RELATIVE CARTIER DIVISORS AND K-THEORY

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Abstract. We study the relative Picard group $\mathrm{Pic}\,(f)$ of a map $f : X \to S$ of schemes. If f is faithful affine, it is the relative Cartier divisor group $\mathcal{I}(f)$. The relative group $K_0(f)$ has a γ-filtration, and $\mathrm{Pic}\,(f)$ is the top quotient for the γ-filtration. When f is induced by a ring homomorphism $A \to B$, we show that the relative "nil" groups $N\mathrm{Pic}\,(f)$ and $NK_n(f)$ are continuous $W(A)$-modules.

Introduction

If $f : X \to S$ is a morphism of schemes, the relative Picard group $\mathrm{Pic}\,(f)$ was defined by Bass in [1], and fits into a natural exact sequence

$$\mathcal{O}^{\times}(S) \xrightarrow{f^*} \mathcal{O}^{\times}(X) \xrightarrow{\partial} \mathrm{Pic}\,(f) \longrightarrow \mathrm{Pic}\,(S) \xrightarrow{f^*} \mathrm{Pic}\,(X). \tag{0.1}$$

The goal of this paper is to study this group as well as $N\mathrm{Pic}\,(f)$, defined to be $\mathrm{Pic}\,(f[t])/\mathrm{Pic}\,(f)$, where $f[t] : X \times \mathbb{A}^1 \to S \times \mathbb{A}^1$.

Our first observation is that when f is $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ for a commutative ring extension $A \hookrightarrow B$, $\mathrm{Pic}\,(f)$ is isomorphic to the relative Cartier divisor group $\mathcal{I}(f)$, defined in [13] as the group of invertible A-submodules of B under multiplication and studied in [15, 14, 16]. This definition of $\mathcal{I}(f)$ also makes sense (and we still have $\mathcal{I}(f) \cong \mathrm{Pic}\,(f)$) for scheme maps $f : X \to S$ for which $\mathcal{O}_S^{\times} \to f_*\mathcal{O}_X^{\times}$ is an injection of sheaves. It then follows from [16] that $\mathrm{Pic}\,(f)$ is a contracted functor in the sense of Bass.

We then relate $\mathrm{Pic}\,(f)$ to the relative group $K_0(f)$, which fits into an exact sequence

$$K_1(S) \xrightarrow{f^*} K_1(X) \xrightarrow{\partial} K_0(f) \longrightarrow K_0(S) \longrightarrow K_0(X).$$

For example, if $f : A \hookrightarrow B$ is subintegral then $K_0(f) \cong \mathrm{Pic}\,(f)$ (Proposition 2.5).

Let $\mathcal{NI}$ denote the Zariski sheaf associated to the presheaf $U \mapsto N\mathrm{Pic}\,(U, f^{-1}U)$ on S. In Theorem 4.1 and Theorem 4.7, we prove the following:

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Theorem 0.2. *Let $f : X \to S$ be a faithful affine morphism of schemes.*

1. *The Zariski sheaf $\mathcal{NI}$ is an étale sheaf on S. Moreover,*
$$N\mathrm{Pic}\,(f) \cong H^0_{\mathrm{et}}(S, \mathcal{NI}) = H^0_{\mathrm{zar}}(S, \mathcal{NI}).$$
2. *If X and S are schemes then $H^*_{\mathrm{et}}(S, \mathcal{NI}) \cong H^*_{\mathrm{zar}}(S, \mathcal{NI})$.*
3. *If X and S are both affine schemes then $H^q_{\mathrm{et}}(S, \mathcal{NI}) = 0$ for $q \neq 0$.*

A secondary goal of this article is to study the relative K-theory groups $K_n(f)$ associated to a morphism of schemes $f : X \to S$. By definition, $K_n(f) = \pi_n K(f)$, where $K(f)$ is the homotopy fiber of $K(S) \to K(X)$. Comparing $X \to S$ to $X[t] \to S[t]$ yields groups $NK_*(f)$.

Theorem 0.3. *For each homomorphism $f : A \to B$:*

1. *$NK_n(f)$ is a continuous $W(A)$-module, for all n.*
2. *$N\mathrm{Pic}\,(f)$ is a continuous $W(A)$-module.*
3. *$\det : NK_0(f) \to N\mathrm{Pic}\,(f)$ is a $W(A)$-module homomorphism.*

(See Theorems 3.3 and 5.6, and Proposition 3.2). This implies that if $\mathrm{char}(A) = p > 0$ then both $NK_n(f)$ and $N\mathrm{Pic}\,(f)$ are p-groups, while if $\mathrm{char}(A) = 0$ the groups have the structure of A-modules.

We conclude with some remarks about $K_n(f)$ when n is negative. If X and S have dimension at most d, then $K_n(S) = K_n(X) = 0$ for $n < -d$ in many cases. In such cases, it follows that $K_n(f) = 0$ for $n < -d-1$. The cohomological interpretation of the negative K-theory of a scheme in terms of the cdh-cohomology of the constant sheaf $\mathbb{Z}$ is given in [4]. In the relative situation, we prove the following (Theorem 6.2 and Theorem 6.3):

Theorem 0.4. *Let $f : X \to S$ be a finite morphism of d-dimensional noetherian schemes.*

1. *If X and S are essentially of finite type over a field k of characteristic 0, $K_{-d-1}(f) \cong H^d_{\mathrm{cdh}}(S, f_*\mathbb{Z}/\mathbb{Z})$.*
2. *If $\dim S = 1$, then $K_{-2}(f) \cong H^1_{\mathrm{nis}}(S, f_*\mathbb{Z}/\mathbb{Z})$ and there is an extension*
$$0 \to H^1_{\mathrm{nis}}(S, f_*\mathcal{O}^\times_X/\mathcal{O}^\times_S) \to K_{-1}(f) \to H^0_{\mathrm{nis}}(S, f_*\mathbb{Z}/\mathbb{Z}) \to 0.$$

1. Relative Pic and invertible submodules

In [1], Bass defined $\mathrm{Pic}\,(f)$ to be the abelian group generated by $[L_1, \alpha, L_2]$, where the L_i are line bundles on S and $\alpha : f^*L_1 \to f^*L_2$ is an isomorphism. The relations are:

(1) $[L_1, \alpha, L_2] + [L_1', \alpha', L_2'] = [L_1 \otimes L_1', \alpha \otimes \alpha', L_2 \otimes L_2']$;
(2) $[L_1, \alpha, L_2] + [L_2, \beta, L_3] = [L_1, \beta\alpha, L_3]$;
(3) $[L_1, \alpha, L_2] = 0$ if $\alpha = f^*(\alpha_0)$ for some $\alpha_0 : L_1 \cong L_2$.

Remark 1.0.1. By (1), every element of $\mathrm{Pic}\,(f)$ has the form $[L, \alpha, \mathcal{O}_S]$. Writing $[L, \alpha]$ for $[L, \alpha, \mathcal{O}_S]$, an alternative presentation for $\mathrm{Pic}\,(f)$ is that it is generated by elements $[L, \alpha]$ satisfying: $[L, \alpha] + [L', \alpha'] = [L \otimes L', \alpha \otimes \alpha']$; $[L, \alpha] = 0$ if (and only if) there is an isomorphism $\alpha_0 : L \cong \mathcal{O}_S$ so that $\alpha = f^*(\alpha_0)$. It is easy to see, and observed by Bass, that the map $\mathrm{Pic}\,(f) \to \mathrm{Pic}\,(S)$ sending $[L, \alpha]$ to $[L]$ fits into an exact sequence (0.1), where $\partial(b) = [\mathcal{O}_S, b]$.

