THE NEGATIVE *K*-THEORY OF NORMAL SURFACES

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Abstract
We relate the negative K-theory of a normal surface to a resolution of singularities. The only nonzero K-groups are K_{-2}, which counts loops in the exceptional fiber, and K_{-1}, which is related to the divisor class groups of the complete local rings at the singularities. We also verify two conjectures of Srinivas about K_0-regularity and K_{-1} of a surface.

This paper gives a geometric interpretation for the negative K-theory of a normal surface X. If R is the semilocal ring of X at its finitely many singularities, then the negative K-theory of X and R are the same by [W3, Theorem 1.2]. Thus we also describe the negative K-theory of any excellent 2-dimensional normal semilocal ring R. As we explain on the next page, our results also give an almost complete classification of projective $R[T]$-modules.

Our interpretation relates the groups $K_{-j}(X)$ to a resolution of singularities $\tilde{X} \to X$ of the surface. For starters (see Theorem 4.4), we show that $K_{-j}(X) = 0$ for all $j \geq 3$, even if X is singular. (This confirms a guess in [W1, p. 180].) If X is normal and $j = 2$, we prove that $K_{-2}(X) \cong \mathbb{Z}^{\lambda}$, where λ denotes the number of "loops" in the exceptional fiber E; the number of "loops" in a curve is made precise in Definition 2.1.

We show in Theorem 5.3 that the group $K_{-1}(R)$ is the quotient of the Picard group of the infinitesimal thickening nE (which is independent of n for large n) by the image of $\mathrm{Pic}(\tilde{X})$. If $\mathrm{Cl}(R)$ denotes the divisor class group of R, $K_{-1}(R)$ is also isomorphic to $\mathrm{Cl}(R^h)/\mathrm{Cl}(R)$, where R^h is the henselization of R; this isomorphism was conjectured by Srinivas [Sr2, p. 597]. We intend to study the intriguing relationship between $K_{-1}(X)$ and the Brauer group in a separate paper [BW].

We also prove several results about K-regularity. Our stable result (Theorem 4.4) is that every 2-dimensional excellent noetherian R is "K_{-2}-regular." This implies that $K_{-j}(R) = K_{-j}(R[t_1, \dots, t_n])$ for every $j \geq 2$, and all n, and settles the case $\dim(R) = 2$ of a second guess we made in [W1, p. 186], at least for excellent rings.

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To explain this implication, recall that a ring R is called K_q*-regular* if $K_q(R) = K_q(R[t_1, \dots, t_n])$ for every n. The passage from K_{-2}-regularity to K_{-j}-regularity is given by Vorst's theorem [Vor1, Corollary 2.1], which states that if R is K_q-regular then R is also K_{q-1}-regular.

We also give a criterion for a surface X to be K_{-1}-regular, and even K_0-regular, involving the vanishing of the cohomology $H^1(E, \mathscr{L})$ of the conormal bundle $\mathscr{L}$ (see Proposition 5.8). This criterion not only sharpens the K_0-regularity results of K. Coombes and Srinivas [CS], but it also verifies [Sr2, Conjecture B].

The key geometric idea behind our results is that one can resolve the singularities of a surface by first blowing up along a locally complete intersection and then normalizing. This is more subtle than the usual resolution of singularities, in which one alternatively blows up closed points and then normalizes. Our naïve technique does not extend to 3-dimensional normal domains that are not Cohen-Macaulay, since there is no regular sequence of length 3 to blow up.

As promised, we now explain why these calculations give the main part of a classification of projective modules over the Laurent polynomial rings $R[T] = R[t_1, t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$, or even polynomial rings $R[S, T] = R[T][s_1, \dots, s_m]$ over $R[T]$. To avoid discussions of extended projective modules, we assume for simplicity that R is a (commutative noetherian) semilocal domain, so that all projective R-modules are free. We are interested in singular rings because if R is regular, then R. Swan proved in [Sw] that every projective $R[S, T]$-module is free.

Let P be a (finitely generated) projective module of rank r over $R[S, T]$, where R is a 2-dimensional semilocal ring. The obstruction to P being free is the class of $[P] - r$ in the Grothendieck group $K_0(R[S, T])$. If $[P] = 1$, it is well known that P is free (see [Bass, p. 466]). If $[P] = r$ and $r \neq 2$, then P is free by [Sw, Theorem 1.1]. In fact, if $r > 2$, then the isomorphism classes of rank r projective modules are in one-to-one correspondence with the elements of $K_0(R[S, T])/\mathbb{Z}$; this cancellation result follows from the Bhatwadekar-Lindel-Rao theorem in [BLR] that $P \cong P_0 \oplus R[S, T]^{r-2}$ for some rank 2 projective module P_0.

If R is essentially of finite type over an algebraically closed field, we expect that this cancellation result extends to rank 2 projective $R[T]$-modules, that is, that the isomorphism classes of rank 2 projectives are also in one-to-one correspondence with the elements of $K_0(R[S, T])/\mathbb{Z}$. For projective R-modules, this is a theorem of M. Murthy and Swan [MS]. S. Bhatwadekar [Bhat] has recently proven that this is also true for $R[s_1]$, that is, when $m = 1$ and $n = 0$.

Now Bass's fundamental theorem in [Bass, p. 669] says that there is a natural decomposition of $K_0(R[S, T])$ involving the groups $K_{-j}(R)$ and the nil-groups $N^i K_{-j}(R)$ for $j = 0, 1, \dots, n$ and $i \geq 0$. Specifically,

$$\begin{aligned} K_0(R[S,T]) = K_0(R) \oplus K_{-1}(R)^n \oplus K_{-2}(R)^{\binom{n}{2}} \oplus \cdots \oplus K_{-n}(R) \\ \oplus NK_0(R)^{2n+m} \oplus \cdots \oplus N^i K_{-j}(R)^{a_{ij}} \oplus \cdots \oplus N^m K_{-n}(R), \end{aligned}$$

where the number a_{ij} of copies of the group $N^i K_{-j}(R)$ equals the coefficient of $N^i L^j$ in the formal expansion of $(1+N)^m(1+2N+L)^n$. Our main result (Theorem 4.4) says that most of these terms are unnecessary when $\dim(R) = 2$.

This paper is organized as follows. In Section 1 we cover the necessary material on resolving singularities by blowing up a locally complete intersection ideal and by normalizing. In Sections 2–3 we cover the modifications necessary to compute the K-theory of these blow-ups. In Sections 4–5 we prove the main theorems for K_{-j}, and in Section 6 we give several examples to illustrate our techniques. Section 7 is a technical continuation of Section 2, concerning SK_1-regularity of nonreduced curves. Finally, Section 8 is devoted to K_0-regularity and a verification of [Sr3, Conjecture B].

Notation. If X is a scheme, we write $X[t]$ and $X[t, t^{-1}]$ for the product of X with $\mathrm{Spec}(\mathbb{Z}[t])$ and $\mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$, respectively. By a *surface* we mean a 2-dimensional quasi-projective scheme over an excellent (noetherian) base ring; this includes the usual surfaces of algebraic geometry.

1. Reduction ideals

In this section we discuss the role that reduction ideals and their Rees algebras play in blow-ups. The following material was developed in discussions with W. Vasconcelos, and we refer the reader to [Vas] for material about reduction ideals. Throughout, R is a commutative noetherian ring.

Definition 1.1

An ideal I is a *reduction* of an ideal J if there is an integer n such that $IJ^n = J^{n+1}$.

We give two alternate interpretations of reduction ideals. First, recall that an element $x \in R$ is called *integral* over I if there are $a_i \in I^i$ such that

$$x^m + a_1 x^{m-1} + \cdots + a_i x^{m-i} + \cdots + a_m = 0. \tag{1.1.1}$$

LEMMA 1.2

Assume that R is noetherian, and assume that $I \subset J$. Then I is a reduction of J if and only if J is integral over I in the sense that every $x \in J$ is integral over I.

Proof

Suppose that I is a reduction of J. If $x \in J$, then $xJ^n \subset IJ^n$. Using the trick of Nakayama's lemma [Mat, Theorem 2.1], it follows that x satisfies equation (1.1.1) modulo $\mathrm{ann}_R(J^n)$. Since the left-hand side of (1.1.1) is in J, its $n+1$st power vanishes, so that x is integral over I.

Conversely, suppose that J is integral over I. Then we can find an integer m such that each generator x of J satisfies $x^m \in IJ^{m-1}$. Letting n be $(m-1)$ times the number of generators of J, this implies that $IJ^n = J^{n+1}$. □

1.3. Zariski's criterion

Here is another interpretation of reduction ideals. Let R be a domain with quotient field F, and let Σ denote the set of all valuations on F which are nonnegative on R. Then the integral closure of I coincides with the intersection of the submodules $R_v I$ of F $(v \in \Sigma)$ by [ZS, p. 350]. This establishes Zariski's criterion:

I is a reduction of J just in case $R_v I = R_v J$ for all valuations $v \in \Sigma$.

LEMMA 1.4 (Vasconcelos)

Let $I \subset J$ be ideals in a domain R. Suppose that for each $x \in I$ there are natural numbers n and r such that $x^r J^{n+1} \subseteq I^{r+1} J^n$. Then I is a reduction of J.

Proof

In order to apply Zariski's criterion, choose a valuation $v \in \Sigma$ on F. Since $I \subset J$, we have $v(I) \geq v(J)$. Pick $x \in I$ such that $v(x) = v(I)$. Applying v to the inclusion $x^r J^{n+1} \subseteq I^r J^{n+1} = I^{r+1} J^n$ yields the inequality

$$r \cdot v(x) + (n+1)v(J) \geq (r+1) \cdot v(x) + n \cdot v(J),$$

so $v(J) \geq v(I)$. Hence $v(I) = v(J)$. This proves that $v(I) = v(J)$ for every valuation v. By Zariski's criterion, I is a reduction of J. □

To prove the next result, we need some basic facts about the blowing up of $\mathrm{Spec}(R)$ along I, that is, the scheme $\mathbf{Proj}(R[It])$. If $x \in I$ and T denotes xt, then $R[It][T^{-1}] \cong R'[T, T^{-1}]$, where $R' = R[I/x]$. Thus the affine open subset $D_+(xt)$ of $\mathbf{Proj}$ $(R[It])$ is isomorphic to $\mathrm{Spec}(R')$.

THEOREM 1.5

Let $I \subset J$ be ideals in a noetherian domain R. Then I is a reduction of J if and only if $\mathbf{Proj}(R[Jt]) \to \mathbf{Proj}(R[It])$ is a finite morphism.

Proof

If I is a reduction of J, then B is finite over A because it is generated over A by the finitely generated R module $Jt \oplus \cdots \oplus J^n t^n$. For each x in I, it follows that the affine open subset $D_+^B(xt)$ of $\mathbf{Proj}(B)$ is finite over the affine open subset $D_+^A(xt)$ of $\mathbf{Proj}(A)$. Since $(Jt)^{n+1} \subset (It)B$, these form a cover of $\mathbf{Proj}(B)$. Hence $\mathbf{Proj}(B)$ is finite over $\mathbf{Proj}(A)$.

Conversely, suppose that $\mathbf{Proj}(B)$ is finite over $\mathbf{Proj}(A)$. Then for each x in I, the restriction to $D_+^B(xt) \to D_+^A(xt)$ is finite; that is, $R[J/x]$ is finite over $R[I/x]$. If $y \in J$, then y/x must satisfy an integral equation over $R[I/x]$ of the form $(y/x)^{n+1} + a_n(y/x)^n + \cdots + a_0$, where all the a_i belong to I^r/x^r for some r. It follows that $x^r y^{n+1} \in I^{r+1}J^n$. Since J is finitely generated, it follows that for some n and r we have $x^r J^{n+1} \subseteq I^{r+1}J^n$. Using Lemma 1.4, we see that I is a reduction of J. □

The following result, which we cite from [Mat, Theorems 14.14 and 17.4], shows that reductions generated by systems of parameters are abundant.

PROPOSITION 1.6

Suppose that $(R, \mathfrak{m})$ is a d-dimensional local domain with infinite residue field. If $J = (x_1, \ldots, x_r)R$ is an $\mathfrak{m}$-primary ideal, then for any sufficiently general $d \times r$ matrix (r_{ij}) over R, the elements $y_i = \sum r_{ij}x_j$ satisfy the following:

(1) *the ideal $I = (y_1, \ldots, y_d)$ is a reduction of J;*
(2) *the elements $y_1, \ldots, y_d$ form a system of parameters of R;*
(3) *if R is Cohen-Macaulay, the y_i form a regular sequence on R.*

Remark 1.6.1

When $R/\mathfrak{m}$ is finite, the conclusions still hold for some power J^e of J, except that the now meaningless term "sufficiently general" must be replaced by some probabilistic statement like "most." This was proven by D. Northcott and D. Rees [NR, Theorem 3.4]; I am grateful to J. Lipman and Vasconcelos for helping me locate this reference.

The Northcott-Rees proof is essentially like the proof in [Mat, Theorem 14.14] for $R/\mathfrak{m}$ infinite. The key step (see [NR, p. 355] or [Mat, Theorem 14.14, step 2]) is to show that, for infinitely many e, some (or most) d-tuples of elements of the finite vector space $J^e/\mathfrak{m}J^e$ generate a primary ideal of $S = \oplus J^e/\mathfrak{m}J^e$. But this is guaranteed by graded Noether normalization (see [BH, Theorem 1.5.17]). The conversion from existence to "most" follows as in [Mat, Theorem 14.14, step 3].

