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The Higher *K*-Theory of Real Curves

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Abstract. If X is a smooth curve defined over the real numbers $\mathbb{R}$, we show that $K_n(X)$ is the sum of a divisible group and a finite elementary Abelian 2-group when $n \geqslant 2$. We determine the torsion subgroup of $K_n(X)$, which is a finite sum of copies of $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Z}/2$, only depending on the topological invariants of $X(\mathbb{R})$ and $X(\mathbb{C})$, and show that (for $n \geqslant 2$) these torsion subgroups are periodic of order 8.

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Introduction

Let X be a smooth curve defined over the real numbers $\mathbb{R}$. The underlying space $X(\mathbb{R})$ of its real points, equipped with the Euclidean topology, is a one-dimensional manifold (possibly empty). Classical calculations dating to 1883 ([Whd]; see Section 1) clearly show the relation between the topological invariants of $X(\mathbb{R})$ and the Picard group (and hence K_0) of X.

In this paper we shall show that there is a strong connection between the higher algebraic *K*-theory of X (see [Q]) and the topological invariants of $X(\mathbb{R})$. The link between these will be forged from recent developments in motivic cohomology.

When X is projective and $X(\mathbb{R})$ is not empty, we have the following satisfactory answer.

MAIN THEOREM 0.1. *Let X be an irreducible, smooth projective curve over $\mathbb{R}$ such that $X(\mathbb{R})$ has $\nu > 0$ components. Then the groups $K_n(X)$ are the direct sum of a divisible group and an elementary Abelian 2-group for $n \neq 0$. Moreover, the torsion subgroups $K_n(X)_{\mathrm{tors}}$ are periodic of period 8. In fact, if X has genus g then* *for $n \geqslant 0$:*

$$K_n(X)_{\text{tors}} \cong \begin{cases} (\mathbb{Q}/\mathbb{Z})^g \oplus (\mathbb{Z}/2)^{\nu-1} & n \equiv 0 \pmod 8, \\ (\mathbb{Z}/2)^{1+\nu} & n \equiv 1 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z})^g \oplus (\mathbb{Z}/2)^{\nu+1} & n \equiv 2 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z})^2 \oplus (\mathbb{Z}/2)^{\nu-1} & n \equiv 3 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z})^g & n \equiv 4 \pmod 8, \\ 0 & n \equiv 5 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z})^g & n \equiv 6 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z})^2 & n \equiv 7 \pmod 8. \end{cases}$$

Note that $K_n(X)$ is divisible for $n \equiv 4, 5, 6, 7 \pmod 8$.

When $X(\mathbb{R}) \neq \emptyset$ and X is affine, the results are similar to the Main Theorem 0.1 but depend upon the real and complex points at infinity. They are stated in Theorem 7.2.

When X is smooth and $X(\mathbb{R}) = \emptyset$, we must distinguish two cases: if X is defined over $\mathbb{C}$ then the torsion is 2-periodic and was given in [PW2]; see 3.1 below. Any component of X which is not defined over $\mathbb{C}$ must be geometrically connected; in this case the torsion is 4-periodic and is recorded in Theorem 6.4. The following theorem extracts the projective case from 6.4, for comparison with 0.1.

THEOREM 0.2. *Let X be a geometrically connected smooth projective curve over $\mathbb{R}$ such that $X(\mathbb{R}) = \emptyset$. Then the groups $K_n(X)$ are the direct sum of a divisible group and an elementary Abelian 2-group for $n \neq 0$.*

Moreover, the torsion subgroups $K_n(X)_{\text{tors}}$ are periodic of period 4. In fact, if X has genus g then for $n \geqslant 0$:

$$K_n(X)_{\text{tors}} \cong \begin{cases} (\mathbb{Q}/\mathbb{Z})^g & n \equiv 0 \pmod 4, \\ (\mathbb{Z}/2) & n \equiv 1 \pmod 4, \\ (\mathbb{Q}/\mathbb{Z})^g \oplus (\mathbb{Z}/2) & n \equiv 2 \pmod 4, \\ (\mathbb{Q}/\mathbb{Z})^2 & n \equiv 3 \pmod 4. \end{cases}$$

When $X(\mathbb{R}) = \emptyset$ and X is affine, the results are similar to Theorem 0.2, and are stated in Theorem 6.4. They depend upon the number r_2 of (complex) points at infinity, except that the periodicity in the torsion starts at $n = 1$ (only because the number of $\mathbb{Q}/\mathbb{Z}$ summands in the Picard group can be fewer than the expected number $g + r_2 - 1$) and the group $K_1(X)$ need not be divisible-by-finite (because the units may have a finitely generated free Abelian factor).

This paper is organized as follows. After recalling the classical story in Section 1, we use Comessatti's theorem to state our transfer argument in Section 2. We use this in Section 3 to deduce the divisible torsion in $K_n(X)$ from the calculations in [PW2]. We use the methods of Merkurjev–Suslin to describe $K_2(X)$ in Section 4, modulo a minor extension problem which is solved in Section 6.

In Section 5 we describe the motivic cohomology of real curves, using results of Suslin and Voevodsky. These are the E_2-terms of a spectral sequence converging to K-theory. We settle the 'no loops' case of our Main Theorem in Section 6. This includes the case $X(\mathbb{R}) = \emptyset$ (Theorems 6.4 and 0.2) as well as the case $X(\mathbb{R}) \neq \emptyset$ (Theorem 6.8).

In Section 7 we study the general case, where loops are present. This includes affine curves (Theorem 7.2) and projective curves (Theorem 0.1). The proof is based upon certain explicit calculations of the E_3-terms of the above-mentioned spectral sequence (Claim 7.6); in order to make the proof easier to follow, we have isolated the proof of 7.6 in the final section (Section 8).

After this paper was written, Karoubi noticed the similarity between our calculations and the KR-theory of M, the manifold with involution underlying the complex curve $X_{\mathbb{C}}$. Recall that if M is a space with involution then Atiyah defined an equivariant cohomology theory $KR^*(M)$ in [A], using the notion of a Real vector bundle on M. If M arises from a real algebraic variety X, Atiyah observed that there is a natural map $K_0(X) \to KR^0(M)$, and Karoubi [K] extended it to maps $K_n(X) \to KR^{-n}(M)$ for all $n \geqslant 0$. Based upon the calculations in this paper, Karoubi and Weibel [KW] have shown that for any smooth real variety X the maps $K_n(X; \mathbb{Z}/2^\nu) \to KR^{-n}(M; \mathbb{Z}/2^\nu)$ are isomorphisms for all $n \geqslant \dim(X)$.

NOTATION. By a *real variety* X we mean a reduced quasi-projective scheme over $\mathbb{R}$ which is not defined over $\mathbb{C}$, i.e., such that the $\mathbb{R}$-algebra $H^0(X, \mathcal{O}_X)$ does not contain $\mathbb{C}$. Note that an irreducible real variety is *geometrically irreducible*. By a *real curve* we mean a real variety of dimension one.

If X is a real curve, its real locus $X(\mathbb{R})$ is a one-dimensional topological space; if X is smooth then each component of $X(\mathbb{R})$ is homeomorphic to either an open line or a circle. We write $\nu(X)$ for the number of path components of its real locus $X(\mathbb{R})$. The number of *loops*, $\lambda(X)$, is the number of circles, i.e., the dimension of $H^1(X(\mathbb{R}))$.

We will write σ for complex conjugation and $G = \{1, \sigma\}$ for the Galois group of $\mathbb{C}/\mathbb{R}$. Note that if A is a G-module, then $(A_{\text{tors}})^G = (A^G)_{\text{tors}}$, so the notation A^G_{tors} is unambigous.

We shall write $\mathbb{Z}/m(i)$ for the motivic complex of sheaves; by [SV] the restriction of $\mathbb{Z}/m(i)$ to the étale topology is quasi-isomorphic to the sheaf $\mu_m^{\otimes i}$.

1. K_0 and K_1 of Real Curves

If X is a connected real curve, it is well known that $K_0(X) \cong \mathbb{Z} \oplus \mathrm{Pic}(X)$, where $\mathrm{Pic}(X)$ is the Picard group (see [SGA6]). In this section we describe how the topological invariants of the underlying space $X(\mathbb{R})$ are reflected in the structure of $\mathrm{Pic}(X)$, and also in the structure of the group $K_1(X)$. For convenience, we begin by defining the basic invariants we shall need. We do not assume that X is smooth or proper.

DEFINITION 1.0. If X is any real curve (not necessarily smooth or proper), let $\nu = \nu(X)$ denote the number of path-connected components of the space $X(\mathbb{R})$, and let $\lambda = \lambda(X)$ denote the number of *loops* of $X(\mathbb{R})$, i.e., the dimension of $H^1(X(\mathbb{R}), \mathbb{Z}/2)$. Let $E(X)$ denote the number of irreducible algebraic components X_i of X which are proper and have no real points, i.e., $X_i(\mathbb{R}) = \emptyset$.

Historically, the first result in this subject is due Weichold and can be found in his 1882 Leipzig thesis [Whd]. We state it here in the form proved later on by Klein (1892) and other authors.

WEICHOLD'S THEOREM 1.1. *Let X be an irreducible, smooth projective real curve, of genus g. Then $X(\mathbb{R})$ is the disjoint union of $\nu = \lambda$ circles, and one has*

$$\mathrm{Pic}(X) \cong \mathbb{Z} \times (\mathbb{R}/\mathbb{Z})^g \times (\mathbb{Z}/2)^{\lambda+E-1}.$$

Thus $\mathrm{Pic}(X) \otimes \mathbb{Z}/2 \cong (\mathbb{Z}/2)^\lambda$ *if* $\nu \neq 0$, *while* $\mathrm{Pic}(X) \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$ *if* $\nu = 0$.

For comparison, recall that $\mathrm{Pic}(X_{\mathbb{C}}) \cong \mathbb{Z} \times (\mathbb{R}/\mathbb{Z})^{2g}$. Weichold obtained his result by computing the effect of complex conjugation on the period matrix for the pairing

$$H_1(X(\mathbb{C}), \mathbb{Z}) \times H^0(X(\mathbb{C}), \Omega^1_X) \to \mathbb{C}.$$

If we represent an element of H_1 as a path γ in the underlying complex space $X(\mathbb{C})$, then its pairing with a global 1-form ω is the path integral $\int_\gamma \omega$.

Weichold's theorem has been generalized in several directions. Here is one such result.

PROPOSITION 1.2 ([PW1, 1.10]). *Let X be a real curve. Then*

$$\mathrm{Pic}(X) \otimes \mathbb{Z}/2 \cong (\mathbb{Z}/2)^{\lambda(X)+E(X)}.$$

Refinement 1.2.1. If X is an affine curve, $\mathrm{Pic}(X)$ is the direct sum of $(\mathbb{Z}/2)^\lambda$ and a divisible group D; this follows from [PW1, 1.2] (or a transfer argument like 2.5 below). However, determining the divisible torsion D_{tors} in $\mathrm{Pic}(X)$ is a delicate arithmetic question, related to the problem of determining the rank of the group $U(X)$ of units; if X is a connected real curve then $U(X)/\mathbb{R}^\times$ is a free Abelian group. We refer the reader to Monnier's paper [Mo] for recent work in this area.

It turns out that the good invariant is the étale cohomology group $H^1_{\mathrm{et}}(X, \mu_\infty)$, which is an extension of $\mathrm{Pic}(X)_{\mathrm{tors}}$ by the divisible group $U(X) \otimes \mathbb{Q}/\mathbb{Z}$. Here μ_∞ is the étale sheaf of all roots of unity on X.

A related invariant is the group $H^2_{\mathrm{et}}(X, \mu_\infty)$, which is an extension of the Brauer group $\mathrm{Br}(X)$ by $\mathrm{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z}$. It is classical [Witt][DK] that $\mathrm{Br}(X)$ is isomorphic to the group $(\mathbb{Z}/2)^{\nu(X)}$ of maps from the set $\pi_0 X(\mathbb{R})$ of path components of $X(\mathbb{R})$ to $\mathbb{Z}/2$.

If X is an irreducible, smooth projective curve, then $U(X) = \mathbb{R}^\times$. By Kummer theory for $\mu_m \subset \mathbb{G}_m$, we have $H^0_{\mathrm{et}}(X, \mu_\infty) \cong \mathbb{Z}/2$ and the group $H^1_{\mathrm{et}}(X, \mu_\infty)$ is isomorphic to $\mathrm{Pic}(X)_{\mathrm{tors}} \cong (\mathbb{Q}/\mathbb{Z})^g \oplus (\mathbb{Z}/2)^{\lambda+E-1}$, the latter isomorphism being part of Weichold's Theorem. The affine case is somewhat different.

THEOREM 1.3. *Let X be an irreducible, smooth affine real curve, obtained by removing r closed points from a smooth projective curve $\bar{X}$ of genus g. If $\nu = \nu(X)$ and $\lambda = \lambda(X)$ then $H^0_{et}(X, \mu_\infty) \cong H^0_{et}(\bar{X}, \mu_\infty) \cong \mathbb{Z}/2$ and there are isomorphisms*:

$$H^1_{et}(X, \mu_\infty) \cong (\mathbb{Q}/\mathbb{Z})^{g+r-1} \oplus (\mathbb{Z}/2)^\lambda; \qquad H^2_{et}(X, \mu_\infty) \cong (\mathbb{Z}/2)^\nu;$$
$$H^1_{et}(\bar{X}, \mu_\infty) \cong (\mathbb{Q}/\mathbb{Z})^{g} \oplus (\mathbb{Z}/2)^{\lambda+E(\bar{X})-1}; \quad H^2_{et}(\bar{X}, \mu_\infty) \cong (\mathbb{Q}/\mathbb{Z}) \oplus (\mathbb{Z}/2)^\nu.$$

Proof. We have already handled $\bar{X}$, so we may suppose that X is an affine real curve. The structure of $\mathrm{Pic}(X)$ implies that $\mathrm{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z} = 0$. Hence, the group $H^2_{et}(X, \mu_\infty)$ is isomorphic to the Brauer group $\mathrm{Br}(X) \cong (\mathbb{Z}/2)^\nu$. Similarly, because $U(X)$ doesn't contain $\mathbb{C}^\times$, we have $H^0_{et}(X, \mu_\infty) = \mu_2 \cong \mathbb{Z}/2$.

Suppose next that $r = 1$, so that $U(X) = \mathbb{R}^\times$ and $H^1_{et}(X, \mu_\infty) \cong \mathrm{Pic}(X)_{\mathrm{tors}}$. It follows easily from Weichold's Theorem 1.1 and 1.2 that $\mathrm{Pic}(X) \cong (\mathbb{Z}/2)^\lambda \oplus (\mathbb{R}/\mathbb{Z})^g$ (see [PW1, 1.2]). The result follows.