Proposition 1.1. *Bass' $\mathrm{Pic}\,(f)$ is the hypercohomology group $H^0(S, \mathcal{O}_S^\times \to f_*\mathcal{O}_X^\times)$.*

Proof. Let C^* denote the mapping cone of $\mathcal{O}_S^\times \to f_*\mathcal{O}_X^\times$. A 0-cocyle of C^* is given by a cover $\{U_i\}$ of S, a unit b_i of $f^{-1}(U_i)$ for each i, and units a_{ij} of $U_i \cap U_j$ for each i, j satisfying the cocycle condition (so that the $\{a_{ij}\}$ define a line bundle L on S) and such that $b_i/b_j = f^{\#}(a_{ij})$ on each $f^{-1}(U_i \cap U_j)$. Since the $\{b_i\}$ define an isomorphism $f^*L \cong \mathcal{O}_X$, each 0-cocyle defines an element $\lambda = [L, \beta, \mathcal{O}_S]$ of $\mathrm{Pic}\,(f)$. A 0-coboundary is given by $a_{ij} = a_i/a_j$ and $b_i = f^{\#}(a_i)$ for units a_i of U_i; adding it to a cocyle does not change λ. Refining the cover does not change λ either. The result follows from the 5-lemma applied to the following diagram with exact rows (which is easily checked to be commutative):

$$\begin{tikzcd}
H^0(S,\mathcal{O}^\times) \arrow[r] \arrow[d, "\cong"'] & H^0(X,\mathcal{O}^\times) \arrow[r] \arrow[d, "\cong"'] & H^0(S,C^*) \arrow[r] \arrow[d] & H^1(S,\mathcal{O}^\times) \arrow[r] \arrow[d, "\cong"'] & H^1(X,\mathcal{O}^\times) \arrow[d, "\cong"'] \\
\mathcal{O}^\times(S) \arrow[r] & \mathcal{O}^\times(X) \arrow[r] & \mathrm{Pic}\,(f) \arrow[r] & \mathrm{Pic}\,(S) \arrow[r] & \mathrm{Pic}\,(X).
\end{tikzcd}$$ □

Now suppose that f is faithful and affine. As observed in [16], $\mathcal{I}(f)$ is isomorphic to $H^0(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times)$. Thus Proposition 1.1 implies that $\mathcal{I}(f) \cong \mathrm{Pic}\,(f)$. Here is a more elementary proof.

Lemma 1.2. *If $f : X \to S$ is a faithful affine map, there is an isomorphism $\rho : \mathcal{I}(f) \xrightarrow{\cong} \mathrm{Pic}\,(f)$, sending L to $[L, i, \mathcal{O}_S]$, where $i : f^*L \cong \mathcal{O}_X$.*

The isomorphism $f^*L \cong \mathcal{O}_X$ is well defined, because in any affine open $U = \operatorname{Spec}(A)$ of S we have $f^{-1}U = \operatorname{Spec}(B)$ with $A \subset B$; it was proven by Roberts and Singh [13] that $L \subset B$ induces $L \otimes_A B \cong B$.

Proof. Since $\rho(LL') = [L \otimes L', i \otimes i', \mathcal{O}_S] = [L, i, \mathcal{O}_S] + [L', i', \mathcal{O}_S]$, ρ is a homomorphism. To define the inverse map, we use the presentation of $\operatorname{Pic}(f)$ and the observation that because $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an injection, so is $L \to L \otimes f_*\mathcal{O}_X$ for every line bundle L. Given a triple $[L_1, \alpha, L_2]$, we set $L = L_2^{-1} \otimes L_1$, so that α induces an isomorphism $f^*L \cong f^*(L_2)^{-1} \otimes f^*(L_1) \cong \mathcal{O}_X$, and define $\psi([L_1, \alpha, L_2])$ to be the submodule L of $L \otimes f_*\mathcal{O}_X \cong f_*\mathcal{O}_X$. Since ψ is compatible with the relations of $\operatorname{Pic}(f)$, it descends to a homomorphism $\psi : \operatorname{Pic}(f) \to \mathcal{I}(f)$. Since $[L_1, \alpha, L_2] = [L_2^{-1} \otimes L_1, \alpha, \mathcal{O}_S]$ in $\operatorname{Pic}(f)$ and $f^*(L) = \mathcal{O}_X$ for all $L \in \mathcal{I}(f)$, ψ is an inverse to ρ. □

2. RELATIVE K_0 AND Pic

Bass gave a presentation of a relative group $K_0(f)$ associated to $f : A \to B$ in [1] and [2, VII.5]; see [29, II.2.10]. It is generated by triples $[P_1, \alpha, P_2]$, where the P_i are finitely generated projective A-modules (or vector bundles on S) and α is an isomorphism $f^*(P_1) \xrightarrow{\cong} f^*(P_2)$, and agrees with the group $\pi_0 K(f)$ of [29, IV.1.11]. The relations are:

(1) $[P_1, \alpha, P_2] + [P_1', \alpha', P_2'] = [P_1 \oplus P_1', \alpha \oplus \alpha', P_2 \oplus P_2']$,
(2) $[P_1, \alpha, P_2] + [P_2, \beta, P_3] = [P_1, \beta\alpha, P_3]$,
(3) $[P_1, \alpha, P_2] = 0$ if $\alpha = f^*(\alpha_0)$ for some $\alpha_0 : P_1 \cong P_2$.

By (1), every element of $K_0(f)$ has the form $[P, \alpha, A^n]$.

Bass showed [2, VII.5.3] that there is an exact sequence for each $f : A \to B$:

$$K_1(A) \xrightarrow{f^*} K_1(B) \xrightarrow{\partial} K_0(f) \longrightarrow K_0(A) \longrightarrow K_0(B), \tag{2.1}$$

where for $g \in GL_n(B)$ we have $\partial([g]) = [A^n, g, A^n]$. Since we do not know if the corresponding sequence is exact for a quasi-projective map $f : X \to S$, we will restrict to the affine case in this section and the next.

Lemma 2.2 (Excision). *Let $f : A \to B$ be a ring homomorphism, and let I be an ideal of A mapping isomorphically onto an ideal of B; write $\bar{f} : A/I \subset B/I$ for the induced map. Then excision holds for K_n for all $n \le 0$: $K_n(f) \cong K_n(\bar{f})$.*

Proof. It suffices to consider the case $n = 0$. Because $K_0(A, I) \cong K_0(B, I)$ [29, Ex. II.2.3] and $K_1(A, I) \to K_1(B, I)$ is onto [29, III.2.2.1], the double-relative group vanishes: $K_0(A,B,I)=0$. Applying contraction, we also have $K_{-1}(A,B,I)=0$. The result now follows from the exact sequence

$$K_0(A,B,I)\to K_0(f)\to K_0(\bar{f})\to K_{-1}(A,B,I). \qquad \square$$

Remark. The failure of Lemma 2.2 in the non-affine setting was investigated in [12, A.5–6]. For example, if X is the normalization of S and the support Y of the conductor $\mathfrak{c}$ is 1-dimensional, the obstruction is $K_0(S,X,Y)\cong H^1(Y,\mathfrak{c}/\mathfrak{c}^2\otimes\Omega_{X/S})$.

