
\wedge\mathfrak{g} & \xrightarrow{\quad \theta \quad} & C^{\lambda}(U\mathfrak{g}, \mathcal{I})[-1] \\
\downarrow{\scriptstyle \rho} & \swarrow & \downarrow \\
C^{\lambda}(A, J)[-1] & \hookrightarrow & C^{\lambda}(A)[-1].
\end{array}$$

The composite $\mathbb{Q}[G] \to \hat{U}\mathfrak{g} \to A$ sends the augmentation ideal $\mathcal{I}_G$ of $\mathbb{Q}[G]$ to J. Consider the resulting map

$$j : HN(\mathbb{Q}[G], \mathcal{I}_G) \to HN(\hat{U}\mathfrak{g}, \hat{\mathcal{I}}) \to HN(A, J).$$

Putting together Theorem 4.2.2 with Proposition 5.2, we get a naturally homotopy commutative diagram (with $G = (1 + J)^{\times}$):

$$\begin{array}{ccccc}
\mathrm{B}(\mathbb{Q}[G], \mathcal{I}_G) & \xrightarrow{c} & HN(\mathbb{Q}[G], \mathcal{I}_G) & & \\
\downarrow{\scriptstyle sw} & & & \searrow^{j} & \\
\wedge^{+}\mathfrak{g} & \xrightarrow{c\circ e} & HN(U\mathfrak{g}, \mathcal{I})_{\mathrm{norm}} & \longrightarrow & HN(A, J)_{\mathrm{norm}} \\
& \searrow^{\theta} & \uparrow{\scriptstyle B} & & \uparrow{\scriptstyle B} \\
& & C^{\lambda}(U\mathfrak{g}, \mathcal{I})[-1] & \longrightarrow & C^{\lambda}(A, J)[-1].
\end{array}$$

6 The relative Chern character of a nilpotent ideal

In this section we establish Theorem 6.5.1, promised in (1.3), that the two definitions (1.1) and (1.2) of the Chern character $K_*(A, I) \to HN_*(A, I)$ agree for a nilpotent ideal I in a unital $\mathbb{Q}$-algebra A. The actual proof is quite short, and most of this section is devoted to the construction of the maps (1.1) and (1.2).

For this it is appropriate to regard a non-negative chain complex C as a simplicial abelian group via Dold-Kan, identifying the homology of the complex with the homotopy groups $\pi_*(C)$ by abuse of notation. If X is a simplicial set, we write $\mathbb{Z}[X]$ for its singular complex, so that $H_*(X; \mathbb{Z})$ is $\pi_*\mathbb{Z}[X]$, and the Hurewicz map is induced by the simplicial map $h : X \to \mathbb{Z}[X]$.

6.1 The absolute Chern character

Let A be a unital $\mathbb{Q}$-algebra, and $\mathrm{BGL}(A)$ the classifying space of $GL(A)$. Now the plus construction $\mathrm{BGL}(A) \to \mathrm{BGL}(A)^+$ is a homology isomorphism, and $K_n(A) = \pi_n\mathrm{BGL}(A)^+$ for $n \geq 1$. In particular, the singular chain complex map $\mathbb{Z}[\mathrm{BGL}(A)] \to \mathbb{Z}[\mathrm{BGL}(A)^+]$ is a quasi-isomorphism. As described in [7, 11.4.1], the absolute Chern character $ch_n : K_n(A) \to HN_n(A)$ (of Goodwillie, Jones et al.) is the composite $ch = ch_A^- \circ h$ of the Hurewicz map $h : \mathrm{B}GL(A)^+ \to \mathbb{Z}[\mathrm{B}GL(A)^+] \simeq \mathbb{Z}[\mathrm{BGL}(A)]$, the identification $\mathbb{Z}[\mathrm{B}G] = \mathrm{B}(\mathbb{Z}[G])$ with the bar complex, and the chain complex map ch_A^-, which is defined as the stabilization (for $GL_n \subset GL_{n+1}$) of the composites:

$$\begin{array}{ccccccc}
\mathrm{B}(\mathbb{Z}[\mathrm{GL}_n(A)]) & \xrightarrow{c} & HN(\mathbb{Z}[\mathrm{GL}_n(A)]) & \longrightarrow & HN(M_n(A)) & \xrightarrow{\mathrm{tr}} & HN(A)_{\mathrm{norm}}. \\
\downarrow & & \downarrow & \nearrow & & & \\
\mathrm{B}(\mathbb{Q}[\mathrm{GL}_n(A)]) & \xrightarrow{c} & HN(\mathbb{Q}[\mathrm{GL}_n(A)]) & & & &
\end{array} \tag{6.1.1}$$

