COMMUNICATIONS IN ALGEBRA, 14(10), 1895-1907 (1986)

# BLOCH'S FORMULA FOR VARIETIES WITH ISOLATED SINGULARITIES: I

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If X is a smooth quasi-projective variety, Bloch's Formula states that the Chow group $CH^p(X)$ of codimension p cycles on X modulo rational equivalence is equal to the cohomology group $H^p(X,\underline{\underline{K}}_p)$ of the algebraic K-theory sheaf $\underline{\underline{K}}_p$. The purpose of this note is to point out that a similar formula holds, modulo torsion, when X is allowed to have a finite number of singularities.

To this end, let X be a quasi-projective variety for which Sing(X) is finite. Let Y be a finite set of closed points of X containing Sing(X). For simplicity, we shall assume that X is connected and that $\dim(X)\geq 2$, which implies in particular that X is reduced. (See [12] for the case $d=1$.)

As in [1],[7], we can define a relative Chow group $CH^p(X,Y)$, generated by those codimension p cycles on X which miss Y, modulo

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[**] Supported by NSF. This author wishes to express his gratitude to the CNR and to the University of Genova for their hospitality during the research stage of this paper.

0092-7872/86/1410-1895$3.50/0

rational equivalence given on codimension $(p-1)$ cycles missing $Y$. The group $CH^1(X,Y)$ is poorly behaved, but the other relative Chow groups satisfy the following version of Bloch's Formula:

Main Theorem 0.1. Let $X$ be a connected $d$-dimensional quasiprojective variety $(d>2)$ whose singular locus is contained in a finite closed set $Y$. Then for $2<p<d$ there are maps $\beta^p: CH^p(X,Y) \to H^p(X,\underline{\underline{K}}_p)$ which are isomorphisms modulo $(d-1)!$ torsion.

More precisely, let $\mathcal{T}_m$ denote the category of all abelian groups $A$ such that $m^rA = 0$ for some $r$. Then:

(a) For $2<p<d-2$ the kernel and cokernel of $\beta^p$ lie in $\mathcal{T}_{(d-2)!}$

(b) $\mathrm{Coker}(\beta^{d-1})$ lies in $\mathcal{T}_{(d-2)!}$ and (if $d>3$) $\mathrm{Ker}(\beta^{d-1})$ lies in $\mathcal{T}_{(d-1)!}$

(c) The kernel and cokernel of $\beta^d$ lie in $\mathcal{T}_{(d-1)!}$

If $Y$ consists in exactly one point, then in fact the maps $\beta^p$ are isomorphisms. This was proven by Collino in [1]. If $X$ is a surface, our result gives the known result that $CH^2(X,Y) \cong H^2(X,\underline{K}_2)$. (See [4] and [7].) However, if $X$ is 3-dimensional, we can only say that the map $\beta^2: CH^2(X,Y) \to H^2(X,\underline{K}_2)$ is onto with kernel a 2-group. We do not know any example in which $\beta^2$ fails to be an isomorphism. Here is the best result about $\ker(\beta^2)$ we can give in general:

Corollary 0.2. Let $X,Y$ be as in the theorem. Assume that $k$ is algebraically closed and that either $\mathrm{char}(k)=0$ or $\mathrm{char}(k)>d$. Then the kernel of $\beta^2: CH^2(X,Y) \to H^2(X,\underline{K}_2)$ is a finite group. If $p$ is a prime dividing the order of this kernel then $p<d-2$ or $p=2$.

Proof. Let $m$ be an integer such that $m\,\mathrm{Ker}(\beta^2) = 0$. By the theorem, $m$ exists and is prime to $\mathrm{char}(k)$. Collino has shown in [2] that for such $m$ and $k$ the kernel of $CH^2(X,Y) \xrightarrow{m} CH^2(X,Y)$ is finite, whence the result.

In a future paper, we will relate the maps $\beta^p$ to the Chern classes $CH^p(X,Y) \to \mathrm{Fil}^pK_0(X)/\mathrm{Fil}^{p+1}K_0(X)$ of Collino [1] and Levine [5], as well as to the spectral sequence $H^p(X,\underline{K}_q) \Longrightarrow K_s(X)$ of [11]. However, the ideas involved in that relation are different in nature from the simple ideas of this note.

