K-THEORY HOMOLOGY OF SPACES

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ABSTRACT. Let KR be a nonconnective spectrum whose homotopy groups give the algebraic K-theory of the ring R. We give a description of the associated homology theory $\mathrm{KR}_*(X)$ associated to KR. We also show that the various constructions of KR in the literature are homotopy equivalent, and so give the same homology theory

0. INTRODUCTION

There is a generalized homology theory E_* associated to every spectrum E, namely

$$E_n(X) = \lim_{i\to\infty} \pi_{n+i}(E_i \wedge X)$$

In particular this is true if E is the nonconnective K-theory spectrum KR of a ring R. In this paper, we give a geometric interpretation of $\mathrm{KR}_n(X)$ for $n \leq 0$ (and a new interpretation for $n > 0$)

Let X be a subcomplex of S^n, and form the open cone $O(X)$ on X inside R^{n+1} (which is the open cone on S^n). There is a category $\mathcal{C} = \mathcal{C}_{O(X)}(R)$, whose objects are based free R-modules parameterized in a locally finite way by $O(X)$, and whose morphisms are linear maps moving the bases a bounded amount. (Compare with Quinn's geometric R-modules in [13]). The group $K_1(\mathcal{C})$ is generated by the automorphisms in $\mathcal{C}$, with well-known relations [1]; our main theorem yields the formula:

$$\mathrm{KR}_0(X) \cong K_1(\mathcal{C})$$

When $R = \mathbb{Z}\pi$, these groups appear as obstruction groups of bounded (or thin) h-cobordisms parameterized by $O(X)$ with constant fundamental group π. (See [8] for a relatively elementary proof of this. This, of course, is in accordance with the basic results of Chapman [2, 3] and Quinn [11, 12]).

The negative KR-homology groups of X can be obtained from the formula $\mathrm{KR}_{-n}(X) = \mathrm{KR}_0(S^n X)$. The positive KR-homology groups are Quillen's higher K-groups of $\mathcal{C}$:

$$\mathrm{KR}_n(X) \cong K_{n+1}(\mathcal{C}) \quad \text{for} \quad n \geq 0.$$

To make the proofs easier, it turns out to be better to generalize the above discussion, replacing the category $\mathcal{F}_R$ of finitely generated based free R-modules by any additive category

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$\mathcal{A}$. We impose the semisimple exact structure of [1], i. e., declaring every short exact sequence split, in order to compute the K-theory of $\mathcal{A}$; this makes Bass' groups $K_1(\mathcal{A})$ the same as Quillen's by [17].

This being said, we can generalize the spectrum KR (for $\mathcal{A} = \mathcal{F}_R$) to the nonconnective spectrum $K\mathcal{A}$ constructed in [9] and ask about $K\mathcal{A}$-homology. There is a category $\mathcal{C}_{O(X)}(\mathcal{A})$, generalizing $\mathcal{C}_{O(X)}(R)$ and described in §1 below. In §2 we make this construction functorial in X. Our Main Theorem is carefully stated in §3 and proved in §4:

Main Theorem. *The $K\mathcal{A}$-homology of X is naturally isomorphic to the algebraic K-theory of the idempotent completion $\mathcal{C}^\wedge$ of $\mathcal{C} = \mathcal{C}_{O(X)}(\mathcal{A})$, with a degree shift:*

$$K\mathcal{A}_{*-1}(X) \cong K_*(\mathcal{C}_{O(X)}(\mathcal{A})^\wedge).$$

Note that $K_*(\mathcal{C}^\wedge) = K_*(\mathcal{C})$ for $* \geq 1$ but that $K_0(\mathcal{C}^\wedge) \neq K_0(\mathcal{C})$ in general. For example if $\mathcal{C} = \mathcal{F}_R$, then $K_0(\mathcal{C})$ is $\mathbb{Z}$ (or a quotient of $\mathbb{Z}$) but $K_0(\mathcal{C}^\wedge) = K_0(R)$. If $* < 0$ and $\mathcal{C} \neq \mathcal{C}^\wedge$, the groups $K_*(\mathcal{C})$ are not even defined. As in [9], the key technical step is an application of Thomason's double mapping cylinder construction from [15].

The knowledgeable reader will wonder about the relationship between the spectrum $K\mathcal{A}$ and other nonconnective spectra in the literature. We pin down this loose end in §6. When $\mathcal{A} = \mathcal{F}_R$ we show that our spectrum $K\mathcal{F}_R$ is homotopy equivalent to the Gersten-Wagoner spectrum of [4, 16]. For general $\mathcal{A}$, we prove that our spectrum $K\mathcal{A}$ is homotopy equivalent to Karoubi's spectrum [6]. Actually, the discussion in [6] only mentions K_0 and K_1, and not spectra, since it was written before higher K-theory emerged. We devote §5 to showing that Karoubi's prescription in [6] actually gives an infinite loop spectrum. The authors want to thank W. Vogell for pointing out an error in an earlier draft of this paper.

1. The functor $\mathcal{C}$

In this section we generalize the functor $\mathcal{C}_i(\mathcal{A})$ considered in [9] to a functor in two variables. We have already described how $\mathcal{C}_i(-)$ is an endofunctor of the category of filtered additive categories [9]. We remind the reader that a filtered additive category is an additive category $\mathcal{A}$, that comes with a filtration of the homsets $\operatorname{Hom}(A, B)$

$$0 \subseteq F_0 \operatorname{Hom}(A, B) \subseteq F_1 \operatorname{Hom}(A, B) \subseteq \ldots \subset Hom(A, B)$$

such that

a) $F_i \operatorname{Hom}(A, B)$ is a subgroup and $\operatorname{Hom}(A, B) = \cup F_i \operatorname{Hom}(A, B)$.
b) $F_0 \operatorname{Hom}(A, A)$ contains 0_A and 1_A for each A, all coherence isomorphisms of $\mathcal{A}$, all projections $A \oplus B \to A$ and all inclusions $A \to A \oplus B$.
c) if $f \in F_i \operatorname{Hom}(A, B)$ and $g \in F_j \operatorname{Hom}(B, C)$ then $g \circ f \in F_{i+j} \operatorname{Hom}(A, C)$.

Note that any additive category may be endowed with "discrete filtration", in which

$$F_0 \operatorname{Hom}(A, B) = \operatorname{Hom}(A, B)$$

for every A, B.

Thinking of the lower index i as the metric space $\mathbb{Z}^i$ or $\mathbb{R}^i$, we shall now turn $\mathcal{C}$ into a functor of that variable. (See [9, Remark 1,2,3]).

Definition 1.1. Let X be a metric space and $\mathcal{A}$ a filtered additive category. We then define the filtered category $\mathcal{C}_X(\mathcal{A})$ as follows:

1) An object A of $\mathcal{C}_X(\mathcal{A})$ is a collection of objects $A(x)$ of $\mathcal{A}$, one for each $x \in X$, satisfying the condition that for each ball $B \subset X$, $A(x) \neq 0$ for only finitely many $x \in B$.
2) A morphism $\phi : A \to B$ is a collection of morphisms $\phi^x_y : A(x) \to B(y)$ in $\mathcal{A}$ such that there exists r depending only on ϕ so that
   (a) $\phi^x_y = 0$ for $d(x, y) > r$
   (b) all ϕ^x_y are in $F_r \operatorname{Hom}(A(x), B(y))$
   (We then say that ϕ has filtration degree $\leq r$.

