A SURVEY OF PRODUCTS IN ALGEBRAIC K-THEORY

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In the last ten years, several authors have defined various kinds of product structures in algebraic K-theory, the idea being that $K_*(R)$ should be a graded commutative ring whenever R is a commutative ring. Such a structure should coincide with the products given by Milnor [Mi] for $* \leq 1$. The purpose of this paper is to survey the two main constructions: Waldhausen's product via the "double Q" construction [Wa] and the Loday product via the $BG\ell^+$ construction [L], as generalized by May in [May 1,2,3]. We also include a discussion of the product in KV-theory, since it is more tractable and so appears useful in understanding products for K-theory (see [S] for example).

It should be pointed out that the material in this survey is well-known to "the experts." It is my hope that condensing the technical matter into the Bibliography will result in a readable introduction for "non-experts." In particular, I have tried to show just how various results are implicit in Waldhausen's fundamental paper [Wa].

It goes without saying that I am enormously grateful to "the experts" for many useful discussions. I would like to thank Fiedorowicz, Loday and May in particular. I would also like to thank Queen's University for their hospitality during the writing stage.

1. Bicategories.

The key idea in Waldhausen's approach to multiplicative structures is that one should consider bicategories (or double categories) as well as categories in K-theory. The best introduction to bicategories is [K-S]; [E] and [Mac, p. 44] also give formal definitions. We will recount and use the viewpoint of §5 of [Wa] in this section.

Associated to every small category $\mathcal{S}$ is a simplicial set $S_\cdot$, called its nerve; S_0 is the set of objects of $\mathcal{S}$, S_1 is the set of morphisms, S_2 is the set of pairs of composable morphisms (i.e., $S_2=S_1\times_{S_0}S_1$), etc. The category $\mathcal{S}$ can be completely recovered from its nerve, and it is possible to write down axioms that describe which simplicial sets are nerves of categories. Identifying small categories and their nerves, we will think of a small category as a special kind of simplicial set.

Thinking of a simplicial set as a functor $S_\cdot:\Delta^{op} \to (\text{Sets})$, where Δ is the category of finite ordinal numbers ([Mac, p. 171]), a bisimplicial set is a functor $C_{\cdot\cdot}:\Delta^{op}\times\Delta^{op} \to (\text{Sets})$. We may visualize $S_{\cdot\cdot}$ as a lattice of sets:

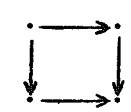
$$\begin{array}{ccccc} \vdots & & \vdots & & \\ S_{01} & \leftleftarrows\!\!\rightarrows & S_{11} & \cdots \\ \uparrow\downarrow\downarrow & & \uparrow\downarrow\downarrow & \\ S_{00} & \leftleftarrows\!\!\rightarrows & S_{10} & \cdots \end{array}$$

A (small) bicategory $S_{\cdot\cdot}$ is then a special kind of bisimplicial set, namely one for which each of the simplicial sets $S_{m\cdot}$ and

$S_{\cdot n}$ are categories. The 'interchange law' is automatic, as it merely states that $d_1^h d_1^v = d_1^v d_1^h : S_{22} \to S_{11}$. We call the sets S_{11}, S_{01} and S_{10}, S_{00} the sets of bimorphisms, horizontal and vertical morphisms, and objects, respectively.

In the notation of [K-S], bimorphisms are called squares. The horizontal source and target maps $d_0^h, d_1^h : S_{11} \to S_{01}$ strip off the left and right edges of the squares, while the vertical source and target maps $d_0^v, d_1^v : S_{11} \to S_{10}$ strip off the top and bottom edges.

Two pertinent constructions of bicategories from categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are $bi(\mathcal{C})$, whose bimorphisms are commutative squares in $\mathcal{C}$



and $\mathcal{A} \otimes \mathcal{B}$, whose objects are pairs (A,B) in $Obj(\mathcal{A}) \times Obj(\mathcal{B})$ and bimorphisms are pairs $(a:A_0 \to A_1, b:B_0 \to B_1)$ in $Mor(\mathcal{A}) \times Mor(\mathcal{B})$. We can represent (a,b) as the square

$$\begin{array}{ccc} (A_0,B_0) & \xrightarrow{(a,1)} & (A_1,B_0) \\ {\scriptstyle (1,b)}\downarrow & & \downarrow{\scriptstyle (1,b)} \\ (A_0,B_1) & \xrightarrow{(a,1)} & (A_1,B_1). \end{array}$$

The content of Lemma 3 on p. 170 of [Wa] is: if we think in terms of bisimplicial sets, then $\mathcal{A} \otimes \mathcal{B}$ is the 'product' of $\mathcal{A}$ and $\mathcal{B}$ in the sense that $(\mathcal{A}\otimes\mathcal{B})_{mn} = \mathcal{A}_m \times \mathcal{B}_n$.

There is a geometric realization functor B from bicategories (through bisimplicial sets) into topological spaces,

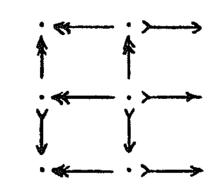
described on [Wa, p. 164]. There is also a diagonalization functor 'diag' from bisimplicial sets into simplicial sets, and it is well known that $B \circ diag \simeq B$. Since $diag(\mathcal{A}\otimes\mathcal{B})$ is the usual product of categories $\mathcal{A}\times\mathcal{B}$, we have $(B\mathcal{A})\times(B\mathcal{B}) \simeq B(\mathcal{A}\otimes\mathcal{B})$. Since there is a map $diag \circ bi(\mathcal{C}) \to \mathcal{C}$, $B\mathcal{C}$ is a retract of $B(bi(\mathcal{C}))$. From these remarks it is clear that for every functor $\mathcal{A}\times\mathcal{B} \to \mathcal{C}$ there is a commutative diagram

$$(1.1) \qquad \begin{array}{ccc} B\mathcal{A} \times B\mathcal{B} & \longrightarrow & B\mathcal{C} \\ \downarrow\simeq & & \uparrow \\ B(\mathcal{A}\otimes\mathcal{B}) & \longrightarrow & B(bi\mathcal{C}). \end{array}$$

2. Waldhausen's Product.

When $\mathcal{A}$ is a small exact category, Waldhausen defines (on p. 194 of [Wa]) a bicategory $QQ\mathcal{A}$ as follows. The bimorphisms are equivalence classes of commutative diagrams

(2.1)



in which the four little squares can be embedded in a 3×3 diagram with short exact rows and columns. Two diagrams are equivalent if they are isomorphic by an isomorphism which restricts to the identity on each corner object. Waldhausen proves (on p. 196) that the loop space $\Omega BQQ\mathcal{A}$ is homotopy equivalent to $BQ\mathcal{A}$ (the category $Q\mathcal{A}$ is defined on [Q, p.100]). Thus by definition ([Q, p.103]) we have $K_p\mathcal{A} = \pi_{p+1}BQ\mathcal{A} = \pi_{p+2}BQQ\mathcal{A}$.