If $r > 1$, set $X' = \bar{X} - \{x_1\}$. Then the Gysin sequence [Mi, p. 244] for $X \subset X'$ is

$$0 \to H^1_{et}(X', \mu_\infty) \to H^1_{et}(X, \mu_\infty) \to (\mathbb{Q}/\mathbb{Z})^{r-1} \xrightarrow{\tau} H^2_{et}(X', \mu_\infty) \to H^2_{et}(X, \mu_\infty).$$

Now $H^1_{et}(X, \mu_\infty)$ is the sum of a divisible group and $(\mathbb{Z}/2)^\lambda$ by 1.2.1. Thus, it suffices to show that the map τ is zero. Since $X(\mathbb{R})$ is obtained from $X'(\mathbb{R})$ by removing points, the map $\pi_0 X(\mathbb{R}) \to \pi_0 X'(\mathbb{R})$ is onto. Hence, $\mathrm{Br}(X') \to \mathrm{Br}(X)$ is an injection, whence $\tau = 0$. □

COROLLARY 1.4. *Let X be an irreducible, smooth real curve. Then there are isomorphisms (with g, r, ν, λ and $E = E(X)$ as in 1.3, and F the function field of X)*:

$$H^1_{et}(X, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{g+r+\lambda+E};$$
$$H^2_{et}(X, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{\nu+\lambda+E};$$
$$H^n_{et}(X, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{\nu+\lambda}, \quad n \geqslant 3.$$

Moreover, the image of $H^n_{et}(X, \mathbb{Z}/2)$ in $H^n_{et}(F, \mathbb{Z}/2)$ is $(\mathbb{Z}/2)^\nu$ for all $n \geqslant 2$.

Proof. The H^1 and H^2 calculations follow from the Kummer sequence for $\mathbb{Z}/2 \subset \mu_\infty$ and Theorem 1.3, since $H^0_{et}(X, \mu_\infty) \cong \mathbb{Z}/2$. The description of H^n for $n \geqslant 2$ is given in [PW1, 1.11] or [CT-S, 2.3.1], and is essentially due to Cox [Cox]. (The last assertion is a particular case of a more general result in [CT-P].) □

Variation 1.4.1. Let $NH^n(X, \mathbb{Z}/2)$ denote the kernel of $H^n_{et}(X, \mathbb{Z}/2) \to H^n_{et}(F, \mathbb{Z}/2)$, where F is the function field of X. That is, $NH^n(X, \mathbb{Z}/2)$ is the union over all open $U \subset X$ of the kernels of $H^n_{et}(X, \mathbb{Z}/2) \to H^n_{et}(U, \mathbb{Z}/2)$.

Since $H^1_{et}(X)$ injects into $H^1_{et}(F)$, $NH^1(X) = 0$. For $n = 2$, it is well known that the image of $H^2_{et}(X, \mathbb{Z}/2) \to H^2_{et}(F, \mathbb{Z}/2) = {}_2\mathrm{Br}(F)$ is the Brauer group

$\mathrm{Br}(X) \cong (\mathbb{Z}/2)^{\nu}$. Hence $NH^2(X, \mathbb{Z}/2) \cong \mathrm{Pic}(X)/2$, a group which is $(\mathbb{Z}/2)^{\lambda+E}$ by 1.2.

For $n > 2$ we claim that $NH^n(X, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{\lambda}$. This is clear from the naturality of 1.4 if $\lambda = 0$. The case $\lambda(X) > 0$ reduces to this because there is an open $U \subset X$ with $\nu(U) = \nu(X)$ but $\lambda(U) = 0$.

LEMMA 1.5. *Let X be a smooth real curve with* $X(\mathbb{R}) \neq \emptyset$. *Multiplication by* $[-1] \in H^1_{\mathrm{et}}(\mathbb{R}, \mathbb{Z}/2)$ *induces the map* $H^n_{\mathrm{et}}(X, \mathbb{Z}/2) \xrightarrow{[-1]} H^{n+1}_{\mathrm{et}}(X, \mathbb{Z}/2)$.

(1) *If* $n \geqslant 2$ *the map is an isomorphism;*
(2) *If* $n = 1$ *and X is affine, the map is a surjection;*
(3) *If* $n = 1$ *and X is projective, the cokernel of the map is* $\mathbb{Z}/2$.

Proof. Write $H^n(X)$ for $H^n_{\mathrm{et}}(X, \mathbb{Z}/2)$, and $[-1]$ for the map $H^n(X) \to H^{n+1}(X)$, $x \mapsto x \cup [-1]$. Colliot-Thélène and Scheiderer observed in [CT-S, (2.4)] that the map $[-1]$ fits into a long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(X) \xrightarrow{\cong} H^0(X_{\mathbb{C}}) \xrightarrow{0} H^0(X) \xrightarrow{[-1]} H^1(X) \longrightarrow H^1(X_{\mathbb{C}}) \\ \longrightarrow H^1(X) \xrightarrow{[-1]} H^2(X) \longrightarrow H^2(X_{\mathbb{C}}) \longrightarrow H^2(X) \\ \xrightarrow{[-1]} H^3(X) \longrightarrow H^3(X_{\mathbb{C}}) \cdots \end{aligned} \tag{1.5.1}$$

If X is affine then $H^n(X_{\mathbb{C}}) = 0$ for all $n > 1$, and the result follows. If X is projective, this argument works for $n > 2$, and shows that $H^2(X) \to H^3(X)$ is onto. Since these vector spaces have the same dimension, this map is an isomorphism. Hence, the preceeding part of (1.5.1) is exact, namely $H^1(X) \to H^2(X) \to H^2(X_{\mathbb{C}}) \to 0$. This implies (3) since $H^2(X_{\mathbb{C}}) \cong \mathbb{Z}/2$. □

Remark 1.5.2. Lemma 1.5 may also be easily determined from Corollary 1.4, by counting dimensions in (1.5.1). Also, part (3) follows from [CT-S, 2.2.2].

COROLLARY 1.6. *If* $E = 0$, *multiplication by the generator* $[-1]^n$ *of* $H^n_{\mathrm{et}}(\mathbb{R}, \mathbb{Z}/2) = \mathbb{Z}/2$ *induces an isomorphism between* $\mathrm{Pic}(X)/2 \cong NH^2(X, \mathbb{Z}/2)$ *and* $NH^{n+2}(X, \mathbb{Z}/2)$ *for all* $n \geqslant 0$.

Proof. As $[-1]^n$ is a power of $[-1]$, this is immediate from Lemma 1.5. □

2. Comessatti's Theorem

In the 1920's, Comessatti used the 'pseudonormal' matrix of periods of real integrals to define what we now call the *Comessatti characters* of the cohomology of a real Abelian variety A. He computed these characters for the Albanese variety $\mathrm{Alb}(X_{\mathbb{C}})$ and its dual, the Picard variety $\mathrm{Pic}^0(X_{\mathbb{C}})$ of a real variety X. In particular his results apply to the Jacobian variety of a real curve. A description of Comessatti's results and their translation into modern language can be found in [CP, 2.5-6].

Here is a modern formulation of Comessatti's key result. We omit its proof, which nowadays is routine; cf. [CP, 2.5.3]. For any Abelian group A and any integer

i, it is convenient to write $A(i)$ for the G-module which is A with σ acting as multiplication by $(-1)^i$. We write $A[G]$ for its induced G-module, which is the group $A \oplus A$ with σ acting as the interchange $\sigma(a, b) = (b, a)$.

COMESSATTI'S THEOREM 2.1. *Let σ be an involution on a free Abelian group H. Then H decomposes as a module over the group $G = \{1, \sigma\}$ as*

$$H \cong \mathbb{Z}(0)^a \oplus \mathbb{Z}(1)^b \oplus \mathbb{Z}[G]^c, \quad \textit{where} \dim(H) = a + b + 2c.$$

The number c is called the *Comessatti character* of H.

COROLLARY 2.2. *Set $D = H \otimes \mathbb{Q}/\mathbb{Z}$. Then*

$$D^G = (\mathbb{Q}/\mathbb{Z})^{a+c} \oplus (\mathbb{Z}/2)^b, \quad H^1(G, D) \cong (\mathbb{Z}/2)^a \quad \textit{and}$$
$$H^2(G, D) \cong (\mathbb{Z}/2)^b.$$

In particular, the numbers a, b and c are determined by D^G and $\dim(H)$.

Now consider Weichold's case $H = H^1(\bar{X}(\mathbb{C}), \mathbb{Z})(1)$, where $\bar{X}$ is smooth projective. Nonequivariantly, $H \cong \mathbb{Z}^{2g}$. There is an equivariant isomorphism $\mathrm{Pic}^0(\bar{X}_\mathbb{C}) \cong H \otimes (\mathbb{R}/\mathbb{Z})$, and a routine calculation shows that $\mathrm{Pic}^0(\bar{X}_\mathbb{C})^G$ equals $(\mathbb{R}/\mathbb{Z})^{a+c} \oplus (\mathbb{Z}/2)^b$. If $\bar{X}(\mathbb{R}) \neq \emptyset$ then this group also equals $\mathrm{Pic}^0(\bar{X})$. Comparing with Weichold's result, we obtain $a + c = g$ and b is $\nu - 1$. If $\bar{X}(\mathbb{R}) = \emptyset$ then b is 0 when g is even and 1 when g is odd; see [PW1, 1.1.2][CP, 4.1.9][Sil, 10]. Since $a + b + 2c = 2g$ we can solve this system to get $a = b$ and $c = g - b$.

The calculation for $H = H^2(\bar{X}(\mathbb{C}), \mathbb{Z})(1) \cong \mathbb{Z}$ is similar. From the exponential sequence, the group $H \otimes \mathbb{Q}/\mathbb{Z} = H^2_{\mathrm{et}}(\bar{X}_\mathbb{C}, \mu_\infty)$ is isomorphic to $\mathrm{Pic}(\bar{X}_\mathbb{C}) \otimes \mathbb{Q}/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}$. Since $H^2(\bar{X}, \mu_\infty) \cong (\mathbb{Q}/\mathbb{Z}) \oplus (\mathbb{Z}/2)^\nu$ by 1.3, a transfer argument shows that $H^2_{\mathrm{et}}(\bar{X}_\mathbb{C}, \mu_\infty)^G$ contains (and hence equals) $\mathbb{Q}/\mathbb{Z}$. Hence, $a = 1$ and $b = c = 0$. We record these observations:

COROLLARY 2.3 (Comessatti). *Let $\bar{X}$ be an irreducible smooth projective real curve, of genus g. Define b to be: $\nu - 1$ if $\bar{X}(\mathbb{R}) \neq \emptyset$; 0 if $\bar{X}(\mathbb{R}) = \emptyset$ and g is even; and 1 if $\bar{X}(\mathbb{R}) = \emptyset$ and g is odd. Then as a G-module we have*

$$\mathrm{Pic}^0(\bar{X}_\mathbb{C}) \cong \left(\mathbb{R}/\mathbb{Z}(0) \oplus \mathbb{R}/\mathbb{Z}(1)\right)^b \oplus (\mathbb{R}/\mathbb{Z})[G]^{g-b};$$
$$H^1_{\mathrm{et}}(\bar{X}_\mathbb{C}, \mu_\infty) \cong \left(\mathbb{Q}/\mathbb{Z}(0) \oplus \mathbb{Q}/\mathbb{Z}(1)\right)^b \oplus (\mathbb{Q}/\mathbb{Z})[G]^{g-b};$$
$$H^2_{\mathrm{et}}(\bar{X}_\mathbb{C}, \mu_\infty) \cong \mathbb{Q}/\mathbb{Z}(0).$$

The affine case is almost identical, except that we must distinguish between the r_1 real points and r_2 complex points removed from $\bar{X}$ to get X, because we must remove $r_1 + 2r_2$ points from $\bar{X}_\mathbb{C}$ to get $X_\mathbb{C}$. Here is the analogue of 2.3.

COROLLARY 2.4. *Let X be an irreducible, smooth affine real curve, which is obtained from a smooth projective curve $\bar{X}$ of genus g by removing r_1 real* *points and r_2 complex points. Let $\nu = \nu(X)$ and $\lambda = \lambda(X)$ denote the number of components and loops of $X(\mathbb{R})$, respectively. Then as an Abelian group $H^1_{\mathrm{et}}(X_{\mathbb{C}}, \mu_\infty) \cong (\mathbb{Q}/\mathbb{Z})^{2g+r_1+2r_2-1}$.*

If $X(\mathbb{R}) \neq \emptyset$ then $\nu = r_1 + \lambda$ and there is a G-module isomorphism

$$H^1_{\mathrm{et}}(X_{\mathbb{C}}, \mu_\infty) \cong \mathbb{Q}/\mathbb{Z}(0)^{\nu-1} \oplus \mathbb{Q}/\mathbb{Z}(1)^{\lambda} \oplus (\mathbb{Q}/\mathbb{Z})[G]^{g+r_2-\lambda}.$$

If $X(\mathbb{R}) = \emptyset$ then there is a G-module isomorphism

$$H^1_{\mathrm{et}}(X_{\mathbb{C}}, \mu_\infty) \cong \mathbb{Q}/\mathbb{Z}(1) \oplus (\mathbb{Q}/\mathbb{Z})[G]^{g+r_2-1}.$$

Note that the exponent of $\mathbb{Q}/\mathbb{Z}(0)$ is $\nu + E(\bar{X}) - 1$ in both cases of 2.4.

Proof. Here we have $H = H^1(X(\mathbb{C}), \mathbb{Z})(1)$ and $H \otimes \mathbb{Q}/\mathbb{Z} \cong H^1_{\mathrm{et}}(X_{\mathbb{C}}, \mu_\infty)$, so we only need to find a, b and c in 2.1. Nonequivariantly, $H \cong \mathbb{Z}^{2g+r_1+2r_2-1}$.

If $X(\mathbb{R}) \neq \emptyset$, we have $H^1_{\mathrm{et}}(X_{\mathbb{C}}, \mu_\infty)^G = H^1_{\mathrm{et}}(X, \mu_\infty)$ by [PW1, 1.1]. Hence, Theorem 1.3 and 2.2 yield $a + c = g + r_1 + r_2 - 1$ and $b = \lambda$. Also, $a + b + 2c = \dim H = 2g + r_1 + 2r_2 - 1$. Using the easily checked fact that $\nu = r_1 + \lambda$, it is easy to solve for a and c.