Example 1.7

Suppose that $(R, \mathfrak{m})$ is a 2-dimensional normal local ring with $R/\mathfrak{m}$ infinite. Then every $\mathfrak{m}$-primary ideal $J = (x_1, \ldots, x_r)R$ has a reduction ideal of the form $I = (y_1, y_2)R$, where y_1, y_2 forms a regular sequence. The blow-up along I is $X' = \mathbf{Proj}(S)$, where $S = R[Y_1, Y_2]/(y_1Y_2 - y_2Y_1)$ is the Rees algebra of I. By Theorem 1.5, the blow-up $X'' = \mathbf{Proj}(R[Jt])$ along J is finite over X'.

1.8. Affine Cone

Let C be a curve in $\mathbb{P}^2_k$ defined by a homogeneous polynomial $F(x, y, z)$ of degree d. Let R be the localization of $k[x, y, z]/(F)$ at $M = (x, y, z)$. Then the following are equivalent: $F(0, 0, 1) \neq 0$; the ideal $I = (x, y)$ is a reduction of M; x, y is a regular sequence in R.

Here is an application of these ideas. Recall that a zero-dimensional subscheme Y of X is a *locally complete intersection* if Y has an affine neighborhood $U = \operatorname{Spec}(R)$ in which the ideal I of R defining Y is generated by a regular sequence y_1, y_2.

THEOREM 1.9

Let R be a 2-dimensional normal domain that is excellent. Then there is a regular surface $\tilde{X}$ birational over $\operatorname{Spec}(R)$*, and any such $\tilde{X}$ is obtained as the normalization of a blow-up along a locally complete intersection ideal.*

Proof

Without loss of generality we can localize R at a singular maximal ideal $\mathfrak{m}$. It is classical (see [Lip2]) that $X = \operatorname{Spec}(R)$ can be desingularized by a sequence of blow-ups at closed points, followed by normalization. By [Hart, Theorem II.7.17] this desingularization $\tilde{X} \to X$ is the blow-up along an ideal J; we can assume that J is $\mathfrak{m}$-primary by [Lip2, p. 155].

If the residue field $R/\mathfrak{m}$ is infinite, we can choose a reduction ideal I for J generated by a regular sequence. Let X' be the blow-up along I; $X' = \mathbf{Proj}(R[It])$. Then $\tilde{X}$ is integral over X', so it is the normalization of X'.

If the residue field $R/\mathfrak{m}$ is finite, we can choose a reduction ideal I for some J^e generated by a regular sequence. Since $\mathbf{Proj}(R[Jt]) \cong \mathbf{Proj}(R[J^e t])$, $\tilde{X}$ is again integral over the blow-up X' along I and hence is the normalization of X'. □

We conclude this section by noting that this technique probably does not work for all 3-dimensional normal domains. For example, there are normal domains that are not Cohen-Macaulay (CM). And even if X is Cohen-Macaulay, once we blow up and normalize we might have a normal scheme X'' that is not Cohen-Macaulay.

If X is not CM, no blow-up along a complete intersection produces an arithmetically Cohen-Macaulay scheme, by the following result of Vasconcelos.

LEMMA 1.10

Suppose that R is a local ring, and suppose that I is an ideal generated by a regular sequence $x_i, \ldots, x_n$. Then

$$R \text{ is } \mathrm{CM} \longleftrightarrow R[It] \text{ is } \mathrm{CM}\,.$$

Proof (Vasconcelos)
Let J be the ideal of $R[X] = R[X_1, \dots, X_n]$ generated by the determinants $X_i x_j - X_j x_i$ $(i, j = 1, \dots, n)$. Then $R[It] = R[X]/J$, the ideal J has height $n-1$, and $pd_{R[X]} R[It] = n - 1$. By the Auslander-Buchsbaum equality (see [W6, Theorem 4.4.15]), we have

$$d - G(R) = d + n - G(R[X]) = (d+1) - G(R[It]),$$

where $d = \dim(R)$. Now observe that R is CM exactly when $d = G(R)$, while $R[It]$ is CM exactly when $d + 1 = G(R[It])$. □

Of course, we do not not know if it is possible for $X' = \mathbf{Proj}(R[It])$ to be CM. Nor do we know if we can arrange for the normalization X'' of X' to be CM.

2. Classical K-theory of projective curves

For the next two sections, we need to collect some facts about the classical K-theory of projective curves and nonnormal surfaces, whose singular locus is a projective curve. These facts differ slightly from the affine case, where one may consult [Bass]. The reader may skip these technical sections without interrupting the flow of ideas.

We first recall what is known for reduced curves. The following elementary definition, due to L. Roberts [Rob, p. 49], is extremely useful.

Definition 2.1
The graph Γ of a noetherian curve Y is the bipartite graph defined as follows. Let S be the singular locus of Y, and let $\pi : \tilde{Y} \to Y$ be the normalization. Then Γ has one vertex for each point of S and one vertex for every component of the normalization $\tilde{Y}$ of Y. There is an edge of Γ for each point of $\pi^{-1}(S)$, connecting the corresponding component of $\tilde{Y}$ to the singular point of Y.

The number $\lambda(Y)$ of loops in a curve Y is defined to be the number of loops in its graph Γ. Because λ is the dimension of $H^1(\Gamma; \mathbb{Z}) \cong \mathbb{Z}^\lambda$, it can be calculated using the Euler characteristic of Γ: if Γ has c connected components, V vertices, and E edges, then $\lambda = c - V + E$. Note that the connected components of Γ correspond to the connected components of Y.

Alternatively, we can calculate λ using the étale cohomology of the curve Y. The following result is proven in [W3, Lemma 2.1] and [W5, Theorem 7.9], and it is implicit in [Rob, p. 47].

LEMMA 2.2
Let Γ be the bipartite graph of a noetherian curve Y with finite normalization. If Γ has λ loops, then

$$H^1_{\text{et}}(Y, \mathbb{Z}) \cong H^1(\Gamma; \mathbb{Z}) \cong \mathbb{Z}^\lambda.$$

The number of loops enters K-theory as the rank of $\text{LPic}(Y)$. Recall that the group $\text{LPic}(Y)$ is defined to be the cokernel of $\text{Pic}(Y[t]) \oplus \text{Pic}(Y[t^{-1}]) \to \text{Pic}(Y[t, t^{-1}])$. [W5, Theorem 7.5] states that $\text{LPic}(Y) \cong H^1_{\text{et}}(Y, \mathbb{Z})$.

LEMMA 2.3

Let Y be a 1-dimensional reduced noetherian scheme. Then

(1) $K_q(Y) = 0$ *for all* $q \leq -2$,

(2) Y *is* K_{-1}*-regular, and*

$$K_{-1}(Y) \cong \text{LPic}(Y) \cong H^1_{\text{et}}(Y, \mathbb{Z}).$$

If Y has finite normalization, then $K_{-1}(Y) \cong \mathbb{Z}^\lambda$, where λ is the number of loops in Y.

Proof

By [W3, Theorem 1.2], we may assume that Y is affine. By [BM, Theorem 8.1] (or [W5, Example 1.7]), $K_{-1}(Y) \cong \text{LPic}(Y)$ and $K_q(Y) = 0$ for $q \leq -2$. □

Examples 2.4

If Y is the node, the graph Γ has two vertices and two parallel edges, forming one loop. Thus $K_{-1}(Y) \cong \mathbb{Z}$. If Y is a cusp, Γ has two vertices connected by one edge; here there are no loops, and $K_{-1}(Y) = 0$.

An interesting class of examples is given by the exceptional fiber Y of a resolution of singularities. For example, resolving the rational double point $x^2 + y^3 + z^5 = 0$ yields a union Y of eight projective lines, intersecting according to the Dynkin diagram E_8 (see [Hart, p. 420]). In this case, the graph Γ is the subdivision of the Dynkin diagram. Since Γ has no loops, $K_{-1}(Y) = 0$.

Now we turn to calculations of $K_*(Y)$. If Y is affine, the following result is proven in [Bass, p. 685]. The last part is standard as well (see [Hart, Exercise III.4.6]). For convenience we write $U(Y) = H^0(Y, \mathcal{O}_Y^\times)$ for the global units of Y.

LEMMA 2.5

Let Y be a 1-dimensional noetherian scheme that is not reduced, and let Y' be a subscheme defined by an ideal $\mathcal{N}$ of $\mathcal{O}_Y$ with $\mathcal{N}^2 = 0$. Then

(1) $K_q(Y) = 0$ *for* $q \leq -2$;

(2) $K_{-1}(Y) \cong K_{-1}(Y') \cong K_{-1}(Y_{\text{red}})$, *which is a free abelian group;*

(3) Y *is* K_{-1}*-regular, and* $K_q(Y[t_1, \dots, t_p]) = K_q(Y)$ *for all* $p > 0$ *and* $q \leq -1$;

(4) $K_0(Y) \cong H^0(Y, \mathbb{Z}) \oplus \text{Pic}(Y)$, *and there is an exact sequence*

$$\begin{aligned} 0 \longrightarrow H^0(Y, \mathscr{N}) \longrightarrow U(Y) \longrightarrow U(Y') \longrightarrow H^1(Y', \mathscr{N}) \\ \longrightarrow \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(Y') \longrightarrow 0; \end{aligned}$$

(5) *if* $H^1(Y', \mathscr{N}) = 0$*, then* $\mathrm{Pic}(Y[t_1, \dots, t_p]) \cong \mathrm{Pic}(Y'[t_1, \dots, t_p])$ *for all* $p \geq 0$*; in particular,* $\mathrm{Pic}(Y) \cong \mathrm{Pic}(Y')$.

Proof
For (1)–(4) it suffices to prove that the relative K-groups $K_n(Y, \mathscr{N})$ vanish for $n < 0$ and for $n = 0$ are isomorphic to the cohomology group $H^1(Y, \mathscr{N})$. To check this we use the Brown-Gersten spectral sequence of [TT]:

$$E_2^{pq} = H^p\big(Y, \mathscr{K}_{-q}\mathscr{N}\big) \Longrightarrow K_{-p-q}(Y, \mathscr{N}).$$

Here $\mathscr{K}_n\mathscr{N}$ denotes the Zariski sheaf associated to the presheaf $K_n(\mathscr{O}_Y, \mathscr{N})$. Now it is well known (see [Bass]) that $\mathscr{K}_n\mathscr{N} = 0$ for $n \leq 0$ and $\mathscr{K}_1\mathscr{N} \cong 1 + \mathscr{N} \cong \mathscr{N}$. This yields $K_0(Y, \mathscr{N}) \cong H^1(Y, \mathscr{N})$, sequence (4), and $K_n(Y) \cong K_n(Y')$ for $n \leq -1$.

To establish K_{-1}-regularity, we consider the functors $K_n(Y[t])$, and so on. The same argument applies here: we replace $\mathscr{K}_n\mathscr{N}$ by $\mathscr{K}_n\mathscr{N}[t]$, the sheaf on Y associated to the presheaf $K_n(\mathscr{O}_Y[t], \mathscr{N}[t])$. Because $\mathscr{N}^2 = 0$, we have $\mathscr{K}_1\mathscr{N}[t] \cong \mathscr{N}[t]$. Hence $K_0(Y[t], \mathscr{N}[t]) \cong H^1(Y, \mathscr{K}_1\mathscr{N}[t]) \cong H^1(Y, \mathscr{N}) \otimes \mathbb{Z}[t]$. □

Porism 2.5.1
More generally, if Y' is defined by a nilpotent ideal $\mathscr{I}$, then the proof shows that $K_n(Y, \mathscr{I}) = 0$ for $n < 0$ and $K_0(Y, \mathscr{I}) \cong H^1(Y, 1 + \mathscr{I})$.

Alternate proof
It is well known that $K_0(Y) \cong H^0(Y, \mathbb{Z}) \oplus \mathrm{Pic}(Y)$. Since $H^0(Y, \mathbb{Z}) = H^0(Y', \mathbb{Z})$, the K_0 part follows from the cohomology sequence associated to the short exact sequence of sheaves on Y:

$$1 \longrightarrow \mathscr{N} \longrightarrow \mathscr{O}_Y^\times \longrightarrow \mathscr{O}_{Y'}^\times \longrightarrow 1.$$

Replacing Y by $Y[T]$ yields the other parts since again we have $K_0(Y[T]) \cong H^0(Y[T], \mathbb{Z}) \oplus \mathrm{Pic}(Y[T])$. □

COROLLARY 2.6
Let Y *be any curve, and let* $\mathscr{I}$ *be a nilpotent ideal sheaf defining a subscheme* Y_0*. If* $H^1(Y_0, \mathscr{I}^i/\mathscr{I}^{i+1})$ *vanishes for all* $i \geq 1$*, then* $K_0(Y, \mathscr{I})$ *vanishes,* $\mathrm{Pic}(Y) \cong \mathrm{Pic}(Y_0)$*, and* $\mathrm{Pic}(Y[t_1, \dots, t_p]) \cong \mathrm{Pic}(Y_0[t_1, \dots, t_p])$ *for all* p.

Recall that a reduced curve Y is *seminormal* if and only if Y is Pic-regular, that is, if and only if$\mathrm{Pic}(Y) \cong \mathrm{Pic}(Y[t_1, \dots, t_p])$ for all p. Of course this is the same as saying that Y is K_0-regular, since $K_0(Y[T]) = H^0(Y, \mathbb{Z}) \oplus \mathrm{Pic}(Y[T])$ for curves.

PROPOSITION 2.7

Let Y be a curve defined over a field k, and let $\mathscr{I}$ denote the nilradical ideal of $\mathcal{O}_Y$.

(a) *If Y is* Pic-*regular, then Y_{red} is a seminormal curve and* $\operatorname{Pic}(Y) \cong \operatorname{Pic}(Y_{\text{red}})$.