As observed by Bass and Murthy long ago [3], the determinant $K_0(S)\to \mathrm{Pic}\,(S)$ induces a surjective homomorphism

$$\det : K_0(f)\to \mathrm{Pic}\,(f), \quad \det[P_1,\alpha,P_2]=[\det(P_1),\det(\alpha),\det(P_2)]. \tag{2.3}$$

Since $SK_0(S)$ is the kernel of $\det : K_0(S)\to \mathrm{Pic}\,(S)$, we write $SK_0(f)$ for the kernel of $\det : K_0(f)\to \mathrm{Pic}\,(f)$.

Recall [29, II.4.2] that a λ-ring $K=\mathbb{Z}\oplus\widetilde{K}$ has a *positive structure* if it contains a λ-semiring P (positive elements) including $\mathbb{N}$, such that every element of $\widetilde{K}$ can be written as a difference of positive elements, the augmentation $\epsilon : K\to\mathbb{Z}$ sends P to $\mathbb{N}$ and, if $p\in P$ has $\epsilon(p)=n$, then $\lambda^i p=0$ for $i>n$ and $\lambda^n p$ is a unit. The *line elements* are $\{p\in P:\epsilon(p)=1\}$; they form a subgroup of the units of K.

Proposition 2.4. *Let $f:A\to B$ be a homomorphism of commutative rings. The operations $\lambda^i[P_1,\alpha,P_2]=[\Lambda^iP_1,\Lambda^i\alpha,\Lambda^iP_2]$ give $\mathbb{Z}\oplus K_0(f)$ the structure of a λ-ring with a positive structure. The top two ideals in the γ-filtration are $F^1_\gamma=\widetilde{K}_0$ and $F^2_\gamma=SK_0(f)$, and the group of its line elements is* $\mathrm{Pic}\,(f)\cong F^1_\gamma/F^2_\gamma$.

Proof. Given $f:A\to B$, choose a surjection $\pi:\mathbb{Z}[X]\to B$ from a polynomial ring $\mathbb{Z}[X]$ in many variables to B; let R be the pullback ring $R=\{(p,a)\in\mathbb{Z}[X]\times A : \pi(p)=f(a)\}$, with $\tilde{f}:R\to\mathbb{Z}[X]$ the projection. Since $K_1(\mathbb{Z}[X])=\pm 1$ and $K_0(\mathbb{Z}[X])=\mathbb{Z}$, we have $K_0(\tilde{f})\xrightarrow{\cong}\widetilde{K}_0(R)$, and this map is compatible with the operations λ^i. Similarly, we have $\mathrm{Pic}\,(\tilde{f})\cong\mathrm{Pic}\,(R)$. By Excision 2.2 for K_0 and Pic, $K_0(\tilde{f})\cong K_0(f)$ and $\mathrm{Pic}\,(\tilde{f})\cong\mathrm{Pic}\,(f)$. Hence $\mathbb{Z}\oplus K_0(f)\cong\mathbb{Z}\oplus\widetilde{K}_0(R)$ is a λ-ring. Thus the result follows from the fact that the operations λ^i make $K_0(R)$ into a λ-ring, with $F^2_\gamma=SK_0(R)$, and $\widetilde{K}_0(R)/SK_0(R)\cong\mathrm{Pic}\,(R)$. $\square$

Recall (Swan [17]) that an extension $A\subset B$ is said to be *subintegral* if B is integral over A, and $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$ is a bijection inducing isomorphisms on all residue fields.

Proposition 2.5. *(Ischebeck) If $f : A \hookrightarrow B$ is subintegral then $K_0(f) \cong \text{Pic}\,(f)$, $K_n(f) = 0$ for all $n < 0$, and there is an exact sequence*

$$1 \to B^\times/A^\times \to K_0(f) \to K_0(A) \to K_0(B) \to 0.$$

Proof. When $A \subset B$ is subintegral, Ischebeck proved in [9, Prop. 7] that the natural map $K_0(A) \to K_0(B)$ is surjective and $SK_1(A) \to SK_1(B)$ is onto, so the cokernel of $K_1(A) \to K_1(B)$ is $B^\times/A^\times$. The exact sequence follows from (2.1). Finally, Ischebeck proved in [9, p. 331] that the determinant (2.3) induces an isomorphism from the kernel of $K_0(A) \to K_0(B)$ onto the kernel of $\text{Pic}\,(A) \to \text{Pic}\,(B)$. The result now follows from (2.1).

Replacing A and B by Laurent polynomial extensions, the Fundamental Theorem of K-theory [29, III.4.1] implies that $LK_n(f) \cong K_{n-1}(f)$ and $K_{-1}(f) \cong L\text{Pic}\,(f)$. Since $A[t, 1/t] \subset B[t, 1/t]$ is subintegral, we have $L\text{Pic}\,(f) = 0$ by Proposition 5.6 of [16]. This shows that that $K_n(f) = 0$ for all $n < 0$. ☐

Given an extension $f : A \hookrightarrow B$, let $i : A \hookrightarrow {}^+A$ be the seminormalization of A in B and ${}^+f : {}^+A \hookrightarrow B$ the induced map. There is an exact sequence

$$\cdots \to K_n(i) \to K_n(f) \to K_n({}^+f) \to K_{n-1}(i) \to \cdots .$$

Corollary 2.6. $K_n(f) \overset{\cong}{\to} K_n({}^+f)$ *for $n < 0$, and the following sequence is exact.*

$$0 \to K_0(i) \to K_0(f) \to K_0({}^+f) \to 0.$$

Proof. By Proposition 2.5 and [16, Lemma 3.3], the map $K_0(i) \cong \text{Pic}\,(i) \to \text{Pic}\,(f)$ is an injection. Since it factors through $K_0(i) \to K_0(f)$, the latter map is an injection. Since $K_n(i) = 0$ for $n < 0$, again by Proposition 2.5, we are done. ☐

3. The $W(A)$-module structure on $NK_0(f)$ and $N\text{Pic}\,(f)$

In this section, we fix a ring homomorphism $f : A \to B$ and show that $NK_0(f)$ and $N\text{Pic}\,(f)$ are continuous modules over the ring $W(A)$ of big Witt vectors, so that

$$NK_1(A) \to NK_1(B) \xrightarrow{\partial} NK_0(f) \to NK_0(A) \to NK_0(B) \tag{3.1}$$

is a sequence of $W(A)$-modules. Recall that $(1+tA[[t]])^\times$ is the underlying abelian group of the ring $W(A)$; a $W(A)$-module is continuous if every element is killed by one of these ideals $(1 + t^nA[[t]])^\times$.