Composition of $\phi : A \to B$ with $\psi : B \to C$ is given by $(\psi\phi)^x_z = \sum_{y \in X} \psi^y_z \phi^x_y$. Notice that the sum makes sense because the category is additive and because the sum will always be finite. The category $\mathcal{C}_{\mathbb{Z}^i}(\mathcal{A})$ is the category $\mathcal{C}_i(\mathcal{A})$ of [9].

We now introduce the category $\mathcal{M}$ of metric spaces and proper, eventually Lipschitz maps.

Definition 1.2. The category $\mathcal{M}$ has objects metric spaces X. A morphism $f : X \to Y$ must be both proper and eventually Lipschitz. We remind the reader that a map $f : X \to Y$ is Lipschitz if there exists a number $k \in \mathbb{R}_+$ such that

$$d(f(x), f(y)) \leq k d(x, y)$$

We say that f is eventually Lipschitz if there exist r and k, only depending on f, so that

$$\forall x, y \in X, \forall s \in \mathbb{R}_+ : \quad \text{if} \quad s > r \quad \text{and} \quad d(x, y) < s \quad \text{then} \quad d(f(x), f(y)) < k \cdot s.$$

Finally, we call f proper if the inverse image of a bounded set is bounded.

Example 1.3. One should note that maps in $\mathcal{M}$ are not necessarily continuous, but any jumps allowed must be universally bounded. For example the map $\mathbb{R} \to \mathbb{Z}$ sending a real number x to the greatest integer smaller than x is a map in $\mathcal{M}$.

Given a proper eventually Lipschitz map $f : X \to Y$ we obtain a functor $f_* : \mathcal{C}_X(\mathcal{A}) \to \mathcal{C}_Y(\mathcal{A})$ by defining $(f_*(A))_y = \oplus_{z \in f^{-1}(y)} A(z)$ for objects A in $\mathcal{C}_X(\mathcal{A})$. Since the inverse of a bounded set is bounded, there are only finitely many nonzero modules in a ball in Y, and $f_*(A))$ is well-defined. On morphisms f_* is induced by the identity. The eventually Lipschitz conditions on f ensures that we indeed do get morphisms in the category $\mathcal{C}_Y(\mathcal{A})$. Hence $\mathcal{C}_-(\mathcal{A})$ is a functor from $\mathcal{M}$ to (semisimple filtered) additive categories.

Lemma 1.4. *Let X and Y be metric spaces and give $X \times Y$ the max metric,*

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \max(d_X(x_1, x_2), d_Y(y_1, y_2)),$$

then $\mathcal{C}_{X \times Y}(\mathcal{A}) = \mathcal{C}_X(\mathcal{C}_Y(\mathcal{A}))$.

Proof. This is where we use that $\mathcal{C}$ takes values in filtered categories. The internal filtration degree will control distances in the Y component, while the external filtration degree will control distances in the X component and thus the max distance will be controlled. □

Remark. Note that the isomorphism class in $\mathcal{M}$ is not affected if we change the metric to a proper Lipschitz equivalent metric, i. e. a metric so that the identity map is a proper Lipschitz equivalence both ways. Therefore Lemma 1.4 remains true up to natural equivalence if the max metric is replaced by the usual product metric

$$d_{X\times Y}((x_1,x_2),(y_1,y_2)) = \sqrt{d_X(x_1,x_2)^2 + d_Y(y_1,y_2)^2}$$

The concept homotopy is introduced in the category $\mathcal{M}$ in a standard fashion using the inclusions $X = X \times \{0\} \to X \times I$, $X = X \times \{1\} \to X \times I$ and the projections $X \times I \to X$ Since $X \times I$ has the max metric, these are maps in the category $\mathcal{M}$. Note that the inclusion $\mathbb{Z} \to \mathbb{R}$ is a homotopy equivalence in $\mathcal{M}$, with homotopy inverse the greatest integer function of example 1.3.

Lemma 1.5. *A compact metric space X is homotopy equivalent to a point in $\mathcal{M}$ and hence $\mathcal{C}_X(\mathcal{A})$ is equivalent to the category $\mathcal{A}$.*

Proof. Since maps are not required to be continuous and a compact metric space is globally bounded, any map $X \times I \to X$ which is constant on the top and the identity on the bottom will be a contracting homotopy. The second assertion follows from the evident fact that $\mathcal{C}_X(\mathcal{A}) = \mathcal{A}$ when X is a point. □

Proposition 1.6. *The functor $\mathcal{C}_{-}(\mathcal{A})$ is homotopy invariant. That is, for each metric space X, the inclusion $X \subset X \times I$ as $X \times 0$ and the projection $X \times I \to X$ induce an equivalence of categories*

$$\mathcal{C}_X(\mathcal{A}) \to \mathcal{C}_{X\times I}(\mathcal{A}).$$

Proof. By 1.4 and 1.5, $\mathcal{C}_{X\times I}(\mathcal{A}) = \mathcal{C}_I(\mathcal{C}_X(\mathcal{A}))$ is homotopic to $\mathcal{C}_X(\mathcal{A})$. □

2. Open Cones

In this section we construct an open cone functor $O(X)$ from finite PL complexes to $\mathcal{M}$, so that $\mathcal{C}_{O(X)}(\mathcal{A})$ depends functorially on X in a homotopy invariant way.
To fix notation let $S^0 = \{-1, 1\} \subset \mathbb{R}$. Then the n-sphere is the join $S^n = S^0 * S^0 * \cdots * S^0$ as a sub PL-complex of $\mathbb{R}^{n+1}$. We shall be considering the category of finite sub PL-complexes of S^∞ (PL complexes that are subcomplexes of some S^n) and PL morphisms. We denote this category $\mathcal{PL}$. This is essentially the category of finite PL-complexes, since any such may be embedded in some S^n, but we need the way the complex sits in S^n as part of the structure. We think of $\mathbb{R}^{n+1}$ as a metric space using the max metric

$$d(\underline{x}, \underline{y}) = \max |x_i - y_i|.$$

This induces a metric on the n-sphere and hence on any subcomplex.

We now construct a functor O from $\mathcal{PL}$ to $\mathcal{M}$.

Definition 2.1. O sends an object $X \subset S^n$ to $O(X) = \{t \cdot x \in \mathbb{R}^{n+1} | t \in [0, \infty), x \in X\}$ with metric induced from $\mathbb{R}^{n+1}$. For example $O(S^n) = \mathbb{R}^{n+1}$. On morphisms $f : X \to Y$ we extend to $O(f) : O(X) \to O(Y)$ by linearity: $f(t \cdot x) = t \cdot f(x)$. One checks easily that $O(f)$ is proper and Lipschitz (and therefore eventually Lipschitz) using the following well-known

Lemma 2.2. *A* PL-*map $f : X \to Y$ between finite complexes is Lipschitz.*

Proof. Triangulate so that f is linear on each simplex. Since there are only finitely many simplices, f will be Lipschitz. □

Remark 2.3. For a PL complex X, $O(X)$ does not really depend on the embedding $X \subset S^n$, since a PL homeomorphism $X_1 \subset S^n$ to $X_2 \subset S^m$ will induce a proper Lipschitz homeomorphism from $O(X_1) \to O(X_2)$.

Lemma 2.4. *Let X and Y be two* PL *complexes and let $*$ denote the join. Then*

$$O(X * Y) \cong O(X) \times O(Y).$$

In particular, $O(\Sigma X) = O(X * S^0) \cong O(X) \times \mathbb{R}$ and $O(CX) \cong O(X) \times [0, \infty)$.

