JOURNAL OF ALGEBRA 105, 465–483 (1987)

Module Structures on the K-Theory of Graded Rings

CHARLES A. WEIBEL*

Department of Mathematics, Rutgers University,
Hill Center, Busch Campus, New Brunswick,
New Jersey 08903

Communicated by Barbara L. Osofsky

Received December 19, 1984

Let R be a commutative ring, $A = A_0 \oplus A_1 \oplus \cdots$ a graded R-algebra, and A_+ the graded ideal $A_1 \oplus A_2 \oplus \cdots$. Then $K_i(A) = K_i(A_0) \oplus K_i(A, A_+)$. We show that the groups $K_i(A, A_+)$ are naturally modules over the ring $W(R)$ of Witt vectors. They also have a natural filtration whose associated graded groups are R-modules. When R contains a field of characteristic zero, $K_i(A, A_+)$ is an R-module, and the filtration is by R-submodules. © 1987 Academic Press, Inc.

Although algebraic K-groups are a priori nothing more than abelian groups, much of our ability to perform calculations rests on module structures which can be imposed on large parts of K-theory. This viewpoint originated in 1971 with van der Kallen's observation in [vdK] that when $\frac{1}{2}$ is in a commutative ring R we have $K_2(R[\varepsilon]/(\varepsilon^2)) = K_2(R) \oplus \Omega_R$, where Ω_R is the R-module of absolute Kahler differentials of R. Four years later, Bloch generalized this, showing that the relative groups $NK_i(R) = K_i(R[x], x)$ and $\varprojlim_n K_i(R[\varepsilon]/(\varepsilon^n), \varepsilon)$ were modules over the ring $W(R)$ of Witt vectors over R. (See [B1, B2, S2]; a summary is given in [W1].) In this note, we fit these phenomena into a more general context.

THEOREM 0.1. *Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded R-algebra, where R is a commutative ring (concentrated in degree 0). If A_+ denotes the graded ideal $A_1 \oplus A_2 \oplus \cdots$, then there is a continuous $W(R)$-module structure on each group $K_i(A, A_+)$. This $W(R)$-module structure is natural on the category of graded R-algebras, and agrees with the known module structures for $A = A_0[x]$ and $A = A_0[\varepsilon]/(\varepsilon^n)$.*

If R contains $\mathbf{Q}$ (the rational numbers), then each $K_i(A, A_+)$ has a natural R-module structure via the ring map $\lambda: R \to W(R)$.

---

* Supported by NSF Grant MCS81-02753.

0021-8693/87 $3.00

The phrase "continuous $W(R)$-module" needs explanation. There is a descending filtration on $W(R)$ by ideals I_n, making $W(R)$ into a topological ring. For example, if R contains $\mathbf{Q}$, then as a ring we have

$$W(R) \cong \prod_{i=1}^{\infty} R, \qquad \text{while} \quad I_m \cong \prod_{i=m}^{\infty} R.$$

Note that $I_m I_n$ does not lie inside I_{m+n}.

Now let M be a $W(R)$-module. We say that M is a *continuous* module if the annihilator of every element of M is open in $W(R)$, i.e., if M is the union of the submodules $F^n M = \{m \in M \mid I_n m = 0\}$. M is *separated* if the intersection of the $I_m M$ is zero. I do not know whether or not the continuous $W(R)$-modules $K_i(A, A_+)$ are separated, although it is clear that the symbol part will be separated. However, if R contains $\mathbf{Q}$, it is clear that every continuous $W(R)$-module is separated. (I am grateful to Wilberd van der Kallen for pointing out the need for care here.) From these general comments we deduce

COROLLARY 0.2. *Suppose that R contains $\mathbf{Q}$. Then the groups $I_n K_i(A, A_+)$ form a natural decreasing filtration on $K_i(A, A_+)$, whose intersection is zero. All groups involved are naturally R-modules, including the associated graded groups.*

EXAMPLE 0.3. Suppose that A_+ is nilpotent and commutative, so that $K_1(A, A_+)$ is the multiplicative group $1 + A_+$. The action of the element $(1 - rt^m)$ of $W(R)$ on $K_1(A, A_+)$ is given by the formula:

$$(1 - rt^m) * (1 - a) = (1 - r^{n/d} a^{m/d})^d, \qquad a \in A_n,\ d = gcd(m, n).$$

The submodule $I_m K_1(A, A_+)$ is contained in $1 + A_{\geqslant m}$, $A_{\geqslant m} = A_m \oplus A_{m+1} \oplus \cdots$. If R contains the rational numbers, the R-module structure is given by the formula:

$$r * (1 + a) = (1 + a)^r = 1 + ra + \frac{r(r-1)}{2} a^2 + \cdots .$$

In fact, the map sending a to e^a is an R-module isomorphism between A_+ and $K_1(A, A_+)$.