(b) *The following are equivalent:*

(1) *Y is* Pic-*regular;*

(2) $N^p \operatorname{Pic}(Y) = 0$ *for some $p > 0$;*

(3) *Y_{red} is seminormal, and $H^1(Y, \mathscr{I}) = 0$.*
If Y is complete and k is perfect, this is equivalent to

(4) *Y_{red} is seminormal, and* $\operatorname{Pic}(Y) \cong \operatorname{Pic}(Y_{\text{red}})$.

Proof

The sequence of Lemma 2.5(4) is a direct summand of the corresponding sequence for $Y[T] = Y[t_1, \dots, t_p]$. The same is true for the variation in Porism 2.5.1 with $Y' = Y_{\text{red}}$. Since $U(Y') = U(Y'[T])$, the complementary exact sequence is

$$0 \longrightarrow H^1\big(Y, 1 + t\mathscr{I}[t]\big) \longrightarrow \frac{\operatorname{Pic}(Y[T])}{\operatorname{Pic}(Y)} \longrightarrow \frac{\operatorname{Pic}(Y'[T])}{\operatorname{Pic}(Y')} \longrightarrow 0.$$

The final term vanishes if and only if Y' is seminormal. Since $H^1(Y, -)$ is right exact, the first term has $H^1(Y, \mathscr{I}/\mathscr{I}^2) \otimes (\mathbb{Z}[T]/\mathbb{Z})$ as a quotient.

For (b), suppose first that Y is defined over a field k of characteristic zero. Then the exponential defines an isomorphism between any nilpotent ideal $\mathscr{I}$ and its multiplicative sheaf $1+\mathscr{I}$. Similarly, we have $\mathscr{I}[t] \cong 1+\mathscr{I}[t]$. If $\mathscr{I}$ is the nilradical of $\mathcal{O}_Y$, we can repeat the above argument (identifying $\mathbb{Z}[T]$ with $t_1 \cdots t_p\mathbb{Z}[T]$) to get exact sequences for all $p > 0$ (and $p = 0$ if Y is complete and k perfect):

$$0 \longrightarrow H^1(Y, \mathscr{I}) \otimes \mathbb{Z}[T] \longrightarrow N^p \operatorname{Pic}(Y) \longrightarrow N^p \operatorname{Pic}\big(Y_{\text{red}}\big) \longrightarrow 0. \qquad (2.7.1)$$

The middle terms vanish exactly when the side terms vanish, and the left sides vanish exactly when $H^1(Y, \mathscr{I})$ does.

If k has positive characteristic, M. Artin has shown in [Art1, Lemma 1.4] that $H^1(Y, \mathscr{I})$ and $H^1(Y, 1 + \mathscr{I})$ have filtrations with isomorphic subquotients; thus $H^1(Y, \mathscr{I}) = 0$ is equivalent to $K_0(Y, \mathscr{I}) = 0$ by Porism 2.5.1. Artin's proof shows that the same is true for $H^1(Y, \mathscr{I}[T])$ and $H^1(Y, 1 + \mathscr{I}[T])$, so the same argument goes through. □

PROPOSITION 2.8

Suppose that C is a (not necessarily reduced) curve on a surface X defined by an invertible ideal $\mathscr{I}$ of $\mathcal{O}_X$, and suppose that Y is a curve on X defined by an ideal $\mathscr{J}$ of $\mathcal{O}_X$, where $\mathscr{I}^n \subseteq \mathscr{J} \subset \mathscr{I}$ for some n.

Suppose that the conormal bundle $\mathscr{L} = \mathscr{I}/\mathscr{I}^2$ *has* $H^1(C, \mathscr{L}^i) = 0$ *for* $i = 1, \dots, n-1$. *Then* $\mathrm{Pic}(Y) \cong \mathrm{Pic}(C)$ *and* $\mathrm{Pic}(Y[t_1, \dots, t_p]) \cong \mathrm{Pic}(C[t_1, \dots, t_p])$ *for all* p.

Proof

First consider the curve $Y = Y_n$ defined by the ideal $\mathscr{I}^n$. Setting $\bar{\mathscr{I}} = \mathscr{I}/\mathscr{I}^n$, the multiplicative sheaf $1 + \bar{\mathscr{I}}$ on Y has a filtration whose associated graded sheaves are the $\mathscr{L}^i = \mathscr{I}^i/\mathscr{I}^{i+1}$ for $i = 1, \dots, n-1$. If $H^1(C, \mathscr{L}^i) = 0$ for $i = 1, \dots, n-1$, then Corollary 2.6 says that the result holds for Y_n.

In the general case, set $\bar{\mathscr{I}} = \mathscr{I}/\mathscr{J}$. Since $H^1(Y, –)$ is right exact, the surjection $1 + \mathscr{I}/\mathscr{I}^n \to 1 + \bar{\mathscr{I}}$ implies that we also have $K_0(Y, \bar{\mathscr{I}}) = H^1(Y, 1 + \bar{\mathscr{I}}) = 0$ and hence $\mathrm{Pic}(Y) = \mathrm{Pic}(C)$, as claimed. □

A curve Y on X is called a *nilpotent thickening* of C if the ideal $\mathscr{J}$ defining Y satisfies $\mathscr{I}^n \subseteq \mathscr{J} \subset \mathscr{I}$ for some n, that is, if the hypothesis of Proposition 2.8 holds.

Variation 2.8.1

Let Y_n be the curve on X defined by the ideal $\mathscr{I}^n$, and set $\mathscr{L} = \mathscr{I}/\mathscr{I}^2$. Suppose that $H^1(C, \mathscr{L}^i) = 0$ for $i \geq n_0$. Then $\mathrm{Pic}(Y_n) \cong \mathrm{Pic}(Y_{n_0})$ for all $n \geq n_0$.

This follows either by induction from Lemma 2.5(5) or from Corollary 2.6 with $Y_0 = Y_{n_0}$, using the proof of Proposition 2.8. The point is that the groups $K_0(Y_n, \mathscr{I}) = H^1(Y, 1 + \mathscr{I}/\mathscr{I}^n)$ are independent of n when $n \geq n_0$.

For example, an ample line bundle $\mathscr{L}$ always has $H^1(C, \mathscr{L}^i) = 0$ for large i (see [Hart, Lemma III.5.2]). Therefore Variation 2.8.1 applies when the conormal bundle $\mathscr{I}/\mathscr{I}^2$ of C is ample.

Example 2.9 (Affine cones)

Suppose that $C \subset \mathbb{P}^n$ is a smooth projective curve of genus $g \leq 2$, or, more generally, that C is embedded in $\mathbb{P}^n$ by a complete linear system of degree greater than or equal to $2g - 1$. Let X be the affine cone of C, and let X'' be the blow-up at the cone point. Then the exceptional divisor E on X'' is isomorphic to C. If Y is any nilpotent thickening of E on X'', then Y is always Pic-regular with $\mathrm{Pic}(Y) = \mathrm{Pic}(C)$.

To see this, note that E is defined by an invertible ideal $\mathscr{I}$ of $\mathscr{O}_{X''}$, and note that the degree of $\mathscr{L} = \mathscr{I}/\mathscr{I}^2$ is $-E \cdot E > 0$ (see [Hart, Example V.1.4.1 and Exercise V.5.7]). By Proposition 2.8, it suffices to show that $H^1(C, \mathscr{L}^i) = 0$ for all $i \geq 1$. For $g \leq 1$ this is easy because $\mathscr{L}$ has positive degree. It holds for $g = 2$ by Riemann-Roch since the degree of $\mathscr{L}$ must be at least 5 (see [Hart, Exercise IV.3.1]).

If C is embedded in $|D|$ for a divisor D of degree $d \geq 2g - 1$, then $|D| \cong \mathbb{P}^{d-g}$

by Riemann-Roch. By Serre duality, the dimension of $H^1(C, \mathscr{L}^i)$ equals $l(K - iD)$, and this is zero because $K - iD$ has degree less than zero.

Example 2.10
Suppose that X' is a projective surface, and suppose that $C \subset X'$ is isomorphic to $\mathbb{P}^1_A$ for some Artinian local ring A. Also suppose that the ideal sheaf $\mathscr{I}$ defining C is invertible, and suppose that $\mathscr{L} = \mathscr{I}/\mathscr{I}^2$ is $\mathscr{O}_{\mathbb{P}^1}(j)$ for some $j \geq 0$.

Let $C \to Y'$ be a nilpotent thickening on X'. Then $\mathrm{Pic}(Y') = \mathrm{Pic}(C) = \mathbb{Z}$ and Y' is Pic-regular. This follows from Proposition 2.8 since $H^1(\mathbb{P}^1_A, \mathscr{O}(i)) = 0$ for all $i \geq -1$.

Example 2.10.1
To show that the hypothesis on $\mathscr{L}$ is critical, consider a line C in $\mathbb{P}^2$ defined, say, by the equation $Z = 0$ in $S = k[X, Y, Z]$. The conormal bundle is $\mathscr{L} \cong \mathscr{O}(-1)$, and its nilpotent thickenings Y_n are defined by $Z^n = 0$.

We claim that $\mathrm{Pic}(Y_2) \cong \mathrm{Pic}(C) = \mathbb{Z}$ but $\mathrm{Pic}(Y_3) \cong \mathbb{Z} \oplus k$ and hence $\mathrm{Pic}(Y_n) \neq \mathrm{Pic}(C) = \mathbb{Z}$ for $n > 2$. This claim follows from the exact sequence in Lemma 2.5(4), which shows that $\mathrm{Pic}(Y_{n+1}) \to \mathrm{Pic}(Y_n)$ is a surjection with kernel $H^1(\mathbb{P}^1, \mathscr{O}(-n)) \cong H^0(\mathbb{P}^1, \mathscr{O}(n-2))$. I am grateful to the referee for pointing out this example.

Example 2.11
Suppose that C is a rational cubic cusp lying on a smooth projective surface X'' over k, and suppose that C is the inverse image of a point under some proper map $X'' \to X$. If $C \to Y$ is a nilpotent thickening on X'', then $\mathrm{Pic}(Y) \cong \mathrm{Pic}(C) \cong k \oplus \mathbb{Z}$. Moreover, because C is not seminormal, the curve Y cannot be Pic-regular.

Indeed, if $\mathscr{L}$ denotes the line bundle $\mathscr{I}/\mathscr{I}^2$ on C and $p : \mathbb{P}^1_k \to C$ is the normalization, then the pullback $p^*\mathscr{L}$ has degree $-C \cdot C > 0$ (see [Hart, Example V.1.4.1]), so $p^*\mathscr{L} \cong \mathscr{O}_{\mathbb{P}^1}(\ell)$ for some $\ell \geq 1$. If ι is the inclusion of the cusp point, then there is an exact sequence of sheaves on C:

$$0 \longrightarrow \mathscr{L}^i \longrightarrow p_*\mathscr{O}_{\mathbb{P}^1}(i\ell) \oplus \iota_*(k) \longrightarrow \iota_*(k[\varepsilon]) \longrightarrow 0.$$

The map $H^0(\mathbb{P}^1_k, \mathscr{O}(i\ell)) \to k[\varepsilon]$ is onto. From this and Example 2.10 we see that $H^1(C, \mathscr{L}^i) = 0$ for all $i \geq 1$. The assertions now follow from Proposition 2.8.

Example 2.12
Suppose that C is a divisor with normal crossings on a projective surface, with ample conormal bundle $\mathscr{L}$. Suppose either that C has no loops in the sense of Definition 2.1 or that each of its irreducible components $C_1, \dots, C_e$ meets at most two other components. Then $H^1(C, \mathscr{L}^i) \cong \oplus_j H^1(C_j, \mathscr{L}^i|_{C_j})$ for all $i \geq 1$. (Use the method of

Example 2.11 and the calculation that $\oplus H^0(C_j, \mathscr{L}^i|_{C_j}) \to \oplus k$ is onto.)

As an illustration, suppose that each C_j is a projective line. Then $H^1(C, \mathscr{L}^i) = 0$ for all $i \geq 1$ as each $\mathscr{L}|_{C_j}$ is ample. If Y is any nilpotent thickening of C, then $\mathrm{Pic}(Y) \cong \mathrm{Pic}(C) \cong \mathbb{Z}^e$ and Y is Pic-regular by Proposition 2.8.

If each C_j is a projective line and there are no loops, then we also have $H^1(C, \mathscr{O}_C) = 0$ (same calculation). Lipman has observed (private communication) that the assumption that $\mathscr{L}$ is ample implies that X has rational singularities. Indeed, it follows from induction and $H^1(C, \mathscr{L}^i) = 0$ that each $H^1(X, \mathscr{O}_X/\mathscr{I}^i)$ vanishes. Taking the inverse limit, the theorem on formal functions [Hart, Theorem III.11.1] implies that $H^1(X, \mathscr{O}_X)_x = 0$ at each singular point x, which is the definition of a rational singularity (cf. [Lip1, Theorem 12.2]).

Example 2.13 (Rational singularities)
Suppose that k is perfect and that X has rational singularities. Choose $X'' \to X$ such that the reduced exceptional divisor C has normal crossings and smooth branches. Then every nilpotent thickening Y of C on X'' is Pic-regular and has $\mathrm{Pic}(Y) \cong \mathrm{Pic}(C)$.

The following proof is due to Lipman (cf. [Lip1, Theorem 12.2]). Artin has shown in [Art2, p. 130] that $H^1(Y, \mathscr{O}_Y) = 0$ for every such Y (since the singularities of X are rational and Y has a thickening Y' which is a divisorial cycle). Since k is perfect, $H^0(Y, \mathscr{O}_Y) \to H^0(C, \mathscr{O}_C)$ is a split surjection. The cohomology sequence for $\mathscr{O}_Y \to \mathscr{O}_C$ shows that $H^1(Y, \mathscr{I}/\mathscr{I}^n) = 0$. As C is seminormal, we see from Proposition 2.7 that Y is Pic-regular and $\mathrm{Pic}(Y) \cong \mathrm{Pic}(C)$.