We first recall the continuous $W(R)$-module structure on $NK_*(A)$ when R is commutative and A is an R-algebra, due to Stienstra [18]. As $NK_*(A)$ is a continuous module, it suffices to describe multiplication by $(1-rt^m)$, $r \in R$. Setting $S = R[s]/(s^m - r)$, the inclusion $i : R \subset S$ induces a base change functor $i^* : \mathbf{P}(A[t]) \to \mathbf{P}(A \otimes_R S[t])$ and a transfer map $i_* : \mathbf{P}(A \otimes_R S[t]) \to \mathbf{P}(A[t])$. If σ denotes the S-algebra map $S[t] \to S[t]$, $\sigma(t) = st$, then the composition $F = i_*\sigma^*i^*$ is an additive self-functor of $\mathbf{P}(A[t])$. As noted in [26, 1.5], the composition $\mathbf{P}(A) \to \mathbf{P}(A[t]) \xrightarrow{F} \mathbf{P}(A[t]) \to \mathbf{P}(A)$ is $\otimes_R S$, so F induces multiplication by m on the summand $K_*(A)$ of $K_*(A[t])$; the restriction of F to $NK_*(A)$ is multiplication by $(1-rt^m)*$. If $A \to B$ is an R-algebra map, $NK_*(A) \to NK_*(B)$ is a homomorphism of continuous $W(R)$-modules.

We can adapt these formulas to define a multiplication by $(1-at^m)*$ on $K_0(f)$ and $NK_0(f)$ when $a \in A$: send $[P_1, \alpha, P_2]$ to $[F(P_1), F(\alpha), F(P_2)]$. It is clear from (2.1) that $(1-at^m)*$ is compatible with the exact sequence (3.1). A priori, though, the maps $(1-at^m)*$ do not fit together to make $NK_0(f)$ into a $W(A)$-module.

Proposition 3.2. *For any homomorphism $f : A \to B$, $NK_0(f)$ is a continuous $W(A)$-module, and* (3.1) *is an exact sequence of continuous $W(A)$-modules.*

Proof. As in the proof of Proposition 2.4, write $B = \mathbb{Z}[X]/I$, where $\mathbb{Z}[X]$ is a polynomial ring. Let R denote the pullback ring $A \times_B \mathbb{Z}[X]$, and write $\tilde{f} : R \to \mathbb{Z}[X]$ for the quotient map. Since $NK_*(\mathbb{Z}[X]) = 0$, we have $NK_n(\tilde{f}) \cong NK_n(R)$ for all n. Since $A = R/I$, Lemma 2.2 and [25] imply that the groups $NK_0(f) \cong NK_0(\tilde{f}) \cong NK_0(R)$ are continuous $W(R)$-modules.

Since $W(A) = W(R)/W(I)$, where $W(I) = 1 + tI[[t]]$, we are reduced to showing that $(1-rt^m)$ acts as zero on $K_0(f)$ whenever $r \in I$. When r is in the kernel I of $R \to A$, the ring $A \otimes_R S$ is just $A[s]/(s^m)$, so $(1-rt^m)$ and $(1-0t^m)$ act identically on $K_0(f[t])$. This shows that $(1-rt^m)$ acts as zero on $K_0(f)$ and proves that the action of $W(A)$ on $K_0(f)$ is well defined and continuous. ☐

Applying N to the determinant described in (2.3), we get an exact sequence

$$0 \to NSK_0(f) \to NK_0(f) \xrightarrow{\det} N\mathrm{Pic}\,(f) \to 0.$$

If $[P, \alpha, A[t]^n]$ is in $NK_0(f)$ then $\det[P, \alpha, A[t]^n] = [\det(P), \det(\alpha), A[t]]$.

Theorem 3.3. *For any homomorphism $f : A \to B$, $N\mathrm{Pic}\,(f)$ is a continuous $W(A)$-module, and* $\det : NK_0(f) \to N\mathrm{Pic}\,(f)$ *is a $W(A)$-module homomorphism.*

Proof. Since the group $NK_0(f)$ is a continuous $W(A)$-module by Proposition 3.2, it is enough to show that $NSK_0(f)$ is closed under multiplication by $W(A)$. Since every element of $W(A)$ can be written as $\prod_{m>0}(1-a_m t^m)$, with $a_m \in A$, and for any element u of $NK_0(f)$ there is an n so that $\prod_{m\geq n}(1-a_m t^m) * u = 0$, it is enough to show that $NSK_0(f)$ is closed under multiplication by $(1-at^m)$ for any $a \in A$ and $m \geq 1$.

It is enough to show that $F = i_*\sigma^* i^*$ sends $SK_0(f[t])$ to itself. We now modify the argument of [5, 4.1]. Fix $u = [P, \alpha, A[t]^n]$ in $SK_0(f[t])$; By Remark 1.0.1, $\det(u) = 0$ implies that $\det(P) = A[t]$ and $\det(\alpha) \in A$. By naturality of det, $\sigma^* i^*(u) = [P \otimes S, \alpha \otimes S, S[t]^n]$, $\det(P \otimes S) = S[t]$, $\det(\alpha \otimes S) \in S$ and $F(u) = [i_*(P \otimes S), i_*(\alpha \otimes S), A[t]^n]$. By Corollary 3.2 of [5] applied to $A[t] \subset S[t]$, $\det(i_*(P \otimes S)) = A[t]$ and $\det(\alpha \otimes S) = \det(\alpha)^m \in A$, so $\det(F(u)) = 0$. ☐

Corollary 3.4. *If* $\operatorname{char}(A) = p$ *then* $N\mathrm{Pic}\,(f)$ *is a* p*-group.*
If $\mathbb{Q} \subseteq A$ *then* $N\mathrm{Pic}\,(f)$ *is an* A*-module.*

Proof. Any continuous $W(A)$-module has these properties; see [25, 3.3]. ☐

4. SHEAF PROPERTIES OF $N\mathrm{Pic}\,(f)$

When $f : X \to S$ is a faithful affine morphism of schemes, let $\mathcal{I}(f)_{\mathrm{zar}}$ denote the Zariski sheaf $f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times$ on the category Sm/S of smooth schemes over S; by [16, 4.4], $\mathcal{I}(f)_{\mathrm{zar}}$ is also an étale sheaf, and $H^0_{\mathrm{et}}(S, \mathcal{I}(f)_{\mathrm{zar}}) = H^0_{\mathrm{nis}}(S, \mathcal{I}(f)_{\mathrm{zar}}) = \mathrm{Pic}\,(f)$. Our choice of Sm/S is dictated by the need to not only include étale extensions but be closed under product with $\mathbb{A}^1_S \xrightarrow{\pi} S$.

Let $\pi^*\mathcal{I}(f)$ denote the restriction of $\mathcal{I}(f)_{\mathrm{zar}}$ to $Sm/\mathbb{A}^1_S$ along π. Its direct image $\pi_*(\pi^*\mathcal{I}(f))$ is the Zariski sheaf $\mathcal{I}(f)_{\mathrm{zar}} \oplus \mathcal{NI}(f)$ on Sm/S, where $\mathcal{NI}(f)$ denotes the Zariski sheaf on Sm/S associated to the presheaf $U \mapsto N\mathrm{Pic}\,(f \times_S U)$.