## §1. Smooth Varieties

Our proof relies on the work of Shektman [9], Merkur'ev-Suslin [6] and Soulé [10] on the degeneration of the (Brown-Gersten) Quillen spectral sequence $H^p(X,\underline{K}_{-q}) \Longrightarrow K_{-p-q}(X)$. They showed that the first three lines of the spectral sequence degenerate modulo torsion, yielding

$$K_s(X) \cong \coprod_{-p+q=s} H^p(X,\underline{K}_q) \quad \text{mod torsion} \quad (s = 0,1,2).$$

In this section we will derive a more careful version, following their work. For the remainder of this section, $X$ will denote a regular scheme, essentially of finite type over a field.

Gillet [3] and Shektman [9] have extended the usual Grothendieck characteristic classes $c_{oq}: K_o(X) \to H^q(X,\underline{\underline{K}}_q)$, obtaining functorial Chern classes $c_{sq}: K_s(X) \to H^{q-s}(X,\underline{\underline{K}}_q)$. Recall (e.g., from [6] (8.6.1)) that $c_{sq}$ vanishes on $\mathrm{Fil}^{q-s+1}K_s(X)$, where $\mathrm{Fil}^{\cdot}K_s(X)$ is the filtration resulting from the Quillen spectral sequence. In particular, if we set $s = -p+q$ then $c_{sq}$ induces a map

$$\overline{c}_{sq}: E_{\infty}^{p,-q} = \mathrm{Fil}^pK_s(X)/\mathrm{Fil}^{p+1}K_s(X) \longrightarrow H^p(X,\underline{\underline{K}}_q).$$

Our first goal will be to show that the maps $\overline{c}_{sq}$ induce the direct sum decomposition of $K_s(X)$ mentioned above.

To give a feeling for what is going on, consider the case $s=0$. Here the edge of the spectral sequence gives a surjection $\beta: H^p(X,\underline{\underline{K}}_q) \to \mathrm{Fil}^pK_o(X)/\mathrm{Fil}^{p+1}K_o(X)$. Composing with $\overline{c}_{op}$ yields an endomorphism $\overline{c}_{op}(\beta)$ of $H^p(X,\underline{\underline{K}}_p)$. The Riemann-Roch Theorem asserts that it is multiplication by $(-1)^{p-1}(p-1)!$.

In order to have a language to discuss this decomposition, we introduce the Serre subcategories $\mathcal{J}_m$ of $\mathcal{A}b$, the category of abelian groups. $\mathcal{J}_m$ consists of all abelian groups A such that, for some r, $m^rA = 0$. To indicate that we are working in the quotient category $\mathcal{A}b/\mathcal{J}_m$, we shall use the phase "mod $\mathcal{J}_m$". For example, $\overline{c}_{op}(\beta)$ is an isomorphism mod $\mathcal{J}_{(p-1)!}$, because the kernel and cokernel of $\overline{c}_{op}(\beta)$ have exponent $(p-1)!$. (A is said to have exponent m if $mA = 0$.) A map f is zero mod $\mathcal{J}_m$ iff some $m^rf = 0$. We will also use the fact that if two abelian groups A and B are isomorphic mod $\mathcal{J}_m$, then there is a map $A \to B$ in $\mathcal{A}b$ whose kernel and cokernel both lie in $\mathcal{J}_m$.

Having introduced the necessary notation, we can now state our main technical result, which is implicit in Merkur'ev and Suslin's proof of [6] (8.6.2). $\{E_r^{p,q}\}$ denotes the Quillen spectral sequence associated to the filtration $\underline{\underline{M}}_p(X)$ of the category $\underline{\underline{M}}(X)$ of coherent sheaves on X. (See [8] (5.11)).

Proposition 1.1. Let X be a regular scheme, essentially of finite type over a field. Fix $m,p,q,s \geq 0$ so that $s = -p-q$ and assume:

(i) For all $r \geq 2$, the differential leaving $E_r^{p,q}$ in the Quillen spectral sequence is zero mod $\mathcal{J}_m$.