Here c is the natural map defined in (3.2.3) for $k = \mathbb{Z}$ and $\mathbb{Q}$; the middle maps in the diagram are induced by the fusion maps $\mathbb{Z}[\mathrm{GL}_n(A)] \subset \mathbb{Q}[\mathrm{GL}_n(A)] \to M_n(A)$, and tr is the trace map. The maps $HN(\mathbb{Q}[\mathrm{GL}_n(A)]) \to HN(A)_{\mathrm{norm}}$ are independent of n by [7, 8.4.5], even though the fusion maps are not.

Remark 6.1.2. If A is commutative and connected, the composition of the rank map $K_0(A) \to \mathbb{Z}$ sending $[A^r]$ to r with the map $H_0(ch_A^-)$ yields the Chern character $ch_0^- : K_0(A) \to HN_0(A)$ of [7, 8.3]. Composing this with the maps $HN(A) \to HC(A)[2n]$ yields the map $ch_{0,n} : K_0(A) \to HC_{2n}(A)$ of [7, 8.3.4]. From Example 3.1.6 above, with $A = k$, we see that $ch([k]) = c(1)$, and $ch_{0,n}([k]) = (-1)^n(2n)!/n!$ in $HC_{2n}(k) \cong k$, in accordance with [7, 8.3.7].

6.2 Volodin models for the relative Chern character of nilpotent ideals

In order to define the relative version ch_* of the absolute Chern character, we need to recall a chunk of notation about Volodin models. For expositional simplicity, we shall assume that I is a nilpotent ideal in a unital $\mathbb{Q}$-algebra A.

Definition 6.2.1. *Let I a nilpotent ideal in a $\mathbb{Q}$-algebra A, and σ a partial order of $\{1,\dots,n\}$. We let $\mathcal{T}_n^\sigma(A,I)$ be the nilpotent subalgebra of $M_n(A)$ defined by:*

$$\mathcal{T}_n^\sigma(A,I) := \{a \in M_n(A) : a_{ij} \in I \text{ if } i \not<_\sigma j\}.$$

We write $\mathfrak{t}_n^\sigma(A,I)$ for the associated Lie algebra, and $T_n^\sigma(A,I)$ for the group

$$T_n^\sigma(A,I) = \exp \mathfrak{t}_n^\sigma(A,I) = 1 + \mathcal{T}_n^\sigma(A,I) \subset GL_n(A).$$

The *Volodin space* $X(A) \subset BGL(A)$ is defined to be the union of the spaces $X_n(A) = \bigcup_\sigma BT_n^\sigma(A,0)$; see [7, 11.2.13]. The *relative Volodin space* $X(A,I)$ is defined to be the union of the spaces $\bigcup_\sigma BT_n^\sigma(A,I)$; see [7, 11.3.3].
The morphism $ch_A^- : \mathbb{Q}[B\mathrm{GL}(A)] \to HN(A)_{\mathrm{norm}}$ of (6.1.1) sends the subcomplex $\mathbb{Q}[X(A,I)]$ to $HN(A,I)_{\mathrm{norm}}$, which is the kernel of $HN(A)_{\mathrm{norm}} \to HN(A/I)_{\mathrm{norm}}$; see [7, 11.4.6]. The chain map ch^- is defined to be the restriction of ch_A^-:

$$ch^- : \mathbb{Q}[X(A,I)] \to HN(A,I)_{\mathrm{norm}}. \tag{6.2.2}$$

It will be useful to have a more detailed description of the restriction of (6.2.2) to $\mathbb{Q}[BT_n^\sigma(A,I)]$. Recall that if Λ is a non-unital subalgebra of R then $\mathbb{Q}+\Lambda$ is a unital subalgebra of R; we write $C(\Lambda)$ for the cyclic submodule $C(\mathbb{Q}+\Lambda,\Lambda)$ of $C(\mathbb{Q}+\Lambda)$ and hence of $C(R)$, following [7, 2.2.16]. When $\Lambda = \mathcal{T}_n^\sigma(A,I)$ and $R = M_n(A)$, we obtain the cyclic submodule $C(\mathcal{T}_n^\sigma(A,I))$ of $C(M_n(A))$. The trace map $C(M_n(A)) \to C(A)$ is a morphism of cyclic modules, sending $C(\mathcal{T}_n^\sigma(A,I))$ to the submodule $C(A,I)$. On the other hand, by Example 5.4, we also have a map