Proof. Embed $X \subset S^n$ and $Y \subset S^m$ so $X * Y \subset S^n * S^m = S^{n+m+1}$. A point in $O(X * Y)$ is $s \cdot (t \cdot x + (1-t) \cdot y) = stx + s(1-t)y$ and stx lies in $O(X) \subset \mathbb{R}^{n+1} \times 0$, whereas $s(1-t)y$ lies in $O(Y) \subset 0 \times \mathbb{R}^{m+1}$. The last sentence follows from the identities $\Sigma X = X * S^0$, $CX = X * (\text{point})$, $O(S^0) = \mathbb{R}$ and $O(\text{point}) = [0, \infty)$. □

Remark 2.5. The functor O is not homotopy invariant. We only obtain homotopy invariance after passing to K-theory.

3. The Main Theorem

So far we have constructed functors

$$O : \mathcal{PL} \to \mathcal{M} \quad \text{and}$$
$$\mathcal{C}_-(\mathcal{A}) : \mathcal{M} \to \text{(semisimple) additive categories}$$

For our main theorem, we need a functor K_* from additive categories to graded abelian groups. For $n \geq 1$, there is no problem: given an additive category $\mathcal{A}$ we take $K_n\mathcal{A} = \pi_n(\Omega BQ\mathcal{A})$ as in [10], using the semisimple exact structure (in which all exact sequences split). However, unless $\mathcal{A}$ is idempotent complete, the groups $K_0\mathcal{A}$ may be wrong for our purposes, and $\mathcal{A}$'s negative K-groups will not even be defined.

For example consider the category $\mathcal{F}_R$ of finitely generated based free R-modules. when R is a group ring, we know that the geometrically interesting group is not $K_0(\mathcal{F}_R) = \mathbb{Z}$, but rather $K_0(R)$, which measures projective modules.

To handle this problem, we pass to the idempotent completion $\mathcal{A}^\wedge$ of $\mathcal{A}$. This provides the correct group $K_0(\mathcal{A}^\wedge)$ and does not change the higher groups, since $K_n(\mathcal{A}) = K_n(\mathcal{A}^\wedge)$

for $n \geq 1$ For example ${\mathcal{F}_R}^\wedge$ is equivalent to the category of finitely generated projective R-modules.

Scholium. $\mathcal{A}^\wedge$ inherits the structure of a filtered additive category from $\mathcal{A}$. The objects of $\mathcal{A}^\wedge$ are pairs (A, p), where $p : A \to A$ is idempotent. A morphism ϕ from (A_1, ϕ_1) to $(A_2.\phi_2)$ is an $\mathcal{A}$-morphism $\phi : A_1 \to A_2$ with $\phi = p_2\phi p_1$. The filtration degree of ϕ is the smallest d such that $\phi = p_2 f p_1$ for some $f \in F_d \operatorname{Hom}(A_1, A_2)$ satisfying $fp_1 = p_2 f$. This filtration should have been stated explicitly in [9, 1.4].

If $\mathcal{A}$ is idempotent complete, the negative K-groups of $\mathcal{A}$ were defined by Karoubi in [6], and agree with the definition in [9] (as we shall see in §6 below). If $\mathcal{A} = \mathcal{F}_R$, they agree with Bass' negative K-groups $K_n(R)$.

The construction in [9] actually gives us slightly more information. If $\mathcal{A}$ is idempotent complete, it yields a nonconnective infinite loop spectrum $K\mathcal{A}$, and the homotopy groups of $K\mathcal{A}$ are the groups $K_*\mathcal{A}$ above.

Now associated to any spectrum such as $K\mathcal{A}$ is a reduced homology theory $K\mathcal{A}_*$. It is defined by

$$K\mathcal{A}_n(X) = \lim \pi_{n+i}((K\mathcal{A})_n \wedge X).$$

The coefficients of this homology theory are the groups

$$K\mathcal{A}_n(S^0) = \pi_n(K\mathcal{A}) = K_n(\mathcal{A}).$$

We can now state our main theorem.

Theorem 3.1. *If $\mathcal{A}$ is an idempotent complete additive category, the functor from $\mathcal{PL}$ to graded abelian groups sending X to $K_*(\mathcal{C}_{O(X)}(\mathcal{A})^\wedge)$ is the $K\mathcal{A}_*$-homology theory of X, with a degree shift:*

$$K_*(\mathcal{C}_{O(X)}(\mathcal{A})^\wedge) \cong K\mathcal{A}_*(X).$$

Remark 3.2. This theorem had previously been known for spheres. For $X = S^i, O(S^i) = \mathbb{R}^{i+1}$, which is homotopy equivalent to $\mathbb{Z}^{i+1}$ in $\mathcal{M}$, so therefore by 1.4

$$K_*(\mathcal{C}_{\mathbb{R}^i}(\mathcal{A})^\wedge) = K_*(\mathcal{C}_{\mathbb{Z}^i}(\mathcal{A})^\wedge),$$

which was shown in [9] to equal $K_{*-i-1}(\mathcal{A}^\wedge)$. The category $\mathcal{C}_{\mathbb{Z}^{i+1}}(\mathcal{A})$ was first studied in [8], where $\mathcal{A}$ was the category of finitely generated R-modules. There it was shown that

$$K_1(\mathcal{C}_{\mathbb{Z}^{i+1}}(R)) = K_{-i}(R)$$

which is equal to $\mathrm{KR}_i(S^0) = \mathrm{KR}_0(S^i)$, thus agreeing with our main theorem.

4. Proof of main theorem

Define the functor f as the composite

$$\mathcal{PL} \xrightarrow{O} \mathcal{M} \xrightarrow{\mathcal{C}_-(\mathcal{A})} \text{filtered add. categ.} \xrightarrow{\square} \text{add. cat.} \xrightarrow{\wedge}$$

$$\text{idempotent complete add. cat.} \xrightarrow{\Omega^\infty K} \text{Top. spaces}$$

Here $\square$ is the forgetful functor, $^\wedge$ is idempotent completion and $\Omega^\infty K$ is the zeroth space of the infinite loop spectrum K giving the K-theory of the category, either ΩBQ or the result of applying an infinite loop machine to the symmetric monoidal category of isomorphisms (see Thomason for a very functorial construction [15]). That is :

$$f(X) = \Omega^\infty K(\mathcal{C}_{O(X)}(\mathcal{A})^\wedge).$$

Lemma 4.1. *If X is a cone, $X = CK$, then $f(X)$ is contractible.*

Proof. $\mathcal{C}_{O(CK)}(\mathcal{A}) = \mathcal{C}_{[0,\infty)\times O(K)}(\mathcal{A}) = \mathcal{C}_1(\mathcal{C}_{O(K)}(\mathcal{A}))$ in the notation of [9]. It was proven in [9, (3.1)] that $\Omega^\infty K\mathcal{C}_+(\mathcal{A})$ is contractible for an arbitrary category, $\mathcal{A}$, and the argument given there applies verbatim to show that

$$f(X) = \Omega^\infty K((\mathcal{C}_+(\mathcal{C}_{O(K)}\mathcal{A})^\wedge)$$

is also contractible.