Theorem 4.1. *Let* $f : X \to S$ *be a faithful affine morphism of schemes. Then* $\mathcal{NI}(f)$ *is an étale sheaf on* S*. Moreover,*

$$H^0_{\mathrm{et}}(S, \mathcal{NI}(f)) = H^0_{\mathrm{zar}}(S, \mathcal{NI}(f)) = N\mathrm{Pic}\,(f).$$

Proof. Since $\pi^*\mathcal{I}(f)$ is an étale sheaf on $\mathbb{A}^1_S$, its direct image $\pi_*\pi^*\mathcal{I}(f)$ is an étale sheaf on S; since $\pi_*\pi^*\mathcal{I}(f) \cong \mathcal{I}(f)_{\mathrm{zar}} \oplus \mathcal{NI}(f)$, $\mathcal{NI}(f)$ is also an étale sheaf. Since

$$H^0_{\mathrm{et}}(S, \pi_*\pi^*\mathcal{I}(f)) = H^0_{\mathrm{et}}(\mathbb{A}^1_S, \pi^*\mathcal{I}(f)) = \mathrm{Pic}\,(f[t]) = \mathrm{Pic}\,(f) \oplus N\mathrm{Pic}\,(f),$$

we see that $H^0_{\text{et}}(S, \mathcal{NI}(f)) = N\text{Pic}\,(f)$. If S_s is a Zariski local scheme of S, this shows that the stalk $\mathcal{NI}(f)_s = H^0_{\text{zar}}(S_s, \mathcal{NI}(f))$ equals $H^0_{\text{et}}(S_s, \mathcal{NI}(f))$. □

Example 4.2. If f is seminormal, the sheaf $\mathcal{NI}(f)$ vanishes and $N\text{Pic}\,(f) = 0$. This follows from Theorem 4.1 and [15, 1.5], which states that $N\text{Pic}\,(A, B) = 0$ when A is seminormal in B.

We now modify an argument of Vorst [22] and van der Kallen [21]. Suppose that $\text{Spec}(A) = \bigcup_{i=0}^r U_i$, where $U_i = \text{Spec}(A_{s_i})$. Given a presheaf F of abelian groups on $\text{Spec}(A)$, we write $C^\bullet(\{U_i\}, F)$ for the augmented Čech complex:

$$0 \to F(A) \xrightarrow{\epsilon} \prod_{i=0}^{r} F(A_{s_i}) \to \prod_{0\le i<j\le r} F(A_{s_is_j}) \to \cdots \to F(A_{s_0s_1\cdots s_r}) \to 0.$$

Given $s \in A$, we have an A-algebra map $\sigma : A[x] \to A[x]$ determined by $\sigma(x) = sx$. We write $NF(A)_{[s]}$ for the direct limit of $F(A[x]) \xrightarrow{\sigma} F(A[x]) \xrightarrow{\sigma} \cdots$. Suppose that for all $0 \le i_0 < i_1 < \cdots < i_p \le r$ and $j \le p$:

$$NF(A_{s_{i_0}\cdots s_{i_j}\cdots s_{i_p}}[x]) \cong NF(A_{s_{i_0}\cdots \hat{s}_{i_j}\cdots s_{i_p}}[x])_{[s_{i_j}]}. \tag{4.3}$$

In this situation, Vorst proved [22, 1.2] that the sequence $C^\bullet(\{U_i\}, NF)$ is always exact. He also proved that $F = NK_n$ satisfied (4.3), so that $C^\bullet(\{U_i\}, NK_n)$ is exact for all n. (See [22, 1.4] or [29, V.8.5]; the nonzerodivisor hypothesis is unnecessary by [20].)

Remark 4.4. It is easy to see (and follows from Vorst's result [22, 1.2]) that the functor $NU(A) = (A[t])^\times/A^\times$ satisfies (4.3). From the exact sequence of complexes

$$0 \to C^\bullet(\{U_i\}, NU) \to C^\bullet(\{U_i\}, NU(-\otimes_A B)) \to C^\bullet(\{U_i\}, NU(-\otimes_A B)/NU) \to 0$$

we see that $C^\bullet(\{U_i\}, F)$ is also exact for the functor $F(A_s) = NU(B_s)/NU(A_s)$.

Lemma 4.5. *$C^\bullet(\{U_i\}, N\text{Pic}\,)$ is always an exact sequence.*

Proof. By Theorem 4.2 of [27], given $s \in A$ we have $N\text{Pic}\,(A_s) \cong N\text{Pic}\,(A)_{[s]}$ and hence $N\text{Pic}\,(A_s[x]) \cong N\text{Pic}\,(A[x])_{[s]}$. This implies that $N\text{Pic}$ satisfies (4.3). Vorst's result shows that $C^\bullet(\{U_i\}, N\text{Pic}\,)$ is an exact sequence. □

We apply these considerations to the presheaf $N\text{Pic}\,(f) : U \mapsto N\text{Pic}\,(f|_U)$.

Lemma 4.6. *Suppose that* $\mathrm{Spec}(A) = \cup_{i=0}^{n} U_i$*, where* $U_i = \mathrm{Spec}(A_{s_i})$*. If* $f : A \hookrightarrow B$ *is a ring extension, the complex* $C^\bullet(\{U_i\}, N\mathrm{Pic}\,(f))$ *is exact.*

$$0 \to N\mathrm{Pic}\,(A,B) \to \prod_{i=0}^{n} N\mathrm{Pic}\,(A_{s_i}, B_{s_i}) \to \prod_{i_1<i_2} N\mathrm{Pic}\,(A_{s_{i_1}s_{i_2}}, B_{s_{i_1}s_{i_2}}) \to \cdots$$

Proof. Let ${}^{+}\!A$ denote the subintegral closure of A in B, so ${}^{+}\!A$ is seminormal in B and we have $A \subset {}^{+}\!A \subset B$. By [14, Prop. 4.1], we have an exact sequence

$$1 \to N\mathrm{Pic}\,(A, {}^{+}\!A) \to N\mathrm{Pic}\,(A,B) \to N\mathrm{Pic}\,({}^{+}\!A, B) \to 1.$$

By Example 4.2, the third term vanishes and we have $N\mathrm{Pic}\,(A, {}^{+}\!A) \cong N\mathrm{Pic}\,(A,B)$. Thus we may assume that B is subintegral over A. In this case, Ischebeck proved [9, Prop. 7] that $N\mathrm{Pic}\,(A) \to N\mathrm{Pic}\,(B)$ is surjective. Now the result follows from Remark 4.4, Lemma 4.5 and the long exact cohomology sequences associated to

$$0 \to C^\bullet(\{U_i\}, F) \to C^\bullet(\{U_i\}, N\mathrm{Pic}\,(f)) \to C^\bullet(\{U_i\}, N\mathrm{Pic}\,(f)/F) \to 0,$$
$$0 \to C^\bullet(\{U_i\}, N\mathrm{Pic}\,(f)/F) \to C^\bullet(\{U_i\}, N\mathrm{Pic}\,) \to C^\bullet(\{f^{-1}(U_i)\}, N\mathrm{Pic}\,) \to 0. \quad \square$$

Theorem 4.7. *Let* $f : A \hookrightarrow B$ *be an extension of rings. Then:*

$$H^q_{\mathrm{et}}(\mathrm{Spec}(A), \mathcal{NI}) = \begin{cases} N\mathrm{Pic}\,(f) & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}$$

Proof. The case $q = 0$ is given by Theorem 4.1. By Lemma 4.6, the Čech cohomology groups $\check{\mathrm{H}}^q(\mathrm{Spec}(A), \mathcal{NI})$ vanish for $q > 0$. Using the Cartan criterion [11, III.2.17], $H^q_{\mathrm{et}}(\mathrm{Spec}(A), \mathcal{NI})$ equals $\check{\mathrm{H}}^q(\mathrm{Spec}(A), \mathcal{NI}) = 0$ for $q > 0$. ☐

Corollary 4.8. *Let* $f : X \to S$ *be a faithful affine morphism of schemes. Then*

$$H^*_{\mathrm{et}}(S, \mathcal{NI}) \cong H^*_{\mathrm{zar}}(S, \mathcal{NI}).$$

Proof. Consider the site change map $\tau : S_{\mathrm{et}} \to S_{\mathrm{zar}}$. Then by Theorem 4.7, the higher direct image sheaves $R^q\tau_*\mathcal{NI}$ vanish for $q > 0$. Therefore the Leray spectral sequence degenerates, yielding the result. ☐

Remark. More generally, if $f : X \to S$ is any morphism of schemes then $\mathcal{O}_S^\times$ may not inject into $f_*\mathcal{O}_X^\times$. In this case, if we interpret $f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times$ as the mapping cone of $\mathcal{O}_S^\times \to f_*\mathcal{O}_X^\times$ (a complex of Zariski sheaves) and use sheaf hypercohomology, then Theorem 4.1 remains valid. However, Theorem 4.7 may fail in this setting.