3. Mayer-Vietoris sequences

In the next section we need to know the lower K-theory of a nonnormal surface X'. As in the affine case, this can be done using a Mayer-Vietoris sequence for finite maps. We say that a closed subscheme $Y' \subset X'$ is *conducting* (for a finite morphism $\pi X'' \to X'$) if the ideal $\mathscr{I}$ of $\mathscr{O}_{X'}$ defining Y' is also an ideal of $\pi_*\mathscr{O}_{X''}$.

THEOREM 3.1 (Mayer-Vietoris)
Let $\pi : X'' \to X'$ be a finite morphism of noetherian schemes such that a conducting subscheme Y' has dimension at most 1. *Setting $Y'' = Y' \times_{X'} X''$, there is an exact "Mayer-Vietoris" sequence*

$$\begin{aligned} K_1(X'') \oplus K_1(Y') \longrightarrow K_1(Y'') \longrightarrow K_0(X') \longrightarrow K_0(X'') \oplus K_0(Y') \longrightarrow K_0(Y'') \\ \longrightarrow K_{-1}(X') \longrightarrow K_{-1}(X'') \oplus K_{-1}(Y') \longrightarrow K_{-1}(Y'') \\ \longrightarrow K_{-2}(X') \longrightarrow K_{-2}(X'') \longrightarrow 0. \end{aligned}$$

Moreover, $K_q(X') \cong K_q(X'')$ for all $q \leq -3$, and $N^p K_q(X') \cong N^p K_q(X'')$ for all $q \leq -2$ and all $p > 0$.

In the affine case, when one considers finite ring extensions, this sequence is due to H. Bass and M. Murthy (see [Bass, p. 677]). The scheme version presented here is slightly more delicate and was proven in [PW2, Theorem A.3] and [BPW, Theorem 3.3]. Note that we are using the vanishing of $K_{-2}(Y')$ and $K_{-2}(Y'')$ to terminate the sequence.

Example 3.2
The blow-up X' of a complete intersection on the affine cone $X = \operatorname{Spec}(R)$ of a smooth plane conic C is an instructive case in point. Specifically, let $R = k[x, y, z]/(xy = z^2)$, $S = R[xt, yt]$, and $X' = \mathbf{Proj}(S)$. Then the normalization of X' is $X'' = \mathbf{Proj}(S'')$, where $S'' = R[xt, yt, zt]$. Now the conductor from S'' to S is the homogeneous prime ideal $I = (x, y, z)S''$, and $S/I = k[X, Y]$, $S''/I = k[X, Y, Z]/(XY = Z^2)$. Hence the associated sheaf $\mathscr{I} = \tilde{I}$ defines the conductor subscheme $Y' = \mathbf{Proj}(S/I) = \mathbb{P}^1_k$, and $Y'' = Y' \times_{X'} X''$ is the plane conic C. In fact, X'' is a line bundle over C, and its zero-section is the inclusion of Y'' in X''.

Since X'', Y', and Y'' are smooth and $Y'' \subset X''$ induces an isomorphism on K-theory, it follows that X' is K_0-regular, $K_q(X') = 0$ for all $q < 0$, and

$$K_0(X') \cong K_0\big(\mathbb{P}^1_k\big) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Example 3.3
Here is a singular (nonnormal) surface with $K_{-2}(X') = \mathbb{Z}/2$. Let Y'' be the configuration of four lines in the plane $X'' = \operatorname{Spec}(k[x, y])$ defined by $f = xy(1-x)(1-y)$. Let Y' be the plane curve $u(v + 1 - u^2) = 0$, and let X' be the surface obtained from X'' by gluing along the evident degree two maps $s : Y'' \to Y'$. That is, $X' = \operatorname{Spec}(R)$, where R is the subring of $k[x, y]$ generated by $u = x(1-x)$, $v = y(1-y) - 1 + u^2$, and $I = fk[x, y]$. The calculation that the map from $K_{-1}(Y') = \mathbb{Z}$ to $K_{-1}(Y'') = \mathbb{Z}$ is multiplication by 2 is left as an exercise for the reader. The claim about $K_{-2}(X')$ follows immediately upon plugging this calculation into the Mayer-Vietoris sequence.

3.4. H^0–LPic sequence
The image of $K_1(Y'') \to K_0(X')$ lies in the kernel $\widetilde{K}_0(X')$ of the rank map $K_0(X') \to H^0(X', \mathbb{Z})$. In fact, the Mayer-Vietoris sequence in Theorem 3.1 surjects onto the "H^0–LPic sequence" of [W5, Proposition 7.8]:

$$\begin{aligned} 0 \longrightarrow H^0(X', \mathbb{Z}) \longrightarrow H^0(X'', \mathbb{Z}) \oplus H^0(Y', \mathbb{Z}) \longrightarrow H^0(Y'', \mathbb{Z}) \\ \longrightarrow \operatorname{LPic}(X') \longrightarrow \operatorname{LPic}(X'') \oplus \operatorname{LPic}(Y') \longrightarrow \operatorname{LPic}(Y''). \end{aligned} \tag{3.4}$$

Since $K_{-1}(Y) \cong \operatorname{LPic}(Y)$ for curves by Lemma 2.3, we see that the H^0–LPic sequence (3.4) may be completed to end in

$$\operatorname{LPic}(X'') \oplus \operatorname{LPic}(Y') \longrightarrow \operatorname{LPic}(Y'') \longrightarrow K_{-2}(X') \longrightarrow K_{-2}(X'') \longrightarrow 0. \tag{3.4.1}$$

Defining $\widetilde{K}_{-1}(X)$ to be the kernel of $K_{-1}(X) \to \mathrm{LPic}(X)$, a diagram chase shows that the kernel sequence is exact:

$$\begin{aligned} K_1(X'') \oplus K_1(Y') \longrightarrow K_1(Y'') \longrightarrow \widetilde{K}_0(X') \longrightarrow \widetilde{K}_0(X'') \oplus \widetilde{K}_0(Y') \\ \longrightarrow \widetilde{K}_0(Y'') \longrightarrow \widetilde{K}_{-1}(X') \longrightarrow \widetilde{K}_{-1}(X'') \longrightarrow 0. \end{aligned} \tag{3.4.2}$$

We saw in Theorem 1.9 that there is a blow-up X' of any normal surface whose normalization is nonsingular. This provides motivation for the following result.

PROPOSITION 3.5

Let X' be a surface whose normalization X'' is regular. Then

(1) *there is an isomorphism $K_{-2}(X') \cong H^2_{\mathrm{nis}}(X', \mathbb{Z})$;*

(2) *X' is K_{-2}-regular, and $K_q(X') = 0$ for all $q \le -3$;*

(3) *there is an exact sequence*

$$\mathrm{Pic}(X') \longrightarrow \mathrm{Pic}(X'') \times \mathrm{Pic}(Y') \longrightarrow \mathrm{Pic}(Y'') \longrightarrow \widetilde{K}_{-1}(X') \longrightarrow 0;$$

(4) *there are isomorphisms*

$$\widetilde{K}_{-1}(X') \cong H^2_{\mathrm{zar}}\big(X', \mathcal{O}^{\times}_{X'}\big) \cong H^2_{\mathrm{nis}}\big(X', \mathcal{O}^{\times}_{X'}\big)$$

and a noncanonical isomorphism $K_{-1}(X') \cong H^1_{\mathrm{nis}}(X', \mathbb{Z}) \oplus H^2_{\mathrm{nis}}(X', \mathcal{O}^{\times}_{X'})$.

Proof

Since $K_{-2}(X'') = 0$ and $H^2_{\mathrm{nis}}(X'', \mathbb{Z}) = 0$, the first assertion is immediate from a comparison of (3.4.1) with the cohomology sequence

$$H^1_{\mathrm{nis}}(X'', \mathbb{Z}) \oplus H^1_{\mathrm{nis}}(Y', \mathbb{Z}) \longrightarrow H^1_{\mathrm{nis}}(Y'', \mathbb{Z}) \longrightarrow H^2_{\mathrm{nis}}(X', \mathbb{Z}) \longrightarrow H^2_{\mathrm{nis}}(X'', \mathbb{Z}).$$

Assertion (2) follows from Theorem 3.1. For assertion (3), we use the "Units-Pic" sequence:

$$\begin{aligned} 1 \longrightarrow U(X') \longrightarrow U(X'') \times U(Y') \longrightarrow U(Y'') \longrightarrow \mathrm{Pic}(X') \\ \longrightarrow \mathrm{Pic}(X'') \times \mathrm{Pic}(Y') \longrightarrow \mathrm{Pic}(Y''). \end{aligned}$$

As pointed out in [W5, Proposition 7.8], this is part of a long cohomology sequence. Since X'' is regular, it is well known that we have $H^2(X'', \mathcal{O}^{\times}_{X''}) = 0$ (for both Zariski and Nisnevich cohomology). So this sequence ends in

$$\mathrm{Pic}(X'') \times \mathrm{Pic}(Y') \longrightarrow \mathrm{Pic}(Y'') \longrightarrow H^2\big(X', \mathcal{O}^{\times}_{X'}\big) \longrightarrow 0.$$

There is a natural surjection from the K_1–K_0 part of (3.4.2) onto the Units-Pic sequence, essentially due to Bass and Murthy (see [Bass, p. 482]). As $K_{-1}(X'') = 0$, this induces an isomorphism of cokernels, from $\widetilde{K}_{-1}(X') = 0$ to $H^2(X', \mathcal{O}^{\times})$. □

4. *K*-theory and resolutions

The following theorem, due to R. Thomason [T], is the key to our calculations in K-theory. We first state it for X of arbitrary dimension and then restrict to surfaces.

THEOREM 4.1 (Thomason [T])
Let X be a quasi-projective scheme, and let $i : Y \to X$ be a locally complete intersection of pure codimension d. Let $f : X' \to X$ be the blow-up of X along Y. Then, for all q,

$$K_q(X') \cong K_q(X) \oplus \big(\mathbb{Z}^{d-1} \otimes K_q(Y)\big).$$

A similar result holds for the functors $N^pK_q(X)$.

Indeed, if Y' denotes the pullback $Y \times_X X'$, then Y' is locally isomorphic to $\mathbb{P}^d_Y$, so we have $K_*(Y') \cong \mathbb{Z}^d \otimes K_*(Y)$. Moreover, the closed immersion $i' : Y' \to X'$ defines a map $i'_* : K_*(Y') \to K_*(X')$. Thomason [T] proves that there is a split exact sequence

$$0 \longrightarrow K_*(Y) \xrightarrow{(-i_*,\lambda\cdot f_Y^*)} K_*(X) \times K_*(Y') \xrightarrow{(f^*,i'_*)} K_*(X') \longrightarrow 0.$$

To obtain the result for $N^pK_q(X)$, use the case $p=0$ with X replaced by $X[t_1, \dots, t_p]$, the product of X with affine space $\mathbb{A}^p$.

Application 4.2 (Murthy [Mur])
Let $R = k[x, y, z]/(xy = z^2)$ be the affine cone of a smooth plane conic. Then R is K_0-regular, $K_q(R) = 0$ for $q < 0$, and $K_0(R) = \mathbb{Z}$. These facts are immediate from Theorem 4.1, given our calculation in Example 3.2 for the blow-up $X = \mathbf{Proj}(R[xt, yt])$ along the complete intersection ideal $(x, y)R$. An alternative proof of these facts was given in 1969 by Murthy [Mur, Proposition 5.2 and Example 6.2].

COROLLARY 4.3
Let R be a 2-dimensional normal local ring, and let x, y be a regular sequence. Set $X' = \mathbf{Proj}(R[U, V]/(xV - yU))$ and $A = R/(x, y)$. Then

$$K_q(X') \cong K_q(R) \oplus K_q(A), \quad \textit{for all } q.$$

In particular, $K_q(X') \cong K_q(R)$ for all $q < 0$, while $K_0(X') \cong \mathbb{Z}\oplus\mathbb{Z}$ and $\mathrm{Pic}(X') \cong \mathbb{Z}$ *on the class of the divisor $Y \times_X X' = \mathbb{P}^1_A$.*

For $q \le 0$ the ring R is K_q-regular if and only if X' is, and $N^pK_q(R) = N^pK_q(X')$ for all $p > 0$. If R is not regular, then X' is not K_1-regular.

Indeed, X' is the blow-up of $\mathrm{Spec}(R)$ along the subscheme $\mathrm{Spec}(A)$, which has codimension 2. And A is K_0-regular by [Bass, p. 685].

Remark 4.3.1

When R is not regular, the maximal ideal $\mathfrak{m}$ of R cannot be generated by x and y, so the nilradical of A is nonzero. By [Bass, p. 671], this means that X' is *never* K_1-regular, even though R might be.

When R is not regular, the blow-up X' is never normal; indeed, it is easy to see that on the affine open $D_+(xt) = \operatorname{Spec}(R[y/x])$ of X' the height-one prime ideal $\mathfrak{m}[y/x]$ is singular. This implies that if $X'' \to X'$ is its normalization, then the conductor subscheme Y' is $Y \times_X X' = \mathbb{P}^1_A$ up to a nilpotent ideal.

THEOREM 4.4

Let X be a 2-dimensional excellent noetherian scheme.

(1) *If $q \leq -3$, then $K_q(X) = 0$.*

(2) *If $q \leq -2$, then X is K_q-regular; that is, $K_q(X) \cong K_q(X[t_1, \dots, t_p])$ for all p.*

(3) *If X is normal, then $K_{-2}(X) \cong \mathbb{Z}^\lambda$, where λ is the number of loops in the exceptional divisors of a resolution of singularities of X.*

Proof

Using the Mayer-Vietoris sequences in Theorem 3.1 and Lemma 2.5 for curves, we see that we may assume that X is normal (cf. [W1, Proposition 2.8]). Since normal surfaces have isolated singularities, [W3, Theorem 0.1] says that we may replace X by Spec of any of its local rings. That is, we may assume $X = \operatorname{Spec}(R)$ for R a normal local ring.

