

Localization for the K -Theory of Noncommutative Rings

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Abstract. If S is denominator set in a ring A , we describe the third term in the long exact sequence relating the K -theory of A and $S^{-1}A$. It is the Waldhausen K -theory of a category $\text{Perf}(A, S)$. If $A \rightarrow B$ is an analytic isomorphism along S , this third term satisfies excision, yielding a long exact Mayer-Vietoris sequence in K -theory.

The recent work [TT] of Thomason and Trobaugh establishes a localization theorem for the K -theory of commutative rings and quasi-compact quasi-separated schemes. This paper is partly an attempt to give a simple exposition of their proof in the important case $A \rightarrow S^{-1}A$, and partly an extension of their proof to the noncommutative case. When S consists of nonzerodivisors, we recover the calculations of [GQ] and [Gr], since in that case our $\text{Perf}(A, S)$ has the same K -theory as the exact category $\mathcal{H}_S(A)$. We include an excision result which is new even in the commutative case.

To understand the statement of our localization theorem, we introduce some terms. Let A be a ring with unit. A *strictly perfect* complex $P = P^\bullet$ is a bounded chain complex of finitely generated projective left A -modules. A chain complex $E = E^\bullet$ of left A -modules is a *perfect complex* if there is a strictly perfect complex P^\bullet and a quasi-isomorphism $P \rightarrow E$. The category $\text{Perf}(A)$ of perfect complexes forms a "Waldhausen" category, i.e., a category with cofibrations (degreewise split monics) and weak equivalences (quasi-isomorphisms). The K -theory of A is the same as the Waldhausen K -theory of $\text{Perf}(A)$. (See 1.1 below.)

We are interested in localizing A at a multiplicatively closed subset S to form a *left quotient ring* $S^{-1}A$ whose elements have the form $s^{-1}a$ ($s \in S, a \in A$). This exists iff S is a *left denominator set*, i.e., it satisfies the following conditions:

- (i) ("Ore condition") $(\forall s \in S, a \in A)(\exists t \in S, b \in A)ta = bs$
- (ii) (Annihilators) $(\forall s \in S, a \in A)$ if $as = 0$ then $(\exists t \in S)ta = 0$.

See [F], 16.9. This hypothesis is sufficient to make $S^{-1}A$ flat as a right A -module, so that $M \mapsto S^{-1}M = S^{-1}A \otimes_A M$ is an exact functor from $A\text{-mod}$ to $S^{-1}A\text{-mod}$. We remark that any *central* multiplicatively closed S will automatically be a left denominator set.

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Suppose that S is a left denominator set in A . The category $\text{Perf}(A, S)$ of perfect A -module complexes E such that $S^{-1}E$ is exact forms a Waldhausen subcategory of $\text{Perf}(A)$, so it makes sense to talk about its algebraic K -theory. We will see below that it has the same K -theory as the subcategories $\text{Perf}^{st}(A, S)$, $\text{Perf}^b(A, S)$ and $\text{Perf}^-(A, S)$ of strictly perfect, bounded, and bounded above complexes, respectively.

Localization Theorem. Let S be a left denominator set in a ring A , for example a central multiplicatively closed set. Then there is a long exact sequence

$$\cdots \rightarrow K_{n+1}(S^{-1}A) \xrightarrow{\partial} K_n \text{Perf}(A, S) \rightarrow K_n(A) \rightarrow K_n(S^{-1}A) \xrightarrow{\partial} \cdots$$

valid for all integers.

Formula. Here is an explicit formula for the boundary map $\partial : K_1(S^{-1}A) \rightarrow K_0 \text{Perf}(A, S)$. If $s \in S$, then s is a unit of $S^{-1}A$ and $\partial(s)$ is represented by the complex

$$0 \rightarrow A \xrightarrow{s} A \rightarrow 0$$

concentrated in degrees 0 and 1. More generally, every matrix $\beta \in GL_n(S^{-1}A)$ is of the form $s^{-1}\alpha$ for some $s \in S$ and some $\alpha \in M_n(A)$. Then $\partial(\beta) = \partial(\alpha) - n\partial(s)$, where $\partial(\alpha)$ is represented by

$$0 \rightarrow A^n \xrightarrow{\alpha} A^n \rightarrow 0.$$

Remark. We will only construct a sequence ending in $K_0(A) \rightarrow K_0(S^{-1}A)$. However, the argument of [C] shows that we can define *negative* K -groups for $\text{Perf}(A, S)$ and continue the above sequence to negative values of n . On a spectrum level, if $\mathcal{K}(A)$ denotes the nonconnective spectrum for the K -theory of A , the fiber \mathcal{F} of $\mathcal{K}(A) \rightarrow \mathcal{K}(S^{-1}A)$ is a nonconnective delooping of the usual connective K -theory spectrum for $\text{Perf}(A, S)$.