As another example, let $A = \mathbf{C}[x_0, \ldots, x_n]/I$ be the homogeneous coordinate ring of a smooth curve X embedded in complex projective n-space. Srinivas proved in [Sr] that when $H^1(X, O(1)) \neq 0$ the group $K_0(A, A_+)$ is an abelian group of uncountable rank. In fact, it is a vector space over $\mathbf{C}$.

A similar remark applies to the 2-dimensional normal domain

$A = \mathbf{C}[x, y, z]/(x^2 + y^3 = z^7)$. We know that both $\tilde{K}_0(A) = K_0(A, A_+)$ and $K_{-1}(A) = K_{-1}(A, A_+)$ are nonzero. (This was originally due to Bloch and Murthy; see [Sr], [W3] and [Reid].) Hence $\tilde{K}_0(A)$ and $K_{-1}(A)$ are both nonzero vector spaces over $\mathbf{C}$. As abelian groups, therefore, they are divisible of uncountable rank.

The outline of this paper is as follows. We define the action of $W(R)$ on $K_*(A, A_+)$ in Section 1 and prove that it is well defined in Section 2. In Section 3 we give another pairing, due to Bloch, and show that it agrees with our module structure. In Section 4 we establish some basic structural results for the module structure. We devote Section 5 to establishing formulas for the action on $K_2(A, A_+)$ when A_+ is nilpotent.

Throughout this paper, R will denote a commutative ring, and $A = A_0 \oplus A_1 \oplus \cdots$ will denote a graded R-algebra. R is to be concentrated in degree zero, and A_+ will denote the ideal $A_1 \oplus \cdots$ of A. The letters a, b (resp. q, r, s) will always denote elements of A (resp., of R), and the letters t, x and y will stand for indeterminates.

I would like to express my gratitude to Jan Stienstra and Wilberd van der Kallen for helpful conversations. In addition, I would like to point out that I presented the calculations in Section 5 in 1981 at the Topology Conference at the University of Western Ontario.

1. THE ACTION OF $W(R)$

The ring $W(R)$ of Witt vectors over R has as its underlying additive group the group $1 + tR[\![t]\!]$. This is a topological group, the subgroups $I_n = 1 + t^n R[\![t]\!]$ forming a basic family of open neighborhoods of the identity. Every element of $W(R)$ has a unique convergent expansion $\omega(t) = \Pi(1 - r_m t^m)$. Using $*$ for the ring product, the ring structure on $W(R)$ is completely determined by the formula:

$$(1 - rt^m) * (1 - st^n) = (1 - r^{n/d} s^{m/d} t^{mn/d})^d, \qquad d = \gcd(m, n). \tag{1.1}$$

We want to make $K_i(A, A_+)$ into a continuous $W(R)$-module in a natural way. It is enough to define natural maps $(1 - rt^m) *: K_i(A, A_+) \to K_i(A, A_+)$ for every r in R and $m \geqslant 1$, and then to verify the following.

AXIOMS 1.2. For every v in $K_i(A, A_+)$:

(a) There is an $M \geqslant 0$ such that $(1 - rt^m) * v = 0$ for every $m \geqslant M$ and every r.

(b) Whenever $\Pi(1 - q_l t^l) \cdot \Pi(1 - r_m t^m) = \Pi(1 - s_n t^n)$ in the group $1 + R[\![t]\!]$, then in $K_i(A, A_+)$:

$$\sum (1 - q_l t^l) * v + \sum (1 - r_m t^m) * v = \sum (1 - s_n t^n) * v.$$

(c) $(1-t)*v=v$. ($1-t$ is the unit of the ring $W(R)$.)

(d) $[(1-rt^m)*(1-st^n)]*v=(1-rt^m)*[(1-st^n)*v]$.