(ii) For all points x of codimension p in X the Chern class

$$c_{ss}(x): K_s(k(x)) \to H^o(\mathrm{Spec}\ k(x),\underline{\underline{K}}_s) = K_s(k(x))$$

coincides mod $\mathcal{J}_m$ with multiplication by $(-1)^{s-1}(s-1)!$.

Then mod $\mathcal{J}_m$ there are natural surjections

$$K_s(\underline{\underline{M}}_p) \xrightarrow{\alpha} H^p(X,\underline{\underline{K}}_q) = E_2^{p,-q} \xrightarrow{\beta} E_{\infty}^{p,-q} = \mathrm{Fil}^pK_s(X)/\mathrm{Fil}^{p+1}K_s(X).$$

Moreover, the composition $\overline{c}_{sq}(\beta)$ coincides mod $\mathcal{J}_m$ with multiplication by $(-1)^{q-1}(q-1)!$ on $H^p(X,\underline{\underline{K}}_q)$.

Proof. Quillen's construction of the spectral sequence shows that the existence mod $\mathcal{J}_m$ of the surjections $\alpha$ and $\beta$ is implied by condition (i). Moreover, $K_s(\underline{\underline{M}}_p)$ is the filtered colimit of the $K_s'(Y)$ as Y ranges over the codimension p subschemes of Y. We can now follow the proof of [6] (8.6.2) almost verbatim to see that condition (ii) and the Riemann-Roch formula imply that $\bar{c}_{sq}(\beta)$ equals $(-1)^{q-1}(q-1)!$ mod $\mathcal{J}_m$.

**Remark.** Condition (ii) always holds for $s = 1,2$. The example $c_{33}: K_3(k) \to K_3(k)$ when $k$ is a finite field shows that condition (ii) need not hold for $s = 3$.

**Corollary 1.2.** Let $X$ be a $d$-dimensional regular scheme, essentially of finite type over a field, and set $m = (d-1)!$. Then:

(1a) For $0<p<d-1$ there are isomorphisms mod $\mathcal{J}_m$ :

$$H^p(X,\underline{\underline{K}}_{p+1}) \cong \mathrm{Fil}^p K_1(X)/\mathrm{Fil}^{p+1}K_1(X)$$

(1b) There is a surjection $H^d(X,\underline{\underline{K}}_{d+1}) \to \mathrm{Fil}^d K_1(X)$ whose kernel is annihilated by $d!$

(2a) For $0<p<d-3$ there are isomorphisms mod $\mathcal{J}_m$ :

$$H^p(X,\underline{\underline{K}}_{p+2}) \cong \mathrm{Fil}^p K_2(X)/\mathrm{Fil}^{p+1}K_2(X).$$

(2b) There are subgroups $B_\infty \subseteq Z_\infty \subseteq H^{d-2}(X,\underline{\underline{K}}_d)$ such that $(d-1)!B_\infty = 0$, $d!\, H^{d-2}(X,\underline{\underline{K}}_d) \subseteq Z_\infty$, and such that

$$Z_\infty/B_\infty \cong \mathrm{Fil}^{d-2}K_2(X)/\mathrm{Fil}^{d-1}K_2(X).$$

(2c) There is an exact sequence (which is split mod $\mathcal{J}_{(d+1)!}$):

$$0 \to \frac{H^d(X,\underline{\underline{K}}_{d+2})}{B_\infty^d} \to \mathrm{Fil}^{d-1}K_2(X) \to \frac{H^{d-1}(X,\underline{\underline{K}}_{d+1})}{B_\infty^{d-1}} \to 0$$

Here $B_\infty^d \subseteq H^d(X,\underline{\underline{K}}_{d+2})$ and $B_\infty^{d-1} \subseteq H^{d-1}(X,\underline{\underline{K}}_{d+1})$ are subgroups of exponent $(d+1)!$ and $d!$, respectively.