Remark. The proof of the Localization Theorem is much easier if we assume that S is central. The difficulty with the general case is that clearing denominators in commutative diagrams is delicate. (See 3.1.)

Remark. It seems probable that the Localization Theorem remains valid if $A \rightarrow S^{-1}A$ is replaced by a flat epimorphism $A \rightarrow B$. However, the techniques of this paper do not immediately extend to that case. Several variants of the Localization Theorem may be found in the second author's thesis [Yao].

In order to compute with the Localization Theorem, we provide an excision result for analytic isomorphisms along S .

Definition. (Cf. [TT] and [We]) Let S be a left denominator set in A . We say that a ring map $f : A \rightarrow B$ is a (left) *analytic isomorphism* along S if

- a) $f(S)$ is a left denominator set in B
- b) $A/As \xrightarrow{\cong} B/Bs$ for all $s \in S$
- c) $\text{Tor}_p^A(B, A/I) = 0$ for $p \neq 0$ and all left ideals I of A meeting S .

Remark. Condition b) implies that $A/I \cong B/BI$ for every ideal I meeting S . If A is commutative, then c) is implied by the condition of [TT] that B_P is flat over A_P for all primes P of A meeting S . If S consists of central nonzerodivisors in A , we will show in 5.5(b) below that c) is equivalent to the assertion that $f(S)$ consists of right (hence left) nonzerodivisors in B . Thus our notion includes the notion of analytic isomorphism used in [K] and [We]. The term “analytic isomorphism” comes from the fact that the S -adic completions $\hat{A} = \varprojlim A/As$ and $\hat{B} = \varprojlim B/Bs$ are isomorphic.

Excision Theorem. Let $A \rightarrow B$ be an analytic isomorphism along S . Then the total tensor product map

$$B \otimes_A^{\mathbf{L}} - : K(\text{Perf}^-(A, S)) \rightarrow K(\text{Perf}^-(B, S))$$

is a homotopy equivalence of spectra. Consequently, there is a long exact Mayer-Vietoris sequence (for all integers n):

$$\cdots \rightarrow K_{n+1}(S^{-1}B) \xrightarrow{\partial} K_n(A) \rightarrow K_n(B) \oplus K_n(S^{-1}A) \rightarrow K_n(S^{-1}B) \xrightarrow{\partial} \cdots$$

Remark. Our proof follows the proof of [TT, 3.19]. If S is a central set of nonzerodivisors on A and B , this result was proven by Karoubi [K] by showing that $\mathcal{H}_S(A) \approx \mathcal{H}_S(B)$. (See [We, 1.1]).

§1. The proof of the Localization Theorem.

The K -theory of A is the K -theory of the category $\mathcal{P}(A)$ of fin. gen. projective left A -modules; either Quillen K -theory or Waldhausen K -theory may be used by [Wa, 1.9]. In order to compare the K -theory of A to that of $\text{Perf}(A, S)$, we invoke the following result.

LEMMA 1.1. (Waldhausen) *The following subcategories of $\text{Perf}(A)$ have the same K -theory (the K -theory of A):*

- a) $\mathcal{P}(A)$, the fin. gen. projective A -modules
- b) $\text{Perf}^{st}(A)$, the strictly perfect complexes
- c) $\text{Perf}^b(A)$, the bounded perfect complexes
- d) $\text{Perf}^-(A)$, the bounded above perfect complexes
- e) $\text{Perf}(A)$, all perfect complexes.

In all cases, cofibrations are degreewise split monics, and the weak equivalences w are the quasi-isomorphisms.

PROOF. By [Gi, 6.2], the categories a) and b) have the same K -theory. The inclusion of strictly perfect complexes in either bounded or bounded above perfect complexes satisfies the approximation property (App) of [Wa, 1.6.7] by a standard exercise (see [SGA6, I.2.7.1] or [TT, 1.9.5]). The inclusion of category d) in category e) satisfies the dual approximation property (App^{op}) since we can truncate complexes. By the Approximation Theorem [Wa, 1.6.7], these categories have the same K -theory. ■

PORISM 1.2. *It follows from the proof of 1.1 that the following subcategories of $\text{Perf}(A, S)$ have the same K -theory:*

- a) $\text{Perf}^{st}(A, S) = \text{Perf}^{st}(A) \cap \text{Perf}(A, S)$
- b) $\text{Perf}^b(A, S) = \text{Perf}^b(A) \cap \text{Perf}(A, S)$
- c) $\text{Perf}^-(A, S) = \text{Perf}^-(A) \cap \text{Perf}(A, S)$
- d) $\text{Perf}(A, S)$

In all cases, cofibrations are degreewise split monics, and the weak equivalences w are the quasi-isomorphisms.