The proof is similar when $X(\mathbb{R}) = \emptyset$, except that (as observed in [PW1, 1.1.1]) we have an extension

$$0 \to H^1_{\mathrm{et}}(X, \mu_\infty) \to H^1_{\mathrm{et}}(X_{\mathbb{C}}, \mu_\infty)^G \to \mathbb{Z}/2 \to 0.$$

Thus Theorem 1.3 yields $H^1_{\mathrm{et}}(X_{\mathbb{C}}, \mu_\infty)^G \cong (\mathbb{Q}/\mathbb{Z})^{g+r_2-1} \oplus \mathbb{Z}/2$. □

We will have frequent recourse to a standard transfer argument, which we formalize like this:

TRANSFER ARGUMENT 2.5. *Let F be a functor with transfers, defined on finite field extensions of $\mathbb{R}$. Suppose that $F(\mathbb{C})$ is divisible, and let $\tilde{D}$ denote the maximal divisible subgroup of $F(\mathbb{C})^G$, $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Then there is an elementary Abelian 2-group E and a divisible subgroup D of $F(\mathbb{R})$ such that $D \cong \tilde{D}$ and $F(\mathbb{R}) \cong D \oplus E$.*

Proof. The hypothesis means that there is a G-module structure on $F(\mathbb{C})$, together with maps $\pi^*\colon F(\mathbb{R}) \to F(\mathbb{C})$ and $\pi_*\colon F(\mathbb{R}) \to F(\mathbb{C})$ such that $\pi_*\pi^*$ is multiplication by 2 and $\pi^*\pi_* = 1+\sigma$. Let D denote the subgroup $\pi_* F(\mathbb{C})$ of $F(\mathbb{R})$, and set $E = F(\mathbb{R})/D$. Since $F(\mathbb{C})$ is divisible, so is D and there is a noncanonical decomposition $F(\mathbb{R}) \cong D \oplus E$. Since D contains $2 \cdot F(\mathbb{R})$, E has exponent 2. Finally, if we set $A = F(\mathbb{C})^G$, then the divisible group $\pi^* D = (1 + \sigma)F(\mathbb{C})$ satisfies $2 \cdot A \subset \pi^* D \subset A$; this readily implies that $\pi^* D$ is the maximal divisible subgroup of $F(\mathbb{C})^G$. Finally, the kernel H of the surjection $\pi^*\colon D \to \pi^* D$ has exponent 2, so the following standard lemma implies that there is a noncanonical isomorphism $\pi^* D \cong D$. □

LEMMA 2.5.1. *If D is a divisible abelian group and H is a subgroup of finite exponent, then the quotient group D/H is (noncanonically) isomorphic to D.*

Proof. We may assume that H has prime exponent p, by induction. Let $I \subseteq D$ be the injective hull of H. Then there is a subgroup D' so that $D \cong I \oplus D'$, and $D/H \cong I/H \oplus D'$. So we may assume that D is the injective hull of H. In this case, the endomorphism $D \xrightarrow{p} D$ induces the isomorphism $D/H \cong D$. □

Here is an example of the usefulness of the transfer argument. Consider the constant étale sheaf $\mathbb{Q}/\mathbb{Z}$ on a real curve X.

PROPOSITION 2.6. *Let X be any irreducible, smooth real curve. Let g, $E = E(X)$, $\nu = \nu(X)$, $\lambda = \lambda(X)$, r_1 and r_2 be as above. Then*:

$$H^1_{\text{et}}(X, \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^{g+r_2} \oplus (\mathbb{Z}/2)^{\nu+E},$$
$$H^2_{\text{et}}(X, \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Z}/2)^{\lambda}.$$

Proof. Set $F(k) = H^1_{\text{et}}(X_k, \mathbb{Q}/\mathbb{Z})$. Then F is a functor with transfers defined on fields over $\mathbb{R}$. Now σ acts as -1 on μ_∞, as $\sigma(\zeta) = \zeta^{-1}$ for all roots of unity ζ. Hence there is a canonical G-module isomorphism $H^1_{\text{et}}(X_{\mathbb{C}}, \mathbb{Q}/\mathbb{Z}) \cong H^1_{\text{et}}(X_{\mathbb{C}}, \mu_\infty)(1)$. It follows that $F(\mathbb{C}) = H^1_{\text{et}}(X_{\mathbb{C}}, \mathbb{Q}/\mathbb{Z})$ is divisible. By inspection of 2.3 and 2.4, we see that in all cases the maximal divisible subgroup of $F(\mathbb{C})^G$ is $D = (\mathbb{Q}/\mathbb{Z})^{g+r_2}$. The transfer argument 2.5 gives everything except the dimension of the elementary Abelian 2-group.

From the Kummer sequence for $\mathbb{Z}/2 \subset \mathbb{Q}/\mathbb{Z}$ and $H^0(X, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$, we get from 1.4 that the exponent 2 subgroup of $F(\mathbb{R})$ is

$${}_2H^1_{\text{et}}(X, \mathbb{Q}/\mathbb{Z}) \cong H^1_{\text{et}}(X, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{g+r_1+r_2+\lambda+E}.$$

Since D contributes $(\mathbb{Z}/2)^{g+r_2}$ and $\nu = r_1 + \lambda$, the description of H^1_{et} follows.

Now suppose that $n = 2$. If X is projective then $H^2_{\text{et}}(X_{\mathbb{C}}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}(1)$ as a G-module by 2.3; if X is affine then $H^2_{\text{et}}(X_{\mathbb{C}}, \mathbb{Q}/\mathbb{Z}) = 0$. In either case, the maximal divisible subgroup of $H^2_{\text{et}}(X_C, \mathbb{Q}/\mathbb{Z})^G$ is zero. Hence, the transfer argument 2.5 implies that $H^2_{\text{et}}(X, \mathbb{Q}/\mathbb{Z})$ is an elementary Abelian 2-group. Therefore it is the quotient of $H^2_{\text{et}}(X, \mathbb{Z}/2) = (\mathbb{Z}/2)^{\nu+\lambda+E}$ by the subgroup $H^1_{\text{et}}(X, \mathbb{Q}/\mathbb{Z})/2 \cong (\mathbb{Z}/2)^{\nu+E}$, i.e., $(\mathbb{Z}/2)^{\lambda}$. □

We now describe the étale cohomology of the sheaves $\mu_\infty^{\otimes i} = \bigcup_m \mu_m^{\otimes i}$. Since σ sends $\zeta \otimes \cdots \otimes \zeta$ to $\zeta^{-1} \otimes \cdots \otimes \zeta^{-1}$, we see that $\mu_\infty^{\otimes i}$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$ for every even i, and isomorphic to μ_∞ for every odd i.

PROPOSITION 2.7. *Let X be any irreducible, smooth real curve. Then for $n \geqslant 3$*:

$$H^n_{\text{et}}(X, \mu_\infty^{\otimes i}) \cong \begin{cases} (\mathbb{Z}/2)^{\lambda}, & n+i \text{ even;} \\ (\mathbb{Z}/2)^{\nu}, & n+i \text{ odd.} \end{cases}$$

If X is affine, this formula also holds for $n = 2$.

Proof. Suppose that $n \geqslant 3$ (and $n \geqslant 2$ when X is affine). We know that $H^n_{\text{et}}(X, \mathbb{Z}/2) = (\mathbb{Z}/2)^{\nu+\lambda}$ by 1.4. We know that $H^n_{\text{et}}(X_{\mathbb{C}}, \mu_\infty^{\otimes i}) = 0$, so $H^n_{\text{et}}(X, \mu_\infty^{\otimes i})$

is an elementary Abelian 2-group by the usual transfer argument. Again by Kummer theory, $H^n_{et}(X, \mathbb{Z}/2)$ is the sum of $H^n_{et}(X, \mu_\infty^{\otimes i})$ and $H^{n-1}_{et}(X, \mu_\infty^{\otimes i})/2$, so the groups may be determined from 1.3 and 2.6 by induction on n. □

Remark 2.7.1. The calculation for i even, i.e., $H^n_{et}(X, \mathbb{Q}/\mathbb{Z})$, is easily derived from Cox' Theorem [Cox], exactly as in [PW1, 1.8]. See also Scheiderer [Sch, 20.1.6–12].

EXAMPLE 2.8. Suppose that X is smooth projective with $X(\mathbb{R}) = \emptyset$. For $n \geqslant 3$ we have $H^n_{et}(X, \mu_\infty^{\otimes i}) = 0$ by 2.7. For $n = 1, 2$ it follows from 1.3 and 2.6 that:

$$H^1_{et}(X, \mu_\infty^{\otimes i}) \cong \begin{cases} (\mathbb{Q}/\mathbb{Z})^g \oplus (\mathbb{Z}/2), & i \text{ even;} \\ (\mathbb{Q}/\mathbb{Z})^g, & i \text{ odd;} \end{cases}$$

$$H^2_{et}(X, \mu_\infty^{\otimes i}) \cong \begin{cases} 0, & i \text{ even;} \\ \mathbb{Q}/\mathbb{Z}, & i \text{ odd.} \end{cases}$$

Remark 2.9. Our transfer argument 2.5 is a simplification of the transfer arguments used in [PW3, 5.8] and [CT-S, 1.3] to calculate to the Chow group $A_0(X)$ of (degree zero) zero cycles on a real variety X. The reader is invited to compare our argument with the ones in loc. cit.

3. Divisible Torsion

In order to extend the above computations to the groups $K_n(X)$ for all n we first recall some known results on the K-theory of complex curves. It is well known that $H^1(X_\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^{b_1}$, where b_1 is the first Betti number of the topological space $X(\mathbb{C})$. In fact, b_1 may be read off from Theorem 2.4.

THEOREM 3.1 ([PW2, 3.2]). *Let $X_\mathbb{C}$ be a smooth irreducible curve over $\mathbb{C}$. Then $K_n(X_\mathbb{C})$ is divisible for $n \geqslant 2$, and the torsion subgroup of $K_n(X_\mathbb{C})$ is given by:*

$$K_n(X_\mathbb{C})_{\text{tors}} = \begin{cases} (\mathbb{Q}/\mathbb{Z})^{b_1}, & \text{if } n \text{ is even,} \\ \mathbb{Q}/\mathbb{Z}, & \text{if } n \text{ is odd and } X_\mathbb{C} \text{ is affine,} \\ (\mathbb{Q}/\mathbb{Z})^2, & \text{if } n \text{ is odd and } X_\mathbb{C} \text{ is projective.} \end{cases}$$

Equivariant Structure 3.2. The action of G on $K_n(X_\mathbb{C})$ is implicit in the proof of loc. cit., given [PW2, 2.3 and 3.1]. One summand in $K_{2i-1}(X_\mathbb{C})_{\text{tors}}$ is the Bott summand $H^0_{et}(X, \mu_\infty^{\otimes i}) \cong \mathbb{Q}/\mathbb{Z}(i)$. When X is projective, the other summand is $H^2_{et}(X, \mu_\infty^{\otimes i+1}) \cong \mathbb{Q}/\mathbb{Z}(i)$, so $K_{2i-1}(X_\mathbb{C})_{\text{tors}} \cong \mathbb{Q}/\mathbb{Z}(i)^2$. When $n = 2i$, there is a G-module isomorphism

$$K_{2i}(X_\mathbb{C})_{\text{tors}} \cong H^1_{et}(X_\mathbb{C}, \mu_\infty^{\otimes i+1}) \cong H^1_{et}(X_\mathbb{C}, \mu_\infty)(i). \tag{3.2.1}$$

THEOREM 3.3. *Let X be a smooth irreducible real curve. Then for all $n \geqslant 2$ we have a noncanonical isomorphism $K_n(X) \cong E_n \oplus D_n$, where $E_n = E_n(X)$ is an* *elementary Abelian 2-group and the torsion part of $D_n = D_n(X)$ is the following divisible group*:

$$(D_n)_{\mathrm{tors}} = \begin{cases} 0, & \text{if } n \equiv 1 \pmod 4; \\ (\mathbb{Q}/\mathbb{Z})^{g+r_2}, & \text{if } n \equiv 2 \pmod 4; \\ \mathbb{Q}/\mathbb{Z}, & \text{if } n \equiv 3 \pmod 4 \text{ and } X \text{ is affine}; \\ \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}, & \text{if } n \equiv 3 \pmod 4 \text{ and } X \text{ is projective}. \\ (\mathbb{Q}/\mathbb{Z})^{g+r_1+r_2-1}, & \text{if } n \equiv 4 \pmod 4 \text{ and } X \text{ is affine}; \\ (\mathbb{Q}/\mathbb{Z})^{g}, & \text{if } n \equiv 4 \pmod 4 \text{ and } X \text{ is projective}. \end{cases}$$

We will see later that $E_n(X)$ is finite, and finish the computation of $K_n(X)_{\mathrm{tors}}$.

Proof. The functor $F(k) = K_n(X_k)$ has transfers, and $F(\mathbb{C}) = K_n(X_{\mathbb{C}})$ is divisible, so the usual transfer argument 2.5 applies: $K_n(X) = D_n \oplus E_n$, where D_n is isomorphic to the maximal divisible subgroup of $K_n(X_{\mathbb{C}})^G_{\mathrm{tors}}$.

Suppose first that $n = 2i - 1$ is odd. Then the G-module $K_n(X_{\mathbb{C}})_{\mathrm{tors}}$ is either one or two copies of $\mathbb{Q}/\mathbb{Z}(i)$ by 3.2, depending upon whether X is affine or projective. The result follows as in 2.2.

Suppose that $n = 2i > 0$ and X is projective. We see from 3.2 and 2.3 that

$$K_{2i}(X_{\mathbb{C}})^G_{\mathrm{tors}} \cong \mathrm{Pic}^0(X_{\mathbb{C}})^G_{\mathrm{tors}} \cong (\mathbb{Q}/\mathbb{Z})^g \oplus (\mathbb{Z}/2)^b,$$

where b is defined in 2.3. Hence $(D_n)_{\mathrm{tors}} \cong (\mathbb{Q}/\mathbb{Z})^g$.

When X is affine and $X(\mathbb{R}) \neq \emptyset$, (3.2.1) and 2.4 yield

$$K_{2i}(X_{\mathbb{C}})^G_{\mathrm{tors}} \cong \begin{cases} (\mathbb{Q}/\mathbb{Z})^{g+r_1+r_2-1} \oplus (\mathbb{Z}/2)^{\lambda}, & i \text{ even}; \\ (\mathbb{Q}/\mathbb{Z})^{g+r_2} \oplus (\mathbb{Z}/2)^{r_1+\lambda-1}, & i \text{ odd}. \end{cases}$$

When X is affine and $X(\mathbb{R}) = \emptyset$, (3.2.1) and 2.4 yield

$$K_{2i}(X_{\mathbb{C}})^G_{\mathrm{tors}} \cong \begin{cases} (\mathbb{Q}/\mathbb{Z})^{g+r_2-1} \oplus (\mathbb{Z}/2), & i \text{ even}; \\ (\mathbb{Q}/\mathbb{Z})^{g+r_2}, & i \text{ odd}. \end{cases}$$

In all these cases, we may read off the maximal divisible subgroup, which is isomorphic to $(D_n)_{\mathrm{tors}}$. □

4. K_1 and K_2

In this section we describe $K_1(X)$ and $K_2(X)$ using off-the-shelf techniques.

First we recall the structure of $K_1(X)$ from [PW1]. Classically, we have $K_1(X) \cong U(X) \oplus SK_1(X)$, where $U(X)$ is the group of global units on X. The structure of $U(X)$ is well known: when X is connected, $U(X)$ is the product of $\mathbb{R}^\times$ and a finitely generated free abelian group (which vanishes if X is proper). If X is affine with r points at infinity, then the rank of $U(X)/\mathbb{R}^\times$ is at most $r - 1$ by 1.3. Here is the structure of $SK_1(X)$:

THEOREM 4.1 ([PW1, 5.7]). *Let X be a real curve. Then there is a uniquely divisible abelian group $V = V_1(X)$ and a natural decomposition:*

$$SK_1(X) \cong (\mathbb{Z}/2)^{\lambda} \oplus V.$$

We now turn to $K_2(X)$. First recall that $K_2(X_{\mathbb{C}})$ is divisible, and that its torsion subgroup is $H^1_{\text{et}}(X_{\mathbb{C}}, \mathbb{Q}/Z)$. This fact is implicit in [Su2, 5.2] and explicit in [PW1, 5.6]. The usual transfer argument 2.5 applied to $F(k) = K_2(X_k)$, combined with Proposition 2.6, shows that $K_2(X)$ is the sum of a divisible group D_2 with $(D_2)_{\text{tors}} \cong (\mathbb{Q}/\mathbb{Z})^{g+r_2}$, and an elementary Abelian 2-group.