If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are small exact categories, a functor $\boxtimes: \mathcal{A}\times\mathcal{B} \to \mathcal{C}$ is called biexact if (i) each partial functor $a\boxtimes(-): \mathcal{B} \to \mathcal{C}$, $(-)\boxtimes b: \mathcal{A} \to \mathcal{C}$ is exact and if (ii) $A\boxtimes 0 = 0\boxtimes B = 0$ for distinguished zero objects 0 of $\mathcal{A}, \mathcal{B}, \mathcal{C}$. Note that we can assume $\mathcal{C}$ skeletal if necessary to obtain the technical condition (ii). Given a biexact functor $\boxtimes$, there is an induced bicategory factorization

$$Q\mathcal{A} \otimes Q\mathcal{B} \to QQ\mathcal{C} \to bi(Q\mathcal{C})$$

of the map of §1. The right-hand map is given on bimorphisms by "forgetting" the middle object in the diagram (2.1). The left-hand map is given on the bimorphism

$$\begin{array}{ccccc} (A_0,B_0) & \twoheadleftarrow & (A_2,B_0) & \rightarrowtail & (A_1,B_0) \\ \uparrow & & \uparrow & & \uparrow \\ (A_0,B_2) & \leftarrow & (A_2,B_2) & \rightarrow & (A_1,B_2) \\ \downarrow & & \downarrow & & \downarrow \\ (A_0,B_1) & \twoheadleftarrow & (A_2,B_1) & \rightarrowtail & (A_1,B_1) \end{array}$$

of $Q\mathcal{A} \otimes Q\mathcal{B}$ by adding the middle object (A_2,B_2) as shown, and then applying $\boxtimes$. This factorization is pointed out in Proposition 9.2 of [Wa], where Waldhausen notes that the resulting map $BQ\mathcal{A}\times BQ\mathcal{B} \to BQQ\mathcal{C}$ of realizations vanishes on the subspace $BQ\mathcal{A}\vee BQ\mathcal{B}$ (because of the technical condition (ii)), and hence induces a map of topological spaces

(2.2) $$BQ\mathcal{A} \wedge BQ\mathcal{B} \to BQQ\mathcal{C}.$$

A lucid account of the product map (2.2) is also given in [Gr].

If we take homotopy groups, we obtain (using, e.g., [Br,(1.6)]) a map

$$K_p(\mathcal{A}) \otimes K_q(\mathcal{B}) \to K_{p+q}(\mathcal{C}).$$

(2.3) In the special case that $\mathcal{A} = \mathcal{C}$ and there is an object b_0 of $\mathcal{B}$ so that $(-)\boxtimes b_0$ is the identity on $\mathcal{A}$, there is a commutative diagram (Lemma 9.2.4 of [Wa]):

$$\begin{array}{ccc} BQ\mathcal{A} = BQ\mathcal{A} \wedge S^0 & \xrightarrow{\simeq} & \Omega BQQ\mathcal{B} \\ \downarrow & & \uparrow \\ BQ\mathcal{A} \wedge \Omega BQ\mathcal{B} & \to & \Omega(BQ\mathcal{A} \wedge BQ\mathcal{B}). \end{array}$$

The left vertical map comes from the inclusion of S^0 into $\Omega BQ\mathcal{B}$ by selection of the loop $[b_0]: 0 \rightarrowtail b_0 \twoheadrightarrow 0$. The fact that the top composite is the natural map is stated on p. 199, line 18 of [Wa].

When there is an associative pairing $\mathcal{B}\times\mathcal{B} \to \mathcal{B}$, $K_*(\mathcal{B})$ becomes a graded ring; $K_*(\mathcal{B})$ has unit $[b_0]$ if $(-)\boxtimes b_0 = b_0\boxtimes(-) = id(\mathcal{B})$. The map $\boxtimes: \mathcal{A}\times\mathcal{B} \to \mathcal{A}$ induces a right $K_*(\mathcal{B})$-module structure on $K_*\mathcal{A}$ when the two evident functors $\mathcal{A}\times\mathcal{B}\times\mathcal{B} \to \mathcal{A}$ agree up to natural isomorphism. These remarks apply notably to the case $\mathcal{B} = \underline{\underline{P}}(k)$, the category of fin. gen. projective k-modules for a commutative ring k: tensor product makes $K_*(k)$ a graded commutative ring with unit, and for every k-algebra Λ the group $K_*(\Lambda)$ is a 2-sided $K_*(k)$-module.

3. Loday's Product

Another approach to products in K-theory is to deal with symmetric monoidal categories, and invoke the "+=Q" theorem. This approach was first used by Loday in [L], using the category $\underset{\sim}{F}(A) = \coprod_{n=0}^{\infty} G\ell_n(A)$, and later generalized by May in [May 1,2]. For simplicity, we first describe Loday's method, and then present the more complicated approach used by May.

The choice of an isomorphism $\theta_{p,q}:A^p \otimes B^q \to (A\otimes B)^{pq}$ for every p and q gives a pairing $\theta:\underset{\sim}{F}(A) \times \underset{\sim}{F}(B) \to \underset{\sim}{F}(A\otimes B)$. Since $0 \otimes \underset{\sim}{F}(B) = \underset{\sim}{F}(A) \otimes 0 = 0$, it induces a map of topological spaces, which is the top row of the following diagram:

$$
(3.1)\qquad
\begin{array}{ccc}
B\underset{\sim}{F}(A)\wedge B\underset{\sim}{F}(B) & \xrightarrow{\quad B\theta \quad} & B\underset{\sim}{F}(A\otimes B) \\
\| & & \| \\
* \amalg \coprod_{p,q\geq 1} BGl_p(A)\times BGl_q(B) & \xrightarrow{\ \coprod B\theta_{pq}\ } & \coprod_{r\geq 0} BGl_r(A\otimes B) \\
\downarrow & & \downarrow \\
* \amalg \coprod_{p,q\geq 1} BGl_p^+(A)\times BGl_q^+(B) & \xrightarrow{\ \coprod f_{pq}=f\ } & \coprod_{r\geq 0} BGl_r^+(A\otimes B) \\
\| & & \\
\coprod_{p\geq 0} BGl_p^+(A)\wedge \coprod_{q\geq 0} BGl_q^+(B) & & \Big\downarrow i_{A\otimes B} \\
\Big\downarrow i_A\wedge i_B & & \\
[\mathbb{Z}\times BGl^+(A)]\wedge[\mathbb{Z}\times BGl^+(B)] & \xrightarrow{\quad \hat{\gamma} \quad} & \mathbb{Z}\times BGl^+(A\otimes B).
\end{array}
$$