In order to prove the Localization Theorem, we introduce a new notion of weak equivalence on the category with cofibrations $\text{Perf}(A)$. We let v denote the class of all maps $E \rightarrow F$ such that $S^{-1}E \rightarrow S^{-1}F$ is a quasi-isomorphism. The subcategory of perfect complexes v -equivalent to zero is $\text{Perf}(A, S)$, so Waldhausen's Fibration Theorem [Wa, 1.6.4] states that there is a homotopy fibration of spectra (yielding a long exact sequence on K -groups):

$$(1.3) \quad K(\text{Perf}(A, S)) \rightarrow K(\text{Perf}(A), w) \rightarrow K(\text{Perf}(A), v).$$

The middle term gives the K -theory of A , so it suffices to compare the right term to the K -theory of $S^{-1}A$.

Since $S^{-1}A$ is flat over A , localization provides an exact functor from $\text{Perf}(A)$ to $\text{Perf}(S^{-1}A)$. This functor not only factors through the change in weak equivalence (from w to v), but it also factors through a category \mathcal{B} , which we now define.

Definition 1.4. Let \mathcal{B} denote the full subcategory of $\text{Perf}(S^{-1}A)$ consisting of those perfect $S^{-1}A$ -module complexes E^\bullet such that the class $[E^\bullet]$ in $K_0(S^{-1}A)$ is in the image of the map $K_0(A) \rightarrow K_0(S^{-1}A)$. We make \mathcal{B} into a category with cofibrations (degreewise split monics) and weak equivalences (quasi-isomorphisms).

Thomason's version of the Cofinality Theorem for Waldhausen K -theory [TT, 1.10.1] applies to the inclusion of \mathcal{B} in $\text{Perf}(S^{-1}A)$, proving that $K_n(\mathcal{B}) \rightarrow K_n(S^{-1}A)$ is an isomorphism for $n \geq 1$, and that $K_0(\mathcal{B})$ is the image of $K_0(A) \rightarrow K_0(S^{-1}A)$. Therefore in order to prove the Localization Theorem it suffices to establish the following assertion:

$$(1.5) \quad K(\text{Perf}(A), v) \rightarrow K(\mathcal{B}) \text{ is a homotopy equivalence.}$$

This will be a consequence of the Thomason-Trobaugh Approximation Theorem [TT, 1.9.8], once we prove (in section 3 below) that the map of derived categories

$$T : v^{-1}\text{Perf}(A) \rightarrow w^{-1}\mathcal{B}$$

is an equivalence. The first step is to show that every complex E^\bullet in \mathcal{B} is quasi-isomorphic to $S^{-1}P^\bullet$ for some perfect A -module complex P^\bullet , i.e., that every object of $w^{-1}\mathcal{B}$ is isomorphic to $T(P^\bullet)$ for some P^\bullet in $v^{-1}\text{Perf}(A)$. This is the subject of the next section.

§2. An Extension Criterion.

In this section we shall assume that S is either central or a left denominator set in A . The following Exercise is trivial when S is central. When S is a left denominator set, it uses the fact that any finite subset $\{b_i\}$ of $S^{-1}A$ has a common denominator, i.e., is of the form $\{t^{-1}a_i\}$.

Exercise 2.1. Fix a left denominator set S in A , and let E^\bullet be a bounded chain complex of fin. gen. free left $S^{-1}A$ -modules. Then there is a bounded complex F^\bullet of fin. gen. free A -modules and an isomorphism $f : S^{-1}F^\bullet \rightarrow E^\bullet$ of $S^{-1}A$ -module complexes. Moreover, if we use a choice of basis to represent the f^i by matrices and assume that $E^i = 0$ for $i > n$, then $f^n = 1$ and every other f^i is right multiplication by an element of S .

COROLLARY 2.2. If P is a fin. gen. projective A -module and

$$0 \rightarrow E^m \rightarrow E^{m+1} \rightarrow \dots \rightarrow E^{n-1} \rightarrow E^n \rightarrow 0$$

is an $S^{-1}A$ -module complex with E^i fin. gen. free for $i \neq n$ and $E^n \cong S^{-1}P$, then there is a bounded chain complex P^\bullet of fin. gen. projective A -modules with $P^n = P$ and an isomorphism $f : S^{-1}P^\bullet \rightarrow E^\bullet$ of $S^{-1}A$ -module complexes.

PROOF. Choose Q so that $P \oplus Q$ is fin. gen. free and apply 2.1 to $E^\bullet \oplus (S^{-1}Q(n))$ to get a free complex F^\bullet with $F^n = P \oplus Q$ and an isomorphism $S^{-1}F^\bullet \cong E^\bullet \oplus (S^{-1}Q(n))$ in which

$$S^{-1}F^n \cong (S^{-1}P) \oplus (S^{-1}Q) \cong E^n \oplus (S^{-1}Q)$$

is the canonical map. Now set $P^\bullet = F^\bullet/Q(n)$. ■

COROLLARY 2.3A. If E^\bullet is a strictly perfect $S^{-1}A$ -module complex, then there is a bounded complex F^\bullet of fin. gen. free A -modules, an $S^{-1}A$ -module complex D^\bullet and an isomorphism $S^{-1}F^\bullet \cong D^\bullet \oplus E^\bullet$ of $S^{-1}A$ -module complexes.