To say more, we must use the extension

$$0 \to H^1(X, \mathcal{K}_3) \to K_2(X) \to H^0(X, \mathcal{K}_2) \to 0.$$

Here $H^0(X, \mathcal{K}_2)$ is the image of $K_2(X) \to K_2(F)$, where F is the function field $\mathbb{R}(X)$, and $H^1(X, \mathcal{K}_3)$ is the image of the transfer map $\oplus_x K_2(x) \to K_2(X)$.

The subgroup $H^1(X, \mathcal{K}_3)$ is the sum of a uniquely divisible group V_{12} and an elementary Abelian 2-group. This follows from the usual transfer argument 2.5, because $H^1(X_{\mathbb{C}}, \mathcal{K}_3)$ is a uniquely divisible group by [PW2, 5.5(1) and 6.5(3)].

Suslin's results in [Su2] yield the following description of the quotient group $H^0(X, \mathcal{K}_2)$.

PROPOSITION 4.2. *Let X be a smooth real curve. Then the Steinberg symbol $\{-1, -1\}$ in $K_2(\mathbb{R})$ remains nonzero in $H^0(X, \mathcal{K}_2)$, and there is a uniquely divisible group V_{02} such that*

$$H^0(X, \mathcal{K}_2) \cong (\mathbb{Q}/\mathbb{Z})^{g+r_2} \oplus (\mathbb{Z}/2)^{\nu+E} \oplus V_{02}.$$

Proof. By [Su2, 3.6], $H^0(X, \mathcal{K}_2)$ contains $K_2(\mathbb{R})$. By [Su2, 4.4] and Theorem 4.1 we have an exact sequence

$$0 \to H^0(X, \mathcal{K}_2)/2 \to H^2_{\text{et}}(X, \mathbb{Z}/2) \to (\mathbb{Z}/2)^{\lambda} \to 0.$$

The middle group is $(\mathbb{Z}/2)^{\nu+\lambda+E}$ by Corollary 1.4, so the left group is $(\mathbb{Z}/2)^{\nu+E}$. □

We now turn to the other group, $H^1(X, \mathcal{K}_3)$. It is convenient to separate out the easy case when X has no real points.

COROLLARY 4.3. *Let X be an irreducible, smooth real curve with $X(\mathbb{R}) = \emptyset$. Then there is a uniquely divisible group V_2 so that*

(1) *if X is projective then $K_2(X) \cong (\mathbb{Q}/\mathbb{Z})^g \oplus (\mathbb{Z}/2) \oplus V_2$, and the $\mathbb{Z}/2$ summand is generated by $\{-1, -1\}$;*
(2) *if X is affine, with r_2 points at infinity, then $K_2(X) \cong (\mathbb{Q}/\mathbb{Z})^{g+r_2} \oplus V_2$. In particular, the symbol $\{-1, -1\}$ is divisible in $K_2(X)$.*

Proof. Since $X(\mathbb{R}) = \emptyset$, $K_2(x) = K_2(\mathbb{C})$ is divisible for every closed point x of X. Hence $H^1(X, \mathcal{K}_3)$ is divisible, and so uniquely divisible. Therefore $K_2(X)_{\text{tors}}$ is $H^0(X, \mathcal{K}_2)_{\text{tors}}$, a group which is given by 4.2, and $K_2(X)/2 \cong (\mathbb{Z}/2)^E$. In particular, if X is affine, then $K_2(X)$ is divisible.

Twisting 2.4 yields a G-module isomorphism $H^1_{\text{et}}(X_{\mathbb{C}}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z} \oplus (\mathbb{Q}/\mathbb{Z})[G]^{g+r_2-1}$; hence $H^1_{\text{et}}(X_{\mathbb{C}}, \mathbb{Q}/\mathbb{Z})^G = (\mathbb{Q}/\mathbb{Z})^{g+r_2}$. From [Su2, 5.4] we see that this is the quotient of $H^0(X, \mathcal{K}_2)_{\text{tors}}$ by $K_2(\mathbb{R})_{\text{tors}}$. It follows from 4.2 that if X is projective (i.e., $E = 1$), then $K_2(\mathbb{R})$ is a summand of $K_2(X)$. □

EXAMPLE 4.3.1 (Brauer–Severi curve). Let $\bar{X}$ be the projective plane curve over $\mathbb{R}$ given by the equation $x^2 + y^2 + z^2 = 0$. Note that $\bar{X}(\mathbb{R}) = \emptyset$. This is the Brauer–Severi variety corresponding to the quaternion $\mathbb{R}$-algebra $\mathbb{H}$. By Quillen [Q341], $K_n(\bar{X}) \cong K_n(\mathbb{R}) \oplus K_n(\mathbb{H})$. It is well known [AD] that $K_2(\mathbb{H})$ is uniquely divisible, so $K_2(\bar{X})_{\text{tors}} \cong (\mathbb{Z}/2)$. By 4.2, the nonzero torsion element of $K_2(\bar{X})$ is $\{-1, -1\}$.

Removing one (complex) point yields the affine real plane curve X given by the equation $x^2 + y^2 = -1$. By 4.3 we see that $K_2(X)_{\text{tors}} \cong \mathbb{Q}/\mathbb{Z}$. Here the element $\{-1, -1\}$, which is still nonzero by 4.2, has become divisible.

If $X(\mathbb{R}) \neq \emptyset$, we need to consider the exact localization sequence

$$K_3(F) \xrightarrow{\partial} \oplus_x K_2(x) \to H^1(X, \mathcal{K}_3) \to 0, \tag{4.4}$$

or rather its reduction modulo 2. By [MS, 11.11], the image of ∂ agrees with the image of the Milnor K-group $K_3^M(F)$. Now Rost as well as Merkurjev and Suslin have proven that $K_3^M(F)/2 \cong H^3_{\text{et}}(F, \mathbb{Z}/2)$ [MS1, 5.1], and we may read this off from 1.4. As the general case is no harder, given Voevodsky's theorem [V], we state our description in that generality.

4.5. *Reciprocity for Milnor K-Theory.* Let F be the function field of a smooth real curve X. By [Su3], the groups $K_n^M(F \otimes \mathbb{C})$ are divisible for $n \geqslant 2$. In fact, they are uniquely divisible for $n \geqslant 3$; see [PW2, 3.1]. The usual transfer argument shows that $K_n^M(F)$ is the sum of a uniquely divisible group and the elementary Abelian 2-group $K_n^M(F)/2$.

Using the norm residue isomorphism $K_n^M(F)/2 \cong H^n_{\text{et}}(F, \mathbb{Z}/2)$ and 1.4, the Gysin sequence for $H^n_{\text{et}}(X, \mathbb{Z}/2)$ [Mi, p. 244] yields the exact sequence for all $n \geqslant 2$:

$$0 \to (\mathbb{Z}/2)^{\nu(X)} \to K_n^M(F)/2 \xrightarrow{\partial} \bigoplus_{x \in X(\mathbb{R})} \mathbb{Z}/2 \to (\mathbb{Z}/2)^{\lambda(X)} \to 0.$$

In particular, if $X = \bar{X}$ is projective, then $\nu(\bar{X}) = \lambda(\bar{X})$ and we have the exact sequence

$$0 \to (\mathbb{Z}/2)^{\nu(\bar{X})} \to K_n^M(F)/2 \xrightarrow{\partial} \bigoplus_{x \in \bar{X}(\mathbb{R})} \mathbb{Z}/2 \to (\mathbb{Z}/2)^{\nu(\bar{X})} \to 0.$$

That is, an element of $K_n^M(F)/2$ determines a subset of $X(\mathbb{R})$ with an even number points on each compact component, and such a subset determines an element of $K_n^M(F)/2$ up to an indeterminacy of $\mathbb{Z}/2^\nu$.

Returning to the exact sequence (4.4), it follows that $H^1(X,\mathcal{K}_3)/2 \cong (\mathbb{Z}/2)^{\lambda(X)}$. Combining this with the calculation of $H^0(X,\mathcal{K}_2)$, we see that we have proven the following result:

THEOREM 4.6. *Let X be a smooth real curve with $X(\mathbb{R}) \neq \emptyset$. The group $K_2(X)$ is the direct sum of $(\mathbb{Q}/\mathbb{Z})^{g+r_2}$, a uniquely divisible group $V_2(X)$ and a finite elementary Abelian group $E_2(X)$. Moreover, there is an exact sequence*

$$0 \to (\mathbb{Z}/2)^\lambda \to K_2(X)_{\mathrm{tors}} \to (\mathbb{Q}/\mathbb{Z})^{g+r_2} \oplus (\mathbb{Z}/2)^\nu \to 0.$$

In particular, if $X(\mathbb{R})$ has no loops then $K_2(X)_{\mathrm{tors}} \cong (\mathbb{Q}/\mathbb{Z})^{g+r_2} \oplus (\mathbb{Z}/2)^\nu$.

Remark 4.6.1. We will solve this extension problem in Section 7 below. If X is affine, then we will show that $K_2(X)_{\mathrm{tors}} \cong (\mathbb{Q}/\mathbb{Z})^{g+r_2} \oplus (\mathbb{Z}/2)^\nu$, and $K_2(X)/2 \cong (\mathbb{Z}/2)^\nu$. However, if X is projective then $K_2(X)_{\mathrm{tors}} \cong (\mathbb{Q}/\mathbb{Z})^g \oplus (\mathbb{Z}/2)^{\nu+1}$, and $K_2(X)/2 \cong (\mathbb{Z}/2)^{\nu+1}$. In this case the sequence of 4.6 splits exactly when $\lambda = 1$.

Remark 4.6.2. It is easy to compare $K_2(X)/2$, described in 4.6.1, with the topological *KO*-theory of $X(\mathbb{R})$. From the Atiyah–Hirzebruch spectral sequence we get $KO^{-2}X(\mathbb{R}) \cong (\mathbb{Z}/2)^\nu$, where $X(\mathbb{R})$ has ν path components.

EXAMPLE 4.7.1 (projective line). It is well known that the projective line $\mathbb{P}^1_{\mathbb{R}}$ has $K_2(\mathbb{P}^1_{\mathbb{R}}) \cong K_2(\mathbb{R}) \oplus K_2(\mathbb{R})$. In this case, $K_2(\mathbb{P}^1_{\mathbb{R}})_{\mathrm{tors}} \cong (\mathbb{Z}/2)^2$ and the extension splits in theorem 4.6.

EXAMPLE 4.7.2 (the circle). Let X be the affine plane curve over $\mathbb{R}$ given by the familiar equation $x^2 + y^2 = 1$. Note that $r_2 = \nu = \lambda = 1$ here. It was computed in [RbW, 2.1] that $K_2(X) = K_2(\mathbb{R}) \oplus \mathbb{R}/\mathbb{Z}$, and that $H^1(X,\mathcal{K}_3)$ is $(\mathbb{Z}/2)$ on the element $\{-1,-1\}\cdot\eta$, where η is the nonzero element of $\mathrm{Pic}(X) \cong \mathbb{Z}/2$. In this case the extension $0 \to \mathbb{Z}/2 \to K_2(X)_{\mathrm{tors}} \to \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}/2$ in 4.6 does not split.

We conclude this section with a description of $K_3(F)$. Since $K_3(F \otimes \mathbb{C})$ is divisible by [MS, 11.6], the usual transfer argument 2.5 shows that $K_3(F)$ is the sum of a divisible group and an elementary 2-group E_3. Since $K_3(F \otimes \mathbb{C})_{\mathrm{tors}} \cong \mathbb{Q}/\mathbb{Z}$ by [PW2, 3.1], and $K_3(F)$ contains $K_3(\mathbb{R})$, we see that the divisible torsion in $K_3(F)$ must be $K_3(\mathbb{R})_{\mathrm{tors}} \cong \mathbb{Q}/\mathbb{Z}$. It is known that $K_3^M(F)$ is a subgroup of $K_3(F)$, and clearly the torsion subgroup of $K_3(\mathbb{R}) \cap K_3^M(F)$ is $K_3^M(\mathbb{R})_{\mathrm{tors}} \cong \mathbb{Z}/2$. By [MS, 11.6], $K_3(F)_{\mathrm{tors}}$ is contained in $K_3^M(F) + K_3(\mathbb{R})$. Using reciprocity 4.5, we have established the following result.

PROPOSITION 4.8. *If F is the function field of a smooth real curve X, there is an exact sequence*:

$$0 \to \mathbb{Q}/\mathbb{Z} \oplus (\mathbb{Z}/2)^{\nu(X)-1} \to K_3(F)_{\mathrm{tors}} \xrightarrow{\partial} \bigoplus_{x \in X(\mathbb{R})} \mathbb{Z}/2 \to (\mathbb{Z}/2)^{\lambda(X)} \to 0.$$

In particular, $H^0(X,\mathcal{K}_3)_{\text{tors}} \cong \mathbb{Q}/\mathbb{Z} \oplus (\mathbb{Z}/2)^{\nu(X)-1}$; *the subgroup* $\mathbb{Q}/\mathbb{Z}$ *is* $K_3(\mathbb{R})_{\text{tors}}$.

COROLLARY 4.9. *If X is affine, then* $K_3(X)_{\text{tors}} \cong \mathbb{Q}/\mathbb{Z} \oplus (\mathbb{Z}/2)^{\nu(X)-1}$. *If X is projective then* $K_3(X)_{\text{tors}} \cong (\mathbb{Q}/\mathbb{Z})^2 \oplus (\mathbb{Z}/2)^{\nu(X)-1}$.

Proof. Since both $K_3(\mathbb{R})$ and $K_3(\mathbb{C})$ are divisible, it follows from the localization sequence that the kernel of the surjection $K_3(X) \to H^0(X,\mathcal{K}_3)$ is divisible. Since we know the divisible torsion subgroup of $K_3(X)$ from Theorem 3.3, the result follows from Proposition 4.8. □

5. Motivic Cohomology of Real Curves

The motivic cohomology $H^*_M(X,\mathbb{Z}(i))$ of X is defined to be its Zariski hypercohomology with coefficients in the complex of sheaves $\mathbb{Z}(i)$, and $H^*_M(X,\mathbb{Z}/m(i))$ is defined using $\mathbb{Z}/m(i) = \mathbb{Z}(i) \otimes^{\mathrm{L}} \mathbb{Z}/m$ (see [Fr, Section 4]). The purpose of this section is to calculate the groups $H^*_M(X,\mathbb{Z}/m(i))$ when X is a real curve and m is a power of 2.

DEFINITION 5.1. Let X be a smooth variety over a field k and let $\mu_m^{\otimes i}$ be the étale sheaf of mth roots of 1, twisted i times, where $1/m \in k$. Write $\mathbb{R}\pi_*(\mu_m^{\otimes i})$ for the total right derived image of the $\mu_m^{\otimes i}$ along the change-of-site morphism π: $X_{\text{et}} \to X_{\text{Zar}}$.

We define the complex $B/m(i)$ of Zariski sheaves to be the good truncation of this complex:

$$B/m(i) = \tau_{\leqslant i}\mathbb{R}\pi_*(\mu_m^{\otimes i})$$

Recall that the good truncation $\tau_{\leqslant i}C$ of a complex $C = C^*$ is the natural subcomplex whose cohomology sheaves satisfy:

$$\mathcal{H}^q(\tau_{\leqslant i}C) = \begin{cases} \mathcal{H}^q(C), & \text{if } q \leqslant i, \\ 0, & \text{if } q > i. \end{cases}$$

In particular, $\mathcal{H}^q(B/m(i))$ vanishes if $q > i$, while for $q \leqslant i$ it is the sheaf $\mathcal{H}^q(\mu_m^{\otimes i})$ associated to the Zariski presheaf $U \mapsto H^q_{\text{et}}(U,\mu_m^{\otimes i})$.