The convention is that $BGl_p^+(A)$ denotes $BGl_p(A)$ for $p \leq 2$; for $p \geq 3$ it denotes the result of the plus construction relative to $E_p(A)$. The maps f_{pq} are the universal maps determined uniquely by the $B\theta_{pq}$. Loday's idea is to define a map $\hat{\gamma}$ making the diagram (3.1) commute up to homotopy. As before, taking homotopy groups yields a map

$$K_p(A) \otimes K_q(B) \longrightarrow K_{p+q}(A\otimes B).$$

If k is a commutative ring, the map $k \otimes k \to k$ makes $K_*(k)$ into a graded ring; the maps $k \otimes A \to A$ make $K_*(A)$ into a $K_*(k)$-algebra for every k-algebra A.

Loday first observes that $B\underset{\sim}{F}(A) = \coprod BGl_n(A)$ is an H-space (see [Wh]) under direct sum, and that $x\theta(-)$, $(-)\theta y$ are H-space maps for each $x \in B\underset{\sim}{F}(A)$, $y \in B\underset{\sim}{F}(B)$. He next observes that $\coprod BGl_p^+$ is an H-space (etc.) and that $f(x,)$, $f(,y)$ are H-space maps; this is (2.1.2)(ii) of [L]. Now $\mathbb{Z}\times BGl^+(A)$ is the "group completion" of $\coprod BGl_p^+(A)$ in the very strong sense that for every x in $\mathbb{Z}\times BGl^+(A)$ there is a positive integer n such that $x+n$ is in the image of $\coprod BGl_p^+(A)$. Since $\mathbb{Z}\times BGl^+(A\otimes B)$ is an H-<u>group</u>, there is a <u>unique</u> extension of f to an H-space map

$$\gamma:[\mathbb{Z}\times BGl^+(A)]\times[\mathbb{Z}\times BGl^+(B)] \to \mathbb{Z}\times BGl^+(A\otimes B).$$

Specifically, if $x \in \mathbb{Z}\times BGl^+(A)$, $y \in \mathbb{Z}\times BGl^+(B)$, we choose $m,n \in \mathbb{N}$, $x_0 \in \coprod BGl_p^+(A)$, $y_0 \in \coprod BGl_q^+(B)$ such that $x+m = i_A(x_0)$, $y+n = i_B(y_0)$, and define

$$\gamma(x,y) = \gamma(i_A(x_0)-m, i_B(y_0)-n)$$

$$= i\circ f(x_0,y_0) - i\circ f(*_m,y_0) - i\circ f(x_0,*_n) + i\circ f(*_m,*_n).$$

Here $*_m, *_n$ are the basepoints of $BGl_m^+(A)$ and $BGl_n^+(B)$, and we have used i for $i_{A\otimes B}$. If we take x and y to be in the respective basepoint components, we recover the map

$$\operatorname{colim} \gamma_{pq}:BGl^+(A)\times BGl^+(B) \to BGl^+(A\otimes B)$$

on the top of p. 332 of [L]. Since γ is homotopically trivial on $[\mathbb{Z}\times BGl^+(A)]\vee[\mathbb{Z}\times BGl^+(B)]$, γ factors through the smash product to give the map $\hat{\gamma}$ of diagram (3.1). The choices used to define γ mean that γ is only well-defined up to weak homotopy type.

In May's generalization, one considers "pairings" $\boxtimes:\mathcal{A}\times\mathcal{B} \to \mathcal{C}$ of symmetric monoidal categories. This means that $A\boxtimes 0 = 0\boxtimes B = 0$ and that there is a coherent natural bidistributivity axiom

$$(a+a')\boxtimes(b+b') \cong (a\boxtimes b) + (a\boxtimes b') + (a'\boxtimes b) + (a'\boxtimes b').$$

Instead of making the technical notion of coherence precise, we refer the reader to §2 of [May 2] and content ourselves with the remark that $\otimes : \underset{\sim}{F}(A)\times\underset{\sim}{F}(B) \to \underset{\sim}{F}(A\otimes B)$ is such a pairing.

At this stage, we need to introduce the "group completion" map $BA \to E_0BA$, defined for every symmetric monoidal category A. For example, $E_0B\underset{\sim}{F}(A)$ is the space $\mathbb{Z}\times BG\ell^+(A)$. One way to construct the group completion is to use the $S^{-1}S$ construction of [GQ]. Another way is to use an infinite loop space machine; for example, one can first obtain a Γ-space $\tilde{BA}$, use Segal's machine to obtain a spectrum EBA, and take the zeroth space E_0BA. This latter approach has both the advantages and disadvantages inherent in infinite loop space machinery.

The point is that a pairing of symmetric monoidal categories functorially determines a pairing $E_0BA \wedge E_0B\mathcal{B} \to E_0B\mathcal{C}$ of infinite loop spaces. This follows, for example, from Theorems 1.6 and 2.1 of [May 2]. More is true: a pairing $EBA \wedge EB\mathcal{B} \to EB\mathcal{C}$ is determined in the stable category of infinite loop spectra, allowing spectrum level work to be performed. There is also a commutative diagram:

$$(3.2)\qquad \begin{array}{ccccc} BA\times B\mathcal{B} & \longrightarrow & BA\wedge B\mathcal{B} & \xrightarrow{B\otimes} & B\mathcal{C} \\ \downarrow & & \downarrow & \hat{\gamma} & \downarrow \\ E_0BA\times E_0B\mathcal{B} & \longrightarrow & E_0BA\wedge E_0B\mathcal{B} & \longrightarrow & E_0B\mathcal{C} \end{array}$$

in which the bottom composite is an infinite loop space map. Commutativity of the diagram is Corollary (6.5) of [May 2]. As remarked in the introduction of [May 2], it is immediate from (3.1) and (3.2) that the product defined by May specializes to Loday's product.