PROOF. Each E^i is a fin. gen. projective $S^{-1}A$ -module, so there are $S^{-1}A$ -modules D^i with $D^i \oplus E^i$ fin. gen. free. Assemble the D^i into a complex (e.g., by 0 maps) and apply 2.1. ■

Remark 2.3.1. This is the elementary analogue of [TT, 5.5.1]. Thomason and Trobaugh need to work harder, invoking the derived category of $S^{-1}A$, because of their geometric context. In *loc. cit.*, they state that “despite the flagrant triviality of the proof, this result is the key point in [TT].”

Extension Criterion 2.4. The following assertions are equivalent for every perfect $S^{-1}A$ -module complex E^\bullet :

- (i) E^\bullet is quasi-isomorphic to $S^{-1}P^\bullet$ for some perfect A -module complex P^\bullet
- (ii) The class $[E^\bullet] \in K_0(S^{-1}A)$ is in the image of $K_0(A) \rightarrow K_0(S^{-1}A)$.

PROOF. That (i) implies (ii) is clear. For the converse, we may suppose that E^\bullet is strictly perfect, so $[E^\bullet] = \sum (-1)^i [E^i]$. By adding short complexes of the form $0 \rightarrow D^i = D^{i+1} \rightarrow 0$, we may assume every E^i is free except E^n , that $E^i = 0$ for $i > n$, and that $[E^n] = [S^{-1}P]$ for some projective A -module P . Hence E^n and $S^{-1}P$ are stably isomorphic $S^{-1}A$ -modules, i.e., $E^n \oplus (S^{-1}A)^r \cong S^{-1}(P \oplus A^r)$ for some r . Adding $(S^{-1}A)^r$ in dimensions $n-1$ and n , we may assume that in fact $E^n \cong S^{-1}P$. Now apply 2.2 to obtain (i). ■

§3. Equivalence of Derived Categories.

If \mathbf{w} is a class of maps in a skeletally small additive category \mathcal{C} , there is an additive category $\mathbf{w}^{-1}\mathcal{C}$ and a functor $Q : \mathcal{C} \rightarrow \mathbf{w}^{-1}\mathcal{C}$ sending \mathbf{w} to isomorphisms which is universal in this respect. If \mathbf{w} is a multiplicative system [H, I.3], this is an especially nice construction, since $\mathbf{w}^{-1}\mathcal{C}$ has the same objects as \mathcal{C} and every morphism is represented by a diagram in \mathcal{C} of the form

$$E \xleftarrow{\mathbf{w}} E' \xrightarrow{\alpha} F.$$

This follows from the calculus of fractions [V, 2.3.2] [H, 3.1]

THEOREM 3.1. *Let $\mathcal{B} \subseteq \text{Perf}(S^{-1}A)$ be as in (1.4), with \mathbf{w} being the quasi-isomorphisms. Let \mathbf{v} be the class of maps $v : E \rightarrow F$ in $\text{Perf}(A)$ such that $S^{-1}v : S^{-1}E \rightarrow S^{-1}F$ is a quasi-isomorphism. Then*

$$T : \mathbf{v}^{-1}\text{Perf}(A) \rightarrow \mathbf{w}^{-1}\mathcal{B}$$

is an equivalence of categories.

Reduction. The Extension Criterion 2.4 shows that every object of \mathcal{B} , hence of $\mathbf{w}^{-1}\mathcal{B}$, comes from an object of $\text{Perf}(A)$. Therefore, it is enough to show that the functor T is full and faithful. The following argument, copied from [TT, 5.2.6], shows that it is enough to prove that T is full, for this implies that T is also faithful. Since \mathbf{v} is a multiplicative system, every map in $\mathbf{v}^{-1}\text{Perf}(A)$ is represented as $E \xleftarrow{\mathbf{v}'} E' \xrightarrow{\alpha} F$ with v' in \mathbf{v} . Suppose that T sends this map (or equivalently, α) to zero in $\mathbf{w}^{-1}\mathcal{B} \subseteq \mathbf{w}^{-1}\text{Perf}(S^{-1}A) \subseteq D(S^{-1}A)$. Let C be the mapping cone of α , so that

$$C(1) \xrightarrow{\pi} E' \xrightarrow{\alpha} F \xrightarrow{\delta} C$$

forms a distinguished triangle of perfect A -module complexes. Since $\text{Hom}(E, -)$ is a cohomological functor [V, 1.2] [H, 1.1] we have a diagram of abelian groups with exact rows:

$$\begin{array}{ccccc} \text{Hom}_{\mathbf{v}}(E, C(1)) & \xrightarrow{\pi} & \text{Hom}_{\mathbf{v}}(E, E') & \xrightarrow{\alpha} & \text{Hom}_{\mathbf{v}}(E, F) \\ \downarrow T & & \downarrow T & & \downarrow \\ \text{Hom}_{\mathbf{w}}(S^{-1}E, S^{-1}C(1)) & \xrightarrow{\pi} & \text{Hom}_{\mathbf{w}}(S^{-1}E, S^{-1}E') & \xrightarrow{0} & \text{Hom}_{\mathbf{w}}(S^{-1}E, S^{-1}F) \end{array}$$