There is a natural map in the derived category of complexes of sheaves in the Zariski topology [SV, Section 6] [Fr, Section 6]:

$$\mathbb{Z}/m(i) \to B/m(i). \tag{5.2}$$

Beilinson and Lichtenbaum conjectured that the map (5.2) is a quasi-isomorphism of complexes of sheaves for all m and i. Voevodsky [V, 5.5] has proven that (5.2) is an isomorphism when m is a power of 2. The case $i = 2$ follows from the Merkurjev–Suslin theorem for K_2 [MS2] and [SV, 7.4]. Suslin and Voevodsky (see [SV, 7.2]) have also verified that it is an isomorphism for $i \leqslant 1$, because $\mathbb{Z}(0) \cong \mathbb{Z}$ and $\mathbb{Z}(1) \cong \mathcal{O}_X^\times[-1]$, while $\mathbb{Z}(i) = 0$ for $i < 0$ by definition.

Remark 5.2.1. Suslin and Voevodsky [SV, 7.4] have shown that the Beilinson–Lichtenbaum Conjecture holds for those fields k (and m, i) such that for any field F over k the natural homomorphism

$$K_i^M(F)/m \to H^i_{\text{et}}(F, \mu_m^{\otimes i})$$

is an isomorphism; this is the so-called *Bloch–Kato conjecture* in weight i (modulo m) over k. Voevodsky has proved the Beilinson–Lichtenbaum conjecture by proving in [V, 5.1] that the Bloch–Kato conjecture holds for all fields k (of characteristic $\neq 2$) and all i in the case $m = 2^\nu$.

The isomorphism (5.2) implies that the Zariski hypercohomology spectral sequence converging to $H_M^{p+q}(X, \mathbb{Z}/m(i))$ has

$$E_2^{p,q} = \begin{cases} H^p_{\text{Zar}}(X, \mathcal{H}^q(\mu_m^{\otimes i})), & \text{if } q \leqslant i \\ 0, & \text{if } q > i. \end{cases} \tag{5.3}$$

For purposes of comparison, the Zariski hypercohomology spectral sequence (also known as the local-to-global spectral sequence, or the Bloch–Ogus spectral sequence) is:

$$\tilde{E}_2^{p,q} = H^p_{\text{Zar}}(X, \mathcal{H}^q(\mu_m^{\otimes i})) \Rightarrow H^{p+q}_{\text{et}}(X, \mu_m^{\otimes i}). \tag{5.3.1}$$

Let $NH^n(X)$ denote the kernel of the map $H^n_{\text{et}}(X) \to H^n_{\text{et}}(F)$, where F is the function field of X. The group $NH^n(X)$ is the first level in the coniveau filtration of $H^n_{\text{et}}(X)$.

LEMMA 5.4. *Let X be a smooth variety of dimension d, defined over a field of characteristic zero. Then for all $m = 2^\nu$:*

$$H^n_M(X, \mathbb{Z}/m(i)) = \begin{cases} H^n_{\text{et}}(X, \mu_m^{\otimes i}), & \text{if } 0 \leqslant n \leqslant i; \\ NH^n(X, \mu_m^{\otimes i}), & \text{if } n = i+1; \\ H^i_{\text{Zar}}(X, \mathcal{H}^d(\mu_m^{\otimes i})), & \text{if } n = i+d; \\ 0, & \text{if } n > i+d. \end{cases}$$

In the missing range $i+1 < n < i+d$ there are surjections from $H^n_M(X, \mathbb{Z}/m(i))$ onto the corresponding level Fil^{n-i} *in the coniveau filtration of $H^n_{\text{et}}(X, \mu_m^{\otimes i})$.*

Proof. This follows from the fact that there is a morphism of spectral sequences from (5.3) to (5.3.1), which is an isomorphism for $q \leqslant i$. □

EXAMPLE 5.5. (Suslin [Su3, 4.3]) If X is a smooth real variety and $i \geqslant d$ then:

$$H^n_M(X_{\mathbb{C}}, \mathbb{Z}/m(i)) \cong H^n_{\text{et}}(X_{\mathbb{C}}, \mu_m^{\otimes i}) \quad \text{for all } m \text{ and } n.$$

Of course, this group vanishes unless $n \leqslant 2d$. If $m = 2^\nu$ this follows from 5.4. Indeed, $\mathcal{H}^q(\mu_m^{\otimes i}) = 0$ for $q > d$, so the spectral sequences (5.3) and (5.3.1) coincide for $i \geqslant d$.

For example, if X is a real curve then $H^1_M(X_{\mathbb{C}}, \mathbb{Q}/\mathbb{Z}(i)) \cong H^1_{\text{et}}(X_{\mathbb{C}}, \mu_m^{\otimes i})$ for $i \geqslant 1$; this group was determined in 2.3–2.6.

LEMMA 5.6. *If X is a smooth real curve, then every group $H^n_M(X, \mathbb{Z}/m(i))$ is finite. This group vanishes if $n \geqslant i+2$, and equals $H^n_{\text{et}}(X, \mu_m^{\otimes i})$ if $n \leqslant i$. Finally,*

$$H^{i+1}_M(X, \mathbb{Z}/2(i)) \cong \begin{cases} 0, & \text{if } i = 0, \\ (\mathbb{Z}/2)^{\lambda+E}, & \text{if } i = 1, \\ (\mathbb{Z}/2)^{\lambda}, & \text{if } i \geqslant 2, \end{cases}$$

$$H^{i+1}_M(X, \mathbb{Q}/\mathbb{Z}(i)) \cong \begin{cases} \mathbb{Q}/\mathbb{Z}, & \text{if } i = 1 \text{ and } X \text{ is projective;} \\ 0, & \text{if } i \neq 1, \text{or if } i = 1 \text{ and } X \text{ is affine.} \end{cases}$$

Proof. If m is odd then $H^n_M(X, \mathbb{Z}/m(i)) = H^n_M(X_{\mathbb{C}}, \mathbb{Z}/m(i))^G$ by the usual transfer argument, so this follows from 5.5. For $m = 2^\nu$ we may proceed by induction on ν. Thus we may suppose that $m = 2$.

Let F be the function field of X, and write H^*_{et} for $H^*_{\text{et}}(-, \mathbb{Z}/2)$. By 5.4 we have $H^{i+1}_M(X, \mathbb{Z}/2(i)) \cong NH^{i+1}(X)$. We observed in 1.4.1 that $NH^1(X) = 0$, $NH^2(X) \cong \text{Pic}(X)/2 \cong (\mathbb{Z}/2)^{\lambda+E}$ and $NH^n(X) \cong (\mathbb{Z}/2)^{\lambda}$ for $n > 2$.

Now consider coefficients $\mathbb{Q}/\mathbb{Z}(i)$. For $i = 0$ we need only observe that $H^1_{\text{et}}(X, \mathbb{Q}/\mathbb{Z})$ injects into $H^1_{\text{et}}(F, \mathbb{Q}/\mathbb{Z})$; this follows from the Gysin sequence, but it could also be obtained from 2.6. For $i = 1$ this is the classical fact that $NH^2(X, \mu_\infty) \cong \text{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z}$ (or one could argue by naturality as in 1.4.1, starting from Theorem 1.3). For $i \geqslant 2$, we see from 2.7 that $H^{i+1}_{\text{et}}(X, \mu_\infty^{\otimes i}) \cong (\mathbb{Z}/2)^{\nu(X)}$. As argued in the proof of Theorem 1.3, this injects into $H^{i+1}_{\text{et}}(F, \mu_\infty^{\otimes i})$, so $H^{i+1}_M(X, \mathbb{Z}/2(i)) = 0$. □

EXERCISE 5.7. For every smooth real curve X, the group $H^n_M(X_{\mathbb{C}}, \mathbb{Z}(i))$ is uniquely divisible for $n \geqslant 4$ and divisible for $n = 3$. If X is affine, it is uniquely divisible for $n = 3$ and divisible for $n = 2$. These assertions follow from 5.5 using universal coefficients, and are implicit in the proof of [PW2, 1.4].

It follows from the usual transfer argument, 2.7 and 5.6 that when $n \geqslant 4$ (or $n \geqslant 3$ if X is affine), $H^n_M(X, \mathbb{Z}(i))$ is the direct sum of a uniquely divisible group and an elementary Abelian 2-group. In fact, since $H^*_M(X, \mathbb{Q}(i))$ is uniquely divisible, $H^n_M(X, \mathbb{Z}(i))_{\text{tors}}$ must be isomorphic to $H^{n-1}_M(X, \mathbb{Q}/\mathbb{Z}(i))$, a finite group which we can read off from 2.7 using 5.6.

If X is affine, we have

$$\begin{aligned} H^n_M(X, \mathbb{Z}(i))_{\text{tors}} &\cong H^{n-1}_M(X, \mathbb{Q}/\mathbb{Z}(i)) \\ &\cong \begin{cases} (\mathbb{Z}/2)^{\nu}, & \text{if } i+1 \geqslant n \geqslant 3 \text{ and } n+i \text{ is even;} \\ (\mathbb{Z}/2)^{\lambda}, & \text{if } i+1 \geqslant n \geqslant 3 \text{ and } n+i \text{ is odd;} \\ 0, & \text{otherwise } (n \geqslant 3). \end{cases} \end{aligned}$$

If X is projective and $n=3$, this fails for odd i, because $H^2_M(X,\mathbb{Q}/\mathbb{Z}(i))\cong\mathbb{Q}/\mathbb{Z}\oplus(\mathbb{Z}/2)^\nu$ by 1.3. If $i>1$, the summand $\mathbb{Q}/\mathbb{Z}$ survives in $H^3_M(X,\mathbb{Z}(i))$ because by [PW2, 3.2] it is detected by the group $H^3_M(X_{\mathbb{C}},\mathbb{Z}(i))\cong\mathbb{Q}/\mathbb{Z}$. However, if $i=1$, then $H^3_M(X,\mathbb{Z}(1))\cong H^2_{\mathrm{Zar}}(X,\mathcal{O}^\times_X)=0$ yet $H^2_M(X,\mathbb{Z}(1))\cong H^1_{\mathrm{Zar}}(X,\mathcal{O}^\times_X)\cong\mathrm{Pic}(X)$.

If X is projective and $X(\mathbb{R})=\emptyset$, then $H^n_M(X,\mathbb{Z}(i))$ is torsionfree for all $n\geqslant 4$ and all i, because $H^n_M(X,\mathbb{Q}/\mathbb{Z}(i))=0$ for all $n\geqslant 3$. This follows from 5.6, using 2.7 when $n\leqslant i$. When $n=2$, we see from 5.6 and 2.8 that

$$H^2_M(X,\mathbb{Q}/\mathbb{Z}(i))=\begin{cases}\mathbb{Q}/\mathbb{Z}, & \text{if } i \text{ is odd;}\\ 0, & \text{otherwise.}\end{cases}$$

As above, $H^3_M(X,\mathbb{Z}(1))=0$, but for each odd $i>1$ we have $H^3_M(X,\mathbb{Z}(1))_{\mathrm{tors}}\cong\mathbb{Q}/\mathbb{Z}$.

6. Curves with No Loops

In this section we introduce the spectral sequence we need, from motivic cohomology to K-theory, and use it to study curves with no loops. The case $X(\mathbb{R})=\emptyset$ is handled first, in 6.4; here there are no differentials in the spectral sequence. The case $X(\mathbb{R})\neq\emptyset$ (and $\lambda=0$) is handled in 6.8; here the differentials are easily determined from the corresponding spectral sequence for $K_*(\mathbb{R})$.

Suslin has identified the motivic cohomology groups of a smooth variety with *Bloch's Higher Chow groups* $CH^i(X,n)$, as defined in [B]. More precisely, the following theorem results from combining [Su3, 2.1] and [FV, 8.2], as in [W, 4.2].

THEOREM 6.1 (Suslin). *Assume X is a smooth variety over a field k with $\mathrm{char}(k)=0$. Then for all i:*

$$CH^i(X,n)\cong H^{2i-n}_M(X,\mathbb{Z}(i))$$

and

$$CH^i(X,n;\mathbb{Z}/m)\cong H^{2i-n}_M(X,\mathbb{Z}/m(i)).$$

Here $CH^i(X,n;\mathbb{Z}/m)$ is the mod $\mathbb{Z}/m$ version of the higher Chow groups.

The results in [PW2] were obtained by considering the *Bloch–Lichtenbaum spectral sequence* [BL] converging to the K-theory of a field F, and the corresponding spectral sequence with finite coefficients (see [RW]).

Friedlander and Suslin [FS, 13.6 and 16.2] have recently proven that a Bloch–Lichtenbaum type spectral sequence also exists for every regular X. A similar spectral sequence was constructed and studied by M. Levine in [L] [L1]. In addition to the existence of this spectral sequence, we also need some information about its multiplicative properties, which are given in [FS, 15.5 and 16.2]:

THEOREM 6.2 (Friedlander–Suslin). *Let X be a smooth variety over a field k of characteristic 0. Then there are spectral sequences converging to K-theory and K-theory with $\mathbb{Z}/m$ coefficients, defined for $q \leqslant 0$:*

$$\begin{aligned} E_2^{p,q}(\mathbb{Z}) &= H_M^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X; \mathbb{Z}); \\ E_2^{p,q}(\mathbb{Z}/m) &= H_M^{p-q}(X, \mathbb{Z}/m(-q)) \Rightarrow K_{-p-q}(X; \mathbb{Z}/m). \end{aligned} \tag{6.2.0}$$

Moreover, there is a multiplicative pairing $E_2^{p,q}(\mathbb{Z}) \otimes E_2^{p',q'}(\mathbb{Z}/m) \to E_2^{p+p',q+q'}(\mathbb{Z}/m)$ induced by the product map in motivic cohomology, whose map on abutments is the usual K-theory product $K_m(X) \otimes K_n(X, \mathbb{Z}/m) \to K_{m+n}(X, \mathbb{Z}/m)$.

Variation. Note that, after reindexing, the Bloch–Lichtenbaum spectral sequence can be written as an Atiyah–Hirzebruch style spectral sequence:

$$'E_2^{p,q} = H_M^p(X, \mathbb{Z}/m(-q/2)) \Rightarrow K_{-p-q}(X; \mathbb{Z}/m), \tag{6.2.1}$$

where $\mathbb{Z}/m(-q/2) = 0$ if q is not an even nonpositive integer.

Remark 6.2.2. Under the 'usual' hypotheses for finite coefficients, the proof of [FS, 16.2] shows that the mod m spectral sequence is multiplicative; cf. [L, 8.14]. However, these 'usual' hypotheses do not hold for coefficients $\mathbb{Z}/2$ or even $\mathbb{Z}/4$ (unless the global functions contain $\sqrt{-1}$).

In general, we claim that the spectral sequence $E_2^{pq}(\mathbb{Z}/m)$ is multiplicative if either m is odd or if $m = 2^\nu$ and $\nu \geqslant 4$. However, we shall not use this result.