5. MODULE STRUCTURES ON $NK_n(f)$

Given an exact functor $F : \mathcal{P} \to \mathcal{Q}$, the relative K-theory groups $K_n(F)$ fit into an exact sequence

$$\cdots \xrightarrow{F} K_{n+1}\mathcal{Q} \xrightarrow{\partial} K_n(F) \to K_n\mathcal{P} \xrightarrow{F} K_n\mathcal{Q} \xrightarrow{\partial} \cdots$$

ending in $K_0\mathcal{Q} \xrightarrow{\partial} K_{-1}(F)$. Waldhausen showed that the $K_n(F)$ are the homotopy groups $\pi_{n+2}|wS_\bullet(S_\bullet F)|$ $(n \geq 0)$, where S_nF denotes the category of pairs

$$(P_*, Q_*) = (P_1 \rightarrowtail P_2 \rightarrowtail \cdots \rightarrowtail P_n, Q_0 \rightarrowtail Q_1 \rightarrowtail \cdots \rightarrowtail Q_n)$$

($P_i \in \mathcal{P}$ and $Q_j \in \mathcal{Q}$), together with choices of Q_i/Q_j for $i > j$, such that $F(P_*)$ is $Q_1/Q_0 \rightarrowtail \cdots \rightarrowtail Q_n/Q_0$. (See [23, 1.5.4–7] or [29, IV.8.5.3].)

Example 5.1. If A is a ring, we write $\mathbf{P}(A)$ for the category of finitely generated projective A-modules. Given a ring homomorphism $f : A \to B$, we have an exact functor $\mathbf{P}(f) : \mathbf{P}(A) \to \mathbf{P}(B)$; by abuse, we write $K_*(f)$ for $K_*\mathbf{P}(f)$. Writing $f[t]$ for $A[t] \to B[t]$, we have $K_*(f[t]) = K_*(f) \oplus NK_*(f)$. The Fundamental Theorem of K-theory easily extends to the relative setting, yielding

$$K_*(f[t, 1/t]) \cong K_*(f) \oplus NK_*(f) \oplus NK_*(f) \oplus K_{*-1}(f).$$

Let A be a commutative ring. As in [29], we write $\mathbf{End}(A)$ for the category of pairs (P, α), where P in $\mathbf{P}(A)$ and $P \overset{\alpha}{\to} P$ is an endomorphism, and write $\mathbf{Nil}(A)$ for the full subcategory of $\mathbf{End}(A)$ consisting of all (P, α) with α nilpotent. As pointed out in [29, II.7.4], $K_*\mathbf{End}(A) \cong K_*(A) \oplus \mathrm{End}_*(A)$ and $K_*\mathbf{Nil}(A) \cong K_*(A) \oplus \mathrm{Nil}_*(A)$, where $\mathrm{End}_*(A)$ is a graded-commutative ring and $\mathrm{Nil}_*(A)$ is a graded $\mathrm{End}_*(A)$-module. By naturality, the exact functors $\mathbf{Nil}(f) : \mathbf{Nil}(A) \to \mathbf{Nil}(B)$ yield relative groups $K_*\mathbf{Nil}(f) \cong K_*(f) \oplus \mathrm{Nil}_*(f)$.

The category $\mathbf{Nil}(A)$ is equivalent to the category $\mathbf{H}_{1,t}(A[t])$ of t-primary torsion $A[t]$-modules M with $pd_{A[t]}M = 1$. Specifically, if (P, ν) is in $\mathbf{Nil}(A)$, and we write P_ν for the $A[t]$-module P on which t acts as ν, then P_ν has projective dimension 1 over $A[t]$. The Fundamental Theorem ([29, V.8.2]) implies that $\mathrm{Nil}_n(A) \cong NK_{n+1}(A)$. We also have $K\mathbf{P}(A[t]) \cong K\mathbf{H}(A[t])$ (see e.g., [29, V.3.2]).

Proposition 5.2. *There is a natural isomorphism* $\mathrm{Nil}_n(f) \cong NK_{n+1}(f)$.

Proof. From the diagram of exact categories

$$\begin{tikzcd}
\mathbf{Nil}(A) \arrow[r, "\cong"] \arrow[d] & \mathbf{H}_{1,t}(A[t]) \arrow[r] & \mathbf{H}(A[t]) & \mathbf{P}(A[t]) \arrow[l, hook', "\cong"'] \arrow[r] \arrow[d] & \mathbf{P}(A[t,1/t]) \arrow[d] \\
\mathbf{Nil}(B) \arrow[r, "\cong"] & \mathbf{H}_{1,t}(B[t]) \arrow[r] & \mathbf{H}(B[t]) & \mathbf{P}(B[t]) \arrow[l, hook', "\cong"'] \arrow[r] & \mathbf{P}(B[t,1/t])
\end{tikzcd}$$

we get a fibration sequence of K-theory spectra

$$\begin{tikzcd}
K\mathbf{Nil}(A) \arrow[r] \arrow[d] & K(A[t]) \arrow[r] \arrow[d, "f[t]^*"] & K(A[t,1/t]) \arrow[d, "f[t,1/t]^*"] \\
K\mathbf{Nil}(B) \arrow[r] & K(B[t]) \arrow[r] & K(B[t,1/t]).
\end{tikzcd}$$

Taking vertical fibers, we see that there is a long exact sequence

$$K_{n+1}(f[t]) \to K_{n+1}(f[t,1/t]) \to K_n\mathbf{Nil}(f) \to K_n(f[t]) \to K_n(f[t,1/t]) \to$$

and (using Example 5.1) an isomorphism $\mathrm{Nil}_n(f) \cong NK_{n+1}(f)$. ☐

Lemma 5.3. *For any ring homomorphism* $f : A \to B$*,* $\mathrm{Nil}_*(f)$ *is a graded* $\mathrm{End}_*(A)$*-module.*

Proof. A typical object in the Waldhausen category $S_n\mathbf{Nil}(f)$ is a pair

$$(\mu_*, \nu_*) = ((M_1, \mu_1) \rightarrowtail \cdots (M_n, \mu_n), (N_0, \nu_0) \rightarrowtail \cdots (N_n, \nu_n)).$$