For clarity, we have written $\text{Hom}_{\mathbf{v}}$ (resp. $\text{Hom}_{\mathbf{w}}$) for Hom in the triangulated category $\mathbf{v}^{-1}\text{Perf}(A)$ (resp. $\mathbf{w}^{-1}\mathcal{B}$). Since we have assumed that T is full, the vertical maps are onto. Hence there is a map $E \xleftarrow{\mathbf{v}''} E'' \xrightarrow{\sigma} C(1)$ in $\mathbf{v}^{-1}\text{Perf}(A)$ such that $T(v^{-1}\sigma\pi)$ is the isomorphism $S^{-1}v''^{-1}$ in $\mathbf{w}^{-1}\mathcal{B}$. But then $S^{-1}(\sigma\pi)$ is a quasi-isomorphism. By definition of \mathbf{v} , $\sigma\pi$ is an isomorphism in $\mathbf{v}^{-1}\text{Perf}(A)$. Since $\sigma\pi\alpha = 0$, this forces α to be zero in $\mathbf{v}^{-1}\text{Perf}(A)$, proving that T is faithful.

PROOF THAT T IS FULL. Note that T is an additive functor between additive categories. As every strictly perfect complex is a direct summand of a bounded

f.g. free complex, we are reduced to showing that if E and F are bounded complexes of fin. gen. free A -modules, then

$$T : \mathrm{Hom}_{\mathbf{v}^{-1}\mathrm{Perf}(A)}(E, F) \rightarrow \mathrm{Hom}_{\mathbf{w}^{-1}\mathcal{B}}(S^{-1}E, S^{-1}F)$$

is onto. By [SGA6, I.2.7] and [V, I.2.4.2], every map in $\mathbf{w}^{-1}\mathcal{B}$ from $S^{-1}E$ to $S^{-1}F$ is represented by a chain map $\beta : S^{-1}E \rightarrow S^{-1}F$. Regarding the β^n as matrices, we can clear denominators in the entries of the β^n to find A -module maps $\alpha^n : E^n \rightarrow F^n$ and $s \in S$ so that $\beta^n = s^{-1}\alpha^n$ for all n . As a warmup we consider the easy case first.

Easy case: S is central. Because over $S^{-1}A$ we have

$$(\alpha^{n-1}d_F - d_E\alpha^n) = s(\beta^{n-1}d_F - d_E\beta^n) = 0,$$

some $t \in S$ annihilates $\alpha^{n-1}d_F - d_E\alpha^n$. Replacing s by ts and α^n by $t\alpha^n$, we have arranged that the $\{\alpha^n\}$ assemble to form a chain map $\alpha : E \rightarrow F$. Multiplication by s is a chain map $E \rightarrow E$ lying in \mathbf{v} , and evidently the map

$$E \xleftarrow{s} E \xrightarrow{\alpha} F$$

in $\mathbf{v}^{-1}\mathrm{Perf}(A)$ maps to β . We are done in this case.

General case. When S is not central, multiplication by s is not even canonically defined, let alone a chain map. To define it, we fix a basis for the free modules E^n and let $s : E \rightarrow E$ be the map represented by the diagonal matrix sI , i.e., multiply the basis elements by s and extend linearly to E . Since $E^n = 0$ for $n > N$, we may use the following lemma, together with descending induction on n , to see that by changing our choice of the α^n (and $s_n \in S$ so that $\beta^n = s_n^{-1}\alpha^n$) we can find a new chain complex

$$E' : \dots \rightarrow E^{n-1} \xrightarrow{e^{n-1}} E^n \xrightarrow{e^n} \dots \rightarrow E^N \rightarrow 0$$

and a diagram of chain maps

$$E \xleftarrow{\{s_n\}} E' \xrightarrow{\{\alpha^n\}} F.$$

Since $\{s_n\}$ is in \mathbf{v} , this represents a map in $\mathbf{v}^{-1}\mathrm{Perf}(A)$ lifting β . This will finish the proof.