Here is an example of the usefulness of Loday's product. Let $\underline{\underline{Ens}}$ denote the skeletal category $\coprod \Sigma_n$ of finite sets and their isomorphisms. For this symmetric monoidal category we have $E_0B\,\underline{\underline{Ens}} = \mathbb{Z}\times B\Sigma^+ = \Omega^\infty\Sigma^\infty$, and a pairing $\underline{\underline{Ens}}\times\underline{\underline{Ens}} \to \underline{\underline{Ens}}$ induced by multiplication, which is discussed on [May 1, p. 161]. Consequently, $K_*\underline{\underline{Ens}} = \pi_*\Omega^\infty\Sigma^\infty = \pi_*^S$ is a graded commutative ring. The map $\underline{\underline{Ens}} \to \underset{\sim}{F}(\mathbb{Z})$ embedding the symmetric group Σ_n into $G\ell_n(\mathbb{Z})$ induces a map of pairings in the senses of [May 1, p. 155] and [May 2, §2], hence a ring map $\pi_*^S \to K_*(\mathbb{Z})$.

4. Agreement of Product Structures

In order to directly compare Waldhausen's product and Loday's product, consider a pairing $A\times\mathcal{B} \to \mathcal{C}$ of exact categories. The subcategories $Is(A)$, $Is(\mathcal{B})$, $Is(\mathcal{C})$ of isomorphisms are all symmetric monoidal categories, and the induced functor $Is(A)\times Is(\mathcal{B}) \to Is(\mathcal{C})$ is a pairing of symmetric monoidal categories. Waldhausen's Lemma 9.2.6 in [Wa] states that the following diagram commutes up to basepoint preserving homotopy:

$$(4.1)\qquad \begin{array}{ccc} BIs(A)\wedge BIs(\mathcal{B}) & \longrightarrow & BIs(\mathcal{C}) \\ \downarrow & & \downarrow \\ (\Omega BQA)\wedge(\Omega BQ\mathcal{B}) & \overset{\hat{\gamma}}{\dashrightarrow} & \Omega BQ\mathcal{C} \\ \downarrow & & \downarrow\simeq \\ \Omega\Omega(BQA\wedge BQ\mathcal{B}) & \longrightarrow & \Omega\Omega(BQQ\mathcal{C}). \end{array}$$

The maps $BIs(A) \to \Omega BQA$, etc., are described on p. 198 of [Wa]. The top arrow in (4.1) is induced from the composite

$$BIs(A)\times BIs(\mathcal{B}) \cong B(Is(\mathcal{B})\otimes Is(\mathcal{B})) \to B(bi(Is(\mathcal{C}))) \to BIs(\mathcal{C}),$$

which by (1.1) is the natural map $B\otimes$. The bottom map is the double looping of Waldhausen's map (2.2), and we have already remarked on the lower right-hand homotopy equivalence (of H-spaces!). There is a unique way, up to homotopy, to fill in the broken arrow so that the diagram remains homotopy commutative.

We point out that the broken arrow is induced from an H-space map $(\Omega BQA)\times(\Omega BQB) \to \Omega BQC$. To see this, note that the functor $QA\otimes QB \to QC$ is a map of symmetric monoidal bicategories (the operation being slotwise direct sum), so that $BQA\times BQB \simeq B(QA\otimes QB) \to BQQC$ is an H-space map (in fact it is an infinite loop space map).

Now suppose that all exact sequences in A split. Then there is a basepoint preserving homotopy equivalence $E_0BIs(A) \to \Omega BQA$ so that

$$\begin{array}{ccc} & BIs(A) & \\ \swarrow & & \searrow \\ E_0BIs(A) & \xrightarrow{\simeq} & \Omega BQA \end{array}$$

commutes (up to basepoint preserving homotopy). This is proven as (9.3.2) of [Wa], modulo the observation that (by (6.3) of [Wa]) the space $\Omega BN_\Gamma(Is(A))$ in (9.3.2) is just $E_0BIs(A)$. Having said this, it follows that the top part of (4.1) induces the following homotopy commutative diagram of H-spaces:

(4.2)
$$\begin{array}{ccc} BIs(A)\times BIs(B) & \longrightarrow & BIs(C) \\ \downarrow & & \downarrow \\ E_0BIs(A)\times E_0BIs(B) & \xrightarrow{\gamma} & E_0BIs(C). \end{array}$$

The H-space map γ in (4.2) is uniquely determined, so it must be the same as the map γ in Loday's construction, as generalized by May. Comparing (4.1) and (4.2), we see that the broken arrow $\hat{\gamma}$ in (4.1) must be the same as the $\hat{\gamma}$ in (3.1) and (3.2). We summarize this as follows:

Theorem 4.3 (Waldhausen). If $A\times B \to C$ is a biexact pairing of exact categories for which all exact sequences split, then the groups $K_*(A) = \pi_*(\Omega BQA)$ agree with the groups $K_*Is(A) = \pi_*E_0BIs(A)$. There are commutative diagrams

$$\begin{array}{ccc} E_0BIs(A)\wedge E_0BIs(B) & \xrightarrow{\hat{\gamma}} & E_0BIs(C) \\ \downarrow & & \downarrow\simeq \\ \Omega BQA\wedge\Omega BQB & \longrightarrow & \Omega\Omega BQQC \end{array}$$

$$\begin{array}{ccc} K_*Is(A)\otimes K_*Is(B) & \longrightarrow & K_*Is(C) \\ \cong\downarrow & & \cong\downarrow \\ K_*(A)\otimes K_*(B) & \longrightarrow & K_*(C) \end{array}$$

in which the top maps are the Loday-May pairings, and the bottom maps are the Waldhausen pairings.