Choose a resolution of singularities $X'' \to X$, with X'' the blow-up along an ideal J of R. By Theorem 1.9, X'' is the normalization of a blow-up X' along a complete intersection. The Mayer-Vietoris sequences in Proposition 3.5 and Corollary 4.3 give $K_q(R) = K_q(X') = 0$ for $q \leq -3$, and $N^p K_q(R) = N^p K_q(X') = 0$ for $p > 0$ and $q \leq -2$.

For $q = -2$, we have $K_{-2}(R) = K_{-2}(X')$. Since X'' has no negative K-theory, the Mayer-Vietoris sequence ends in $K_{-1}(Y') \to K_{-1}(Y'') \to K_{-2}(R) \to 0$. Now $Y'_{\text{red}} \cong \mathbb{P}^1_k$, so $K_{-1}(Y') = 0$ by Lemma 2.3. Since Y''_{red} is the exceptional fiber E over the singular point of $X = \operatorname{Spec}(R)$, we have $K_{-1}(Y'') = K_{-1}(E) \cong \mathbb{Z}^\lambda$ by Proposition 3.5. □

Exercise 4.5

Suppose that X is a surface that is not normal. Use the Mayer-Vietoris sequence (3.4.1) for the normalization to show that the group $K_{-2}(X)$ is a finitely generated group. As we saw in Example 3.3, this group need not be torsion free. For extra credit, find a surface X' proper over X so that $K_{-2}(X)$ is isomorphic to $H^2_{\text{nis}}(X', \mathbb{Z})$.

Example 4.6 (Affine cones)
Suppose that R is the homogeneous coordinate ring of a reduced curve C in $\mathbb{P}^n$. Then the blow-up X'' of $\operatorname{Spec}(R)$ at the maximal ideal $\mathfrak{m}$ is an $\mathbb{A}^1$-bundle over C, whose exceptional fiber is $C = \mathbf{Proj}(R)$. From Theorem 4.4 we have $K_q(R) = 0$ for all $q \le -3$.

We claim that we also have $K_{-2}(R) = 0$. This follows easily from Theorem 4.4 only when C is *projectively normal*, that is, when R is normal and C is a smooth curve. Suppose now that R is not normal; it might not even be Cohen-Macaulay. In this case, we pass to the ring $R' = \bigcup \mathfrak{m}^{-i}$, which is not only Cohen-Macaulay but also finite over R (see [GW, Theorem 1.5]). Since the conductor ideal is $\mathfrak{m}$-primary, we see from the Mayer-Vietoris sequence in Section 3.4 that $K_{-1}(R) \cong K_{-1}(R')$ and $K_{-2}(R) \cong K_{-2}(R')$.

Since R' is Cohen-Macaulay, the scheme X'' is finite over a blow-up X' along a local complete intersection by Proposition 1.6. Using Lemma 2.5, we see that $K_{-2}(X'') \cong K_{-2}(C) = 0$ and that $K_{-1}(X'') \to K_{-1}(Y'') \cong K_{-1}(C)$ is an isomorphism. Since $K_{-2}(R') \cong K_{-2}(X')$ by Theorem 4.1, the claim now follows from sequence (3.4.1).

An interesting nonnormal family of examples arises when C is a union of projective lines. If C is connected, then R' is the seminormalization of R (see [GW, Corollary 5.9]). Otherwise, R' is the product of the seminormalizations of the affine cones of the connected components of C (see [GW, Lemma 6.4]).

We describe $K_{-1}(R)$ in Example 5.8.1, at least in the case when C is a smooth curve.

5. K_{-1} and Class groups

We can also describe $K_{-1}(X)$ and say when X is K_{-1}-regular. Let X'' be a resolution of singularities of a normal surface X, with exceptional fibers E_i over the singular points y_i of X. Choose a blow-up X' along a local complete intersection Y so that X'' is finite over X'. Choose a conductor subscheme Y' which is a nilpotent thickening of $Y \times_X X' \cong \mathbb{P}^1_Y$, and set $Y'' = Y' \times_{X'} X''$. Then Y'_{red} is a disjoint union of copies of $\mathbb{P}^1_k$, and $E = Y''_{\text{red}}$ is the disjoint union of the E_i.

PROPOSITION 5.1
Let X be a normal surface, with resolution of singularities X''. Suppose that X has e singularities, and suppose that Y' and Y'' are as above. Then $K_{-1}(X)$ is presented by the exact sequences

$$\operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(X'') \longrightarrow \operatorname{Pic}(Y'') \longrightarrow K_{-1}(X) \longrightarrow 0.$$

Moreover, there is a natural isomorphism $N\operatorname{Pic}(Y'') \cong NK_{-1}(X)$.

Proof
By [W5], we have $\mathrm{LPic}(X') = \mathrm{LPic}(X) = 0$, so $K_{-1}(X) \cong K_{-1}(X') = \widetilde{K}_{-1}(X')$ by Theorem 4.1. The result now follows from Proposition 3.5, given the vanishing $N\,\mathrm{Pic}(X') = N\,\mathrm{Pic}(X) = N\,\mathrm{Pic}(Y') = 0$, Example 2.10 for Y', and the calculation in Lemma 5.2. □

LEMMA 5.2
Let X be a normal surface, and let $X' \xrightarrow{\pi} X$ be a blow-up along a local complete intersection Y of dimension zero. Then the fiber $Y \times_X X'$ is isomorphic to $\mathbb{P}^1_Y$, with conormal bundle $\mathscr{O}(1)$, and the inclusion $j : \mathbb{P}^1_Y \hookrightarrow X'$ induces a split surjection $j^* : \mathrm{Pic}(X') \to \mathrm{Pic}(\mathbb{P}^1_Y)$ with kernel $\mathrm{Pic}(X)$. More precisely, the following composition is an isomorphism:

$$K_0(Y) \xrightarrow{\pi^*} K_0\big(\mathbb{P}^1_Y\big) \xrightarrow{j_*} \widetilde{K}_0(X') \xrightarrow{j^*} \widetilde{K}_0\big(\mathbb{P}^1_Y\big) = \mathrm{Pic}\left(\mathbb{P}^1_Y\right).$$

Proof
The question being local on X, we may assume that $X = \mathrm{Spec}(R)$ for a local ring R, and we may assume that $Y = \mathrm{Spec}(R/I)$ for $I = (x, y)R$. The structure sheaf $\mathscr{O}_{\mathbb{P}^1}$ of $\mathbb{P}^1_Y$ is defined by the invertible ideal $\mathscr{I}$ on $X' = \mathbf{Proj}(R[It])$ associated to the ideal $IR[It]$, so the map $K_0(Y) \to \widetilde{K}_0(X')$ sends the generator to $[\mathscr{O}_{\mathbb{P}^1}] = 1 - [\mathscr{I}]$. Applying j^* yields $1 - [\mathscr{I}/\mathscr{I}^2]$. But $\mathscr{I}/\mathscr{I}^2$ is the structure sheaf $\mathscr{O}(1)$ on $\mathbb{P}^1_Y$, and this sheaf generates $\mathrm{Pic}(\mathbb{P}^1_Y) \cong \mathbb{Z}$. □

The following theorem verifies [Sr2, Conjecture A]. Note that if R is semilocal, then the henselization and completion of R are, respectively, equal to the product (over the maximal ideals $\{\mathfrak{m}_i\}$ of R) of the henselizations and completions of the $R_{\mathfrak{m}_i}$. Since $\mathrm{Cl}(R) \subseteq \oplus\,\mathrm{Cl}(R_{\mathfrak{m}_i}) \subseteq \mathrm{Cl}(R^h)$, our result is compatible with the description of $K_{-1}(R)$ in [W3, Theorem 3.7].

THEOREM 5.3
Let R be a normal semilocal ring of dimension 2, essentially of finite type over a field (or over an excellent Dedekind domain). Write R^h and $\widehat{R}$ for the henselization and completion of R. Then we have the following:
(1) $K_{-1}(R^h) = K_{-1}(\widehat{R}) = 0$;
(2) *the class groups of R^h and $\widehat{R}$ are isomorphic:* $\mathrm{Cl}(R^h) \cong \mathrm{Cl}(\widehat{R})$;
(3) $K_{-1}(R) \cong \mathrm{Cl}(R^h)/\mathrm{Cl}(R) \cong \mathrm{Cl}(\widehat{R})/\mathrm{Cl}(R)$.

Proof
We first consider $\widehat{S} = \mathrm{Spec}(\widehat{R})$. Then $\widehat{X}'' = X'' \times_R \widehat{S}$ is a resolution of singularities

of $\widehat{S}$. Grothendieck's existence theorem implies that the map $\mathrm{Pic}(\widehat{X}'') \to \varprojlim \mathrm{Pic}(Y''_n)$ is an isomorphism (see [Art3, (3.6)] or [Hart, Exercise II.9.6]). By Proposition 5.1, we have $K_{-1}(\widehat{R}) = 0$.

Artin approximation implies that $\mathrm{Cl}(R^h) \cong \mathrm{Cl}(\widehat{R})$ and $K_{-1}(R^h) = 0$. In somewhat more detail, let X''_h denote $X'' \times_R \mathrm{Spec}(R^h)$. By [Art3, Theorem 3.5], $\mathrm{Pic}(X''_h)$ is a dense subgroup of $\varprojlim \mathrm{Pic}(Y''_n)$, so we also have $\mathrm{Pic}(X''_h) \cong \mathrm{Pic}(Y''_{n_0})$. Again by Proposition 5.1, we have $K_{-1}(R^h) = 0$.

The isomorphism $K_{-1}(R) \xrightarrow{\simeq} \mathrm{Cl}(\widehat{R})/\,\mathrm{Cl}(R)$ also follows from Proposition 5.1, given Srinivas's observation in [Sr2, p. 597] that for $n \gg 0$ we have an exact sequence

$$\mathrm{Pic}(X'') \longrightarrow \mathrm{Pic}(Y''_n) \longrightarrow \frac{\mathrm{Cl}(\widehat{R})}{\mathrm{Cl}(R)} \longrightarrow 0. \qquad \square$$

Remark 5.3.1

If we knew that $K_{-1}(R^h) = 0$, we could prove (3) as follows. The map $R \to \prod R^h = R^h_i$ is an analytic isomorphism along $\prod \mathfrak{m}_i$, so (as in [W3, Theorem 3.7], but using [TT, Theorem 7.1]) there is an exact sequence

$$\widetilde{K}_0\big(\mathrm{Spec}(R) - \{\mathfrak{m}_i\}\big) \longrightarrow \oplus \widetilde{K}_0\big(\mathrm{Spec}(R^h_i) - \{\mathfrak{m}_i\}\big) \longrightarrow K_{-1}(R) \longrightarrow \oplus K_{-1}\big(R^h_i\big).$$

The first two terms are class groups, and the final term is zero.

COROLLARY 5.4

Let X be a normal surface, and let R^h_i ($i = 1, \dots, e$) be the Hensel local rings of X at its singular points. Then there is an exact sequence

$$\mathrm{Cl}(X) \longrightarrow \oplus\, \mathrm{Cl}\left(R^h_i\right) \longrightarrow K_{-1}(X) \longrightarrow 0.$$

Proof

Let R be the semilocal ring of X at the singular locus; clearly $\mathrm{Cl}(X) \to \mathrm{Cl}(R)$ is onto. By [W3, Theorem 1.2], $K_{-1}(X) \cong K_{-1}(R)$, so the result follows from Theorem 5.3. □

Remark 5.4.1

Since $K_{-1}(X) \cong H^2(X', \mathcal{O}^\times_{X'})$ by Proposition 3.5(4), the Leray spectral sequence for $X' \to X$ and $\mathcal{O}^\times_{X'}$ degenerates to yield the exact sequence

$$0 \longrightarrow H^2\big(X, \mathcal{O}^\times_X\big) \longrightarrow K_{-1}(X) \longrightarrow \oplus_i K_{-1}\big(\mathcal{O}_{X,y_i}\big) \longrightarrow 0.$$

This is the local-to-global sequence of [W3, Theorem 0.2]; the group $H^2(X, \mathcal{O}^\times_X)$ is zero unless X has more than one singular point.

Example 5.5 (The E_s singularity)
Let $R = k[x, y, z]/(x^2 + y^3 + z^5)$. It is well known that R and R^h are unique factorization domains with an isolated singularity at the origin (see [Hart, p. 420], for example). By Corollary 5.4, we see that $K_{-1}(R) = 0$.

In this case, we also have $K_{-2}(R) = 0$ by Theorem 4.4 and Example 2.4. The singularity is resolved by 8 successive blow-ups of closed points, and the exceptional fiber is a union of 8 projective lines, which intersect each other according to the Dynkin diagram E_8. From this it is easy to calculate that $\mathrm{Pic}(Y'') = \mathbb{Z}^8$, as in Example 2.13 or as in [Art1, Theorem 1.7].

This surface was discovered by Klein; the binary icosahedral group $G = \mathrm{SL}_2(\mathbb{F}_5)$ acts on the plane, and R is the invariant subring $k[u, v]^G$.

Following Lipman [Lip1, p. 225], the Néron-Severi group $NS(E) = \oplus NS(E_i)$ is the free abelian group on the irreducible components C_i of E. Let H denote the cokernel of the endomorphism of $NS(E)$ given by the intersection matrix $(C_i \cdot C_j)$. A lemma of P. Du Val (see [dVal], [Mum, p. 6], or [Lip1, Lemma 14.1]) states that the intersection matrix is negative definite, so H is a finite group.