To verify the Axioms (1.2) for every A, it is enough to verify that (1.2) holds when A is the polynomial ring $A_0[x]$ with x in degree one. To see this, let B denote the R-algebra $A[x]$. Grade B by setting A in degree zero and x in degree one, so that the ring homomorphism $\phi: A \to B$ which sends a_i in A_i to $a_i x^i$ is a degree-preserving map. The induced map $\phi^*: K_i(A, A_+) \to K_i(B, B_+)$ is an injection, because it is a summand of the map $\phi^*: K_i(A) \to K_i(B)$, and this map is split by the nongraded map $B \to A$ sending x to 1. If the Axioms (1.2) hold for $K_i(B, B_+)$, then they must hold for $K_i(A, A_+)$ as well.

In the remainder of this section, we define the map $(1-rt^m)*$ on $K_i(A, A_+)$. In the next section, we will verify the Axioms (1.2) for the special case $A=A_0[y]$, proving that the $K_i(A, A_+)$ are continuous $W(R)$-modules.

We will work with the category $\mathbf{P}(B)$ of finitely generated projective right B-modules. If $F: \mathbf{P}(B) \to \mathbf{P}(C)$ is an additive functor, $F(B)$ is a left B-module via the isomorphism $B \cong \operatorname{Hom}(B, B)$, and therefore a $B-C$ bimodule, i.e., an object of B-**mod**-C. The possibility of going back and forth between F and $F(B)$ is made possible by the following elementary result, whose proof we omit (cf. [Bass, p. 57]).

LEMMA 1.3. *If B and C are rings, there is an equivalence of categories*:

$$\left\{\begin{array}{c}\text{additive functors } \mathbf{P}(B) \to \mathbf{P}(C) \\ \text{and natural transformations}\end{array}\right\} \cong \left\{\begin{array}{c}B-C \text{ bimodules in } \mathbf{P}(C) \\ \text{and bimodule maps}\end{array}\right\}.$$

Under this equivalence, F corresponds to $F(B)$ and the $B-C$ bimodule P corresponds to the functor $F_P(M)=M \otimes_B P$.

For the rest of this section, we fix r in R and an integer $m \geqslant 1$. We want to define an additive functor $F: \mathbf{P}(A) \to \mathbf{P}(A)$, and we do this by defining an A-bimodule P. As a right module, P is free on basis $\{e_0, ..., e_{m-1}\}$. For $j \geqslant m$, we make the convention that e_j means $e_{j-m}r$, and we define the left A-module structure by

$$a_i e_j = e_{i+j} a_i \qquad \text{for } a_i \text{ in } A_i.$$

Remark 1.4. Here is another way to understand the functor F. Set $S=R[s]/(s^m-r)$, and let $\sigma: A \otimes S \to A \otimes S$ be the graded S-algebra map sending $a_i \otimes 1$ in $A_i \otimes S$ to $a_i \otimes s^i$. If $j: A \to A \otimes S$ denotes the inclusion, then F is the functor

$$\mathbf{P}(A) \xrightarrow{j^*} \mathbf{P}(A \otimes S) \xrightarrow{\sigma^*} \mathbf{P}(A \otimes S) \xrightarrow{j_*} \mathbf{P}(A).$$

In fact, $P = F(A) = j * \sigma^* j^* A$ is just $A \otimes S$, with e_i in P corresponding to $1 \otimes s^i$ in $A \otimes S$.

For example, when $m = 1$ the ring map $\sigma: A \to A$ is $\sigma(a_i) = a_i r^i$ and F is the base-change map σ^*. If $m = 1$ and $r = 1$, F is the identity map. If $m = 1$ and $r = 0$, F is $\iota^* p^*$, where

$$p: \; A \to A/A_+ = A_0 \qquad \text{and} \qquad \iota: \; A_0 \to A$$

induce the functors

$$p^*: \; \mathbf{P}(A) \to \mathbf{P}(A_0) \qquad \text{and} \qquad \iota^*: \; \mathbf{P}(A_0) \to \mathbf{P}(A).$$

On K-theory, the functors F, p^*, and ι^* induce maps which we abusively write as $K_i F: K_i(A) \to K_i(A)$, $p^*: K_i(A) \to K_i(A_0)$, and $\iota^*: K_i(A_0) \to K_i(A)$. Since $p\iota: A_0 \to A_0$ is the identity, we obtain a direct sum decomposition $K_i(A) = K_i(A_0) \oplus K_i(A, A_+)$.