LEMMA. Suppose we are given $s_n \in S$ and $\alpha^n : E^n \rightarrow F$ so that $\beta^n = s_n^{-1}\alpha^n$, and a map $e^n : E^n \rightarrow E^{n+1}$ such that $s_n d_E = e^n s_{n+1}$. Then there is a map $e : E^{n-1} \rightarrow E^n$, an $s_{n-1} \in S$ and an $\alpha : E^{n-1} \rightarrow F^{n-1}$ so that $ee^n = 0$, $\beta^{n-1} = s_{n-1}^{-1}\alpha$ and such that the following diagram commutes:

$$\begin{array}{ccccc} E^{n-1} & \xleftarrow{s_{n-1}} & E^{n-1} & \xrightarrow{\alpha} & F^{n-1} \\ d_E \downarrow & & \downarrow e & & \downarrow d_F \\ E^n & \xleftarrow{s_n} & E^n & \xrightarrow{\alpha^n} & F^n. \end{array}$$

PROOF. Recall that $\beta^{n-1} = s^{-1}\alpha^{n-1}$ is given. Choose $s' \in S$ and $e' : E^{n-1} \rightarrow E^n$ so that $e's_n = s'd_E$. Then choose $t \in S$ and $a \in A$ so that $as' = ts$. Set $e'' = ae'$ and $\alpha'' = t\alpha^{n-1}$, so that

$$(e''e^n)_{s_{n+1}} = (ae')(s_n d_E) = a(s'd_E)d_E = 0$$

and over $S^{-1}A$ we have

$$\begin{aligned} (\alpha''d_F - e''\alpha^n) &= t(s\beta^{n-1})d_F - (ae')(s_n\beta^n) \\ &= as'(\beta^{n-1}d_F - d_E\beta^n) \\ &= 0. \end{aligned}$$

Therefore there is an $s'' \in S$ so that $s''e''e^n = 0$ and

$$s''(\alpha''d_F - e''\alpha^n) = 0.$$

Set $e = s''e''$, $s_{n-1} = s''as' = s''ts$ and $\alpha = s''\alpha''$. ■

§4. Criteria for perfectness and pseudocoherence.

The following two results are straightforward modifications of results in [TT, 2.4]. We need them for the excision result in the next section.

Recall from [SGA6, I.2] that an A -module chain complex P^\bullet is said to be *strictly pseudo-coherent* if it is a bounded above complex of fin. gen. projective A -modules. A complex E^\bullet is said to be *pseudo-coherent* if there is a quasi-isomorphism $P^\bullet \rightarrow E^\bullet$ with P^\bullet strictly pseudo-coherent. Recall also that $\tau^n E$ is the good truncation

$$\dots \rightarrow 0 \rightarrow d(E^{n-1}) \rightarrow E^n \rightarrow E^{n+1} \rightarrow \dots$$

THEOREM 4.1. ([TT, 2.4.2]) *Let E be an A -module chain complex. The following are equivalent:*

- a) E is pseudo-coherent
- b) For all integers n and k , and all directed systems $\{F_\alpha\}$ of A -module complexes, the canonical map (4.1.1) is an isomorphism.

$$(4.1.1) \quad \varinjlim_{\alpha} H^k(\mathbf{R} \operatorname{Hom}(E, \tau^n F_\alpha)) \xrightarrow{\cong} H^k(\mathbf{R} \operatorname{Hom}(E, \varinjlim_{\alpha} \tau^n F_\alpha))$$

- c) Same as b) except we require the F_α to be strictly perfect
- d) Same as c) except we require the F_α to be uniformly bounded above, and we require E to be cohomologically bounded above.
- e) For all integers n , and all directed systems $\{F_\alpha\}$ of A -module complexes, the canonical map (4.1.2) is an isomorphism.

$$(4.1.2) \quad \varinjlim_{\alpha} \operatorname{Hom}_{D(A)}(E, \tau^n F_\alpha) \xrightarrow{\cong} \operatorname{Hom}_{D(A)}(E, \varinjlim_{\alpha} \tau^n F_\alpha)$$

- f) Same as e) except we require the F_α to be strictly perfect
- g) Same as f) except we require the F_α to be uniformly bounded above, and we require E to be cohomologically bounded above.

PROOF. We merely note the changes that are needed to modify the proof of [TT, 2.4.2]. Note that the meaning of “perfect” is slightly different in *op. cit.* In the proof that b) \Rightarrow e) we cite [H, I.6.4] instead of [TT, 2.4.1] to see that

$$H^0 \mathbf{R} \operatorname{Hom}(E, \tau^n F) = \operatorname{Hom}_{D(A)}(E, \tau^n F).$$

In the proof that g) \Rightarrow a) we cite [SGA6, I.2.12 and I.2.7] instead of [TT, 2.2.13] to see that if $E \oplus E'$ is n -pseudo-coherent then so is E , and if E is n -pseudo-coherent for all n then E is pseudo-coherent. ■

THEOREM 4.2. ([TT, 2.4.3]) *Let E be an A -module chain complex. The following are equivalent:*

- a) E is perfect
- b) E is cohomologically bounded below, and for any directed system $\{F_\alpha\}$ of A -module complexes, the canonical map (4.2.1) is an isomorphism.

$$(4.2.1) \quad \varinjlim_{\alpha} \operatorname{Hom}_{D(A)}(E, F_{\alpha}) \xrightarrow{\cong} \operatorname{Hom}_{D(A)}(E, \varinjlim_{\alpha} F_{\alpha})$$

- c) E is cohomologically bounded, and (4.2.1) is an isomorphism for any directed system $\{F_\alpha\}$ of strictly perfect complexes which is uniformly cohomologically bounded above.