We note in passing that H arose in D. Mumford's 1961 study of the topology of normal singularities on a complex surface X: if M is the intersection of $X(\mathbb{C})$ with a small sphere about the singularity, then H is the torsion subgroup of $H_1(M, \mathbb{Z})$ (see [Mum, p. 11]).

Now consider the map $\theta : \mathrm{Pic}(X'') \to \mathrm{Pic}(E) \to NS(E)$, and set $\mathrm{Pic}^0(X'') = \ker\theta$ and $G = \operatorname{coker}\theta$. Since the map $\mathrm{Pic}(X'') \to NS(E)$ sends $[C_i]$ to the ith row of this matrix, G is a quotient of H. Hence G is also a finite group. In fact, Lipman's main exact sequence [Lip1, Proposition 14.2] is

$$0 \longrightarrow \mathrm{Pic}^0(X'') \longrightarrow \mathrm{Cl}(R) \longrightarrow H \longrightarrow G \longrightarrow 0.$$

5.6. Rational singularities

If R has a rational singularity, then the group $K_{-1}(R)$ is the finite group G studied by Lipman. Indeed, $\mathrm{Cl}(R^h) \cong H$ by [Lip1, Proposition 17.1], so Lipman's main sequence shows that G is the cokernel of $\mathrm{Cl}(R) \to \mathrm{Cl}(R^h)$, which by Theorem 5.3 is $K_{-1}(R)$.

Conversely, if k is algebraically closed and $\mathrm{Cl}(R^h)$ is finite, then R must have a rational singularity by [Lip1, Theorem 17.4]. However, P. Salmon showed in [Salm] that $R = k_0(u)[x, y, z]/(x^2 + y^3 + uz^6)$ has $\mathrm{Cl}(R^h) = 0$, and hence $K_{-1}(R) = 0$, even though R does not have a rational singularity.

PROPOSITION 5.7
Let X be a normal surface over an algebraically closed field of characteristic zero.

Then $K_{-1}(X)$ *is the direct sum of a divisible group and the finite group* G*. In particular,* $K_{-1} \otimes \mathbb{Q}/\mathbb{Z}$ *vanishes.*

Proof

By Lemma 2.5, $\mathrm{Pic}(Y'') \to NS(Y'') = NS(E)$ is a surjection whose kernel is a divisible group. Also, $\mathrm{Pic}(X'') \to NS(X'')$ is a surjection whose kernel is divisible. By the 5-lemma, the cokernel of $\mathrm{Pic}(X'') \to \mathrm{Pic}(Y'')$ contains a divisible group, and the quotient is the the cokernel G. □

We now turn to the question of K_{-1}-regularity.

PROPOSITION 5.8

Let $X'' \to X$ *be a resolution of singularities of a normal surface* X*, and let* $\mathscr{L}$ *be the conormal bundle of the exceptional fibers* E_j*. That is,* $\mathscr{L} = \mathscr{I}/\mathscr{I}^2$*, where* $\mathscr{I}$ *is the invertible ideal defining the (reduced) exceptional divisors* E_j *on* X*.*

If X *is* K_{-1}*-regular, then*

(1) *every exceptional fiber* E_j *must be a seminormal curve, and*

(2) *each* $H^1(E_j, \mathscr{L})$ *must vanish.*

Conversely, if each E_j *is seminormal and* $H^1(E_j, \mathscr{L}^i) = 0$ *for all* $i \geq 1$*, then* X *is* K_{-1}*-regular.*

Proof

Because K-regularity is a local question, we may assume that X has an isolated singularity, and we may assume $Y''_{\mathrm{red}} = E$. Because $NK_{-1}(X) \cong N\,\mathrm{Pic}(Y'')$ by Proposition 5.1, we are reduced to the assertions about $N\,\mathrm{Pic}(Y'')$ made in Propositions 2.7 and 2.8. □

5.8.1. Affine cones

Let R be the homogeneous coordinate ring of a smooth curve C in $\mathbb{P}^n$. As noted in Example 4.6, the blow-up X'' of $\mathrm{Spec}(R)$ at the vertex point y is an $\mathbb{A}^1$-bundle over C, and the inverse image of y is isomorphic to C. If C is not projectively normal, then the normalization of R is the ring $R' = \oplus\Gamma(C, \mathscr{O}_C(n))$ (see [Hart, Exercise II.5.14]), and we have $K_{-1}(R) \cong K_{-1}(R')$ by Theorem 3.1. By Proposition 5.8, the K_{-1}-regularity of R and R' is connected with the vanishing of $H^1(C, \mathscr{O}_C(1))$.

Suppose first that C has genus $g \leq 2$, or suppose that C is embedded in $\mathbb{P}^n$ by a complete linear system D of degree greater than or equal to $2g - 1$. By Example 2.9, the cohomology groups vanish, and so R is K_{-1}-regular. Moreover, from Proposition 5.1 we have $K_{-1}(R) = 0$. If $\mathrm{char}(k) = 0$ and $\deg(D) \geq 2g + 1$, this follows from the Srinivas-Varley theorem that $K_0(R) = \mathbb{Z}$ (see [Sr1], [Var]), which implies that R

is K_0-regular by [MP, Corollary 1.4].

At the other extreme, suppose that $H^1(C, \mathscr{O}_C(1)) \neq 0$. Then R is not K_{-1}-regular by Proposition 5.8. A fortiori, R cannot be K_0-regular. This recovers the theorem of Coombes and Srinivas, who proved in [Sr1] and [CS] that if $H^1(C, \mathscr{O}_C(1))$ is nonzero then $NK_0(R) \neq 0$.

If R is K_0-regular, then $K_0(R) = \mathbb{Z}$ because R is a graded ring with $R_0 = k$. The converse need not hold; Srinivas also proved in [Sr1] that if R is normal and k is algebraically closed of finite characteristic, then $K_0(R) = \mathbb{Z}$ always holds.

THEOREM 5.9

Let X be a normal surface over a perfect field. Let $X'' \to X$ be a resolution of singularities such that the exceptional divisor E has smooth components and normal crossings. Then the following are equivalent:

(1) *X is K_{-1}-regular;*
(2) $NK_{-1}(X) = 0$*;*
(3) $N^pK_{-1}(X) = 0$ *for some $p > 0$;*
(4) $H^1(E, \mathscr{I}/\mathscr{I}^n) = 0$ *for every n, where $\mathscr{I}$ is the ideal of $\mathscr{O}_{X''}$ defining E;*
(5) $\operatorname{Pic}(Y'') \cong \operatorname{Pic}(E)$ *for every curve Y'' on X'' with $Y''_{\text{red}} = E$.*

Proof

This result is immediate from Propositions 5.1 and 2.7 because any reduced curve with smooth components and normal crossings is seminormal. □

Remark 5.9.1

When $k = \mathbb{C}$, Srinivas proved in [Sr2, Theorem 2] that if a normal surface X is K_0-regular (e.g., if $NK_0(X) = 0$), then $\operatorname{Pic}(Y'') \to \operatorname{Pic}(E)$ is an isomorphism for every curve Y'' on X'' with $Y''_{\text{red}} = E$. Since K_0-regularity implies K_{-1}-regularity, Srinivas's result is a consequence of Theorem 5.9.

6. Examples

We begin with an example of some historical importance.

Example 6.1 (Bloch and Murthy)

The ring $R = k[x, y, z]/(z^2 - x^3 - y^7)$ is a 2-dimensional normal domain. Bloch and Murthy discovered in November 1979 (while we were writing up [W1]) that $NK_0(R) \neq 0$ when $k = \mathbb{C}$; their argument is sketched in [Sr1, p. 259]. For any field k, a desingularization $\pi : X'' \to \operatorname{Spec}(R)$ can be chosen so that the curve $E = \pi^{-1}(0)$ is a rational cubic cusp. By Example 2.4 and Theorem 4.4, we have $K_{-2}(R) = 0$.

Since R is not K_{-1}-regular by Theorem 5.9, it is not K_0-regular either. This recovers the Bloch-Murthy result for all fields k. To compute $K_{-1}(R)$, we proceed as follows. By Example 2.11 we have $\operatorname{Pic}(Y'') \cong \operatorname{Pic}(E) = k \oplus \mathbb{Z}$. Moreover, the image of $\operatorname{Pic}(X'') \to \operatorname{Pic}(Y'')$ is the $\mathbb{Z}$ summand. Hence the sequence in Proposition 5.1 yields

$$K_{-1}(R) \cong k.$$

Example 6.2 (Mumford [Mum, p. 16])
Let C_0 be a smooth cubic in $\mathbb{P}^2$, and let $p_1, \dots, p_{15}$ be points on C_0 which are in general position except that on C_0 the divisor $\sum p_i \equiv 5H$, where H is a hyperplane section. Blow up these points on $\mathbb{P}^2$ to get a smooth surface X'', with E_i the exceptional divisor over p_i. On X'' the proper transform C of C_0 has $C \cdot C = -6$, and the linear system of quintics through the p_i contracts C on X'' to yield a normal projective surface X having one singular point y. By Theorem 4.4, $K_{-2}(X) = 0$.

In this case, the map from $\operatorname{Pic}(X'') \cong \mathbb{Z}^{16}$ to $\operatorname{Pic}(C)$ sends the $E_i - E_j$ to nonzero points x_{ij} in the abelian variety $\operatorname{Pic}^0(C) \cong \operatorname{Pic}(C)/[E_1]$. Because the conormal bundle $\mathscr{L}$ of the elliptic curve C on X'' has degree 6, $H^1(C, \mathscr{L}^i) = 0$ for all $i \geq 0$ by Riemann-Roch. By Proposition 2.8, $\operatorname{Pic}(Y'') = \operatorname{Pic}(C)$ for every nilpotent thickening Y'' of C. Thus Proposition 5.1 yields

$$K_{-1}(X) \cong \operatorname{Pic}^0(C)/\text{subgroup generated by the } x_{ij}.$$

This example shows that $K_{-1}(X)$ does not have a reasonable algebraic structure.

This example was originally given by Mumford in order to show that the homology of a normal singularity was not reflected by the ideal class group $\operatorname{Cl}(R)$ of the local ring R. Mumford conjectured in [Mum] that $\operatorname{Cl}(R)$ was equal to $\operatorname{Cl}(\widehat{R})$; A. Grothendieck observed in [GB, p. 75] that this was false. We can now see another reason why the class groups are not equal, since $\operatorname{Cl}(\widehat{R})/\operatorname{Cl}(R) \cong K_{-1}(X)$ by Theorem 5.3.

Example 6.3 (Srinivas [Sr2])
Let C be an irreducible sextic in $\mathbb{P}^2$ with 10 nodes, let $X'' \to \mathbb{P}^2$ be the blow-up at these 10 nodes, and let E be the strict transform of C. Then $E^2 = -4$, and E contracts on X'' to yield a normal projective surface X having one singular point y of multiplicity 4. Since $E \cong \mathbb{P}^1$, we have $K_{-2}(X) = 0$.

In this case, $K_{-1}(X) \cong \mathbb{Z}/2$. One can either see this by calculating class groups as in [Sr2, p. 597]—the local ring $R = \mathscr{O}_{X,y}$ has $\operatorname{Cl}(R) = \mathbb{Z}/2$ and $\operatorname{Cl}(\hat{R}) \cong \mathbb{Z}/4$—or see it geometrically by observing that the image of $\operatorname{Pic}(X'') \cong \mathbb{Z}^{11}$ in $\operatorname{Pic}(E) \cong \mathbb{Z}$ has index two. The 10 exceptional curves E_i of $X'' \to \mathbb{P}^2$ have $E \cdot E_i = 2$, and the inverse image L of the general line in $\mathbb{P}^2$ has $E \cdot L = 6$ (see [Sr2, p. 629]).

6.4. Rational double points

Suppose for simplicity that k has characteristic zero. We say that R has a *rational double point* if $\widehat{R}$ is isomorphic to $k[[x,y,z]]/(f)$ for one of the following f:

$$\begin{array}{lll}
(A_n) & f = z^{n+1} + xy & \mathrm{Cl}(\widehat{R}) \cong \mathbb{Z}/(n+1), \\
(D_n),\ n \geq 4 & f = z^2 + xy^2 + x^{n-1} & \mathrm{Cl}(\widehat{R}) \cong \begin{cases} (\mathbb{Z}/2)^2 & \text{for } n \text{ even}, \\ \mathbb{Z}/4 & \text{for } n \text{ odd}, \end{cases} \\
(E_6) & f = z^2 + y^3 + x^4 & \mathrm{Cl}(\widehat{R}) \cong \mathbb{Z}/3, \\
(E_7) & f = z^2 + x^3 y + y^3 & \mathrm{Cl}(\widehat{R}) \cong \mathbb{Z}/2, \\
(E_8) & f = z^2 + x^3 + y^5 & \mathrm{Cl}(\widehat{R}) = 0
\end{array}$$

(see [Durf, Characterization A5]). In each case, the exceptional curve C consists of n copies of $\mathbb{P}^1$, meeting each other according to the associated Dynkin diagram [Durf, Characterization A3]. As in Example 2.4, C has no loops because its graph Γ is the subdivision of the Dynkin diagram. Thus $K_{-2}(R) = K_{-1}(C) = 0$. We also know that the group $K_{-1}(R)$ is finite by Theorem 5.3 because it is a quotient of $\mathrm{Cl}(\widehat{R})$, which is listed above. Examples when R is a UFD are given in [Sr3]; in these cases $K_{-1}(R) = \mathrm{Cl}(\widehat{R})$.