□

Proposition 4.2. *For each X, $\Omega f(\Sigma X)$ is homotopy equivalent to $f(X)$. In particular $f(X)$ is an infinite loop space.*

Proof. By 1.6 and 2.4 we have

$$\mathcal{C}_{O(\Sigma X)} = \mathcal{C}_{O(X)\times\mathbb{R}} = \mathcal{C}_{\mathbb{R}}(\mathcal{C}_{O(X)}(\mathcal{A})) = \mathcal{C}_1(\mathcal{C}_{O(X)}(\mathcal{A})).$$

By [9, Theorem (3.2)], applied to the filtered additive category $\mathcal{C}_{O(X)}(\mathcal{A})^\wedge$, we know that the loop space of $\Omega^\infty K\mathcal{C}_{O(\Sigma X)}(\mathcal{A})$ is homotopy equivalent to $f(X)$. But by the cofinality theorem, the spaces $\Omega^\infty K(\mathcal{C})$ and $\Omega^\infty K(\mathcal{C}^\wedge)$ have homotopy equivalent connected components, and hence homotopy equivalent loop spaces, for any $\mathcal{C}$. □

Theorem 4.3. *The functor f sends cofibrations to fibrations.*

Before proving this result , let us draw a quick consequence. It follows from Lemma 4.1 that f is homotopy invariant, since $X \to X\times I \to CX$ is a cofibration. The homotopy groups of the spectrum $\{f(X), f(\Sigma X), f(\Sigma^2 X), \ldots\}$ coincide with the groups $K_*(\mathcal{C}_{O(X)}(\mathcal{A})^\wedge)$. These groups are homotopy invariants of X which vanish when X is contractible by 4.1. From 4.3 we immediately obtain

Corollary 4.4. *The functor from $\mathcal{PL}$ to graded abelian groups sending X to $K_*(\mathcal{C}_{O(X)}(\mathcal{A})^\wedge)$ is a reduced homology theory.*

Proof of Theorem 4.3. The special case $X \subset CX \subset \Sigma X$ follows from 4.1 and 4.2. To do the general case, we proceed in a manner very much like the proof of Theorem 3.4 in [9]. Consider a cofibration $A \subset X \to X \cup CA$. Then we get

$$O(A) \subset O(X) \subset O(X) \cup_{O(A)} O(CA)$$

and a diagram

$$\begin{array}{ccc} \mathcal{C}_{O(A)}(\mathcal{A}) & \longrightarrow & \mathcal{C}_{O(CA)}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathcal{C}_{O(X)}(\mathcal{A}) & \longrightarrow & \mathcal{C}_{O(X\cup CA)}(\mathcal{A}) \end{array}$$

By 1.6 and 2.4 we have $O(CA) = O(A) \times [o, \infty)$ and

$$\mathcal{C}_{O(CA)}(\mathcal{A}) = \mathcal{C}_{[0,\infty)}(\mathcal{C}_{O(A)}(\mathcal{A})) = \mathcal{C}_{+}(\mathcal{C}_{O(A)}(\mathcal{A})).$$

To simplify notation, let us write $\underline{\underline{C}}_X$ for the category of isomorphisms in $\mathcal{C}_{O(X)}(\mathcal{A})^{\wedge}$. Using the "double mapping cylinder" pushout construction $\underline{\underline{P}}$ of Thomason [15, (5.1)], we get

$$\begin{array}{cccc} \underline{\underline{C}}_A & \longrightarrow & \underline{\underline{C}}_{CA} & \\ \downarrow & & \downarrow & \\ \underline{\underline{C}}_X & \longrightarrow & \underline{\underline{P}} & \\ & & & \searrow^{\Sigma} \\ & & & \underline{\underline{C}}_{X \cup CA} \end{array}$$

We wish to show that Σ induces a homotopy equivalence on certain components, i. e. that Σ induces a π_0-monomorphism and π_i-isomorphism. That π_0 behaves correctly is then obtained by applying the argument to the suspension of this cofibration. It is therefore enough to show that for every object Y of $\mathcal{C}_{O(X\cup CA)}(\mathcal{A})$, considered as an object of $\underline{\underline{C}}_{X \cup CA}$, that the category $Y \downarrow \Sigma$ is a contractible category. At this point we follow the proof of [9, Theorem (3.4)] very closely. We use the bound d to filter $Y \downarrow \Sigma$ as the increasing union of subcategories fil_d and show each of these has an initial object $*_d$.

Fil_d is the full subcategory of all iso's $\alpha : Y \to \Sigma(B^A, B^X, A^+)$, where B^X is an object of $\underline{\underline{C}}_X$, B^A and object of $\underline{\underline{C}}_A$ and A^+ an object of $\underline{\underline{C}}_{CA}$. We define Y_d^A, Y_d^X and Y_d^+ in $\underline{\underline{C}}_A$, $\underline{\underline{C}}_X$ and $\underline{\underline{C}}_{CA}$ as follows: Let

$$N_d = \{x \in O(X \cup CA) | \exists y \in O(A) : d(x, y) \le d\}.$$

Since $A \subseteq$ is a cofibration it is easy to see that N_d is proper Lipschitz homotopy equivalent to $O(A)$. Choose some proper Lipschitz homotopy equivalence $h : N_d \to O(A)$ and proceed as follows:

$$Y_d^X(x) = \begin{cases} Y(x) & \text{for} \quad x \in O(X) - N_d \\ 0 & \text{otherwise} \end{cases}$$

$$Y_d^+(x) = \begin{cases} Y(x) & \text{for} \quad x \in O(A) \times (d, \infty) = O(CA) - N_d \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_d^A = h_*(Y_d^A) \text{ where}$$

$$Y_d^A = \begin{cases} Y(x) & \text{for} \quad x \in N_d \\ 0 & \text{otherwise} \end{cases}$$

There is now an obvious isomorphism

$$\sigma : Y \cong Y_d^X \oplus Y_d^A \oplus Y_d^+ = \Sigma(Y_d^A, Y_d^X, Y_d^+)$$

bounded by d (essentially the identity) and we may proceed to prove this is an initial object in Fil_d exactly as in the proof of [9, Theorem 3.4]. □

We finish off the proof of the main theorem as follows:

Proof of main theorem 3.1. We need to identify the homology theory of 4.4 as the homology theory associated to th spectrum $K(\mathcal{A})$. It was proven by Thomas Gunnarson [5] that when a homotopy functor f sends cofibrations to fibrations and contractible spaces to contractible spaces, then the homology theory obtained by applying homotopy groups has as its representing spectrum $\{f(S^0), f(S^1), f(S^2), \ldots\}$. This is a well-known fact when the homology theory is connective - see e. g. [18, theorem 1.14], [14] or [7] - but in the general case we need to use [5] Now we have by Lemma 2.4 that

$$f(S^i) = \Omega^\infty K(\mathcal{C}_{i+1}(\mathcal{A})^\wedge).$$

The space $f(S^i)$ is therefore the $(i+1)^{\mathrm{st}}$-space of the spectrum $K(\mathcal{A})$ constructed in Theorem B of [9]. The representing spectrum for the homology theory of 4.4 is therefore $\Omega^{-1}K(\mathcal{A})$, and the main theorem follows. □

We are finished, except we need to show that the delooping given in [9] agrees with the other deloopings in the literature. This is the subject of the final 2 sections.

5. Karoubi's nonconnective spectra

In this section, we follow Karoubi's ideas in [6] and construct nonconnective K-theory spectra, whose negative homotopy groups are the negative K-groups defined by Karoubi in op. cit. We then show that these agree with the spectra constructed in [9] (and in special cases, in [4], [16]). More explicitly, but also more technically, we show that $\mathcal{C}_1(\mathcal{A}) \to \mathcal{C}_+(\mathcal{A})/\mathcal{A}$

induces an isomorphism on Quillen K-theory. We are indebted to Karoubi, who suggested this possibility to us in 1983.