We can apply this result to the exact category $\underline{\underline{P}}(A) = A$ of fin. gen. projective A-modules, etc. Since $E_0BIs\underline{\underline{P}}(A)$ is $K_0(A)\times BG\ell^+(A)$, we obtain:

Corollary 4.4 (Waldhausen). Let A,B be rings with unit. There is a homotopy commutative diagram

$$\begin{array}{ccc} [K_0(A)\times BG\ell^+(A)]\wedge[K_0(B)\times BG\ell^+(B)] & \xrightarrow{\hat{\gamma}} & [K_0(A\otimes B)\times BG\ell^+(A\otimes B)] \\ \downarrow & & \downarrow \\ [\Omega BQ\underline{P}(A)]\wedge[\Omega BQ\underline{P}(B)] & \longrightarrow & \Omega\Omega[BQQ\underline{P}(A\otimes B)], \end{array}$$

where the top arrow is Loday's pairing and the bottom arrow is Waldhausen's product. Thus the two pairings agree on homotopy to give the same graded map

$$K_*(A) \otimes K_*(B) \longrightarrow K_*(A\otimes B).$$

Remark. Waldhausen gave the argument for Theorem (4.3) on p.235 of [Wa] for the special case $A = \mathbb{Z}$, $B = \mathbb{Z}G$, in order to show that the map $\pi_* K(BG;\mathbb{Z}) \longrightarrow K_*(\mathbb{Z}G)$ constructed on [Wa, p. 227] agrees with Loday's map on [L, p. 226].

If we had used the skeletal subcategories $\underline{\underline{F}}(A) \subseteq \underline{\underline{P}}(A)$ of free modules, we would get $\underset{\sim}{F}(A) = \mathrm{Is}(\underline{\underline{F}}(A)$ when A is commutative, or more generally when $A^m \neq A^n$ for $m \neq n$. In this case we could write (4.4) as the commutative diagram

$$\begin{array}{ccc} [\mathbb{Z}\times BG\ell^+(A)]\wedge[\mathbb{Z}\times BG\ell^+(B)] & \xrightarrow{\hat{\gamma}} & [\mathbb{Z}\times BG\ell^+(A\otimes B)] \\ \downarrow & & \downarrow \\ [\Omega BQ\underline{\underline{F}}(A)]\wedge[\Omega BQ\underline{\underline{F}}(B)] & \longrightarrow & \Omega\Omega[BQQ\underline{\underline{F}}(A\otimes B)] \end{array}$$

5. Relative K-theory.

When I is an ideal in a ring A, we will construct a pairing $K_*(A,I) \otimes K_*(B) \to K_*(A\otimes B, I\otimes B)$ so that:

If B is commutative and A is a B-algebra, the map

(5.1) $K_*(A,I) \otimes K_*(B) \to K_*(A\otimes B, I\otimes B) \to K_*(A,I)$ makes $K_*(A,I)$ a graded $K_*(B)$-module.

There are two approaches to this problem, corresponding to the two types of product. We will first describe May's approach, which is more conceptually straightforward, and then describe Waldhausen's more subtle method.

In May's approach, the basic object is the category $\underset{\sim}{F}(B)$ of free B-modules. One has a "morphism of pairings of symmetric monoidal categories" (q.v. §2 of [May 2]):

$$\begin{array}{ccc} \underset{\sim}{F}(A)\times\underset{\sim}{F}(B) & \longrightarrow & \underset{\sim}{F}(A\otimes B) \\ \downarrow & & \downarrow \\ \underset{\sim}{F}(A/I)\times\underset{\sim}{F}(B) & \longrightarrow & \underset{\sim}{F}(A/I\otimes B). \end{array}$$

The machinery described in Theorem 1.6 of [May 2] produces a "morphism of pairings in the stable category" which is adequately represented by commutativity of the right-hand portion of the following diagram:

(5.2)
$$\begin{array}{ccccc} \mathrm{Fiber}(A,I)\wedge E_0B\underset{\sim}{F}B & \longrightarrow & E_0B\underset{\sim}{F}A\wedge E_0B\underset{\sim}{F}B & \longrightarrow & E_0B\underset{\sim}{F}(A/I)\wedge E_0B\underset{\sim}{F}B \\ \vdots\downarrow & & \downarrow & & \downarrow \\ \mathrm{Fiber}(A\otimes B, I\otimes B) & \longrightarrow & E_0BF(A\otimes B) & \longrightarrow & E_0B\underset{\sim}{F}(A/I\otimes B). \end{array}$$

In (5.2) we have written Fiber(A,I) for the homotopy fiber of the map $E_0B\underset{\sim}{F}(A) \to E_0B\underset{\sim}{F}(A/I)$, and similarly for Fiber(A⊗B,I⊗B). This being said, it is standard that there is a broken arrow making (5.2) a map of (infinite loop space) fibrations. Since $K_p(A,I) = \pi_p\mathrm{Fiber}(A,I)$, the homotopy groups of (5.2) yield a map of long exact sequences:

$$\begin{array}{ccccc} \cdots K_{p+1}(A/I)\times K_q(B) & \longrightarrow & K_p(A,I)\times K_q(B) & \longrightarrow & K_p(A)\times K_q(B)\cdots \\ \downarrow & & \vdots\downarrow & & \downarrow \\ \cdots K_{p+q+1}(A/I\otimes B) & \longrightarrow & K_{p+q}(A\otimes B, I\otimes B) & \longrightarrow & K_{p+q}(A\otimes B)\cdots \end{array}$$

The problem with this approach is that the broken arrow in (5.2) is not unique and, unless care is taken, will not make (5.1) hold.

Happily, there is enough structure in the categories involved to save the day. The functoriality of May's approach makes the right-hand square in (5.2) commute on the nose, and the details of associativity can be checked directly. This is done in detail in [May 3].

The second approach to relative pairings is due to Waldhausen and is implicit in §7 of [Wa]. The idea is to use simplicial exact categories (SEC's) to produce a model for $\Omega^{-2}\mathrm{Fiber}(A,I)$, and do all work at the category-theoretic level.

To an exact category A, Waldhausen associates an SEC denoted $S_.A$, and proves in [Wa, (7.1)] that $\Omega BQS_.A \simeq BQA$. An object of the exact category S_nA is a sequence

$$A: A_{01} \rightarrowtail A_{02} \rightarrowtail \cdots \rightarrowtail A_{0n}$$

of admissible monics in A, together with choices of objects $A_{ij}\,(i>j)$ and isomorphisms $A_{ij} \cong A_{0i}/A_{0j}$. The i^{th} face map $d_i : S_nA \to S_{n-1}A$ is induced by "dropping the index i". For example, the sequence for $d_0(A)$ is

$$A_{12} \rightarrowtail A_{13} \rightarrowtail \cdots \rightarrowtail A_{1n}.$$

Next, we suppose given an exact functor $f: A \to A'$. Waldhausen constructs on [Wa, p. 182] an SEC denoted $F_.(f)$ fitting into a sequence of SEC's:

(5.3) $$A \to A' \to F_.(f) \to S_.A \to S_.A'.$$

Of course, in (5.3) we consider A and A' to be constant simplicial exact categories.