There is a pairing $\mathbf{End}(A) \times S.\mathbf{Nil}(f) \to S.\mathbf{Nil}(f)$ of simplicial Waldhausen categories, sending $(P, \alpha) \times (\mu_*, \nu_*)$ to

$$((P\otimes M_1, \alpha\otimes\mu_1) \rightarrowtail \cdots \rightarrowtail (P\otimes M_n, \alpha\otimes\mu_n), (P\otimes N_0, \alpha\otimes\nu_1) \rightarrowtail \cdots \rightarrowtail (P\otimes N_n, \alpha\otimes\nu_n)).$$

It induces a pairing $K_*\mathbf{End}(A) \otimes K_*\mathbf{Nil}(f) \to K_*\mathbf{Nil}(f)$. Since the tensor product $(\alpha\otimes\beta)\otimes\mu \cong \alpha\otimes(\beta\otimes\mu)$ is associative up to natural isomorphism, the two pairings

$$\mathbf{End}(A) \times \mathbf{End}(A) \times S.\mathbf{Nil}(f) \to S.\mathbf{Nil}(f)$$

agree up to natural isomorphism, making $K_*\mathbf{Nil}(f)$ a graded $K_*\mathbf{End}(A)$-module. In particular, $\mathrm{Nil}_*(f)$ is a graded module over $\mathrm{End}_*(A)$. ☐

Recall that the ring $W(A)$ of big Witt vectors has underlying abelian group $(1+tA[[t]])^\times$. Almkvist's theorem [29, II.7.4.3] states that $[P, \alpha] \mapsto \det(1-t\alpha)$ maps $\mathrm{End}_0(A)$ isomorphically onto the subring of $W(A)$ whose underlying abelian group consists of all quotients $f(t)/g(t)$ of polynomials in $1+tA[t]$. The intersection of the ring $\mathrm{End}_0(A)$ with the ideal $(1+t^mA[[t]])$ of $W(A)$ is the ideal $I_m = \{1+t^m(f/g)\}$

of $\mathrm{End}_0(A)$, and $\mathrm{End}_0(A)/I_m \cong W(A)/(1+t^m A[[t]])$. In particular, $W(A)$ is the completion of $\mathrm{End}_0(A)$ with respect to the t-adic filtration.

We say that an $\mathrm{End}_0(A)$-module M is *continuous* if for every $x \in M$ there is an m so that $I_m \cdot x = 0$. Thus every continuous $\mathrm{End}_0(A)$-module M is also continuous as a $W(A)$-module: for every $x \in M$ we have $(1+t^m A[[t]]) \cdot x = 0$ for some m.

The exact functors $F_n, V_n : \mathbf{Nil}(A) \to \mathbf{Nil}(A)$, defined by $F_n(P,\nu) = (P,\nu^n)$ and $V_n(Q,\nu) = (Q[t]/(t^n-\nu), t)$, commute with $\mathbf{Nil}(A) \to \mathbf{Nil}(B)$. Hence they induce exact endofunctors F_n, V_n on $S.\mathbf{Nil}(f)$ by $F_n(\mu_*,\nu_*) = (F_n(\mu_*), F_n(\nu_*))$ and $V_n(\mu_*,\nu_*) = (V_n(\mu_*), V_n(\nu_*))$. For $a \in A$ and $n > 0$, and ν in $\mathrm{Nil}_*(f)$, Almkvist's theorem associates $(1-at^n)$ to $V_n([A,a]-[A,0])$ and yields the product formula

$$(1-at^n) * \nu = V_n([A,a]-[A,0]) * \nu. \tag{5.4}$$

Stienstra proved in [18, 19] that the $\mathrm{Nil}_n(A)$ are continuous $\mathrm{End}_0(A)$-modules, and hence $W(A)$-modules. The key step [18, 2.12] was showing that the projection formula holds:

$$(V_n\alpha) * \nu = V_n(\alpha * F_n(\nu)) \quad \text{for} \quad \alpha \in \mathrm{End}_0(A) \text{ and } \nu \in \mathrm{Nil}_*(A).$$

Here is the corresponding projection formula in the relative setting; we will postpone its proof in order to get to the main result.

Lemma 5.5. *For all $\alpha \in \mathrm{End}_0(A)$ and $\beta \in \mathrm{Nil}_*(f)$,*

$$(V_n\alpha) * \beta = V_n(\alpha * F_n(\beta)).$$

Theorem 5.6. *Let $f : A \to B$ be a ring map. Then the product* (5.4) *makes $\mathrm{Nil}_n(f) \cong NK_{n+1}(f)$ into a continuous $W(A)$-module for every integer n.*

Proof. For each $m > 0$, let $\mathbf{Nil}^m(A)$ denote the exact subcategory of all (P,ν) in $\mathbf{Nil}(A)$ such that $\nu^m = 0$. Thus we have relative groups $K_*\mathbf{Nil}^m(f)$ associated to $K_*\mathbf{Nil}^m(A) \to K_*\mathbf{Nil}^m(B)$, and $K_*\mathbf{Nil}(f)$ is the direct limit of the $K_*\mathbf{Nil}^m(f)$.

Suppose that $n \geq m$. Clearly, F_n acts as zero on $\mathbf{Nil}^m(f)$. By the projection formula 5.5, $V_n(\alpha)$ acts as zero on the image $\mathrm{Nil}^m_*(f)$ of $K_*\mathbf{Nil}^m(f) \to K_*\mathbf{Nil}(f) \to \mathrm{Nil}_*(f)$. By (5.4), $(1-at^n)$ acts as zero on $\mathrm{Nil}^m_*(f)$. Since $\mathrm{Nil}_*(f)$ is the union of the $\mathrm{Nil}^m_*(f)$, for any $\beta \in \mathrm{Nil}_*(f)$ there is an m such that $(1-at^n) \cdot \beta = 0$ for all $n \geq m$ and $a \in A$. This shows that $\mathrm{Nil}_*(f)$ is a continuous $\mathrm{End}_0(A)$-module, and hence a continuous $W(A)$-module. □

Proof of Lemma 5.5. Following Stienstra [18, §6], set $R = \mathbb{Z}[y_1, y_2]$, and set $\mathbf{E} = \mathbf{End}(R; S_6)$, where S_6 is the multiplicative subset of $R[x]$ generated by x and $x^n - y_1^n y_2$. As pointed out in *loc. cit.*, there is a multi-exact pairing

$$\Theta : \mathbf{E} \times \mathbf{End}(A) \times \mathbf{Nil}(B) \to \mathbf{Nil}(B)$$

sending (E, ω), (P, α) and (N, ν) to $(E \otimes_R (P \otimes_A N), \omega \otimes 1)$, where $P \otimes_A N$ is ragarded as an R-module by letting y_1 and y_2 act as $\alpha \otimes 1$ and $1 \otimes \nu$. As this pairing is natural in B, we may replace $\mathbf{Nil}(B)$ by $S.\mathbf{Nil}(f)$. This yields (among other things) a product

$$\Theta_* : K_0\mathbf{E} \otimes \mathrm{End}_0(A) \otimes \mathrm{Nil}_*(f) \to \mathrm{Nil}_*(f).$$

Stienstra proves in *loc. cit.* that the elements $[R^n, \omega]$ and $[R^n, \omega']$ agree in $K_0\mathbf{E}$, where

$$\omega = \begin{pmatrix} 0 & & & y_1^n y_2 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{pmatrix} \quad \text{and} \quad \omega' = \begin{pmatrix} 0 & & & y_1 y_2 \\ y_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & y_1 & 0 \end{pmatrix}.$$