We shall use Karoubi's delooping construction from [6], so we begin by recalling the construction. Let $\mathcal{A}$ be a full subcategory of an additive (hence semisimple exact) category $\mathcal{U}$. We shall use the notation that letters $A-F$ (resp $U-Z$) denote the objects of $\mathcal{A}$ (resp. $\mathcal{U}$), and that $U = E_\alpha \oplus U_\alpha$ means an internal direct sum decomposition of U with $E_\alpha \in \mathcal{A}$. We say that $\mathcal{U}$ is $\mathcal{A}$- filtered if every object U has a family of decompositions $\{U = E_\alpha \oplus U_\alpha\}$ (called a filtration) of U) satisfying the following axioms (cf. [6, pp. 114 ff.]):

- (F1) For each U, the decompositions form a poset under the partial order that $E_\alpha \oplus U_\alpha \leq E_\beta \oplus U_\beta$ whenever $U_\beta \subseteq U_\alpha$ and $E_\alpha \subseteq E_\beta$.
- (F2) Every map $A \to U$ factors $A \to E_\alpha \to E_\alpha \oplus U_\alpha = U$ for some α.
- (F3) Every map $U \to A$ factors $U = E_\alpha \oplus U_\alpha \to E_\alpha \to A$ for some α.
- (F4) For each U, V the filtration on $U \oplus V$ is equivalent to the sum of the filtrations $\{U = E_\alpha \oplus U_\alpha\}$ and $\{V = F_\beta \oplus V_\beta\}$ i. e., to $\{U \oplus V = E_\alpha \oplus F_\beta) \oplus (U_\alpha \oplus V_\beta)\}$.

We shall assume each filtration is saturated in the sense that if $U = E_\alpha \oplus U_\alpha$ is in the filtration and $E_\alpha = A \oplus B$ in $\mathcal{A}$, then $U = A \oplus (B \oplus U_\alpha)$ is also in the filtration. Finally, we say that $\mathcal{U}$ is flasque if there is a functor $\infty : \mathcal{U} \to \mathcal{U}$ and a natural transformation $U^\infty \cong U \oplus U^\infty$ ([6, p. 147]).
Our favorite selection of $\mathcal{U}$ is the following:

Example 5.1. The category $\mathcal{C}_+(\mathcal{A})$ of [9, (1.2.4)] is flasque and $\mathcal{A}$-filtered. Objects of $\mathcal{C}_+(\mathcal{A})$ are sequences $(A_0, A_1, \ldots)$ of objects in $\mathcal{A}$, and the morphisms are given by "bounded" matrices. $\mathcal{A}$ is the full subcategory of objects $(A_0, 0, 0, \ldots)$. The $\mathcal{A}$-filtration on an object $U = (A_0, A_1, \ldots)$ contains the decompositions

$$U \cong \mathrm{Fil}_n(U) \oplus (0, \ldots, 0, A_{n+1}, \ldots)$$
$$\mathrm{Fil}_n(U) = A_0 \oplus \ldots \oplus A_n \text{ in } \mathcal{A}.$$

We proved that $\mathcal{C}_+(\mathcal{A})$ was flasque in [9, (1.3)], using the translation $t(A_0, A_1, \ldots) = (0, A_0, A_1, \ldots)$ on $\mathcal{C}_+(\mathcal{A})$.

We now suppose given an $\mathcal{A}$-filtered category $\mathcal{U}$. Call a map $U \to V$ completely continuous (cc) if it factors through an object of $\mathcal{A}$. Karoubi defined $\mathcal{U}/\mathcal{A}$ to be the category with the same objects as $\mathcal{U}$, but with $\mathrm{Hom}_{\mathcal{U}/\mathcal{A}}(U, V) = \mathrm{Hom}_{\mathcal{U}}(U, V)/\{\text{cc maps}\}$.

Lemma 5.2. *Suppose $\mathcal{A}$ is idempotent complete and that $\overline{\phi} : U \to V$ is an isomorphism in $\mathcal{U}/\mathcal{A}$. Then there are decompositions $U = E_\beta \oplus U_\alpha$ and $V = F_\alpha \oplus V_\alpha$ in the (saturated) filtrations and a $\mathcal{U}$-isomorphism $U_\alpha \cong V_\alpha$ such that $\overline{\phi}$ is represented by $U \to U_\alpha \cong V_\alpha \to V$.*

Proof. Choose representatives ϕ, ψ for $\overline{\phi}, \overline{\phi}^{-1}$. Since $1_U - \psi\phi$ is cc, there is a decomposition $U = E_\alpha \oplus U_\beta$ with $1 = \psi\phi$ on U_β. Replace U by U_β to assume $1_U = \psi\phi$. Similarly, write $V = F \oplus W$ with $\phi\psi = 1_W$. Write $\phi = (\epsilon, i) : U \to F \oplus W$ and $\psi = (\delta, j) : F \oplus W \to U$.

Observe that $ij = 1_W$ and that

$$(*) \quad \delta\epsilon + ji = 1_U$$

Multiplying (*) by i, j and by (*) yields the equations $i\delta\epsilon = 0$, $\delta\epsilon j = 0$ and $(\delta\epsilon)^2 = \delta\epsilon$. Replacing ϵ by $\epsilon\delta\epsilon$ makes $(\epsilon\delta)^2 = \epsilon\delta$ without affecting $\overline{\phi}$ or (*). Set $F_\alpha = \ker(\epsilon\delta)$ and $V_\alpha = (\epsilon\delta F) \oplus W$; $F_\alpha \in \mathcal{A}$ because $\mathcal{A}$ is idempotent complete. The rest of the proof is straightforward, and left to the reader. □

Remark. The failure of this lemma when $\mathcal{A}$ is not idempotent complete is the Bass-Heller-Swan phenomenon. See [8, (1.16)].

Theorem 5.3. *Let $\mathcal{A}$ be semisimple and idempotent complete. Then for every $\mathcal{A}$-filtered category $\mathcal{U}$ the sequence*

$$K^Q(\mathcal{A}) \to K^Q(\mathcal{U}) \to K^Q(\mathcal{U}/\mathcal{A})$$

is a homotopy fibration, where $K^Q(\mathcal{A})$ denotes the space whose homotopy groups give the algebraic K-theory of $\mathcal{A}$.

Before proving this theorem, we draw our main conclusion. Suppose that in addition $\mathcal{U}$ is flasque; from additivity and the equation $\infty \cong 1 + \infty$ we conclude that $K^Q(\mathcal{U}) \simeq *$. Since $\mathcal{U}/\mathcal{A}$ shares the same objects as $\mathcal{U}$, we conclude that $K_0(\mathcal{U}/\mathcal{A}) = K_0(\mathcal{U}) = 0$. Finally, applying 5.3 to the diagram $\mathcal{U} \to \mathcal{C}_+(\mathcal{U}) \leftarrow \mathcal{C}_+(\mathcal{A})$ shows that $K^Q(\mathcal{U}/\mathcal{A}) \simeq K^Q(\mathcal{C}_+(\mathcal{A})/\mathcal{A})$. This proves:

Theorem/Definition 5.4. Let $\mathcal{A}$ be semisimple and idempotent complete. Choose a flasque, $\mathcal{A}$-filtered category $\mathcal{U}$ and define $S\mathcal{A}$ to be $\mathcal{U}/\mathcal{A}$. Then $K^Q(S\mathcal{A})$ is a connected space with $\Omega K^Q(S\mathcal{A}) \simeq K^Q(\mathcal{A})$, and the homotopy type of $K^Q(S\mathcal{A})$ is independent of the choice of $\mathcal{U}$.