LEMMA/DEFINITION 1.5. *The induced functor* $K_i F: K_i(A) \to K_i(A)$ *respects the direct sum decomposition* $K_i(A) = K_i(A_0) \oplus K_i(A, A_+)$ *and is multiplication by* m *on the summand* $K_i(A_0)$. *The map* $(1 - rt^m)*$: $K_i(A, A_+) \to K_i(A, A_+)$ *is defined to be the restriction of* $K_i F$ *to the summand* $K_i(A, A_+)$.

Proof. Let $F_0: \mathbf{P}(A_0) \to \mathbf{P}(A_0)$ by $F_0(M) = M \otimes A_0^m = M \oplus \cdots \oplus M$. Since $P \cong A^m$ as left A_0-modules we have $F\iota = \iota F_0$. Since $(A/A_+) \otimes_A P \cong P \otimes_A (A/A_+) \cong A_0^m$ as A_0-bimodules, we have $F_0 p \cong pF$. This implies that $K_i F$ respects the decomposition of $K_i(A)$, and is $K_i F_0$ on $K_i(A_0)$. The fact that $K_i F_0$ is multiplication by m is standard.

Before moving on, we should clear up an apparent notational problem, namely the case $r = 0$. For clarity, let us write P_m and F_m for the A-bimodule and functor constructed for $r = 0$, and our chosen integer m.

LEMMA 1.6. *If* $r = 0$, *the map* $(1 - 0t^m)*: K_i(A, A_+) \to K_i(A, A_+)$ *induced from* F_m *is the zero map for all* m.

Proof. We have already observed that $F_1 = \iota^* p^*$, so the case $m = 1$ follows from Lemma 1.5. Inductively, note that the subbimodule $e_m A$ of P_{m+1} is isomorphic to P_1, and that the quotient bimodule is P_m. This yields a short exact sequence of functors $\mathbf{P}(A) \to \mathbf{P}(A)$,

$$0 \to F_1 \xrightarrow{e_m} F_{m+1} \to F_m \to 0.$$

By the additivity theorem [Q, p. 106], $K_i F_{m+1} = K_i F_m + K_i F_1$. Hence we have $(1 - 0t^{m+1})* = (1 - 0t^m)* + (1 - 0t)* = 0$.

2. THE CASE $A = A_0[x]$.

In this section, A will denote the polynomial ring $A_0[x]$ with x in degree one, and we will write $NK_i(A_0)$ for $K_i(A_0[x], x)$. It is a result of Bloch and Stienstra that the groups $NK_i(A_0)$ are continuous $W(R)$-modules; in this section we shall write $\omega \circ v$ for the Bloch–Stienstra product of $\omega \in W(R)$ and $v \in NK_i(A_0)$. We will show that the map $(1-rt^m)*$ of the last section produces the same endomorphism of $NK_i(A_0)$ as the Bloch-Stienstra map $(1-rt^m)\circ$. This will prove that the maps $(1-rt^m)*$ satisfy the axioms (1.2) for every graded R-algebra A, since the $(1-rt^m)\circ$ satisfy (1.2) for $A = A_0[x]$. These axioms imply that the $K_i(A, A_+)$ are naturally continuous $W(R)$-modules.

Under the Bloch-Stienstra module structure on $NK_i(A_0)$, multiplication by $(1-rt^m)$ is induced from the functor

$$\mathbf{P}(A_0[x]) \xrightarrow{i_*} \mathbf{P}(A_0[y]) \xrightarrow{\rho^*} \mathbf{P}(A_0[y]) \xrightarrow{i^*} \mathbf{P}(A_0[x]),$$

where $i: A_0[y] \to A_0[x]$ and $\rho: A_0[y] \to A_0[y]$ are the A_0-algebra maps given by $i(y) = x^m$, $\rho(y) = ry$. (The fact that this yields a $W(R)$-module structure on $NK_i(A_0)$ is asserted on p. 316 of [B2] and proven in [S2]. A discussion may be found in [W1].)