By Example 2.13, $\mathrm{Pic}(Y'') = \mathrm{Pic}(C)$ for every nilpotent thickening Y'' of C. Hence rational double point singularities are all K_{-1}-regular by Theorem 5.9. If $k = \mathbb{C}$, Srinivas showed in [Sr2, Corollary 4.4] that R is K_0-regular by applying [MP] to the calculation of $K_0(X)$ in [Sr3] and [MK].

6.5. Quotient singularities

If G is a finite subgroup of $\mathrm{SL}_2(k)$ acting on the plane (hence on $k[u,v]$), we can form the invariant subring $k[u,v]^G$. A rational surface singularity $\mathrm{Spec}(R)$ is called a *quotient singularity* if its completion $\widehat{R}$ is isomorphic to the completion of some $k[u,v]^G$. By [Durf, Characterization A5], a rational double point is a quotient singularity with embedding dimension 3.

Now suppose that $k = \mathbb{C}$. Srinivas showed in [Sr2, Corollary 4.4] that every quotient singularity R is K_0-regular. To do this, he observed that R has a finite cover S, étale over $\mathrm{Spec}(R) - \{\mathfrak{m}\}$, which is a rational double point. Since $N^p K_q(R)$ is a $\mathbb{C}$-vector space for $p > 0$, and the kernel of $N^p K_q(R) \to N^p K_q(S)$ has exponent n for $q \leq 0$, $N^p K_q(R)$ is a subgroup of $N^p K_q(S)$. The result now follows from Example 6.4.

Since the kernel of $K_q(R) \to K_q(S)$ also has exponent n, it follows from Theorem 4.4 that $K_{-2}(R) = 0$ and that $K_{-1}(R)$ is finite. Example 6.3, which is a quotient singularity, shows that $K_{-1}(R)$ need not be zero.

6.6. Reid's method

This example clarifies the calculations by L. Reid [R, p. 198]. Let $F(x, y, z)$ be a homogeneous polynomial of degree d, defining a reduced curve C in $\mathbb{P}^2_k$. Then the equation $f(x, y) = F(x, y, 1)$ defines an affine curve C_0. Consider the subring $R = k[f, xf, yf]$ of $k[x, y]$. Then the map from $k[u, v, w]$ to $k[x, y]$ sending u, v, and w to xf, yf, and f determines an isomorphism

$$R \cong \frac{k[u, v, w]}{\bigl(w^{d+1} - F(u, v, w)\bigr)}.$$

If C has d distinct points at infinity, then R is a 2-dimensional normal domain whose only singularity is at the maximal ideal $\mathfrak{m}$. This is easy to see from the Jacobian criterion, using Euler's formula $u(\partial F/\partial u) + v(\partial F/\partial v) + w(\partial F/\partial w) = d \cdot F$ to see that the singularities lie on the locus $w = F(u, v, 0) = 0$.

Let X'' denote the blow-up of $X = \operatorname{Spec}(R)$ along $\mathfrak{m}$. The affine open $D_+(wt)$ of X'' is just $\operatorname{Spec}(k[x, y])$. From this it is easy to see that X'' is nonsingular and that the exceptional fiber of $X'' \to X$ is the curve $C = \mathbf{Proj}(k[u, v, w]/F)$. Hence

$$K_{-2}(R) \cong K_{-2}(X') \cong K_{-1}(C).$$

If we choose F so that C is a node, or any other plane curve with $K_{-1}(C) = \mathbb{Z}$, then $K_{-2}(R) = \mathbb{Z}$.

Suppose that $F(1, 0, 0) \neq 0$. Then v, w is a regular sequence, and the ideal $I = (v, w)$ is a reduction of $\mathfrak{m}$, since $u^d \in I\mathfrak{m}^{d-1}$ and hence $\mathfrak{m}^d = I\mathfrak{m}^{d-1}$. Set $A = R/I = k[u]/(u^d)$, and let X' be the blow-up along I. Not only is X'' finite over X', but the ideal $\mathfrak{m}^{d-1}R[\mathfrak{m}t]$ lies in the conductor from $R[\mathfrak{m}t]$ to $R[It]$. Therefore we may analyze $\operatorname{Pic}(Y'')$ as in Section 2 in order to describe $K_{-1}(R)$.

7. SK_1-regularity for nonreduced curves

This section is in some sense a technical continuation of Section 2, where we considered K_0-regularity for nonreduced curves. It is used in Section 8 to discuss K_0-regularity of normal surfaces.

Recall that there is a natural decomposition $K_1(Y) = U(Y) \oplus SK_1(Y)$, where $U(Y)$ denotes the global units of Y. Using Lemma 2.5 and Porism 2.5.1, the argument of [PW1, p. 369] with $K_i(\bar{D})$ replaced by $K_i(Y, \mathcal{N})$ yields an exact sequence for every closed subscheme Y' defined by an ideal $\mathcal{N}$ of $\mathcal{O}_Y$:

$$K_2(Y) \longrightarrow K_2(Y') \longrightarrow SK_1(Y, \mathcal{N}) \longrightarrow SK_1(Y) \longrightarrow SK_1(Y') \longrightarrow 0. \tag{7.0}$$

Replacing Y with the product $Y[T]$ of Y with $\operatorname{Spec}(k[t_1, \dots, t_p])$, this sequence is

$$\cdots K_2\bigl(Y'[T]\bigr) \longrightarrow SK_1\bigl(Y[T], \mathcal{N}[T]\bigr) \longrightarrow SK_1(Y[T]) \longrightarrow SK_1(Y'[T]) \longrightarrow 0. \tag{7.0[T]}$$

We begin with a result that could have been included in Lemma 2.5. Recall (say, from [W2]) that if N is a nilpotent ideal of a ring A, then $K_2(A, N)$ is generated by Dennis-Stein symbols $\langle x, a\rangle$, where $x \in N$ and $a \in A$. Write $\otimes_Y$ for $\otimes_{\mathcal{O}_Y}$.

LEMMA 7.1

For any scheme Y, let $\mathcal{N}$ be an ideal of $\mathcal{O}_Y$ with $\mathcal{N}^2 = 0$, defining a subscheme Y'. Then there is an exact sequence of sheaves

$$\mathcal{N} \otimes_Y \mathcal{N} \xrightarrow{\psi} \mathcal{K}_2\mathcal{N} \xrightarrow{\varphi} \mathcal{N} \otimes_Y \Omega_{Y'} \longrightarrow 0.$$

Here $\psi(x \otimes y) = \langle x, y\rangle$ and $\varphi(\langle x, a\rangle) = x \otimes da$ for sections x, y of $\mathcal{N}$ and a of $\mathcal{O}_Y$.

If Y is a curve, then $SK_1(Y, \mathcal{N}) \cong H^1(Y, \mathcal{K}_2\mathcal{N})$ and there is an exact sequence

$$H^1\big(Y, \mathcal{N} \otimes_Y \mathcal{N}\big) \xrightarrow{\psi} H^1\big(Y, \mathcal{K}_2\mathcal{N}\big) \longrightarrow H^1\big(Y, \mathcal{N} \otimes_Y \Omega_{Y'}\big) \longrightarrow 0.$$

In fact, ψ factors through the quotient $\mathcal{N}\tilde{\wedge}\mathcal{N} = (\mathcal{N} \otimes_Y \mathcal{N})/\{x\otimes y + y\otimes x\}$ because $\langle x, y\rangle + \langle y, x\rangle = 0$ in $K_2(A, N)$ (see [W2, Theorem 1.3]).

Proof

The first assertion is just the sheafification of [W2, Theorem 1.3]. Now suppose that Y is a curve. The Brown-Gersten spectral sequence used in the proof of Lemma 2.5 yields $SK_1(Y, \mathcal{N}) \cong H^1(Y, \mathcal{K}_2\mathcal{N})$ because of the sequence in Lemma 2.5(4) and the extension

$$0 \longrightarrow H^1\big(Y, \mathcal{K}_2\mathcal{N}\big) \longrightarrow K_1(Y, \mathcal{N}) \longrightarrow H^0(Y, \mathcal{N}) \longrightarrow 0.$$

Applying the right exact $H^1(Y, -)$ yields the final sequence. □

Porism 7.1.1

Here is a variation, for $\pi : Y[T] \to Y$. Let $\mathcal{K}_i\mathcal{N}[T]$ denote the sheaf on Y associated to the presheaf $U \mapsto K_i(U[T], \mathcal{N}[T])$. Then $SK_1(Y[T], \mathcal{N}[T])$ equals $H^1(Y, \mathcal{K}_2\mathcal{N}[T])$, by the same argument applied to the Brown-Gersten spectral sequence $H^p(Y, \mathcal{K}_{-q}\mathcal{N}[T]) \Rightarrow K_{-p-q}(Y[T], \mathcal{N}[T])$ of [TT]. Since $H^i(Y, \mathcal{N}[T])$ is $H^i(Y, \mathcal{N}) \otimes k[T]$ and since $\psi(xt^i \otimes yt^j) = \psi(xt^{i+j}, y)$ when $xy = 0$, the proof of Lemma 7.1 yields an exact sequence

$$H^1\big(Y, \mathcal{N}\otimes_Y\mathcal{N}\big)\otimes_k k[T] \xrightarrow{\psi} H^1\big(Y, \mathcal{K}_2\mathcal{N}[T]\big)\longrightarrow H^1\big(Y, \mathcal{N}\otimes_Y\pi_*\Omega_{Y'[T]}\big)\longrightarrow 0.$$

Moreover, $\mathcal{N} \otimes_Y \pi_*\Omega_{Y'[T]} \cong \mathcal{N} \otimes_Y \Omega_{Y'} \otimes_k k[T] \oplus \mathcal{N} \otimes_k \Omega_{k[T]}$.

COROLLARY 7.2

Assume that Y is a curve over a field k, and assume that $\mathcal{N}$ is a locally principal ideal of $\mathcal{O}_Y$ with $\mathcal{N}^2 = 0$, defining a subscheme Y'. If $\operatorname{char}(k) = 2$, *assume in addition* *that either* $\mathscr{N} \subseteq (\operatorname{ann} \mathscr{N})^2$ *or* $H^1(Y, \mathscr{N} \otimes_Y \mathscr{N}) = 0$. *Then*

$$SK_1(Y, \mathscr{N}) \cong H^1\big(Y', \mathscr{N} \otimes_Y \Omega_{Y'}\big),$$
$$SK_1\big(Y[T], \mathscr{N}[T]\big) \cong SK_1(Y, \mathscr{N}) \otimes k[T] \oplus H^1(Y, \mathscr{N}) \otimes \Omega_{k[T]}.$$

Thus $SK_1(Y[T], \mathscr{N}[T]) \cong SK_1(Y, \mathscr{N})$ *if and only if both* $SK_1(Y, \mathscr{N})$ *and* $H^1(Y, \mathscr{N})$ *vanish.*

Proof
By [W2, Lemma 1.2 and Theorem 1.3], the assumptions imply that the sheaf map $\psi = 0$ and that $\mathscr{K}_2\mathscr{N} \cong \mathscr{N} \otimes_Y \Omega_{Y'}$. Similarly, $\mathscr{K}_2\mathscr{N}[T] \cong \mathscr{N}[T] \otimes \Omega_{Y'[T]} \cong \mathscr{N} \otimes \Omega_{Y'}[T] \oplus \mathscr{N} \otimes \Omega_{k[T]}$. The rest follows from Porism 7.1.1. □

COROLLARY 7.3
Let Y, Y', *and* $\mathscr{N}$ *be as in Corollary 7.2. If* $H^1(Y, \mathscr{N}) = 0$, *then* $SK_1(Y, \mathscr{N}) \cong H^1(Y', \mathscr{N} \otimes_Y \Omega_{Y'/k})$ *and* $SK_1(Y[T], \mathscr{N}[T]) \cong SK_1(Y, \mathscr{N}) \otimes k[T]$.

Proof
Tensor the “first fundamental exact sequence” [W6, Sequence 9.2.6] with $\mathscr{N}$ to get

$$\mathscr{N} \otimes_k \Omega_k \longrightarrow \mathscr{N} \otimes_Y \Omega_{Y'} \longrightarrow \mathscr{N} \otimes_Y \Omega_{Y'/k} \longrightarrow 0.$$

Now apply the right exact functor $H^1(Y', -)$, and use Corollary 7.2. □

Porism 7.3.1
More generally, the proof shows that the quotient of $SK_1(Y, \mathscr{N})$ by the image of ψ and $H^1(Y, \mathscr{N}) \otimes \Omega_k$ is $H^1(Y', \mathscr{N} \otimes \Omega_{Y'/k})$.

Remark 7.3.2
If $H^1(Y, \mathscr{N})$ is nonzero and if Y is defined over k_0, the proof yields a surjection from $NSK_1(Y)$ to $H^1(Y, \mathscr{N}) \otimes \Omega_{k/k_0} \otimes tk[t]$. This map was constructed by Srinivas [Sr2, Section 3].

Example 7.4 (Fat $\mathbb{P}^1$*)*
(a) Suppose that $C = \mathbb{P}^1_A$ for some Artinian A, and suppose that $C \to Y$ is a nilpotent thickening defined by an ideal $\mathscr{I}$ such that $\mathscr{L} = \mathscr{I}/\mathscr{I}^2$ is an ample line bundle on C. We claim that $SK_1(Y) \cong SK_1(\mathbb{P}^1_A) \cong A^\times$.

To see this, recall that $\Omega_C \cong \mathscr{O}_C(-2) \oplus (\Omega_A \otimes_A \mathscr{O}_C)$. Inductively, the claim holds for any subscheme Y' of Y defined by a nonzero ideal $\mathscr{N} \subseteq \mathscr{I}^i$ with $\mathscr{I}\mathscr{N} = 0$. Then $\mathscr{N}$ is a quotient of $\mathscr{L}^i$, so $H^1(Y, \mathscr{N}) = H^1(Y, \mathscr{N} \otimes_Y \mathscr{N}) = 0$. Moreover, we have $\mathscr{N} \otimes_Y \Omega_{Y'} \cong \mathscr{N} \otimes_Y \Omega_C$. It follows from Corollary 7.3 that $SK_1(Y, \mathscr{N}) \cong H^1(Y, \mathscr{N} \otimes \Omega_{C/k}) = 0$, yielding the inductive step that $SK_1(Y) \cong SK_1(Y')$.