Our proof of Theorem 5.3 follows the proof of [9, (3.4)]. Let $\underline{\underline{A}}$, $\underline{\underline{U}}$ and $\underline{\underline{S}}$ denote the categories of isomorphisms of $\mathcal{A}$, $\mathcal{U}$ and $\mathcal{U}/\mathcal{A}$. It is well-known that $K^Q(\mathcal{A})$ is the group completion of $B\underline{\underline{A}}$, and that $K^Q(\mathcal{A})$, $B\underline{\underline{A}}^{-1}\underline{\underline{A}}$ and $\mathrm{Spt}_0(\underline{\underline{A}})$ are homotopy equivalent. Let $\underline{\underline{Q}}$ denote the double mapping cylinder of $(0 \leftarrow \underline{\underline{A}} \to \underline{\underline{U}})$ given by Thomason in [15, (5.1)]. Thus objects of $\underline{\underline{Q}}$ are pairs (A, U), and a $\underline{\underline{Q}}$-map from (A, U) to (B, V) is an equivalence class of data $(E, F, A \cong E \oplus B \oplus F, F \oplus U \cong V)$. Thomason proves in [15, (5.2) and (5.5)] that $\mathrm{Spt}_0(\underline{\underline{A}}) \to \mathrm{Spt}_0(\underline{\underline{U}}) \to \mathrm{Spt}_0(\underline{\underline{Q}})$ is a homotopy fibration. Since $\underline{\underline{U}} \to \underline{\underline{S}}$ factors through $\underline{\underline{Q}}$, we see that 5.3 follows from [15, (2.3)] and the following result:

Proposition 5.5. *The functor $\Sigma : \underline{\underline{Q}} \to \underline{\underline{S}}$ given by $\Sigma(A, U) = A \oplus U$ is a homotopy equivalence when $\mathcal{A}$ is idempotent complete.*

Proof. Fix an object S of $\underline{\underline{S}}$; we will show that $S \downarrow \Sigma$ is a contractible category. The desired result will the follow from Quillen's Theorem A [10]. In order to do this, we need to thicken $S \downarrow \Sigma$ up a bit. Let $\mathcal{S}$ denote the category whose objects are tuples

$$\alpha = (A, U, S \cong D_\alpha \oplus S_\alpha, U \cong E_\alpha \oplus U_\alpha, f_\alpha : S_\alpha \cong U_\alpha)$$

where $A \in \mathcal{A}$, $U \in \mathcal{U}$, f_α is a $\mathcal{U}$-isomorphism, and the direct sum decompositions belong to the $\mathcal{A}$-filtrations of S and U. A map from α to

$$\beta = (B, V, S \cong D_\beta \oplus S_\beta, V \cong F_\beta \oplus V_\beta, f_\beta : S_\beta \cong V_\beta)$$

is just a map in $\underline{\underline{Q}}$ from (A, U) to (B, V), say given by

$$e = (E, F, A \cong E \oplus B \oplus F, F \oplus U \cong V)$$

such that $D_\alpha \oplus S_\alpha \geq D_\beta \oplus S_\beta$, $(F \oplus E_\alpha) \oplus U_\alpha \geq F_\beta \oplus V_\beta$ in the filtrations of S and V, and such that there is a commutative square in $\mathcal{U}$:

$$\begin{array}{ccc} S_\beta & \cong & V_\beta \\ \uparrow & & \uparrow \\ S_\alpha & \cong & U_\alpha \end{array}$$

There is a natural functor $\mathrm{pr} : \mathcal{S} \to (S \downarrow \Sigma)$ sending α to the object $\mathrm{pr}_\alpha : S \to S_\alpha \cong U_\alpha \to U \to A \oplus U = \Sigma(A, U)$ and e to itself. By Lemma 5.2, pr is onto. In fact pr is cofibered. We assert that pr is a homotopy equivalence, which follows from [10, p. 93] and the following: □

Sublemma 5.6. Given an isomorphism $\overline{\phi} : S \to \Sigma(A, U)$ in $\mathcal{U}/\mathcal{A}$, the fiber category $\mathrm{pr}^{-1}(\overline{\phi})$ is a cofiltered poset, and hence contractible.

Proof. Set $\Phi = \mathrm{pr}^{-1}(\overline{\phi})$; it is clear from the definition of $\mathcal{S}$ that Φ is a poset. We need only show that for each α, β in Φ there is a diagram $\alpha \leftarrow \gamma \to \beta$ in Φ. To do this, we introduce some notation. Write $\alpha = (A, U, S \cong D_\alpha \oplus S_\alpha, U \cong E_\alpha \oplus U_\alpha, f_\alpha)$ and $\beta = (A, U, S \cong D_\beta \oplus S_\beta, U \cong E_\beta \oplus U_\beta, f_\beta)$. If $D_\alpha \oplus S_\alpha \leq D_\gamma \oplus S_\gamma$ we set $D_{\gamma\alpha} = D_\gamma \cap S_\alpha$, $E_{\gamma\alpha} = f_\alpha(D_{\gamma\alpha})$ and $\alpha^1 = (A, U, S \cong D_\gamma \oplus S_\gamma, U \cong (E_\alpha \oplus E_{\gamma\alpha}) \oplus f_\alpha(S_\gamma), f_\alpha|S_\gamma : S_\gamma \cong f_\alpha(S_\gamma))$. We shall refer to α^1 as "α cut down to $D_\gamma \oplus S_\gamma$"; note that there is a map $\alpha^1 \to \alpha$ in Φ.

By axion (F1), there is a decomposition $S \cong D_0 \oplus S_0$ larger than both $D_\alpha \oplus S_\alpha$ and $D_\beta \oplus S_\beta$. Cutting α and β down, we can assume that $D_\alpha = D_\beta = D_0$ and $S_\alpha = S_\beta = S_0$. Now $f_\alpha - f_\beta : S_0 \to D$ is completely continuous, so after cutting α and β down further we can assume that $f_\alpha = f_\beta$. Note that we still may have $E_\alpha \neq E_\beta$. Consider the maps

$$E_\alpha \to U \to U_\beta \cong S_0 \quad \text{and} \quad E_\beta \to U \to U_\alpha \cong S_0.$$

By axiom (F2) there is a decomposition $S \cong D_\gamma \oplus S_\gamma$ for which these two maps factor through D_γ. Let γ be α cut down to $D_\gamma \oplus S_\gamma$; evidently γ is also β cut down to $D_\gamma \oplus S_\gamma$.

The resulting maps $\alpha \leftarrow \gamma \rightarrow \beta$ in Φ were what we needed to prove sublemma 5.6, so we are done. □

Resuming the proof of 5.5 we let $e : \alpha \rightarrow \beta$ be the map described at the proof's outset. Let $D_{\alpha\beta} = D_\alpha \cap S_\beta$ and $E_{\alpha\beta} = f_\beta(D_{\alpha\beta})$, so that $D_{\alpha\beta} \oplus D_\beta = D_\alpha$ and $D_{\alpha\beta} \oplus S_\alpha = S_\beta$. We first observe that from the definition of the map e there is a natural identification of subobjects of V:

$$(*) \quad F_\beta \oplus E_{\alpha\beta} = F \oplus E_\alpha.$$

Using this, there is a natural isomorphism in $\mathcal{A}$:

$$s_e : D_\alpha \oplus A \oplus E_\alpha \cong (D_{\alpha\beta} \oplus D_\beta) \oplus (E \oplus B \oplus F) \oplus E_\alpha \cong (D_{\alpha\beta} \oplus E) \oplus (D_\beta \oplus B \oplus F_\beta) \oplus E_{\alpha\beta}.$$

Now define q_α to be the object $S \rightarrow S_\alpha \rightarrow \Sigma(D_\alpha \oplus A \oplus E_\alpha, S_\alpha)$ of $S \downarrow \Sigma$ defined naturally by α, and let $q_e : q_\alpha \rightarrow q_\beta$ be the map

$$(D_{\alpha\beta} \oplus E, E_{\alpha\beta}, s_e, E_{\alpha\beta} \oplus S_\alpha \cong S_\beta).$$

It is easy to see that q is a functor from $\mathcal{S}$ to $S \downarrow \Sigma$.