By 1.3, the functor $i^*\rho^*i_*$ is determined by the A-bimodule $Q = i^*\rho^*i_*(A)$. As a right A-module, Q is free on basis $\{f_0, ..., f_{m-1}\}$. Making the convention that f_{j+m} means $f_j(rx^m)$, the left A-module structure on Q is given by the formula:

$$(a_0x^i)f_j = f_{i+j}a_0 \qquad \text{for } a_0 \text{ in } A_0.$$

Fixing $m \geqslant 1$, let us write P_r and Q_r for the A-bimodules corresponding to $(1-rt^m)*$ and $(1-rt^m)\circ$, respectively. Write F_r and G_r for the respective functors $\mathbf{P}(A) \to \mathbf{P}(A)$ they induce. With respect to the bases $\{e_0, ..., e_{m-1}\}$ and $\{f_0, ..., f_{m-1}\}$ of P_r and Q_r, left multiplication by x is represented by the respective matrices

$$\begin{pmatrix} 0 & & & rx \\ x & 0 & & \\ & \ddots & \ddots & \\ & & x & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & & & rx^m \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Define the right module map $\eta_A: P_r \to Q_r$ by the formula $\eta_A(e_j) = f_jx^{m-1-j}$, $0 \leqslant j \leqslant m-1$. For example, $\eta_A(e_0) = f_0x^{m-1}$ and $\eta_A(e_{m-1}) = f_{m-1}$. Since $x\eta_A(e_j) = \eta_A(xe_j)$ for all j, it follows that η_A is a left A-module map as well. By 1.3, η_A induces a natural transformation $\eta: F_r \to G_r$.

PROPOSITION 2.1. *The two functors* $F_r = j_* \sigma^* j^*$ *and* $G_r = i^* \rho^* i_*$ *induce the same maps* $K_i(A) \to K_i(A)$ *and* $NK_i(A_0) \to NK_i(A_0)$.

Proof. Let C_r be the cokernel of the injection $\eta_A\colon P_r \to Q_r$. As a right A-module, it has finite homological dimension. The functor $H_r(M) = M \otimes_A C_r$ maps $\mathbf{P}(A)$ to the exact category $\mathbf{H}(A)$ of finitely generated right A-modules with finite homological dimension. There is a short exact sequence

$$0 \to F_r \to G_r \to H_r \to 0$$

of exact functors from $\mathbf{P}(A)$ to $\mathbf{H}(A)$. By the additivity theorem [Q, p. 106],

$$K_i H_r = K_i G_r - K_i F_r\colon \qquad K_i(A) \to K_i(\mathbf{H}(A)) = K_i(A).$$

However, it is easy to see that the A-bimodule C_r is independent of the choice of r. Taking $r = 0$, Lemma 1.6 yields the desired equation,

$$K_i H_r = K_i H_0 = K_i G_0 - K_i F_0 = 0 - 0 = 0.$$

COROLLARY 2.2. *The* $W(R)$*-module structure on* $NK_i(A_0)$ *given in Section* 1 *agrees with the Bloch–Stienstra module structure.*

3. BLOCH'S PAIRING

There is another way to define a $W(R)$-module structure on the groups $K_i(A, A_+)$, implicitly due to Bloch [B2, p. 315]. Bloch begins with the biexact functor

$$\mathbf{P}(R[\![t]\!]) \otimes \mathbf{Nil}(A) \to \mathbf{P}(A), \qquad M \otimes (N, v) = M \otimes_{R[\![t]\!]} N,$$

where N is considered to be a left $R[\![t]\!]$-module with t acting via the nilpotent endomorphism v. This produces a map from $K_q(R[\![t]\!]) \otimes K_p \mathbf{Nil}(A)$ to $K_{p+q}(A)$. Identifying $W(R)$ with $K_1(R[\![t]\!], t)$ and $K_{p+1}(A[x], x)$ with the kernel of the forgetful map $K_p\mathbf{Nil}(A) \to K_p(A)$ we obtain a pairing

$$W(R) \otimes K_{p+1}(A[x], x) \to K_{p+1}(A).$$

Now suppose that A is graded. Using the injection $\phi^*\colon K_i(A, A_+) \to K_i(A[x], x)$ of Section 1, we obtain pairings for each $i \geq 1$,

$$W(R) \otimes K_i(A, A_+) \to K_i(A, A_+).$$

This is Bloch's pairing.

PROPOSITION 3.1. *Bloch's pairing agrees with the action of* $W(R)$ *on* $K_i(A, A_+)$ *that we defined above. In particular, Bloch's pairing makes* $K_i(A, A_+)$ *into a continuous* $W(R)$*-module for every graded R-algebra* A.