(b) Now suppose that X' is the blow-up of a surface X along a local complete intersection $Y = \operatorname{Spec}(A)$. We claim that for every nilpotent thickening Y' of the fiber $C = Y \times_X X' \cong \mathbb{P}^1_A$ we have $SK_1(Y') \cong A^\times$ and $SK_1(Y'[T]) \cong A[T]^\times$.

To see this, note that the fiber is defined by an invertible ideal $\mathscr{I}$ of $\mathscr{O}_{X'}$ with conormal bundle $\mathscr{I}/\mathscr{I}^2 \cong \mathscr{O}(1)$. If $Y'_n \subset X'$ is defined by $\mathscr{I}^n$, then $Y'_n \subset Y'_{n+1}$ is defined by $\mathscr{N} = \mathscr{I}^n/\mathscr{I}^{n+1} \cong \mathscr{O}_{\mathbb{P}^1_Y}(n)$. Thus $\mathscr{N} \otimes \Omega_{Y'_n} \cong \mathscr{O}_{\mathbb{P}^1_Y}(n-2)$. Hence $SK_1(Y_{n+1}, \mathscr{N}) = 0$, and hence $SK_1(Y'_{n+1}) \cong SK_1(Y'_n) \cong SK_1(Y'_1)$ for all $n \geq 1$ by Corollary 7.2. By Example 2.10 and Corollary 7.3, we even have $SK_1(Y_{n+1}[T], \mathscr{N}[T]) = 0$, and hence $SK_1(Y'_n[T]) \cong SK_1(Y'_1[T]) \cong A[T]^\times$.

PROPOSITION 7.5

Let Y be a curve, proper over a perfect field k. Let Y' be a subscheme defined by a nilpotent ideal $\mathscr{N}$ of $\mathscr{O}_Y$ such that $\mathscr{I}\mathscr{N} = 0$, where $\mathscr{I}$ is the nilradical. Assume that $\operatorname{Pic}(Y) \cong \operatorname{Pic}(Y')$. *Then the map $SK_1(Y, \mathscr{N}) \to SK_1(Y)$ factors through the quotient $H^1(Y, \mathscr{N} \otimes \Omega_{Y'/k})$ of $SK_1(Y, \mathscr{N})$.*

Similarly, the map $SK_1(Y[T], \mathscr{N}[T]) \to SK_1(Y[T])$ factors through the quotient $H^1(Y, \mathscr{N} \otimes \Omega_{Y'/k}) \otimes k[T]$ of $SK_1(Y[T], \mathscr{N}[T])$.

Proof

Let W denote the subgroup of $SK_1(Y, \mathscr{N})$ generated by the image of ψ and $H^1(Y, \mathscr{N}) \otimes \Omega_k$. We will show that W is in the image of $K_2(Y') \to SK_1(Y, \mathscr{N})$; by (7.0) this implies that W vanishes in $SK_1(Y)$. The first assertion will then follow from Remark 7.3.1; the second assertion will be proven analogously, using (7.0[T]).

Since Y is proper, the three rings $k' = H^0(Y, \mathscr{O}_Y/\mathscr{I})$, $A = H^0(Y, \mathscr{O}_Y)$, and $B = H^0(Y, \mathscr{O}_Y/\mathscr{N})$ are finite-dimensional over k. As k is perfect, k' is smooth and $A \to k'$ is a split surjection. The assumption on $\operatorname{Pic}(Y)$ implies that B/A surjects onto $H^1(Y, \mathscr{N})$.

In order to lift elements of $H^1(Y, \mathscr{N})$, cover Y by two affine opens $U_i = \operatorname{Spec}(A_i)$ with affine intersection $U_{12} = \operatorname{Spec}(A_{12})$, and set $N_{12} = H^0(U_{12}, \mathscr{N})$. Then each $\lambda \in H^1(Y, \mathscr{N})$ is represented by an $n \in N_{12}$; the assumption that λ comes from B/A implies that there are $a_i \in A_i$ such that $a_2 = a_1 + n$ in A_{12}. These a_i then define an element b of B. Subtracting an element of k' if necessary, we can assume that a_1 and b are nilpotent. Hence $a_1 n = 0$.

For each $s, x \in A[T]$, the Dennis-Stein symbols $\langle a_i s, x\rangle$ of $K_2(A_i[T])$ both map to $\langle bs, x\rangle$ in $K_2(B[T])$, while in $K_2(A_{12}[T])$ we have

$$\langle a_1 s, x\rangle \langle ns, x\rangle = \langle a_2 s, x\rangle$$

because $a_1 n = 0$. From this we see that the composition

$$K_2(B[T]) \longrightarrow K_2(Y'[T]) \longrightarrow SK_1\big(Y[T], \mathcal{N}[T]\big) \cong H^1\big(Y[T], \mathcal{K}_2\mathcal{N}[T]\big)$$

sends $\langle bs, x\rangle$ to the class $\sigma \in H^1(Y[T], \mathcal{K}_2\mathcal{N}[T])$ represented by $\langle ns, x\rangle$.

When $x \in H^0(Y, \mathcal{N})$, then σ corresponds to $\psi(\lambda s \otimes x)$ in Lemma 7.1 and Porism 7.1.1. When $x, s \in k[T]$, σ corresponds to $\lambda \otimes s\,dx$ in $H^1(Y, \mathcal{N}) \otimes \Omega_{k[T]}$. Since the $s\,dx$ form a k-basis of $\Omega_{k[T]}$, the result follows from Porism 7.3.1 or from Lemma 7.1. □

PROPOSITION 7.6

Let C be a seminormal curve over a perfect field k, and let $\mathcal{N}$ be an ample line bundle. Then $H^1(C, \mathcal{N} \otimes \Omega_{C/k}) = 0$.

Proof

Write ι for the inclusion of the singular locus $S = \coprod \operatorname{Spec}(k_i)$ in C, $p : \coprod C_j \to C$ for the normalization, and $p^{-1}S = \coprod \operatorname{Spec}(k_{ij})$ for the fiber. Because C is seminormal, the k_i and k_{ij} are fields. Because k is perfect, each $\Omega_{k_i/k}$ and $\Omega_{k_{ij}/k}$ vanishes. Then for certain k_i-vector spaces T_i we have exact sequence

$$0 \longrightarrow \oplus\iota_*(T_i) \longrightarrow \Omega_{C/k} \longrightarrow \oplus p_*\Omega_{C_j/k} \longrightarrow 0. \tag{7.6.1}$$

Tensor this sequence with $\mathcal{N}$, and write $\mathcal{N}_j$ for the pullback of $\mathcal{N}$ to C_j. We have $H^1(C, \mathcal{N} \otimes \Omega_{C/k}) \cong \oplus H^1(C_j, \mathcal{N}_j \otimes \Omega_{C_j/k})$. Since $\mathcal{N}$ is ample and since the C_j are smooth, Serre duality yields $H^1(C_j, \mathcal{N}_j \otimes \Omega_{C_j/k}) = 0$. We are done. □

Recall that by T. Vorst's theorem [Vor2], every seminormal curve C over a perfect field is also SK_1-regular.

COROLLARY 7.7

Let Y be a Pic*-regular curve over a perfect field k such that $C = Y_{\text{red}}$ is defined by an ideal $\mathcal{N}$ of $\mathcal{O}_Y$ with $\mathcal{N}^2 = 0$ and such that $\mathcal{N}$ is (a quotient of) an ample line bundle over C.*

Then $SK_1(Y) \cong SK_1(C)$, and Y is SK_1-regular.

Proof

By Proposition 2.7, the Pic-regularity of Y implies that C is seminormal, so $SK_1(C) = SK_1(C[T])$ by Vorst. By Proposition 7.6, $H^1(C, \mathcal{N} \otimes \Omega_{C/k}) = 0$. But then Theorem 7.5 implies that $SK_1(Y) \cong SK_1(C)$, and $SK_1(Y[T]) \cong SK_1(C[T])$. □

Remark 7.7.1

If $\operatorname{char}(k) \neq 2$, a stronger assertion holds: $SK_1(Y, \mathcal{N}) = 0$ and $SK_1(Y[T], \mathcal{N}[T]) = 0$.

This follows from Corollary 7.3, which applies because (by Proposition 2.7) the Pic-regularity of Y implies that $H^1(C, \mathscr{N}) = 0$.

THEOREM 7.8

Let Y be a Pic*-regular curve, proper over a perfect field k, such that $C = Y_{\text{red}}$ is defined by a locally principal ideal $\mathscr{I}$ of $\mathscr{O}_Y$. Assume that $\mathscr{L} = \mathscr{I}/\mathscr{I}^2$ is ample on C. Then $SK_1(Y) \cong SK_1(C)$, and Y is SK_1-regular.*

Proof

The case $\mathscr{I}^2 = 0$ is just Corollary 7.7. In general, suppose $\mathscr{I}^{n+1} = 0$, and let Y_n denote the subscheme of Y defined by $\mathscr{N} = \mathscr{I}^n$; Y_n is Pic-regular by Lemma 2.5. Now $\mathscr{N}$ is a quotient of $\mathscr{L}^n$, which is ample on C, so Proposition 7.6 yields $H^1(C, \mathscr{N} \otimes \Omega_{C/k}) = 0$. By Proposition 7.5, $SK_1(Y[T]) \cong SK_1(Y_n[T])$. The result now follows by induction on n. □

8. K_0-regularity equals K_{-1}-regularity for normal surfaces

In this section we show that a normal surface X over a perfect field is K_0-regular if and only if it is K_{-1}-regular. Combined with Theorem 5.9, this verifies [Sr2, Conjecture B] that X is K_0-regular if and only if $\mathrm{Pic}(nE) \cong \mathrm{Pic}(E)$ for all n.

Recall that $K_0(X) = H^0(X, \mathbb{Z}) \oplus \widetilde{K}_0(X)$, and recall that $SK_0(X)$ denotes the kernel of the determinant map $\widetilde{K}_0(X) \to \mathrm{Pic}(X)$. It is well known that normal schemes are Pic-regular, so $N^pK_0(X) = N^pSK_0(X)$ for $p > 0$.

If Y is a curve, the determinant map is an isomorphism, so $SK_0(Y) = 0$. Combining (3.4.2) with the "units-Pic" sequence yields the following exact sequence, where $X'' \to X'$ and $Y'' \to Y'$ are as in Theorem 3.1:

$$SK_1(X'') \oplus SK_1(Y') \longrightarrow SK_1(Y'') \longrightarrow SK_0(X') \longrightarrow SK_0(X'') \longrightarrow 0. \tag{8.1}$$

(When X'' is affine, this is the sequence in [Bass, p. 490].)

There are of course similar exact sequences, which we refer to as N^p(8.1), in which "SK" is formally replaced by "N^pSK." Note that if X'' is regular and $p > 0$, then the two terms $NSK_i(X'')$ vanish.

THEOREM 8.2

Suppose that $\pi : X'' \to X$ is a resolution of singularities of a normal surface X, with exceptional fibers E_j over the singular points of X. Then for sufficiently large nilpotent thickenings Y''_j of the exceptional fibers E_j we have

$$N^pK_0(X) \cong \oplus_j N^pSK_1(Y''_j).$$

Proof

By Theorem 1.9, X'' is finite over some blow-up X' of X along a local complete intersection $Y = \mathrm{Spec}(A)$. Choosing a conducting subscheme $j : Y' \subset X'$, we have $SK_1(Y') \cong SK_1(\mathbb{P}^1_Y) \cong A^\times$ by Example 7.4(b). Now the map $j^* : SK_1(X') \to SK_1(\mathbb{P}^1_Y)$ is onto because it is split by $j_*\pi^*$ as in the proof of Lemma 5.2. Hence the map $N^pSK_1(Y') \to N^pSK_1(Y'')$ in sequence $N^p(8.1)$ factors through $N^pSK_1(X'')$, which is zero for $p > 0$. The result now follows from sequence $N^p(8.1)$. □

Example 8.3

Let k be a nonperfect field of characteristic 2, and set $R = k[x, y, w]/(w^3 - y^2 + \alpha x^2)$, where $\alpha \in k$ is an element with no square root. Using [Hart, Exercise III.10.1], one can show that $R[1/x]$ and $R[1/y]$ are regular, so by Serre's criterion R is a normal domain with one singular point. Applying Reid's method (see Section 6.6) to the projective curve C defined by $f(x, y, z) = y^2 - \alpha x^2$, we see that C is the exceptional fiber of a resolution of singularities $X'' \to X = \mathrm{Spec}(R)$. In particular, its conormal bundle is ample.

Now C is seminormal but not SK_1-regular, by [Vor2, Theorem A]. Since the normalization of C is a projective line $\mathbb{P}^1_\ell$ over $\ell = k(\sqrt{\alpha})$, the argument of Example 2.12 shows that every nilpotent thickening Y'' of C on X'' is Pic-regular. By Proposition 5.1 and Theorem 8.2, R is K_{-1}-regular but not K_0-regular.

We can now establish [Sr3, Conjecture B].

THEOREM 8.4

Suppose that X is a K_{-1}-regular surface over a perfect field. Then X is K_0-regular.

Proof

By Proposition 5.1, our K_{-1}-regularity assumption implies that every conductor subscheme Y'' is Pic-regular. Now combine Theorems 7.8 and 8.2. □

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