$$C(\mathbb{Q}[G],\mathcal{I}_G) \xrightarrow{j} C(G), \quad \text{for } G = \mathcal{T}_n^\sigma(A,I).$$

From the definition of ch_A^- in (6.1.1) and the naturality of c, we obtain the promised description of ch^-, which we record.

Lemma 6.2.3. *Set $G = T_n^\sigma(A,I)$. The restriction of ch^- to* $\mathrm{B}(\mathbb{Q}[G],\mathcal{I}_G)$ *is the composition*

$$\mathrm{B}(\mathbb{Q}[G],\mathcal{I}_G) \xrightarrow{c} HN(\mathbb{Q}[G],\mathcal{I}_G) \xrightarrow{j} HN(\mathcal{T}_n^\sigma(A,I)) \xrightarrow{tr} HN(A,I)_{\mathrm{norm}}.$$

6.3 The relative Chern character for rational nilpotent ideals

When I is a nilpotent ideal in an algebra A, we define $K(A,I)$ to be the homotopy fiber of $B\mathrm{GL}(A)^+ \to B\mathrm{GL}(A/I)^+$; $K(A,I)$ is a connected space whose homotopy groups are the relative K-groups $K_n(A,I)$ for all n. We now cite Theorem 6.1 of [9] for nilpotent I; the proof in [9] is reproduced on page 361 of [7].

Theorem 6.3.1. *If I is a nilpotent ideal in A, there are homotopy fibrations*

$$X(A,I) \to B\mathrm{GL}(A) \to B\mathrm{GL}(A/I)^+,$$
$$X(A) \to X(A,I) \to K(A,I).$$

Moreover, $X(A,I)^+ \xrightarrow{\sim} K(A,I)$ and $\mathbb{Z}[X(A,I)] \xrightarrow{\sim} \mathbb{Z}[K(A,I)]$ are homotopy equivalences (i.e., $X(A,I) \to K(A,I)$ is a homology isomorphism).

Definition 6.3.2. *(See [7, 11.4.7]) The relative Chern character for the ideal I of a $\mathbb{Q}$-algebra A is the composite of the Hurewicz map, the inverse of the homotopy equivalence of Theorem 6.3.1 and the map ch^- of* (6.2.2)*:*

$$ch : K(A,I) \xrightarrow{h} \mathbb{Q}[K(A,I)] \underset{6.3.1}{\xleftarrow{\sim}} \mathbb{Q}[X(A,I)] \xrightarrow{ch^-} HN(A,I)_{\mathrm{norm}}.$$

6.4 The rational homotopy theory character for nilpotent ideals

For a nilpotent ideal I, consider the chain subcomplex of the Chevalley-Eilenberg complex $\wedge\mathfrak{gl}(A)$,

$$x(A,I) = \sum_{n,\sigma} \wedge\mathfrak{t}_n^\sigma(A,I).$$

Because sw is natural in G, the family of maps $\mathrm{B}(\mathbb{Q}[T_n^\sigma(A,I)]) \xrightarrow{sw} \wedge\,\mathfrak{t}_n^\sigma(A,I)$ induces a morphism of complexes

$$sw_X : \mathbb{Q}[X(A,I)] \to x(A,I).$$

On the other hand, for each n and σ, the composite of the map $B\rho : \wedge\mathfrak{t}_n^\sigma(A,I) \to HN(\mathcal{T}_n^\sigma(A,I))$ of Example 5.4 with the inclusion and the trace, i.e., with

$$HN(\mathcal{T}_n^\sigma(A,I)) \subset HN(M_n(A)) \xrightarrow{tr} HN(A),$$

sends $\wedge\mathfrak{t}_n^\sigma(A,I)$ into the subcomplex $HN(A,I)$. All of these maps are natural in n and σ; by abuse of notation, we write $\mathrm{tr}(B\,\rho)$ for the resulting map:

$$\mathrm{tr}(B\,\rho) : x(A,I) \to HN(A,I).$$

Definition 6.4.1. *The map $ch^-_{\mathrm{rht}} : \mathbb{Q}[X(A,I)] \to x(A,I) \to HN(A,I)_{\mathrm{norm}}$ is defined to be $tr(B\,\rho)\circ sw_X$, followed by $HN(A,I) \to HN(A,I)_{\mathrm{norm}}$. The rational homotopy theory character of [7, 11.3.1], cited in* (1.2)*, is the composite $ch' : K(A,I) \to HN(A,I)_{\mathrm{norm}}$ defined by:*

$$K(A,I) \xrightarrow{h} \mathbb{Q}(K(A,I)) \overset{\simeq}{\leftarrow} \mathbb{Q}[X(A,I)] \xrightarrow{ch^-_{\mathrm{rht}}} HN(A,I)_{\mathrm{norm}}.$$

Remark 6.4.2. We will not need the unnormalized version of ch'. By construction, ch' is the map $K(A,I) \to C^\lambda(A,I)[-1]$ of [2, A.13], followed by Connes' operator B.

6.5 Main theorem

Let I be a nilpotent ideal in a $\mathbb{Q}$-algebra A. The relative Chern character ch of Definition 6.3.2 induces the relative Chern character ch_* of (1.1) on homotopy groups, and the rational homotopy character ch' of Definition 6.4.1 induces the character ch'_* of (1.2) on homotopy groups. Therefore the equality $ch_* = ch'_*$ of (1.3) is an immediate consequence of our main theorem.

Theorem 6.5.1. *The maps ch^- and ch^-_{rht} are naturally chain homotopic. Hence the maps ch and ch' are homotopic for each A and I.*

Proof. We first consider the restriction of ch^- and ch^-_{rht} to $\mathrm{B}(\mathbb{Q}[T_n^\sigma(A,I)])$ for some fixed n and σ. By Lemma 6.2.3, the restriction of ch^- to $\mathrm{B}(\mathbb{Q}[T_n^\sigma(A,I)])$ is the map $\mathrm{tr}(jc)$; by Example 5.4, there is a natural chain homotopy from jc to $B\,\rho\circ sw$. Since $\mathrm{tr}(B\,\rho)\circ sw$ is the restriction of ch^-_{rht} to $\mathrm{B}(\mathbb{Q}[T_n^\sigma(A,I)])$, the chain homotopies glue together by naturality to give the desired chain homotopy from ch^-_{rht} to ch^-. □

6.6 Naturality

In order to formulate a naturality result for the homotopy between ch and ch', it is necessary to give definitions for the maps ch and ch' which are natural in A and I. Contemplation of Definitions 6.3.2 and 6.4.1 shows that we need to find a natural inverse for the backwards quasi-isomorphism of Theorem 6.3.1. One standard way is to fix a small category of pairs (A,I) (say all pairs with a fixed cardinality bound on A) and consider the global model structure on the category of covariant functors from this category to Simplicial Sets; this will yield naturality with respect to all morphisms in the small category.

Let $K(A,I)'$ be the cofibrant replacement of $K(A,I)$, and factor the backwards map as $\mathbb{Q}[K(A,I)] \overset{\simeq}{\twoheadleftarrow} C \overset{\simeq}{\hookleftarrow} \mathbb{Q}[X(A,I)]$. Then h lifts to a map $h' : K(A,I)' \to C$ and, since $HN(A,I)$ is fibrant, ch^- and ch^-_{rht} both lift to maps $C \to HN(A,I)$. We can then define ch to be the composite $K(A,I)' \xrightarrow{h'} C \xrightarrow{ch^-} HN(A,I)$; ch' is defined similarly using ch^-_{rht}. By Theorem 6.5.1, there is a homotopy between the two maps $C \to HN(A,I)$, and hence between the two maps $K(A,I)' \to HN(A,I)$. Since this is a homotopy of functors from the small category of pairs (A,I) to Simplicial Sets, it provides a natural homotopy between ch and ch' as maps $K(A,I)' \to HN(A,I)$.

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