**Proof.** We can apply (1.1) with $m = 1$ for (1b) and (2c), and can use $m = (d-1)!$ for (1a), (2a). This leaves (2b). By (1b) it suffices to show that the natural surjection $Z_\infty \to Z_\infty/B_\infty = E_\infty^{d-2,-d}$ has kernel of exponent $(d-1)!$. By the construction of the spectral sequence we know that $K_2(\underline{\underline{M}}_{d-2})$ maps onto $Z_\infty$, so in order to prove (2b) it is enough to show that for $Y$ of pure codimension $d-2$ the composition

$$K_2'(Y) \xrightarrow{\alpha} Z_\infty \to E_\infty^{d-2,-d} \xrightarrow{c_{2d}} H^{d-2}(X,\underline{\underline{K}}_d)$$

coincides with the the natural map $\alpha$, multiplied by $(-1)^{d-1}(d-1)!$. This follows from the Riemann-Roch formula, using the argument in [6] (8.6.2).

**Remark.** Soulé actually has a slightly stronger version of (1a) and (2a) in [10], Theorem 5, but the improvement is not useful for our purposes.

**Corollary (1.3).** Let $X$ be a $d$-dimensional regular scheme essentially of finite type over a field, and suppose that $SK_1(X) = 0$. Then:

(1a) For $1<p<d-1$ the groups $H^p(X,\underline{\underline{K}}_{p+1})$ lie in $\mathcal{J}_{(d-1)!}$

(1b) If $d \neq 0$, the group $H^d(X,\underline{\underline{K}}_{d+1})$ has exponent $d!$

(2a) For $0<p<d-3$ the Chern class map

$c_{2,p+2}: K_2(X) \to H^p(X,\underline{\underline{K}}_{p+2})$ is surjective mod $\mathcal{J}_{(d-1)!}$

(2b) The cokernel of $c_{2,d}: K_2(X) \to H^{d-2}(X,\underline{\underline{K}}_d)$ has exponent $d!$

(2c) The cokernel of $c_{2,d+1}: K_2(X) \to H^{d-1}(X,\underline{\underline{K}}_{d+1})$ has exponent $d!$

(2d) The cokernel of $c_{2,d+2}: K_2(X) \to H^d(X,\underline{\underline{K}}_{d+2})$ has exponent $(d+1)!$

Proof. $SK_1(X)$ is the kernel of the map $K_1(X) \to K_1(k(X))$, whose image is $H^0(X,\underline{\underline{K}}_1)$, so $SK_1(X) = \mathrm{Fil}^1K_1(X)$. Hence $\mathrm{Fil}^pK_1(X) = 0$ for all $p>1$, and 1a), 1b) follow from (1.2). (2a) follows from (1.2.2a). Next, (1.2.2b) implies for every $x \in H^{d-2}(X,\underline{\underline{K}}_d)$ that $d!x$ is in $\mathrm{Fil}^{d-2}K_2(X)$. So $\mathrm{coker}(K_2(X) \to H^{d-2}(X,\underline{\underline{K}}_d))$ has exponent $d!$. For $p = d-1$ or $d$, all differentials leaving $E_2^{p,-(p+2)}$ are zero; hence we can apply (1.1), with $m = 1$, to get (2c) and (2d).

## §2. The Semilocal Singular Case

If A is a semilocal ring, it is well-known that $\tilde{K}_0(A) = SK_1(A) = 0$. If the spectral sequence of [11] degenerated completely, we would have $H^p(\mathrm{Spec}A,\underline{\underline{K}}_p) = H^p(\mathrm{Spec}A,\underline{\underline{K}}_{p+1}) = 0$ for $p \neq 0$. This is trivial if $\dim(A) = 0$, and proven in [12] for 1-dimensional A. If A is normal of dimension 2, then $H^1(\mathrm{Spec}A,\underline{\underline{K}}_2) = H^2(\mathrm{Spec}A,\underline{\underline{K}}_2) = 0$ by [7] (6.5). In this section we extend these results to higher dimensions. Because of our lack of knowledge concerning the higher K-theory of nilpotent ideals, we must assume A is reduced.