To see this claim, recall that the multiplicative structure arises from a map ρ: $\mathbf{M} \to \mathbf{M} \wedge \mathbf{M}$, where $\mathbf{M}$ is a Moore space. We may now cite [O]; see [Wp, 2.5]: If $4|m$ there is a product, compatible under reduction with the product on $H_M^*(X)$ and $K_*(X)$; it is commutative if $8|m$ and associative if $m \not\equiv 3 \pmod 9$.

PROPOSITION 6.3. *If X is a smooth real curve, then $K_n(X; \mathbb{Z}/2)$ is finite for every n. In particular the groups $E_n(X) = K_n(X)/2$ are finite for every n.*

Proof. From Theorem 6.2 and Lemmas 5.4 and 5.6 it follows that the group $K_n(X; \mathbb{Z}/2)$ is a finite group. Since $K_n(X)/2$ is a subgroup of $K_n(X; \mathbb{Z}/2)$, it is also finite. □

THEOREM 6.4. *Suppose that X is an irreducible, smooth real curve, with $X(\mathbb{R}) = \emptyset$. Then $K_n(X)$ is a divisible-by-finite group for all $n \geqslant 1$, and its torsion is 4-fold periodic:*

$$K_n(X)_{\text{tors}} \cong \begin{cases} \mathbb{Z}/2, & n \equiv 1 \pmod 4; \\ (\mathbb{Q}/\mathbb{Z})^{g+r_2} \oplus (\mathbb{Z}/2)^E, & n \equiv 2 \pmod 4; \\ (\mathbb{Q}/\mathbb{Z})^{1+E}, & n \equiv 3 \pmod 4; \\ (\mathbb{Q}/\mathbb{Z})^{g+r_2+E-1}, & n \equiv 4 \pmod 4). \end{cases}$$

Proof. By 3.3 we have $K_{n+1}(X)\otimes\mathbb{Q}/\mathbb{Z}=0$, so $K_{n+1}(X,\mathbb{Q}/\mathbb{Z})\cong K_n(X)_{\text{tors}}$. Because $X(\mathbb{R})=\emptyset$, we see from 5.7 that $H^n_M(X,\mathbb{Q}/\mathbb{Z}(i))=0$ for all $n\geqslant 3$, and that $H^2_M(X,\mathbb{Q}/\mathbb{Z}(i))$ is zero or $\mathbb{Q}/\mathbb{Z}$. Hence, there are no differentials in the spectral sequence 6.2, and every extension of H^0 by H^2 is split. Thus we have

$$K_n(X,\mathbb{Q}/\mathbb{Z})\cong\begin{cases} H^0(X,\mathbb{Q}/\mathbb{Z}(i))\oplus H^2(X,\mathbb{Q}/\mathbb{Z}(i+1)), & n=2i;\\ H^1(X,\mathbb{Q}/\mathbb{Z}(i)), & n=2i-1.\end{cases}$$

Now plug in the values of these groups from 5.6, 5.7, 1.3, 2.6 and 2.7. □

Remark 6.4.1. It is helpful to compare this periodicity with the known values of $K_n(X)$ for low n. The group $K_0(X)\cong\mathbb{Z}\oplus\operatorname{Pic}(X)$ is never divisible-by-finite, and its torsion is not always given by the formula in 6.4 unless $r_2\leqslant 1$ (see the proof of 1.3). The group $K_1(X)$ decomposes as $U(X)\oplus SK_1(X)$, and the group $U(X)$ of global units of X is the product of $\mathbb{R}^\times$ and a finitely generated free Abelian group. The group $SK_1(X)$ is uniquely divisible by 4.1. If X is projective, or even if $r_2=1$, then $U(X)=\mathbb{R}^\times$ and $K_1(X)$ is divisible-by-finite. The calculation of $K_2(X)$ in 6.4 agrees with the value found in 4.3 above.

EXAMPLE 6.5 (Brauer–Severi curve). Let $\bar{X}$ be the projective plane curve over $\mathbb{R}$ given by the equation $x^2+y^2+z^2=0$, i.e., the Brauer–Severi variety corresponding to the quaternion $\mathbb{R}$-algebra $\mathbb{H}$. By Quillen [Q341], $K_n(\bar{X})\cong K_n(\mathbb{R})\oplus K_n(\mathbb{H})$.

The group structure of $K_*(\mathbb{H},\mathbb{Z}/m)$ was determined by Suslin [Su1, 3.5]; it is isomorphic with finite coefficients to the homotopy groups of $\mathbb{Z}\times BSp\simeq\Omega^4BO$ because the symplectic group Sp is homotopy equivalent to $BGL(\mathbb{H}^{\text{top}})$. In particular, we have $K_n(\mathbb{H})_{\text{tors}}\cong K_{n+4}(\mathbb{R})_{\text{tors}}$ for all $n\geqslant 0$.

It is an amusing exercise to check that our Proposition 6.4 agrees with the Quillen–Suslin calculation for $\bar{X}$.

Now suppose that $X(\mathbb{R})\neq\emptyset$, i.e., $\nu>0$, and pick ν real points of X, one on each component of $X(\mathbb{R})$. The inclusion of these points in X induces a map from $H^n(X,\mathbb{Z}/2)$ to $\oplus_{i=1}^{\nu}H^n_{\text{et}}(\mathbb{R},\mathbb{Z}/2)\cong(\mathbb{Z}/2)^\nu$.

The choice of these ν points yields a morphism of spectral sequences (6.2), from the one converging to $K_*(X,\mathbb{Z}/2)$ to the sum of ν copies of the one converging to $K_*(\mathbb{R},\mathbb{Z}/2)$. By Lemma 6.6, the morphism has a surjection in the (p,q) spot for all $p>q$, $p\leqslant 0$.

LEMMA 6.6. *If X is affine, then $H^n(X,\mathbb{Z}/2)\to(\mathbb{Z}/2)^\nu$ is onto for all $n\geqslant 1$.*

Proof. Since X is smooth affine, the maps $H^n_{\text{et}}(X,\mathbb{Z}/2)\to H^{n+1}_{\text{et}}(X,\mathbb{Z}/2)$ are onto by Lemma 1.5. Since $H^n_{\text{et}}(\mathbb{R},\mathbb{Z}/2)\to H^{n+1}_{\text{et}}(\mathbb{R},\mathbb{Z}/2)$ are isomorphisms, we may assume that $n>2$. The result now follows from the natural isomorphisms $H^n_{\text{et}}(X,\mathbb{Z}/2)\cong\oplus H^i(X(\mathbb{R}),\mathbb{Z}/2)$, $n>2\dim(X)$ in Cox' Theorem [Cox], formalized in [CT-S, 2.3.1]. Indeed, the following commutative diagram is a special case:

$$\begin{array}{ccc} H^n_{\text{et}}(X,\mathbb{Z}/2) & \xrightarrow{\cong} & H^0(X(\mathbb{R}),\mathbb{Z}/2)\bigoplus H^1(X(\mathbb{R}),\mathbb{Z}/2) \\ \downarrow & & \downarrow{\scriptstyle \text{onto}} \\ \bigoplus_1^{\nu} H^n_{\text{et}}(\mathbb{R},\mathbb{Z}/2) & \xrightarrow{\cong} & \bigoplus_1^{\nu} H^0(\mathbb{R},\mathbb{Z}/2). \end{array}$$

□

We now turn to real curves such that $X(\mathbb{R})$ has no loops, and $X(\mathbb{R}) \neq \emptyset$. That is, $\lambda = 0$ and $\nu > 0$. Clearly, X is affine. If $n \geqslant 2$ then $H^n(X,\mathbb{Z}/2) \cong (\mathbb{Z}/2)^\nu$ by 1.4, so the map $H^n(X,\mathbb{Z}/2) \to (\mathbb{Z}/2)^\nu$ of 6.6 is an isomorphism. That is, the morphism of spectral sequences has an isomorphism in the (p,q) spot for all $p \geqslant q+2$.

By [RW, 5.3] the classes in bidegrees (p,q) with $p = -4k-2$ or $p = -4k-3$ support nontrivial differentials in the spectral sequence for $\mathbb{R}$. It follows that when $p \equiv -3$ and $p \equiv -2$ and $p > q$ the differentials $E_2^{pq} \to E_2^{p+2,q-1}$ are onto in the spectral sequence for X. When $p \equiv -3$ and $p \equiv -2$ and $p = q$ the differential (for X) goes from $E_2^{pp} \cong \mathbb{Z}/2$ to $E_2^{p+2,p-1} \cong (\mathbb{Z}/2)^\nu$ and must be injective by the same argument. When $p \equiv -1$ and $p = q$ the differential is zero; this follows by the same argument because by [RW, 5.3] the corresponding differential in the spectral sequence for $\mathbb{R}$ vanishes.

If we write $\tilde{H}^1$ for the kernel of the surjection $H^1_{\text{et}}(X,\mathbb{Z}/2) \to (\mathbb{Z}/2)^\nu$, then the first few columns of the spectral sequence look like this:

-3	-2	-1	$p=0$	$+1$
			$\mathbb{Z}/2$	0
		$\mathbb{Z}/2$	H^1	$\mathrm{Pic}(X)/2$
	$\mathbb{Z}/2$	H^1	ν	0
$\mathbb{Z}/2$	H^1	ν	ν	0
H^1	ν	ν	ν	0
ν	ν	ν	ν	0
ν	ν	ν	ν	0

E_2 terms for $K_*(X,\mathbb{Z}/2)$

-3	-2	-1	$p=0$	$+1$
			$\mathbb{Z}/2$	0
		$\mathbb{Z}/2$	H^1	$\mathrm{Pic}(X)/2$
	0	H^1	ν	0
0	$\tilde{H}^1$	ν	$\nu-1$	0
$\tilde{H}^1$	0	$\nu-1$	0	0
0	0	0	0	0
0	0	0	0	0

E_3 terms for $K_*(X,\mathbb{Z}/2)$

(6.7)

Here the symbol 'ν' represents $(\mathbb{Z}/2)^\nu$, and similarly with '$\nu - 1$'. Every set of four columns to the left is the same as the set from $p = 0$ to $p = -3$, shifted left 4 and down 4. By inspection, there are no further differentials, and $E_3 = E_\infty$. Thus the associated graded groups for the filtration of $K_*(X,\mathbb{Z}/2)$ may be read off from (6.7).

Warning 6.7.1. There are nontrivial extensions in the filtration of $K_*(X,\mathbb{Z}/2)$ coming from (6.7). For example, $K_{8k+2}(\mathbb{R},\mathbb{Z}/2) \cong \mathbb{Z}/4$ is a subgroup of $K_{8k+2}(X,\mathbb{Z}/2)$, which must equal $\mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\nu-1}$. Thus the extensions

$$0 \to (\mathbb{Z}/2)^\nu \to K_{8k+2}(X,\mathbb{Z}/2) \to \mathbb{Z}/2 \to 0$$

do not split. Similarly, it can be shown that $K_{8k+3}(X, \mathbb{Z}/2)$ contains $\nu - 1$ copies of $\mathbb{Z}/4$. We omit the argument as we do not need this fact.

THEOREM 6.8. *Let X be a smooth real curve such that $X(\mathbb{R})$ has $\nu > 0$ components and no loops. Then the groups $K_n(X)_{\text{tors}}$ are periodic of period 8 for $n > 0$. In fact:*

$$K_n(X)_{\text{tors}} \cong \begin{cases} \mathrm{Pic}(X)_{\text{tors}} & n = 0, \\ \mathbb{Z}/2 & n \equiv 1 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z})^{g+r_2} \oplus (\mathbb{Z}/2)^{\nu} & n \equiv 2 \pmod 8, \\ \mathbb{Q}/\mathbb{Z} \oplus (\mathbb{Z}/2)^{\nu-1} & n \equiv 3 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z})^{g+r_2+\nu-1} & n \equiv 4 \pmod 8, \\ 0 & n \equiv 5 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z})^{g+r_2} & n \equiv 6 \pmod 8, \\ \mathbb{Q}/\mathbb{Z} & n \equiv 7 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z})^{g+r_2+\nu-1} & n \equiv 0 \pmod 8, \quad n \geqslant 8. \end{cases}$$

Note that $K_n(X)$ is divisible for $n \geqslant 2$ unless $n \equiv 1, 2, 3 \pmod 8$.

Before beginning the proof, we make some preliminary observations. Because there are no loops, X is affine and $r_1 = \nu$ (see 2.4). By 1.4, $\dim H^1_{\text{et}}(X, \mathbb{Z}/2) = g + r_2 + \nu$ and $\dim \tilde{H}^1 = g + r_2$. The groups $(D_{n-1})_{\text{tors}} \cong (\mathbb{Q}/\mathbb{Z})^a$ given in Theorem 3.3 must contribute $(\mathbb{Z}/2)^a$ to $K_n(X, \mathbb{Z}/2)$ by the universal coefficient theorem. For $n \equiv 1, \dots, 4$ the numbers a are 0, $g + r_2$, 1 and $g + r_2 + \nu - 1$, respectively.

Proof. It is easiest to begin at $K_6(X, \mathbb{Z}/2)$; it is 0 by (6.7). This implies that $K_6(X)$ is divisible and that $K_5(X)$ is torsionfree. By 3.3, we have $K_6(X) = D_6$ and $K_5(X) = D_5$. Hence $K_6(X)_{\text{tors}} = (D_6)_{\text{tors}} = (\mathbb{Q}/\mathbb{Z})^{g+r_2}$, and $K_5(X)$ is uniquely divisible.

Working backwards from K_5, this implies that the exponent 2 subgroup of $K_4(X)$ is $K_5(X, \mathbb{Z}/2)$, which by (6.7) is $\tilde{H}^1 \oplus (\mathbb{Z}/2)^{\nu-1}$. But D_4 accounts for all of this torsion, whence $K_4(X) = D_4(X)$. This implies that the exponent 2 subgroup of $K_3(X)$ is $K_4(X, \mathbb{Z}/2) = (\mathbb{Z}/2)^{\nu}$; since $K_3(X)$ contains $K_3(\mathbb{R})_{\text{tors}} \cong \mathbb{Q}/\mathbb{Z}$, we see that $K_3(X)_{\text{tors}}$ is $\mathbb{Q}/\mathbb{Z} \oplus (\mathbb{Z}/2)^{\nu-1}$.

Working forwards from K_6, $K_6(X)_{\text{tors}}$ accounts for all of $K_7(X, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{g+r_2}$, so $K_7(X)$ is divisible and $K_7(X)_{\text{tors}} = (D_7)_{\text{tors}} \cong \mathbb{Q}/\mathbb{Z}$. This in turn accounts for $K_8(X, \mathbb{Z}/2) \cong \mathbb{Z}/2$, so $K_8(X)$ is divisible. In turn, $K_8(X)_{\text{tors}} \cong (D_8)_{\text{tors}}$ accounts for $g + r_2 + \nu - 1$ terms in $K_9(X, \mathbb{Z}/2) \cong H^1$. Therefore, $K_9(X)_{\text{tors}} = K_9(X)/2 \cong \mathbb{Z}/2$.