Now let z denote the object $1 : S \rightarrow \Sigma(0, S)$ of $S \downarrow \Sigma$. There is a map $\theta_\alpha : q_\alpha \rightarrow z$ given by the data

$$(A \oplus E_\alpha, D_\alpha, D_\alpha \oplus A \oplus E_\alpha \cong (A \oplus E_\alpha) \oplus 0 \oplus D_\alpha, D_\alpha \oplus S_\alpha \cong S)$$

and a map $\eta_\alpha : q_\alpha \rightarrow \mathrm{pr}_\alpha$ given by the data

$$(D_\alpha, E_\alpha, D_\alpha \oplus A \oplus E_\alpha = D_\alpha \oplus A \oplus E_\alpha, E_\alpha \oplus S_\alpha \cong E_\alpha \oplus U_\alpha \cong U).$$

Using (*), it is a straightforward matter to check that θ and η are natural transformations. This proves that the maps z, q and pr from $B\mathcal{S}$ to $B(S \downarrow \Sigma)$ are homotopic. Since pr is a homotopy equivalence, this shows that $B(S \downarrow \Sigma)$ is contractible. This finishes the proof of Proposition 5.5, and hence of Theorem 5.3.

Next we show how to remove the hypothesis that $\mathcal{A}$ is idempotent complete from Theorem 5.4 Let $\mathcal{U}$ be a flasque $\mathcal{A}$-filtered category. Let $\mathcal{A}^\wedge$, and let $\mathcal{U}^\wedge$ be the full subcategory of the idempotent completion of $\mathcal{U}$ on objects $P \oplus U$, P in $\mathcal{A}^\wedge$, and U in $\mathcal{U}$. Then $\mathcal{U}^\wedge$ is $\mathcal{A}^\wedge$-filtered but not flasque. However it is easy to see that $K^Q(\mathcal{U}^\wedge)$ is contractible, and Theorem 5.3 applies to show that $\Omega K^Q(\mathcal{U}^\wedge/\mathcal{A}^\wedge) \cong K^Q(\mathcal{A}^\wedge)$. On the other hand, $\mathcal{U}^\wedge/\mathcal{A}^\wedge \cong \mathcal{U}/\mathcal{A}$, so we have proven:

Corollary 5.7. *If $\mathcal{A}^\wedge$ denotes the idempotent completion of $\mathcal{A}$, and $\mathcal{U}$ is a flasque $\mathcal{A}$-filtered category, then*

(i) $K^Q(\mathcal{U}/\mathcal{A}) \simeq K^Q(\mathcal{U}^\wedge/\mathcal{A}^\wedge)$
(ii) $\Omega K^Q(\mathcal{U}/\mathcal{A}) \simeq K^Q(\mathcal{A}^\wedge)$
(iii) $K^Q(\mathcal{A}) \rightarrow K^Q(\mathcal{U}) \rightarrow K^Q(\mathcal{U}/\mathcal{A})$ *is a homotopy fibration if and only if* $K_0(\mathcal{A}) \cong K_0(\mathcal{A}^\wedge)$

Definition 5.8. (Karoubi [6]) Given a semisimple exact category $\mathcal{A}$, we define $K_{-n}(\mathcal{A})$ to be $K_1(S^{n+1}\mathcal{A})$. Note that $K_{-0}(\mathcal{A}) = K_0(\mathcal{A}^\wedge)$ by 5.7 (ii). This is well-defined because by 5.4 the homotopy type of $S^{n+1}\mathcal{A}$ is independent of the choice of flasque category $\mathcal{U}$ used to construct $S\mathcal{A} = \mathcal{U}/\mathcal{A}$. Note that $K_0(S^n\mathcal{A}) = 0$ for $n \geq 1$ because $S^n\mathcal{A}$ need not be idempotent complete. In fact $K_0((S^n\mathcal{A})^\wedge) = K_{-n}(\mathcal{A})$. (Cf. [6, p.151]

Definition 5.9. By 5.4, 5.7 and 5.8, there is an Ω-spectrum

$$|K_{-n}(\mathcal{A}) \times K^Q(S^n\mathcal{A})| \simeq |K^Q((S^n\mathcal{A})^\wedge)|$$

We shall call it Karoubi's non-connective K-theory spectrum for $\mathcal{A}$, since Karoubi gave the prescription for this spectrum in [6].

6. Agreement of spectra

Our task is now to show that Karoubi's spectrum agrees with the other spectra in the literature. We first recall Wagoner's construction in [16]. Given a ring R, let lR denote the ring of locally finite $\mathbb{N}$- indexed matrices over R, i. e. matrices (r_{ij}) with $1 \leq i, j < \infty$ such that each row and each column has only finitely many nonzero entries. The finite matrices form an ideal mR of lR, and we let $\mu R = lR/mR$. Wagoner's spectrum is

$$\{K_0(\mu^n R) \times BGL^+(\mu^n R)\}.$$

We shall show that this spectrum is the same as Karoubi's spectrum for the category $\mathcal{F}_R$ of (based) finitely generated free R-modules, Note that $\mathcal{F}^\wedge(R)$ is equivalent to the category of finitely generated projective R-modules, so $K(\mathcal{F}^\wedge(R))$ is the usual space $K_0^Q(R) \times BGL^+(R)$. The following argument was shown to us by H. J. Munkholm and A. A. Ranicki.

Proposition 6.1. *Let $\mathcal{U}$ denote the category of countably (but not necessarily infinitely) generated based free R-modules and locally finite matrices over R. Then $\mathcal{U}/\mathcal{F}(R)$ is equivalent to the category $\mathcal{F}(\mu R)$. Consequently, Wagoner's spectrum for R is homotopy equivalent to Karoubi's K-theory spectrum for $\mathcal{F}(R)$.*

Proof. (Munkholm-Ranicki) Choose an infinitely generated based R-module R^∞ in $\mathcal{U}$, and observe that $\mathrm{End}_{\mathcal{U}}(R^\infty) = lR$. Now $\mathcal{U}$ is $\mathcal{F}(R)$-filtered, and the completely continuous endomorphisms of R^∞ form the ideal mR. Thus $\mathrm{End}_{\mathcal{U}/\mathcal{F}}(R^\infty) \cong \mu R$. The additive functor $\mathcal{F}(\mu R) \to \mathcal{U}/\mathcal{F}(R)$ which sends $1_{\mu R}$ to 1_{R^∞} is therefore full and faithful. But every object of $\mathcal{U}/\mathcal{F}(R)$ is either isomorphic to 0 or to R^∞, so this functor is also an equivalence. Done. □

Gersten has also constructed a nonconnective spectrum for the K-theory of a ring in [4]. Since Wagoner showed in [16] that it agreed with Wagoner's spectrum, Gersten's spectrum is also homotopy equivalent to Karoubi's spectrum.