Theorem (2.1). Let A be a reduced semilocal ring of dimension $d \geq 1$, essentially of finite type over a field. Assume that $\mathrm{Sing}(A) \subseteq \mathrm{Max}(A)$. Then for all $p \neq 0$ the groups $H^p(\mathrm{Spec}A,\underline{\underline{K}}_p)$ and $H^p(\mathrm{Spec}A,\underline{\underline{K}}_{p+1})$ lie in $\mathcal{T}_{d!}$. More precisely:

$$H^p(\mathrm{Spec}A,\underline{\underline{K}}_p) \text{ is } \begin{cases} \text{in } \mathcal{T}_{(d-2)!} & \text{if } 1 \leq p \leq d-1 \\ \text{in } \mathcal{T}_{(d-1)!} & \text{if } p=d \\ 0 & \text{if } p \geq d+1 \end{cases}$$

$$H^p(\mathrm{Spec}A,\underline{\underline{K}}_{p+1}) \text{ is } \begin{cases} \text{in } \mathcal{T}_{(d-2)!} & \text{if } 1 \leq p \leq d-3 \\ \text{in } \mathcal{T}_{(d-1)!} & \text{if } p=d-2 \text{ or } d-1 \ (p \neq 0) \\ \text{in } \mathcal{T}_{d!} & \text{if } p=d \\ 0 & \text{if } p \geq d+1 \end{cases}$$

Proof. We will write $H^p(A,\underline{\underline{K}}_q)$ for $H^p(\mathrm{Spec}(A),\underline{\underline{K}}_q)$ in this proof. If A is regular, then we know by [8] that the spectral sequence for $K_s(A)$ degenerates completely, yielding $K_s(A) \xrightarrow{\cong} H^0(A,\underline{\underline{K}}_s)$ for all s and $H^p(A,\underline{\underline{K}}_q) = 0$ for $p \neq 0$. If A is local, then $H^p(A,-) = 0$ for all $p \neq 0$. We can therefore assume that $\mathrm{Max}(A) = \{m,p_1,\dots p_r\}$ with $r \neq 0$ and induct on the size of Max(A). Let $\{q_1,\dots,q_s\}$ be the minimal primes of A; since $\dim(A)>1$ we can assume that m is not minimal, so that $T = A - \cup p_i - \cup q_j$ intersects m and consists of nonzero divisors. As in [7] (6.2) or [13], Proof of (1.1), the ring $R = T^{-1}A_m$ is regular, has dimension at most d-1, and $SK_1(R) = 0$. By induction, the theorem holds for the semilocal rings $A_m$ and $A_T$. Employing the argument used in [7] (6.5), there is an exact sequence

$$H^{p-1}(R,\underline{\underline{K}}_p) \longrightarrow H^p(A,\underline{\underline{K}}_p) \longrightarrow H^p(A_T,\underline{\underline{K}}_p) \oplus H^p(A_m,\underline{\underline{K}}_p).$$

By (1.3.1), the left-hand group lies in $\mathcal{J}_{(d-2)!}$ for $2<p<d-1$, has exponent $(d-1)!$ if $p=d$, and is zero if $p>d+1$. By induction, the theorem holds for $H^p(A,\underline{\underline{K}}_p)$, since $H^1(A,\underline{\underline{K}}_1) = \mathrm{Pic}(A) = 0$ always holds.

Again as in [7] (6.5), there is a commutative diagram with exact rows ($p>1$):

$$\begin{tikzcd}
H^{p-1}(A_m,\underline{\underline{K}}_q) \oplus H^{p-1}(A_T,\underline{\underline{K}}_q) \arrow[r] & H^{p-1}(R,\underline{\underline{K}}_q) \arrow[r] & H^p(A,\underline{\underline{K}}_q) \arrow[r] & H^p(A_T,\underline{\underline{K}}_q) \\
K_2(A_m) \oplus K_2(A_T) \arrow[r] \arrow[u, "c_{2q}"] & K_2(R) \arrow[r] \arrow[u, "c_{2q}(R)"] & 0 &
\end{tikzcd}$$

Now apply (1.3.2) with $q = p+1$. This proves that the cokernel of $c_{2,p+1}(R)$ lies in $\mathcal{J}_{(d-2)!}$ if $1<p<d-3$, has exponent $(d-1)!$ if $p=d-2$ or $d-1$, has exponent $d!$ if $p=d$, and is zero if $p>d+1$. Since inductively the same is true for $H^p(A_T,\underline{\underline{K}}_{p+1})$, it also holds for $H^p(A,\underline{\underline{K}}_{p+1})$.