Briefly, an object of the exact category $F_p(f)$ is a triple $(A',A,\cong)$: an object A' of $S_{p+1}(A')$, an object A of $S_p(A)$, and an isomorphism $f(A) \cong d_0(A')$ in $S_p(A')$. The map $A' \to F_p(f)$ sends A' in A' to the object $(A'=A'=\cdots=A',0,0)$ in $F_p(f)$, and the map $F_p(f) \to S_p(A)$ sends $(A',A,\cong)$ to A.

The content of Propositions (7.1) and (7.2) of [Wa] is that

$$BQA \longrightarrow BQA' \longrightarrow BQF_.(f) \longrightarrow BQS_.A \longrightarrow BQS_.A'$$

is a fibration sequence up to homotopy. Thus $\Omega BQF_.(f)$ is a model for the fiber of $BQA \to BQA'$; if we set $K_p(f) = \pi_{p+2}BQF_.(f)$, there is a long exact sequence

(5.4) $$\cdots K_{*+1}A \longrightarrow K_{*+1}A' \longrightarrow K_*(f) \longrightarrow K_*A \longrightarrow K_*A' \cdots$$

In particular: if $A = \underline{\underline{P}}(A)$, $A' = \underline{\underline{P}}(A/I)$ then $K_p(A,I) = \pi_{p+2}BQF_.(f)$

Given this category-theoretic encoding of the relative term in K-theory, we can construct relative pairings with ease. One starts with a commutative diagram

(5.5) $$\begin{array}{ccc} A \times B & \xrightarrow{\otimes} & C \\ {\scriptstyle f\times B}\downarrow & \otimes & \downarrow{\scriptstyle f'} \\ A'\times B & \longrightarrow & C' \end{array}$$

in which the horizontal arrows are biexact. The functors $\otimes$ induce simplicial biexact functors $S_.A\times B \to S_.C$ and $F_.(f)\times B \to F_.(f')$ in an obvious manner. The result is the commutative diagram of SEC's, analogous to (5.3):

$$\begin{array}{ccccccccc} \mathcal{A}\times\mathcal{B} & \longrightarrow & \mathcal{A}'\times\mathcal{B} & \longrightarrow & F_{\bullet}(f)\times\mathcal{B} & \longrightarrow & S_{\bullet}\mathcal{A}\times\mathcal{B} & \longrightarrow & S_{\bullet}\mathcal{A}'\times\mathcal{B} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{C}' & \longrightarrow & F_{\bullet}(f) & \longrightarrow & S_{\bullet}\mathcal{C} & \longrightarrow & S_{\bullet}\mathcal{C}'. \end{array}$$

Following §2 above, there is a commutative diagram of simplicial bicategories, the middle of which is

$$(5.6)\qquad \begin{array}{ccccc} Q\mathcal{A}'\otimes Q\mathcal{B} & \longrightarrow & QF_{\bullet}(f)\otimes Q\mathcal{B} & \longrightarrow & QS_{\bullet}\mathcal{A}\otimes Q\mathcal{B} \\ \downarrow & & \downarrow & & \downarrow \\ QQ\mathcal{C}' & \longrightarrow & QQF_{\bullet}(f') & \longrightarrow & QQS_{\bullet}(\mathcal{C}). \end{array}$$

Geometric realization yields a map of fibration sequences, the middle of which is

$$\begin{array}{ccccc} BQ\mathcal{A}'\wedge BQ\mathcal{B} & \longrightarrow & BQF_{\bullet}(f)\wedge BQ\mathcal{B} & \longrightarrow & BQS_{\bullet}\mathcal{A}\wedge BQ\mathcal{B} \\ \downarrow & & \downarrow & & \downarrow \\ BQQ\mathcal{C}' & \longrightarrow & BQQF_{\bullet}(f') & \longrightarrow & BQQS_{\bullet}(\mathcal{C}). \end{array}$$

Taking homotopy groups yields the map of long exact sequences

$$\begin{array}{ccccc} \cdots K_{p+1}\mathcal{A}'\times K_q(\mathcal{B}) & \longrightarrow & K_p(f)\times K_q(\mathcal{B}) & \longrightarrow & K_p(\mathcal{A})\times K_q(\mathcal{B})\cdots \\ \downarrow & & \downarrow & & \downarrow \\ \cdots K_{p+q+1}(\mathcal{C}') & \longrightarrow & K_{p+q}(f') & \longrightarrow & K_{p+q}(\mathcal{C})\cdots, \end{array}$$

the middle vertical arrow being the desired pairing.

Now suppose that we are in the situation of (2.3) above, i.e., that $\mathcal{A}=\mathcal{C}$ and $\mathcal{A}'=\mathcal{C}'$. We assume that there is an associative pairing $\mathcal{B}\times\mathcal{B}\to\mathcal{B}$, $(-)\otimes b_0$ is the identity functor on $\mathcal{A},\mathcal{A}'$ and $\mathcal{B}$, and that (5.5) fits into an "associativity axiom" cube (going from $\mathcal{A}\times\mathcal{B}\times\mathcal{B}$ to $\mathcal{A}'$). Then the two evident functors $F_{\bullet}(f)\times\mathcal{B}\times\mathcal{B}\to F_{\bullet}(f)$ agree (up to natural isomorphism), so that $K_*(f)$ is a graded

$K_*(\mathcal{B})$-module in such a way that (5.4) is a sequence of $K_*(\mathcal{B})$-modules.

The above paragraph applies to the situation of (5.1). We take (5.5) to be the diagram

$$\begin{array}{ccc} \underline{\underline{P}}(A)\times\underline{\underline{P}}(B) & \xrightarrow{\otimes} & \underline{\underline{P}}(A\otimes B) \\ \downarrow & & \downarrow \\ \underline{\underline{P}}(A/I)\times\underline{\underline{P}}(B) & \xrightarrow{\otimes} & \underline{\underline{P}}(A/I\otimes B). \end{array}$$

All hypotheses are met, and $K_*(A,I)$ is a graded $K_*(B)$-module.