Therefore the two maps

$$\Theta_*([R^n, \omega], -), \Theta_*([R^n, \omega'], -) : \mathrm{End}_0(A) \otimes \mathrm{Nil}_*(f) \to \mathrm{Nil}_*(f)$$

agree. Stienstra also observes that these maps send $[P, \alpha] \otimes \beta$ to $V_n(\alpha * F_n\beta)$ and $(V_n\alpha) * \beta$, respectively; see also [19, p.14]. The projection formula follows. ☐

6. Negative relative K-theory

Let $f : X \to S$ be a morphism of schemes. Then we have a long exact sequence of negative K-groups, part of which is:

$$(6.1) \qquad \cdots \to K_{-d}(f) \to K_{-d}(S) \to K_{-d}(X) \to K_{-d-1}(f) \to K_{-d-1}(S) \to \cdots .$$

Theorem 6.2. *Let $f : X \to S$ be a morphism of d-dimensional schemes, essentially of finite type over a field k of characteristic 0. Then for all $r > 0$:*

(1) $K_n(f) = K_n(f \times \mathbb{A}^r) = 0$ *for* $n \le -d - 2$.
(2) $K_{-d-1}(f) \cong K_{-d-1}(f \times \mathbb{A}^r)$ *("f is K_{-d-1}-regular.")*
(3) *If f is a finite map then* $K_{-d-1}(f) \cong H^d_{\mathrm{cdh}}(S, f_*\mathbb{Z}/\mathbb{Z})$.

Proof. By Corollary 5.9 and Theorem 6.2 of [4], $K_n(S) \cong K_n(S \times \mathbb{A}^r)$ for all $n \le -d$, $K_n(S) = 0$ for $n < -d$ and $K_{-d}(S) \cong H^d_{\mathrm{cdh}}(S, \mathbb{Z})$; the analogous assertions hold for X. The exact sequence (6.1) for S and $S \times \mathbb{A}^r$ implies the first two assertions. For (3), we have a distinguished triangle cdh sheaves on S,

$$\mathbb{Z} \to f_*\mathbb{Z} \to f_*\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}[1].$$

Since the cdh-cohomological dimension of S is at most d, $H^{d+1}_{\mathrm{cdh}}(S, \mathbb{Z}) = 0$. Thus the long exact sequence on cdh-cohomology ends in

$$\to H^d_{\mathrm{cdh}}(S, \mathbb{Z}) \to H^d_{\mathrm{cdh}}(S, f_*\mathbb{Z}) \to H^d_{\mathrm{cdh}}(S, f_*\mathbb{Z}/\mathbb{Z}) \to 0.$$

Since f is finite, we have $H^*_{\mathrm{cdh}}(S, f_*\mathbb{Z}) \xrightarrow{\cong} H^*_{\mathrm{cdh}}(X, \mathbb{Z})$; assertion (3) follows. □

Remark 6.2.1. Let k be a perfect field of characteristic p. Kerz and Strunk have shown in [10] that $K_n(S)$ is a p-primary torsion group for $n < -d$. Then Theorem 6.2 holds for k up to p-torsion.

If in addition k is a perfect field, over which weak resolution of singularities holds, then Theorem 6.2(1,2) holds for k. This also follows from [10]; if strong resolution of singularities holds, (1) also follows from the Geisser–Hesselholt theorem in [6] that $K_n(S) = 0$ for $n < -d$.

When S is a curve, not necessarily defined over $\mathbb{Q}$, we have a similar result.

Theorem 6.3. *Let $f : X \to S$ be a finite map of 1-dimensional noetherian schemes. Then $K_{-1}(f)$ fits into an exact sequence*

$$0 \to H^1_{\mathrm{nis}}(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times) \to K_{-1}(f) \to H^0_{\mathrm{nis}}(S, f_*\mathbb{Z}/\mathbb{Z}) \to 0.$$

In addition, $K_{-2}(f) \cong H^1_{\mathrm{nis}}(S, f_*\mathbb{Z}/\mathbb{Z})$ and $K_n(f) = 0$ for $n < -2$.

Proof. By Thomason-Trobaugh [20, 10.8], we have a spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{nis}}(S, \mathcal{K}_{-q}(f)) \Rightarrow K_{-p-q}(f),$$

where $\mathcal{K}_n(f)$ is the Nisnevich sheafification of the presheaf $U \mapsto K_n(U, f^{-1}U)$. Each stalk $\mathcal{K}_n(f)$ is $K_n(A, B)$, where A is a hensel local ring of dimension ≤ 1. By Lemma 6.4 below, we have

$$\mathcal{K}_n(f) = \begin{cases} 0 & \text{if } n \le -2 \\ f_*\mathbb{Z}/\mathbb{Z} & \text{if } n = -1 \\ f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times & \text{if } n = 0. \end{cases}$$

Since $cd_{\text{nis}}(S) \leq 1$, $E_2^{p,q} \neq 0$ only for $p = 0, 1$ and $q \leq 1$. Thus the spectral sequence degenerates to yield $K_{-2}(f) \cong H^1_{\text{nis}}(S, f_*\mathbb{Z}/Z)$ and $K_n(f) = 0$ for $n < -2$. ☐

Lemma 6.4. *Let A be a 1-dimensional hensel local ring and $f : A \hookrightarrow B$ a finite extension. If B has r components, then*

$$K_0(f) \cong B^\times/A^\times, \quad K_{-1}(f) \cong \mathbb{Z}^{r-1} \quad \textit{and} \quad K_n(f) = 0 \textit{ for } n < -1.$$

Proof. Since B is a finite A-algebra, B is a finite product of r hensel local rings. By [24, 2.8], $K_n(A) = K_n(B) = 0$ for $n < -1$. By a result of Drinfeld [29, III.4.4.3], we have $K_{-1}(A) = K_{-1}(B) = 0$. The result now follows from (6.1). ☐

Remark 6.5. A necessary condition for $K_{-1}(f) = 0$ is that the ring extension $f : A \hookrightarrow B$ is *anodal*, i.e., if every $b \in B$ such that $(b^2 - b) \in A$ and $(b^3 - b^2) \in A$ belongs to A. (See [27, 3.1].) This is because (2.3) induces a surjection $L\det : K_{-1}(f) \to L\text{Pic}\,(f)$, and we showed in [16] that $L\text{Pic}\,(f) = 0$ implies that $A \subset B$ is anodal. The converse does not hold, even if f is a birational extension of domains, as Example 3.5 in [27] shows.

Example 6.6. Here is an example to show why we assume S affine in Proposition 2.5. For each n, the scheme $S = \mathbb{P}^1_k$ has a sheaf of algebras $\mathcal{O}_B = \mathcal{O}_S \oplus \mathcal{O}(n)$ with $\mathcal{O}(n)$ a square-zero ideal; fix $n \leq -2$ and set $X = \text{Spec}(\mathcal{O}_B)$. Then $H = H^1(\mathbb{P}^1, \mathcal{O}(n))$ is nonzero and $\text{Pic}\,(X) = \text{Pic}\,(S) \oplus H$, $K_0(X) \cong K_0(S) \oplus H$. In particular, $K_{-1}(f) = H \neq 0$.

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