We now turn to $K_{10}(X, \mathbb{Z}/2)$. Since $K_{10}(\mathbb{R}, \mathbb{Z}/2) \cong \mathbb{Z}/4$ is a summand, we must have $K_{10}(X, \mathbb{Z}/2) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\nu-1}$. Since $K_9(X)_{\text{tors}} \cong \mathbb{Z}/2$, it follows that $K_{10}(X)/2 \cong (\mathbb{Z}/2)^{\nu}$. Hence $K_{10}(X)_{\text{tors}}$ is the sum of $(\mathbb{Z}/2)^{\nu}$ and $(D_{10})_{\text{tors}} \cong (\mathbb{Q}/\mathbb{Z})^{g+r_2}$.

Since the columns in the spectral sequence are 8-fold periodic, the same argument works for all other values of $n \geqslant 3$. One could argue similarly for K_1 and K_2, but we prefer to cite 4.1 and 4.6. □

6.9. FOLKLORE CALCULATION. If X is obtained from the affine line Spec $(\mathbb{R}[t])$ by removing $\nu - 1$ real points and r_2 complex points, then the localization sequence for $X \subset \mathrm{Spec}(\mathbb{R}[t])$ yields a decomposition:

$$K_n(X) \cong K_n(\mathbb{R}) \oplus K_{n-1}(\mathbb{R})^{\nu-1} \oplus K_{n-1}(\mathbb{C})^{r_2}.$$

This result agrees with our formulas for the torsion, as one can easily check.

7. Curves with Loops

In this section we assemble the information from the previous sections into a calculation of the torsion in $K_*(X)$ for all real curves X. By Theorem 6.8, we may assume that $\lambda \neq 0$, i.e., that $X(\mathbb{R})$ has at least one loop. The computation for affine curves is given in Theorem 7.2, modulo the proof of 'Claim 7.6' which is given in Section 8. The calculation for projective curves (Theorem 0.1) follows from the affine case by the following trick.

LEMMA 7.1. *Let $\bar{X}$ be an irreducible, smooth projective real curve with $\bar{X}(\mathbb{R}) \neq \emptyset$. If p is any real point and $X = \bar{X} - \{p\}$ then for all n:*

$$K_n(\bar{X}) \cong K_n(X) \oplus K_n(\mathbb{R}).$$

Proof. Let $\pi\colon \bar{X} \to \mathrm{Spec}(\mathbb{R})$ denote the structure map and $\iota\colon \mathrm{Spec}(\mathbb{R}) = p \to \bar{X}$ the inclusion. Then the composition $\pi\iota$ is the identity, and both maps are proper. Hence the map $\iota_*\colon K_n(\mathbb{R}) \to K_n(\bar{X})$ is an injection, split by π_* for all n. Therefore the localization sequence breaks up into split exact sequences $0 \to K_n(\mathbb{R}) \xrightarrow{\iota_*} K_n(\bar{X}) \to K_n(X) \to 0$. □

THEOREM 7.2. *Let X be an smooth affine real curve such that $X(\mathbb{R})$ has $\nu > 0$ components and λ loops. Then the groups $K_n(X)_{\mathrm{tors}}$ are periodic of period 8 for $n > 0$. In fact:*

$$K_n(X)_{\mathrm{tors}} \cong \begin{cases} \mathrm{Pic}(X)_{\mathrm{tors}} & n = 0, \\ (\mathbb{Z}/2)^{1+\lambda} & n \equiv 1 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z})^{g+r_2} \oplus (\mathbb{Z}/2)^{\nu} & n \equiv 2 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z}) \oplus (\mathbb{Z}/2)^{\nu-1} & n \equiv 3 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z})^{g+r_1+r_2-1} & n \equiv 4 \pmod 8, \\ 0 & n \equiv 5 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z})^{g+r_2} & n \equiv 6 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z}) & n \equiv 7 \pmod 8, \\ (\mathbb{Q}/\mathbb{Z})^{g+r_1+r_2-1} \oplus (\mathbb{Z}/2)^{\lambda} & n \equiv 0 \pmod 8, \quad n \geqslant 8. \end{cases}$$

Note that $K_n(X)$ is divisible for $n \equiv 4, 5, 6, 7 \pmod 8$.

Proof of Main Theorem 0.1. Let $\bar{X}$ be an irreducible, smooth projective real curve such that $\bar{X}(\mathbb{R})$ has $\nu > 0$ components. If X is obtained by removing a real point from $\bar{X}$ then X has ν components and $\lambda = \nu - 1$ loops. Now we use

Lemma 7.1 to deduce the result from Theorem 7.2. The extension to K_0 is just a rephrasing of Weichhold's Theorem 1.1. □

The idea of the proof of 7.2 is to compute with the spectral sequence (6.2.0), using Lemma 8.2 to reduce to the no loops case in 6.8 and the calculations in [RW, 5.3]. The result will then follow from Theorem 3.3. For simplicity, we shall write $H^{n,i}$ for $H^n_M(X, \mathbb{Z}/2(i))$, which is the term $E_2^{n-i,-i}(\mathbb{Z}/2)$. Here is a very useful trick.

LEMMA 7.4. *Fix $p < 0$ and set $i = 2 - p$. Suppose that the d_2-differential $E_2^{p,p-2} = H^{2,i} \to H^{5,i+1}$ has rank r. Then:*

(1) *The d_2-differential $E_2^{p,p-1} = H^{1,i-1} \to H^{4,i}$ has rank r;*
(2) *The d_2-differential $E_2^{p,p-2-j} = H^{2+j,i+j} \to H^{5+j,i+j+1}$ has rank r for all $j \geqslant 0$.*

Proof. The element $[-1] \in E_2^{0,-1}(\mathbb{Z}) = H^0(X, \mathbb{G}_m)$ is a permanent cycle. By Theorem 6.2, multiplication by this element commutes with the differentials in the spectral sequence $E_2^{**}(\mathbb{Z}/2)$. Hence we have a commutative diagram

$$\begin{array}{ccccc} H^{1,i-1} & \xrightarrow[\text{onto}]{[-1]} & H^{2,i} & \xrightarrow[\cong]{[-1]^j} & H^{2+j,i+j} \\ d\big\downarrow & & d\big\downarrow & & d\big\downarrow \\ H^{4,i} & \xrightarrow[\cong]{[-1]} & H^{5,i+1} & \xrightarrow[\cong]{[-1]^j} & H^{5+j,i+j+1} \end{array}$$

The horizontal maps are obtained by multiplication by $[-1]$; by 1.5, they are all isomorphisms except for the upper left one, which is onto. The result follows. □

Now every term in the spectral sequence (6.2.0) is a group of exponent two. We shall find it useful to abbreviate $(\mathbb{Z}/2)^n$ with the integer 'n', as in 'ν' and 'λ'. The terms in the fourth quadrant of (6.2.0) are given by 1.4, while the terms in the column $p = 1$ come from 5.6. The left side of figure (7.5) gives the first few columns of the spectral sequence at E_2; every set of four columns to the left of this (from $p = -4k$ to $-4k - 3$) is the same as the set depicted from $p = 0$ to $p = -3$, shifted left 4 and down 4.

-3	-2	-1	$p = 0$	$+1$
			$\mathbb{Z}/2$	0
		$\mathbb{Z}/2$	H^{11}	λ
	$\mathbb{Z}/2$	H^{12}	$\nu+\lambda$	λ
$\mathbb{Z}/2$	H^{13}	$\nu+\lambda$	$\nu+\lambda$	λ
H^{14}	$\nu+\lambda$	$\nu+\lambda$	$\nu+\lambda$	λ
$\nu+\lambda$	$\nu+\lambda$	$\nu+\lambda$	$\nu+\lambda$	λ
$\nu+\lambda$	$\nu+\lambda$	$\nu+\lambda$	$\nu+\lambda$	λ

E_2 terms for $K_*(X, \mathbb{Z}/2)$

-3	-2	-1	$p = 0$	$+1$
			$\mathbb{Z}/2$	0
		$\mathbb{Z}/2$	H^{11}	λ
	0	$\bar{H}^{12}$	$\nu+\lambda$	λ
0	$\bar{H}^{13}$	ν	$\nu+\lambda-1$	0
$g+r_2$	0	$\nu-1$	0	0
λ	0	0	0	0
λ	0	0	0	0

E_3 terms for $K_*(X, \mathbb{Z}/2)$

(7.5)

CLAIM 7.6. *We claim that the E_3 terms of the spectral sequence vanish for $p-q \geqslant 4$, and are given by the right side of* (7.5), *where the notation is as follows. The symbol $\tilde{H}^{12}$ denotes the kernel of the surjection $H^1_{et}(X, \mathbb{Z}/2) \to (\mathbb{Z}/2)^{\lambda}$, which has dimension $g + r_1 + r_2$. The symbol $\tilde{H}^{13}$ denotes the kernel of the surjection $H^1_{et}(X, \mathbb{Z}/2) \to (\mathbb{Z}/2)^{\nu+\lambda}$ of Lemma* 1.5, *a vector space of dimension $g + r_2 - \lambda$.*

We will prove this claim in the next section. Assuming this claim, we can now prove Theorem 7.2. We will then devote the next section to the column-by-column proof of our Claim 7.6; the key results are 8.1.1 ($p = 1$), 8.3 ($p \leqslant 0$ even) and 8.4 ($p < 0$ odd).

Proof of Theorem 7.2. By inspection, there are no further differentials, and we may read off the associated graded groups for the filtration on $K_*(X, \mathbb{Z}/2)$. We will focus on $K_n(X)$ for $2 \leqslant n \leqslant 9$, since the cases $n + 8k$ are entirely similar.

Clearly $K_6(X, \mathbb{Z}/2) = 0$, which implies that $K_5(X)$ is torsionfree and that $K_6(X)$ is divisible. By Theorem 3.3, the $(\mathbb{Q}/\mathbb{Z})^{g+r_2}$ in $K_6(X)$ accounts for all of $K_7(X, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{g+r_2}$. Thus $K_7(X)$ is divisible. After accounting for the torsion subgroup $\mathbb{Q}/\mathbb{Z}$ of $K_7(X)$, the extension $K_8(X, \mathbb{Z}/2)$ of $\mathbb{Z}/2$ by $(\mathbb{Z}/2)^{\lambda}$ yields $K_8(X)/2 \cong (\mathbb{Z}/2)^{\lambda}$. By Theorem 3.3, this yields $K_8(X)_{\text{tors}} \cong (\mathbb{Q}/\mathbb{Z})^{g+r_1+r_2-1} \oplus (\mathbb{Z}/2)^{\lambda}$.

From the extension $0 \to (\mathbb{Z}/2)^{\lambda} \to K_9(X, \mathbb{Z}/2) \to H^{15} \to 0$ and the knowledge that the dimension $g + r + \lambda - 1$ of ${}_2(K_8X)$ is one less than the dimension of H^{15}, it follows that $K_9(X)/2 \cong (\mathbb{Z}/2)^{1+\lambda}$. From 3.3, the torsion subgroup of $K_9(X)$ has exponent 2, so $K_9(X)_{\text{tors}} \cong K_9(X)/2 \cong (\mathbb{Z}/2)^{1+\lambda}$.

Now we move in the other direction. Since $K_5(X)$ is uniquely divisible, $K_5(X, \mathbb{Z}/2) \cong {}_2(K_4X)$; by (7.5), $\dim K_5(X, \mathbb{Z}/2)$ is $(g+r_2-\lambda)+(\nu-1) = g+r_1+r_2-1$, which exactly matches the dimension of ${}_2(D_4)$. Hence, $K_4(X)$ is divisible. But then the exponent 2 subgroup of $K_3(X)$ equals $K_4(X; \mathbb{Z}/2)$, a group which is $(\mathbb{Z}/2)^{\nu}$ by (7.5). By Theorem 3.3 again, this yields $K_3(X)_{\text{tors}} \cong \mathbb{Q}/\mathbb{Z} \oplus (\mathbb{Z}/2)^{\nu-1}$, agreeing with the computation in 4.9.

To compute $K_2(X)_{\text{tors}}$, we turn to the extension

$$0 \to (\mathbb{Z}/2)^{\nu+\lambda-1} \to K_3(X, \mathbb{Z}/2) \to \tilde{H}^{12} \to 0.$$

From the computation of $K_3(X)_{\text{tors}}$ it follows that $K_3(X)/2 \cong (\mathbb{Z}/2)^{\nu-1}$, and hence ${}_2(K_2X)$ has dimension $\dim \tilde{H}^{12} + \lambda = g + r_2 + \nu$ (as $\nu = r_1 + \lambda$). By 3.3, we have $K_2(X) = D_2 \oplus E_2$ with $(D_2)_{\text{tors}} \cong (\mathbb{Q}/\mathbb{Z})^{g+r_2}$. A dimension counts shows that the elementary Abelian 2-group must be $E_2 \cong (\mathbb{Z}/2)^{\nu}$. Thus $K_2(X)_{\text{tors}} \cong (\mathbb{Q}/\mathbb{Z})^{g+r_2} \oplus (\mathbb{Z}/2)^{\nu}$.

Continuing in this direction, we come to $K_2(X, \mathbb{Z}/2)$; this is a group of cardinality $2^{1+\nu+\lambda}$ containing $K_2(\mathbb{R}, \mathbb{Z}/2) \cong \mathbb{Z}/4$. Since $K_2(X)/2$ has dimension ν, a count shows that ${}_2(K_1X)$ has dimension $1 + \lambda$ (cf. Theorem 4.1).

We have seen that the groups $K_n(X)_{\text{tors}}$ are determined for $2 \leqslant n \leqslant 9$ by (7.5) and Theorem 3.3. By periodicity of (7.5), we may add a multiple of 8 to all subscripts and use the same argument to determine the groups $K_n(X)_{\text{tors}}$ for all n. □

8. Proof of Claim 7.6

We now turn to the proof of the Claim 7.6. The vanishing in column $p = +1$ will follow from 8.1.1; the even columns ($p \leqslant 0$) will be described in 8.3, and the odd columns ($p < 0$) will be described in 8.4.

Recall that NH^{ni} is the kernel of $H^{ni}(X) \to H^{ni}(F)$. By 1.4 and 5.4, $NH^{ni} \cong (\mathbb{Z}/2)^{\lambda}$ whenever $2 \leqslant n \leqslant i+1$.

LEMMA 8.1. *If $i > 0$ and $i \equiv 3, 4 \pmod 4$, the differential $d_X\colon H^{2,i} \to H^{5,i+1}$ induces an isomorphism $NH^{2,i} \cong NH^{5,i+1}$.*

Proof. The differential $H^{0,i-1}(\mathbb{R}) \xrightarrow{d} H^{3,i}(\mathbb{R})$ is an isomorphism in the spectral sequence for Spec($\mathbb{R}$) by [RW, 5.3]. By naturality and Theorem 6.2, we can multiply by $E_2^{1,-1}(\mathbb{Z}) = H^2_M(X, \mathbb{Z}(1)) = \mathrm{Pic}(X)$; because all differentials vanish on Pic(X), this multiplication commutes with the differentials in the spectral sequence for X. This yields the commutative diagram:

$$\begin{array}{ccc} H^{0,i-1}(\mathbb{R}) \otimes \mathrm{Pic}(X) & \xrightarrow[d\otimes 1]{\cong} & H^{3,i}(\mathbb{R}) \otimes \mathrm{Pic}(X) \\ \cong\downarrow & & \cong\downarrow \\ NH^{2,i}(X) & \xrightarrow{Nd_X} & NH^{5,i+1}(X). \end{array} \tag{8.1.0}$$

Now $\mathrm{Pic}(X)/2 \cong NH^2_{\mathrm{et}}(X, \mathbb{Z}/2)$ and $H^{p,i}(\mathbb{R}) \cong H^p_{\mathrm{et}}(\mathbb{R}, \mathbb{Z}/2) \cong \mathbb{Z}/2$ on $[-1]^p$ for $0 \leqslant p \leqslant i$. Hence the vertical maps are isomorphisms by 1.6, and this implies that the bottom arrow Nd_X in (8.1.0) is also an isomorphism. □

In particular, when $i = 3$ the differential from $E_2^{-1,-3}(\mathbb{Z}/2) = H^{23}$ to $E_2^{1,-4} = H^{54}_M \cong NH^{54} \cong (\mathbb{Z}/2)^{\lambda}$ is onto. Now $E_2^{1q} \cong (\mathbb{Z}/2)^{\lambda}$ for all $q < 0$. Combining these facts with lemma 7.4 for $p = -1$, we obtain the following corollary.