6. Products for KV-theory

There is another type of K-theory with product: the Karoubi-Villamayor groups $KV_*(A)$. This theory makes sense for any ring (with or without "one"), and is uniquely determined by the axioms given in [K-V]. One way to define them is to set $KV_0(A) = K_0(A)$, and for $p > 0$ to define $KV_p(A)$ by exactness of

$$0 \longrightarrow KV_p(A) \longrightarrow K_0(\Omega^p A) \longrightarrow K_0(E\Omega^{p-1}A).$$

Here we have used the notation $\Omega A = (t^2-t)A[t] \subset tA[t] = EA$ for any ring A, and defined $\Omega^p A$ by iteration: $\Omega^p A = \Pi(t_i^2-t_i)A[t_1,\ldots,t_p]$.

In [K], Karoubi constructs a pairing for KV-theory from the following "usual" pairing for K_0: When A is a ring without "one" we can define $K_0(A) = K_0(R\oplus A, A)$ for any ring R with "one" with an R-algebra structure on A; this definition is independent of the choice of R. Since $(\mathbb{Z}\oplus A)\otimes(\mathbb{Z}\oplus B) = R\oplus(A\otimes B)$ for $R = \mathbb{Z}\oplus A\oplus B$, we can define the bilinear map $\theta_{0,0}$ as the composite:

$$K_0(A)\otimes K_0(B) \longrightarrow K_0(\mathbb{Z}\oplus A)\otimes K_0(\mathbb{Z}\oplus B) \longrightarrow K_0(R\oplus(A\otimes B)) \longrightarrow K_0(A\otimes B).$$

One interpretation of Karoubi's construction of pairings is this: It is easy to show that $K_0(EA\otimes\Omega B) \to K_0(EA\otimes EB)$ and $K_0(\Omega A\otimes EB) \to K_0(EA\otimes EB)$ are injections for every A,B. We then have the following commutative square with exact rows (for $p,q\geq 1$):

$$\begin{array}{ccccccc} 0 \to & KV_p(A)\times KV_q(B) & \to & K_0(\Omega^p A)\times K_0(\Omega^q B) & \to & K_0(E\Omega^{p-1}A)\times K_0(E\Omega^{q-1}B) \\ & \Big\downarrow \theta_{p,q} & & \Big\downarrow & & \Big\downarrow \\ 0 \longrightarrow & KV_{p+q}(A\otimes B) & \longrightarrow & K_0(\Omega^p A\otimes \Omega^q B) & \longrightarrow & K_0(E\Omega^{p-1}A\otimes E\Omega^{q-1}B). \end{array} \tag{6.1}$$

It follows that the broken arrow $\theta_{p,q}$ is defined when $p, q \geq 1$ When $p = 0$, $q \geq 1$ and $p \geq 1$, $q = 0$ it is also easy to induce maps $\theta_{p,q}$. Karoubi then proves the following theorem in [K, p. 78]:

Theorem (6.2) The maps $\theta_{p,q}: KV_p(A) \otimes KV_q(B) \to KV_{p+q}(A\otimes B)$ are the unique natural bilinear maps satisfying the following axioms:

(i) $\theta_{0,0}$ is the "usual" product $K_0(A) \otimes K_0(B) \to K_0(A\otimes B)$

(ii) Every map of GL-fibrations (see [K-V])

$$\begin{array}{ccccccccc} 0 \longrightarrow & A\times B & \longrightarrow & A\times B & \longrightarrow & A''\times B & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow 0 \end{array} \tag{6.2a}$$

gives rise to a commutative diagram

$$\begin{array}{ccccc} KV_{p+1}(A'') \otimes KV_q(B) & \longrightarrow & KV_{p+q+1}(C'') & \longleftarrow & KV_p(B) \otimes KV_{q+1}(A'') \\ \Big\downarrow \partial\times 1 & & \Big\downarrow \partial & & \Big\downarrow (-1)^p\times\partial \\ KV_p(A') \otimes KV_q(B) & \longrightarrow & KV_{p+q}(C') & \longleftarrow & KV_p(B) \otimes KV_q(A'). \end{array}$$

In the remainder of this section, we show that the natural map $K_*A \to KV_*(A)$ sends the K_*-pairing to the KV_*-pairing. In order to do this, it is necessary to recall the construction of the map $K_* \to KV_*$.

For a ring A with "one" we can define a simplicial ring $A_\bullet$, which in degree n is the coordinate ring $A[x_0,\ldots,x_n]/(\Sigma x_i=1)$ of the "standard n-simplex," the face and degeneracy maps being dictated by the geometry. Applying $BG\ell^+$ gives the simplicial topological space $p \mapsto BG\ell^+(A_p)$, and we have for $* \geq 1$ that $KV_*(A) = \pi_*|BG\ell^+(A_\bullet)|$. This is proven in [A, p. 65]. The map of simplicial spaces $BG\ell^+(A) \to BG\ell^+(A_\bullet)$ induces the map $K_*(A) \to KV_*(A)$ of homotopy groups.

Loday's pairing now induces a pairing in KV - theory in a completely canonical way: the choice of isomorphisms $A^p \otimes B^q \cong (A\otimes B)^{pq}$ completely determines a simplicial pairing $\underline{F}(A_\bullet)\times\underline{F}(B_\bullet) \to \underline{F}(A\otimes B)_\bullet$, and this in turn yields a map of simplicial topological spaces

$$\hat{\gamma}_\bullet : [\mathbb{Z}\times BG\ell^+(A_\bullet)]\wedge[\mathbb{Z}\times BG\ell^+(B_\bullet)] \to \mathbb{Z}\times BG\ell^+(A\otimes B)_\bullet.$$

Applying geometric realization yields a map

$$|\hat{\gamma}_\bullet| = [\mathbb{Z}\times|BG\ell^+(A_\bullet)|]\wedge[\mathbb{Z}\times|BG\ell^+(B_\bullet)|] \to \mathbb{Z}\times|BG\ell^+(A\otimes B)_\bullet|.$$

Another way to proceed is to use Waldhausen's pairing and a Q-version of the map $K_* \to KV_*$. Define the simplicial subcategory $\underline{P}_\bullet(A)$ of $\underline{P}(A_\bullet)$ by letting $\underline{P}_p(A)$ denote the full subcategory of $\underline{P}(A_p)$ of projective A_p-modules extended from A (i.e., isomorphic to some $P \otimes_A A_p$). We then have

$$(6.3)\qquad \begin{aligned} K_0(A)\times|BG\ell^+(A_\bullet)| &\simeq \Omega|BQ\underline{P}_\bullet(A)|. \\ KV_*(A) &= \pi_{*+1}|BQ\underline{P}_\bullet(A)|. \end{aligned}$$