Finally we must compare Karoubi's K-theory spectrum with the spectrum $\{f(S^n)\}$ of section 4 above, constructed in [9] using the categories $\mathcal{C}_n(\mathcal{A})$. We assume that $\mathcal{A}$ is filtered in the sense of [9, (1.1)], or §2 above, so that in the notation of op. cit. we have $\mathcal{C}_{n+1}(\mathcal{A}) =$

$\mathcal{C}_1(\mathcal{C}_n(\mathcal{A}))$. The choice of the filtration affects the morphisms allowed in $\mathcal{C}_n(\mathcal{A})$ and $\mathcal{C}_+(\mathcal{A})$, but not the homotopy type of $K^Q(\mathcal{C}_n(\mathcal{A}))$, as [9, (3.2)] shows. The real point of the filtration on $\mathcal{A}$ is to reduce the discussion to the case $n = 1$.

Recall the objects of $\mathcal{C}_1(\mathcal{A}) = \mathcal{C}_{\mathbb{Z}}(\mathcal{A})$ are $\mathbb{Z}$-graded sequences $A = (\ldots, A_{-1}, A_0, A_1, \ldots)$ in $\mathcal{A}$. A map $\phi : A \to B$ is a matrix of maps $\phi_{ij} : A_i \to B_j$ such that for some bound $b = b(\phi)$ we have $\phi_{ij} = 0$ whenever $|i-j| > b$. Composition is given by matrix multiplication. Define $\operatorname{trunc}(A)$ to be the object $(A_0, A_1, \ldots)$ of $\mathcal{C}_+(\mathcal{A})$ and $\operatorname{trunc}(\phi)$ to be the submatrix of ϕ_{ij} with $i, j > 0$. If $\psi : B \to C$, then $\operatorname{trunc}(\psi\phi) - \operatorname{trunc}(\psi)\operatorname{trunc}(\phi)$ is completely continuous being bounded by $b(\psi) + b(\phi)$. Hence trunc defines a functor from $\mathcal{C}_1(\mathcal{A}) \to \mathcal{C}_+(\mathcal{A})/\mathcal{A}$.

Theorem 6.2. *The functor* trunc *induces a homotopy equivalence*

$$K^Q(\mathcal{C}_1(\mathcal{A})) \xrightarrow{\simeq} K^Q(\mathcal{C}_+(\mathcal{A})/\mathcal{A}) \simeq K^Q(S\mathcal{A})$$

Assuming this result, it follows directly from the above remarks that we have $K^Q(\mathcal{C}_{n+1}(\mathcal{A}))) \xrightarrow{\simeq} K^Q(\mathcal{C}_1(\mathcal{C}_n(\mathcal{A})) \simeq K^Q(S\mathcal{C}_n(\mathcal{A})) \simeq K^Q(S^{n+1}\mathcal{A})$. Hence the spaces $\hat{B}_n$ of [9] are $K_{-n}(\mathcal{A}) \times K^Q(\mathcal{C}_n(\mathcal{A}))$, and we have

Corollary 6.3. *The nonconnective spectra of* [9] *agree with Karoubi's. In fact,* trunc *induces a homotopy equivalence of spectra:*

$$\{K_{-n}(\mathcal{A}) \times K^Q(\mathcal{C}_n\mathcal{A})\} \xrightarrow{\simeq} \{K_{-n}(\mathcal{A}) \times K^Q(S^n\mathcal{A})\}$$

Proof of Theorem 6.2. We shall use the notation of [9] , only remarking that $\underline{\underline{C}}_{\epsilon}$ is the category of isomorphisms of $\mathcal{C}_\epsilon(\mathcal{A})$, and that $\operatorname{Spt}_0(\underline{\underline{C}}_{\epsilon} \simeq K^Q(\mathcal{C}_\epsilon(\mathcal{A}))$. By 5.7 and [9, §3], we can assume that $\mathcal{A}$ is idempotent complete. There is a map of squares

$$\begin{array}{ccc} \underline{\underline{A}} & \longrightarrow & \underline{\underline{C}}_+ \\ \downarrow & & \downarrow \\ \underline{\underline{C}}_- & \longrightarrow & \underline{\underline{C}}_1 \end{array} \quad \to \quad \begin{array}{ccc} \underline{\underline{A}} & \longrightarrow & \underline{\underline{C}}_+ \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \underline{\underline{S}} \end{array}$$

By Theorem 5.3 and [9, §3], applying Spt_0 yields homotopy cartesian squares with $\operatorname{Spt}_0(\underline{\underline{C}}_1) \simeq K^Q(\mathcal{C}_1(\mathcal{A}))$ and $\operatorname{Spt}_0(\underline{\underline{S}}) \simeq K^Q(S\mathcal{A})$ connected. Since $\operatorname{Spt}(\underline{\underline{C}}_-)$ and $\operatorname{Spt}(\underline{\underline{C}}_+)$ are contractible, it follows that $K^Q(\mathcal{C}_1(\mathcal{A})) \to K^Q(S\mathcal{A})$ is a weak homotopy equivalence, hence a homotopy equivalence. □

References

1. H. Bass, *Algebraic K-theory*, Benjamin, 1968.
2. T. A. Chapman, *Controlled boundary and h-cobordism theorems*, Trans. Amer. Math. Soc. **280** (1983), 73–95.

3. ______, *Controlled Simple Homotopy Theory and Applications*, Lecture Notes in Mathematics, vol. 1009, Springer, 1983.
4. S. Gersten, *On the spectrum of algebraic K-theory*, Bull. Amer. Math. Soc. (N.S.) **78** (1972), 216–219.
5. T. Gunnarson, *Algebraic K-theory of spaces as K-theory of monads*, Aarhus preprint series, 21, 1981/1982.
6. M. Karoubi, *Foncteur dérivés et K-théorie*, Lecture Notes in Mathematics, vol. 136, Springer, 1970.
7. P. May, *E^∞-spaces, group completions and permutative categories, new developments in topology*, Proc. Sympos. Algebraic Topology, (Oxford, 1972), London Math. Soc. Lecture Notes, vol. 11, Cambridge Univ. Press, Cambridge, 1974, pp. 61–94.
8. E. K. Pedersen, *On the K_{-i} functors*, J. Algebra **90** (1984), 461–475.
9. E. K. Pedersen and C. Weibel, *A nonconnective delooping of algebraic K-theory*, Algebraic and Geometric Topology, (Rutgers, 1983), Lecture Notes in Mathematics, vol. 1126, Springer, Berlin, 1985, pp. 166–181.
10. D. G. Quillen, *Higher algebraic K-theory I*, Algebraic K-theory, I: Higher K-theories, (Battelle Memorial Inst., Seattle, Washington, 1972), Lecture Notes in Mathematics, vol. 341, Springer, Berlin, 1973, pp. 85–147.
11. F. Quinn, *Ends of maps, I*, Ann. of Math. (2) **110** (1979), 275–331.
12. ______, *Ends of maps, II*, Invent. Math. **68** (1982), 353–424.
13. ______, *Geometric algebra*, Algebraic and Geometric Topology, (Rutgers, 1983), Lecture Notes in Mathematics, vol. 1126, Springer, Berlin, 1985, pp. 182–198.
14. G.B. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.
15. R. W. Thomason, *First quadrant spectral sequences in algebraic K-theory via homotopy colimits*, Comm. Algebra **10** (1982), 1589–1668.
16. J. Wagoner, *Delooping classifying spaces in algebraic K-theory*, Topology **11** (1972), 349–370.
17. C. A. Weibel, *K-theory of Azumaya Algebras*, Proc. Amer. Math. Soc. **81** (1981), 1–7.
18. R. Woolfson, *Hyper γ-spaces and hyperspectra*, Quart. J. Math. Oxford Ser. (2) **30** (1979), 229–255.

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