COROLLARY 8.1.1. *The d_2-differentials $E_2^{-1,q} \to E_2^{1,q-1} \cong (\mathbb{Z}/2)^{\lambda}$ are onto for all $q \leqslant -2$.*

To describe the differentials for $p < -1$ we resort to a comparison with the no loops case, which was described in (6.7) and 6.8.

LEMMA 8.2. *Let U be the subscheme of X obtained by removing one point from each of the λ loops on X. Then $H^n_{\mathrm{et}}(X, \mathbb{Z}/2) \to H^n_{\mathrm{et}}(U, \mathbb{Z}/2)$ is onto for all n, and is an isomorphism for $n \leqslant 1$.*

Proof. Note that $\lambda(U) = 0$ but $\nu(U) = \nu(X)$ and $r_2(U) = r_2(X)$. The lemma now follows from the localization sequence

$$\begin{aligned} 0 &\to H^1_{\mathrm{et}}(X, \mathbb{Z}/2) \to H^1_{\mathrm{et}}(U, \mathbb{Z}/2) \to (\mathbb{Z}/2)^{\lambda} \to H^2_{\mathrm{et}}(X, \mathbb{Z}/2) \\ &\to H^2_{\mathrm{et}}(U, \mathbb{Z}/2) \cdots \end{aligned}$$

and a dimension count from 1.4: $\dim H^1_{\mathrm{et}}(X, \mathbb{Z}/2) = \dim H^1_{\mathrm{et}}(U, \mathbb{Z}/2) = g + r_2 + \nu$ and $\dim H^n_{\mathrm{et}}(X, \mathbb{Z}/2) = \nu + \lambda = \dim H^n_{\mathrm{et}}(U, \mathbb{Z}/2) + \lambda$ for all $n \geqslant 2$. □

There is a morphism of spectral sequences which on the E_2 level are the maps $H^*_M(X, \mathbb{Z}/2(i)) \to H^*_M(U, \mathbb{Z}/2(i))$.

COROLLARY 8.2.1. *Suppose that* $p < 0$. (a) *When either* $p \equiv 0$ *or* $p \equiv -1 \pmod 4$, *the differential* $E_2^{pp} \to E_2^{p+2,p-1}$ *is zero. Hence* $E_3^{pp} = \mathbb{Z}/2$.

(b) *When either* $p \equiv -2$ *or* $p \equiv -3 \pmod 4$, *the differential* $E_2^{pp} \to E_2^{p+2,p-1}$ *is an injection. Hence,* $E_3^{pp} = 0$.

Proof. This follows from chasing the following commutative diagram with $i = -p$:

$$\begin{array}{ccccc} \mathbb{Z}/2 = H^{0i}(\mathbb{R}) & = & H^{0i}(X) & = & H^{0i}(U) \\ {\scriptstyle d_{\mathbb{R}}}\downarrow & & {\scriptstyle d_X}\downarrow & & {\scriptstyle d_U}\downarrow \\ H^{3,i+1}(\mathbb{R}) & \longrightarrow & H^{3,i+1}(X) & \longrightarrow & H^{3,i+1}(U). \end{array}$$

The maps in the bottom row are induced by the closed embeddings $\operatorname{Spec}\mathbb{R} \hookrightarrow X$ (for each of the λ points of $X - U$) and by the open inclusion $U \subset X$.

When $p \equiv 0$ or $p \equiv -1 \pmod 4$, the left vertical $d_{\mathbb{R}}$ is zero by [RW, 5.3], so $d_X = 0$. When $p \equiv -2$ or $p \equiv -3 \pmod 4$, we saw in (6.7) that the right vertical d_U is an injection, so d_X must also be an injection. □

COROLLARY 8.2.2. *If* $i \geqslant 3$, *the* d_2*-differential* $H^{2,i}(X) \xrightarrow{d_X} H^{5,i+1}(X)$ *is:*

(1) *an isomorphism if* $i \equiv 0 \pmod 4$;
(2) *of rank* ν *if* $i \equiv 1 \pmod 4$;
(3) *of rank* λ *if* $i \equiv 3 \pmod 4$.

We will see in 8.3 that $d_X = 0$ in the missing case $i \equiv 2 \pmod 4$.

Proof. Using lemma 8.2 and 1.4, the rows are exact in the commutative diagram:

$$\begin{array}{ccccc} 0 \to NH^{2,i}(X) & \longrightarrow & H^{2,i}(X) & \longrightarrow & H^{2,i}(U) \to 0 \\ \downarrow{\scriptstyle Nd_X} & & \downarrow{\scriptstyle d_X} & & \downarrow{\scriptstyle d_U} \\ 0 \to NH^{5,i+1}(X) & \longrightarrow & H^{5,i+1}(X) & \longrightarrow & H^{5,i+1}(U) \to 0. \end{array}$$

By Lemma 8.1, the left vertical map Nd_X is an isomorphism when $i \equiv 3, 4 \pmod 4$, and zero if $i \equiv 1, 2 \pmod 4$. By (6.7), the right vertical d_U is an isomorphism when $i \equiv 0, 1 \pmod 4$, and zero if $i \equiv 2, 3 \pmod 4$. When $i \equiv 0 \pmod 4$, both Nd_X and d_U are isomorphisms; hence d_X is an isomorphism. When $i \equiv 1 \pmod 4$, the map Nd_X is zero and hence d_X has rank ν. When $i \equiv 3 \pmod 4$, the map d_U is zero and hence d_X has rank λ. □

Since we are interested in the columns of (7.5), it is convenient to index things by $p = 2 - i$. We first handle the even columns.

LEMMA 8.3. *When* $p \equiv 2 \pmod 4$ *and* p *is negative,*

(1) $E_3^{pq} = 0$ *for all* $q \neq p - 1$;

(2) $E_3^{p,p-1}$ *is the kernel* $\tilde{H}^{1,1-p}$ *of the surjection* $H^{1,1-p} \xrightarrow{d} H^{4,2-p} \cong (\mathbb{Z}/2)^{\nu+\lambda}$. *Hence* $E_3^{p,p-1} \cong (\mathbb{Z}/2)^{g+r_2-\lambda}$.

When $p \equiv 0 \pmod 4$ *and* $p \leqslant 0$,

(1) $E_3^{pq} = 0$ *for all* $q \leqslant p - 4$;

(2) $E_3^{p,p-3} = H^{3,3-p}/(\mathbb{Z}/2)$ *has rank* $\nu + \lambda - 1$.

(3) $E_3^{pp} = \mathbb{Z}/2$, $E_3^{p,p-1} = E_2^{p,p-1} = H^{1,1-p}$ *and* $E_3^{p,p-2} = E_2^{p,p-2} = (\mathbb{Z}/2)^{\nu+\lambda}$.

Proof. First assume that $p \equiv 2 \pmod 4$ and set $i = 2 - p \geqslant 4$. We saw in 8.2.2(1) that the differential $E_2^{p,p-2} = H^{2,i} \to H^{5,i+1}$ is an isomorphism, so it has full rank $\nu + \lambda$. By 7.4, $E_2^{pq} \to E_2^{p+2,q-1} \cong (\mathbb{Z}/2)^{\nu+\lambda}$ also has full rank $\nu + \lambda$ (i.e., is onto) for all $q < p$. Hence $E_3^{p+2,q-1} = 0$ for $q \leqslant p - 1$. For $q = p - 1$ we have $E_2^{pq} = H^{1,1-p}$ so $E_3^{p,p-1} = \tilde{H}^{1,1-p}$. For $q \leqslant p - 2$ we have $E_2^{pq} \cong (\mathbb{Z}/2)^{\nu+\lambda}$ so $E_3^{pq} = 0$. Finally, the case $q = p$ is handled in 8.2.1.

When $p \equiv 0 \pmod 4$ we argue as follows. We have already seen that $E_3^{pq} = 0$ if $q \leqslant p - 4$ in the last paragraph. If $p = 0$ and $q > -4$, the description of E_3^{0q} is immediate from (7.5) and 8.2.1(b). For $p < 0$ we have chain complexes

$$0 \to E_2^{pp} \xrightarrow{d} E_2^{p+2,p-1} \xrightarrow{d \cong} E_2^{p+4,p-2}$$
$$0 \to E_2^{p,p-1} \xrightarrow{d} E_2^{p+2,p-2} \xrightarrow{d \cong} E_2^{p+4,p-3}$$
$$0 \to E_2^{p,p-2} \xrightarrow{d} E_2^{p+2,p-3} \xrightarrow{d \cong} E_2^{p+4,p-4}$$

We have seen that the second differential is an isomorphism, so the first differentials must be zero. Thus $E_3^{pq} = E_2^{pq}$ for $q = p, p - 1, p - 2$. This proves (3). Finally, we use the chain complex

$$0 \to E_2^{p-2,p-2} \xrightarrow[\text{into}]{d} E_2^{p,p-3} \xrightarrow{d} E_2^{p+2,p-4} \xrightarrow{d \cong} E_2^{p+4,p-5}.$$

Again, we have seen that the third differential is an isomorphism, so the second differential must be zero. We have also shown in 8.2.1 that the first differential is an injection. Since $E_2^{p,p-3} \cong (\mathbb{Z}/2)^{\nu+\lambda}$, we get $E_3^{p,p-3} \cong (\mathbb{Z}/2)^{\nu+\lambda-1}$, as claimed in (2). □

LEMMA 8.4. *When* $p \equiv -3 \pmod 4$ *and p is negative, then* $E_3^{pq} = 0$ *for all* $q \leqslant p - 4$. *Also* $E_3^{pp} = 0$ *by* 8.2.1, $E_3^{p,p-1} \cong (\mathbb{Z}/2)^{g+r_2}$, *and* $E_3^{p,p-2} \cong E_3^{p,p-3} \cong (\mathbb{Z}/2)^{\lambda}$.

When $p \equiv -1 \pmod 4$ *and p is negative, then* $E_3^{pq} = 0$ *for all* $q \leqslant p - 4$. *Also* $E_3^{pp} \cong \mathbb{Z}/2$, $E_3^{p,p-1} \cong \tilde{H}^{1,1-p}$, $E_3^{p,p-2} \cong (\mathbb{Z}/2)^{\nu}$ *and* $E_3^{p,p-3} \cong (\mathbb{Z}/2)^{\nu-1}$.

Proof. When $p = 4k - 1$ and $q \leqslant p - 4$ the group $E_2^{pq} \cong (\mathbb{Z}/2)^{\nu+\lambda}$ fits into a chain complex

$$E_2^{p-2,q+1} \xrightarrow{\nu} E_2^{pq} \xrightarrow{\lambda} E_2^{p+2,q-1}. \tag{8.4.1}$$

The map labeled 'λ' has rank λ; for $p = -1$ this is 8.1.1, while for $p \leqslant -5$ this follows from 8.2.2 (with $i = 2 - p$) and 7.4. The map labeled 'ν' has rank ν, by 8.2.2 (with $i = p - 4$) and 7.4. Similarly, when $p = 4k - 3$ and $q \leqslant p - 4$ the group $E_2^{pq} \cong (\mathbb{Z}/2)^{\nu+\lambda}$ fits into a chain complex

$$E_2^{p-2,q+1} \xrightarrow{\lambda} E_2^{pq} \xrightarrow{\nu} E_2^{p+2,q-1}. \tag{8.4.2}$$

Again, the map labeled 'ν' has rank ν, by 8.2.2 (with $i = 2 - p$) and 7.4. The map labeled 'λ' has rank λ, by 8.2.2 (with $i = 4 - p$) and 7.4. It follows that $E_3^{pq} = 0$ in both of these cases.

We need more ad hoc arguments when $q \geqslant p - 3$. The case $q = p$ is given by 8.2.1. When $p = -1$, we know from 8.1.1 that $E_2^{-1,q} \to E_2^{1,q-1} = (\mathbb{Z}/2)^{\lambda}$ is onto for $q \leqslant -2$. The calculations of $E_3^{-1,q}$ are immediate for $q = -2, -3$, and for $q = -4$ using 8.2.1 with $p = -3$.

Now suppose that $p < -1$ and p is odd. When $q = p - 2$, we know from 8.2.2 (with $i = 2 - p$) that there is a chain complex like (8.4.1) or (8.4.2), except that the leftmost term is zero; this yields the given descriptions of $E_3^{p,p-2}$. When $q = p - 1$ and $p \equiv -1$ the extended version of (8.4.1) is

$$0 \to E_2^{p,p-1} \xrightarrow{d} E_2^{p+2,p-2} \xrightarrow{\nu} E_2^{p+4,p-3}. \tag{8.4.3}$$

The map d has rank λ, by 8.2.2 (with $i = 2 - p$) and 7.4(1), and the map 'ν' has rank ν by 8.2.2 (with $i = -p$) and 7.4(2). This yields $E_3^{p,p-1} \cong \bar{H}^{1,1-p}$. Since $E_2^{p+2,q-1} \cong (\mathbb{Z}/2)^{\nu+\lambda}$ it follows that d has rank exactly λ, giving $E_3^{p,p-1} = \bar{H}^{1,1-p}$. The argument for $q = p - 1$ and $p \equiv -3$ is identical, with ν and λ interchanged.

Finally, to compute $E_3^{p,p-3}$ we consider the chain complex

$$E_2^{p-2,p-2} \xrightarrow{d} E_2^{p,p-3} \xrightarrow{d} E_2^{p+2,p-4}.$$

If $p \equiv -1 \pmod 4$, we know from 8.2.1 that the left differential d is an injection, and from 8.2.2 (with $i = 2 - p$) and 7.4 that the right differential d has rank λ. Since $E_2^{p,p-3}$ has rank $\nu + \lambda$ and $E_2^{p-2,p-2} = \mathbb{Z}/2$, we have $E_3^{p,p-3} \cong (\mathbb{Z}/2)^{\nu-1}$.

If $p \equiv -3 \pmod 4$, we know from 8.2.1 that the left differential d is zero, and from 8.2.2 (with $i = 2 - p$) and 7.4 that the right differential d has rank ν. Since $E_2^{p,p-3}$ has rank $\nu + \lambda$ we have $E_3^{p,p-3} \cong (\mathbb{Z}/2)^{\lambda}$. □

This completes the verification of Claim 7.6, and hence the proofs of both Theorem 7.2 and the Main Theorem 0.1.

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