This is Theorem 2.1 of [We]; the technical reason for using $\underline{\underline{P}}_\cdot(A)$ instead of $\underline{\underline{P}}(A_\cdot)$ is that $\Omega|BQ\underline{\underline{P}}(A_\cdot)| \simeq |K_0(A_\cdot)|\times|BG\ell^+(A_\cdot)|$, and the space $|K_0(A_\cdot)|$ need not be $K_0(A)$. We now induce a pairing from Waldhausen's product:

External $\otimes$ gives a biexact functor $\underline{\underline{P}}_\cdot(A)\times\underline{\underline{P}}_\cdot(B) \to \underline{\underline{P}}_\cdot(A\otimes B)$, and so a morphism of simplicial bicategories

$$Q\underline{\underline{P}}_\cdot(A) \otimes Q\underline{\underline{P}}_\cdot(B) \longrightarrow QQ\underline{\underline{P}}_\cdot(A\otimes B),$$

which realizes to a map of topological spaces:

(6.4) $$|BQ\underline{\underline{P}}_\cdot(A)|\wedge|BQ\underline{\underline{P}}_\cdot(B)| \longrightarrow |BQQ\underline{\underline{P}}_\cdot(A\otimes B)|.$$

Since each $BQQ\underline{\underline{P}}_p$ is connected, we have that

$$\Omega|BQQ\underline{\underline{P}}_\cdot| \simeq |\Omega BQQ\underline{\underline{P}}_\cdot| \simeq |BQ\underline{\underline{P}}_\cdot|,$$

and therefore we have an induced map of homotopy groups

(6.5) $$\begin{array}{ccc} \pi_{p+1}|BQ\underline{\underline{P}}_\cdot(A)| \otimes \pi_{q+1}|BQ\underline{\underline{P}}_\cdot(B)| & \longrightarrow & \pi_{p+q+2}|BQQ\underline{\underline{P}}_\cdot(A\otimes B)| \\ \| & & \| \\ KV_p(A) \otimes KV_q(B) & \longrightarrow & KV_{p+q}(A\otimes B), \end{array}$$

defined for $p, q \geq 0$. In view of (6.3) and (4.4), it is clear that the map (6.4) agrees with Loday's $|\hat{\gamma}_\cdot|$.

In view of Theorem (6.2), we can show that the pairing (6.5) agrees with Karoubi's pairing (6.1) by checking the two axioms. Since Waldhausen's map $K_0(A) \otimes K_0(B) \to K_0(A\otimes B)$ agrees with the classical external product used by Karoubi, we only have to check axiom (ii). We can assume that A,A", C,C", and B have a "one",

so that the commutative diagram **(6.2a)** gives rise to a commutative diagram of bisimplicial bicategories analogous to (5.6):

$$\begin{array}{ccccc} Q\underline{\underline{P}}_\cdot(A'') \otimes Q\underline{\underline{P}}_\cdot(B) & \longrightarrow & QF_\cdot(f_\cdot) \otimes Q\underline{\underline{P}}_\cdot(B) & \longrightarrow & QS_\cdot\underline{\underline{P}}_\cdot(A) \otimes Q\underline{\underline{P}}_\cdot(B) \\ \downarrow & & \downarrow & & \downarrow \\ QQ\underline{\underline{P}}_\cdot(A''\otimes B) & \longrightarrow & QQF_\cdot(f_\cdot\otimes B) & \longrightarrow & QQS_\cdot\underline{\underline{P}}_\cdot(A\otimes B). \end{array}$$

Applying geometric realization gives a map of fibrations <u>at each level</u>, and the assumption that the exact sequences of rings were Gl-fibrations implies that we have <u>global</u> fibrations, i.e., that the rows in the following diagram are fibrations of topological spaces:

$$\begin{array}{ccccc} |BQ\underline{\underline{P}}_\cdot(A'')|\wedge|BQ\underline{\underline{P}}_\cdot(B)| & \longrightarrow & |BQF_\cdot(f_\cdot)|\wedge|BQ\underline{\underline{P}}_\cdot(B)| & \longrightarrow & |BQS_\cdot\underline{\underline{P}}_\cdot(A)|\wedge|BQ\underline{\underline{P}}_\cdot(B)| \\ \downarrow & & \downarrow & & \downarrow \\ |BQQ\underline{\underline{P}}_\cdot(A''\otimes B)| & \longrightarrow & |BQQF_\cdot(f_\cdot\otimes B)| & \longrightarrow & |BQQS_\cdot\underline{\underline{P}}_\cdot(A\otimes B)|. \end{array}$$

The fact that $\pi_{*+2}(BQF_\cdot(f_\cdot)) = KV_*(A')$, $\pi_{*+3}(BQQF_\cdot(f_\cdot\otimes B)) = KV_*(A'\otimes B)$ follows from consideration of the long exact homotopy sequences of the rows (and §5), and the commutative diagram

$$\begin{array}{ccc} \pi_{p+1}\Omega|BQ\underline{\underline{P}}_\cdot A''| \otimes \pi_q\Omega|BQ\underline{\underline{P}}_\cdot B| & \longrightarrow & \pi_{p+1}\Omega|BQF_\cdot(f_\cdot)| \otimes \pi_q\Omega|BQ\underline{\underline{P}}_\cdot B| \\ \downarrow & & \downarrow \\ \pi_{p+q+1}\Omega\Omega|BQQ\underline{\underline{P}}_\cdot(A''\otimes B)| & \longrightarrow & \pi_{p+q+1}\Omega\Omega|BQQF_\cdot(f_\cdot\otimes B)| \\ \uparrow & & \uparrow \\ \pi_p\Omega|BQ\underline{\underline{P}}_\cdot B| \otimes \pi_{q+1}\Omega|BQ\underline{\underline{P}}_\cdot A''| & \xrightarrow{(-1)^p} & \pi_p\Omega|BQ\underline{\underline{P}}_\cdot B| \otimes \pi_{q+1}\Omega|BQF_\cdot(f_\cdot)| \end{array}$$

translates into the diagram of axiom (ii). We summarize this:

<u>Proposition (6.6)</u> The pairing (6.5) on KV_*-theory induced from the pairing on K_*-theory satisfies the axioms of Theorem (6.2), and so agrees with Karoubi's product.

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