Remark (2.2). The hypotheses that A be reduced can be relaxed to requiring that A has at most one embedded prime (m), without changing the proof.

Remark (2.3). For the groups $H^d(A,\underline{\underline{K}}_d)$, $H^{d-1}(A,\underline{\underline{K}}_d)$, $H^{d-2}(A,\underline{\underline{K}}_{d-1})$ and $H^d(A,\underline{\underline{K}}_{d+1})$ it is possible to give a more precise result, i.e., to find an upper bound for their exponents depending on the number s of singular primes in Spec A and the number m of maximal ideals of A. In fact, let n be such that $2^n$ is the smallest power of 2 greater than or equal to $\min\{m,2s\}$. Then $(d-1)!^n$ is an upper bound for the exponent of $H^d(A,\underline{\underline{K}}_d)$, $H^{d-1}(A,\underline{\underline{K}}_d)$, and $H^{d-2}(A,\underline{\underline{K}}_{d-1})$, while $(d!)^n$ is an upper bound for the exponent of $H^d(A,\underline{\underline{K}}_{d+1})$.

The proof of this uses induction on n, arguing as in (2.1): divide Spec A into two pieces $U_i$, one containing $2^{n-1}$ singular primes of A, the other containing the remaining maximal primes of A. Consider the Mayer-Vietoris sequence for $H^*(-,\underline{\underline{K}}_q)$, as in the proof of (2.1), and use induction to get the exponents of the groups $H^p(U_i,\underline{\underline{K}}_q)$, hence the exponent for $H^d(A,\underline{\underline{K}}_d)$. The result follows for the other groups by looking at the commutative diagrams of exact sequences, as in (2.1), using (1.3) for the exponent of $\mathrm{coker}(K_2(U_1\cap U_2) \to H^{p-1}(U_1\cap U_2,\underline{\underline{K}}_2))$.

## §3. Proof of the Main Theorem.

In this section, X will denote a connected quasiprojective variety of dimension $d>2$. We will also assume that Sing(X) is contained in a finite set Y of closed points on X. As in [1],[7], we define the relative Chow groups $CH^p(X,Y)$ to be the cokernel of the cycle map:

$$\coprod_{\substack{\mathrm{codim}\ x=p-1\\ V(x)\cap Y=\phi}} k(x)^* \longrightarrow \coprod_{\substack{\mathrm{codim}\ x=p\\ V(x)\cap Y=\phi}} \underline{Z}.$$

Since X is connected and $\dim(X) \neq 0$, X is reduced and Y contains no component of X. Embedding X in a suitable projective space, there is a hyperplane H missing Y with X-H affine, say X-H = Spec(B). Localizing B at the primes in Y yields a reduced semilocal ring A, essentially of finite type over a field, with Sing(A) $\subseteq$ Y = Max(A). In fact, $X_Y$ = Spec(A) is the intersection of all affine open neighborhoods U of Y such that X-U is a divisor. Thus the "Standing Assumptions" of [7] are met.

By Corollary (4.2) of [7], there is an exact sequence for p>2:

$$H^{p-1}(\mathrm{Spec}A,\underline{\underline{K}}_p) \rightarrow CH^p(X,Y) \xrightarrow{\beta^p} H^p(X,\underline{\underline{K}}_p) \rightarrow H^p(\mathrm{Spec}A,\underline{\underline{K}}_p) \rightarrow 0.$$

By (2.1), the right-hand term lies in $\mathcal{J}_{(d-2)!}$ if 2<p<d-1, and in $\mathcal{J}_{(d-1)!}$ if p=d. Also by (2.1), the left-hand term lies in $\mathcal{J}_{(d-2)!}$ if 2<p<d-2 and in $\mathcal{J}_{(d-1)!}$ if p=d-1 or d. It follows that $\beta^p$ is an isomorphism mod $\mathcal{J}_{(d-2)!}$ when 2<p<d-2, onto mod $\mathcal{J}_{(d-2)!}$ when p=d-1, and finally an isomorphism mod $\mathcal{J}_{(d-1)!}$ for 2<p<d. This proves the Main Theorem.

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Received: July 1985