PROOF. We merely note the changes needed for the proof of [TT, 2.4.3] to go through. For a) \Rightarrow b) we cite [H, I.6.4] instead of [TT, 2.4.1], as above. For c) \Rightarrow a), the proof in [TT] shows that some $E \oplus E'$ is isomorphic in $D(A)$ to a strictly perfect complex. As in the proof of 4.1 above, this implies that E is pseudo-coherent. By [SGA6, I.5.8.1] E is perfect. ■

§5. Excision.

In this section, we assume that $f : A \rightarrow B$ is an analytic isomorphism along a left denominator set S in A . In order to compare $\operatorname{Perf}(A, S)$ and $\operatorname{Perf}(B, S)$, we first compare the derived categories of A and B .

Recall the construction of the total tensor product

$$\mathbf{L}f^* = B \otimes_A^{\mathbf{L}} - : D^-(A) \rightarrow D^-(B).$$

Any bounded above A -module complex E has a quasi-isomorphism $P \rightarrow E$ with a bounded above projective complex P , and $\mathbf{L}f^*(E)$ is $B \otimes_A P$. The choice of P may be made functorial—use the total complex of the canonical free resolution—and therefore defines a functor from the category of bounded above A -module complexes to the category of bounded above B -module complexes. Restricting still further, but retaining the notation, we get functors

$$\begin{aligned} \mathbf{L}f^* &: \operatorname{Perf}^-(A) \rightarrow \operatorname{Perf}^-(B) \\ \mathbf{L}f^* &: \operatorname{Perf}^-(A, S) \rightarrow \operatorname{Perf}^-(B, S). \end{aligned}$$

The former induces the map $f^* : K(A) \rightarrow K(B)$. The latter map is the focal point of this section: we shall prove that it induces an isomorphism on K -theory.

PROPOSITION 5.1. *Let $f : A \rightarrow B$ be an analytic isomorphism along S .*

- a) *If E is a bounded above complex of A -modules such that $S^{-1}E$ is exact, then the canonical map $E \rightarrow B \otimes_A^L E$ is an isomorphism in the derived category $D^-(A)$.*
- b) *If F is a bounded above complex of B -modules such that $S^{-1}F$ is exact, then the canonical map (obtained by thinking of F as an A -module complex)*

$$B \otimes_A^L F \rightarrow F$$

is an isomorphism in the derived category $D^-(B)$.

PROOF. (Cf. [TT, 2.6.3 (a,b)]) For purposes of checking we may assume that E and F are also bounded below. The usual devissage argument (see *op. cit.*) now reduces to the case in which E and F are concentrated in one degree, i.e., S -torsion (left) modules. Since for every A -module M we have

$$H_*(B \otimes_A^L M) = \mathrm{Tor}_*^A(B, M),$$

we are done by the following lemma. ■

LEMMA 5.2. *Let $A \rightarrow B$ be an analytic isomorphism along S . Then for every S -torsion left A -module M we have $M \cong B \otimes_A M$ and $\mathrm{Tor}_p^A(B, M) = 0$ for $p \neq 0$.*

PROOF. If $M \cong A/I$ then S meets I and we are done by the definition of analytic isomorphism. An induction on the number of generators of M proves this result if M is finitely generated. As every M is the union of its fin. gen. submodules, and Tor commutes with filtered colimits, the result holds for infinitely generated M as well. ■

PROPOSITION 5.3. *If E is a perfect B -module complex with $S^{-1}E$ exact, then E is also perfect as an A -module complex.*

PROOF. (Cf. [TT, 2.6.3 (d)].) By truncating, we may assume that E is bounded above. We appeal to criterion 4.2 (c) to see that E is perfect, so let $\{F_\alpha\}$ be a uniformly cohomologically bounded above directed system of strictly perfect A -module complexes. Given any bounded above complex F , let $\Gamma_S F$ denote the mapping cone of $F \rightarrow S^{-1}F$, translated by $+1$, so that $S^{-1}(\Gamma_S F)$ will be exact, and the natural map

$$\mathrm{Hom}_{D(A)}(E, \Gamma_S F) \rightarrow \mathrm{Hom}_{D(A)}(E, F)$$

is an isomorphism. (To see this, use the long exact Hom sequence and note that $\mathrm{Hom}(E, S^{-1}F)$ is trivial because $S^{-1}E \cong 0$ in $D(A)$.) Using (5.1) and the adjointness property [V, 2.3.3], we see that if E and F are bounded above

$$\begin{aligned} \mathrm{Hom}_{D(A)}(E, \Gamma_S F) &\cong \mathrm{Hom}_{D(A)}(E, B \otimes_A^L \Gamma_S F) \\ &\cong \mathrm{Hom}_{D(B)}(B \otimes_A^L E, B \otimes_A^L \Gamma_S F) \\ &\cong \mathrm{Hom}_{D(B)}(E, B \otimes_A^L \Gamma_S F). \end{aligned}$$

Now set $F = \varinjlim_{\alpha} F_{\alpha}$, and note that $\varinjlim_{\alpha} (\Gamma_S F_{\alpha}) \cong \Gamma_S F$.

Using criterion 4.2 (c) in $D(B)$ and the isomorphism above (first putting $F = F_{\alpha}$ and then $F = \varinjlim_{\alpha} F_{\alpha}$), we therefore have

$$\begin{aligned} \varinjlim_{\alpha} \mathrm{Hom}_{D(A)}(E, \Gamma_S F_{\alpha}) &\cong \varinjlim_{\alpha} \mathrm{Hom}_{D(B)}(E, B \otimes_A^{\mathbf{L}} \Gamma_S F_{\alpha}) \\ &\cong \mathrm{Hom}_{D(B)}(E, \varinjlim_{\alpha} B \otimes_A^{\mathbf{L}} \Gamma_S F_{\alpha}) \\ &\cong \mathrm{Hom}_{D(B)}(E, B \otimes_A^{\mathbf{L}} \Gamma_S F) \cong \mathrm{Hom}_{D(A)}(E, \Gamma_S F). \end{aligned}$$

Hence

$$\varinjlim_{\alpha} \mathrm{Hom}_{D(A)}(E, F_{\alpha}) \cong \mathrm{Hom}_{D(A)}(E, F).$$

Using 4.2 (c), this proves that E is perfect in $D(A)$. ■

COROLLARY 5.4. *The forgetful functor from B -modules to A -modules induces a functor $u : \mathrm{Perf}(B, S) \rightarrow \mathrm{Perf}(A, S)$ and an equivalence of derived categories*

$$\mathbf{w}^{-1} \mathrm{Perf}(B, S) \approx \mathbf{w}^{-1} \mathrm{Perf}(A, S),$$

whose inverse is the total tensor product $B \otimes_A^{\mathbf{L}} -$.

PROOF OF EXCISION THEOREM. By 5.1 and [Wa, 1.3.1] the compositions

$$\begin{aligned} \mathrm{Perf}^{-}(A, S) &\xrightarrow{\mathbf{L}f^*} \mathrm{Perf}^{-}(B, S) \xrightarrow{u} \mathrm{Perf}^{-}(A, S) \\ \mathrm{Perf}^{-}(B, S) &\xrightarrow{u} \mathrm{Perf}^{-}(A, S) \xrightarrow{\mathbf{L}f^*} \mathrm{Perf}^{-}(B, S) \end{aligned}$$

induce maps on K -theory which are homotopy equivalent to the identity. The existence of the Mayer-Vietoris sequence is a formal consequence of the homotopy equivalence

$$K(\mathrm{Perf}^{-}(A, S)) \xrightarrow{\sim} K(\mathrm{Perf}^{-}(B, S)),$$

given the Localization Theorem. (See, e.g., [We, 1.2]). ■

We conclude with the following promised result, that our notion of analytic isomorphism generalizes both the notion of “isomorphism infinitely near Y ” of [TT] and the notion of analytic isomorphism used in [K] and [We].

LEMMA 5.5. *Let S be central in A and $f : A \rightarrow B$ a map such that $A/As \cong B/Bs$ for all $s \in S$, and $f(S)$ is a left denominator set in B .*

- a) f is an analytic isomorphism iff $\mathrm{Tor}_p^A(B, A/As) = 0$ for $p \neq 0$ and all $s \in S$
- b) If S consists of nonzerodivisors in A , then f is an analytic isomorphism iff S consists of right nonzerodivisors in B .

PROOF. If s is a nonzerodivisor on A , then $\mathrm{Tor}_p^A(B, A/As) = 0$ for $p > 1$ and $\mathrm{Tor}_1^A(B, A/As) \cong \{b \in B : bs = 0\}$. Therefore a) implies b). To see a), let I be a left ideal of A containing $s \in S$ and set $J = As$. As $A/I \cong A/J \otimes_A A/I$ there is a spectral sequence

$$E_{pq}^2 = \mathrm{Tor}_p^{A/J}(\mathrm{Tor}_q^A(B, A/J), A/I) \Rightarrow \mathrm{Tor}_{p+q}^A(B, A/I).$$

If $\mathrm{Tor}_q^A(B, A/As) = 0$ for $q \neq 0$ and $B/Bs \cong A/As$, the spectral sequence collapses to give

$$\mathrm{Tor}_p^A(B, A/I) \cong \mathrm{Tor}_p^{A/J}(A/J, A/I).$$

This vanishes for $p \neq 0$, proving (a). ■

Remark. The proof goes through if, instead of assuming S central, we assume that As is a 2-sided ideal of A for all $s \in S$.

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