Proof. By the trick of section 1 involving $\phi: A \rightarrow A[x]$, it is enough to prove the result for $A = A_0[x]$. In this case, Stienstra showed in [S2, (9.23)] that Bloch's pairing agrees with the Bloch–Stienstra pairing we cited in Section 2. We are done by Corollary 2.2.

Remark 3.2. Bloch defined his pairing only for $A = R[x]$ and $R[\varepsilon]/(\varepsilon^n)$, but the construction in [B1, B2] extends word for word to graded A. In [B1, B2], Bloch asserted, but did not prove, that his pairing made these rings into $W(R)$-modules. In [B1, (II.2.1.4)], Bloch proved that $\varprojlim_n K_i(R[\varepsilon]/(\varepsilon^n), \varepsilon)$ was a $W(R)$-module by verifying axioms (1.2). Using the presentation for K_2 of a radical ideal, Stienstra showed in [S0] that the $K_2(R[\varepsilon]/\varepsilon^n, \varepsilon)$ were $W(R)$-modules. We can now see that Bloch's assertion was correct.

4. STRUCTURAL RESULTS

In this section, we collect several results that are useful in calculations. First note that the group $K_*(A, A_+)$ is a graded module over the graded ring $K_*(R)$. We have

PROPOSITION 4.1 (Product formula). *For* $\gamma \in K_m(R)$, $v \in K_n(A, A_+)$, *and* $\omega(t) \in W(R)$ *we have the formula in* $K_{m+n}(A, A_+)$,

$$\omega(t) * \{\gamma, v\} = \{\gamma, \omega(t) * v\}.$$

Proof. By additivity, we can assume that $\omega(t)$ is $1 - rt^m$. Since the $K_*(R)$-module structure arises from the biexact pairing $\otimes: \mathbf{P}(R) \times \mathbf{P}(A) \rightarrow \mathbf{P}(A)$, sending (L, M) to $L \otimes_R M$, the product formula follows from the equation $(L \otimes_R M) \otimes_A P \cong L \otimes_R (M \otimes_A P)$, i.e., from commutativity up to natural isomorphism of the diagram:

$$\begin{tikzcd}
\mathbf{P}(R) \times \mathbf{P}(A) \arrow[r, "1 \times (\otimes_A P)"] \arrow[d, "\otimes"'] & \mathbf{P}(R) \times \mathbf{P}(A) \arrow[d, "\otimes"] \\
\mathbf{P}(A) \arrow[r, "\otimes_A P"] & \mathbf{P}(A).
\end{tikzcd}$$

Next, we consider the effect of changing the grading on A. For our purposes, a grading on A is a decomposition $A = \amalg A_i$; we say that A is *regraded by a factor of n* if we give it the decomposition $A = \amalg B_j$, where

$$B_j = \begin{cases} A_i & \text{if } j = ni, \\ 0 & \text{if } j \equiv 0 \pmod{n}. \end{cases}$$

PROPOSITION (4.2) (Change of grading). *If A is regraded by a factor of m, the resulting $W(R)$-module structure $*'$ on $K_i(A, A_+)$ is the pullback of the original $W(R)$-module structure $*$ along the Frobenius ring map $F_m\colon W(R) \to W(R)$. That is, we have*

$$\omega *' v = (F_m \omega) * v \qquad \textit{for} \quad \omega \in W(R) \textit{ and } v \in K_i(A, A_+).$$

Proof. For clarity, let us write B for the graded ring $A = \bigsqcup B_j$. We grade $A[y]$ and $B[x]$ with A, B in degree 0 and x, y in degree 1, and let ϕ be the graded map of Section 1. Finally, let i be the nongraded A-algebra map $A[y] \to B[x]$ given by $i(y) = x^m$. We have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, equal] & A[y] \arrow[r, "x=1"] \arrow[d, "i"] & A \arrow[d, equal] \\
B \arrow[r, "\phi"] & B[x] \arrow[r, "y=1"] & B.
\end{tikzcd}$$

The induced map $i^*\colon K_i(A[y], y) \to K_i(B[x], x)$ is called V_m in [B2] and [S2], where they show that $V_m((F_m\omega) * v) = \omega * (V_m v)$ for every $\omega \in W(R)$ and $v \in K_i(A[y], y)$. (Beware the typo in (2.7.1) of [B2].) This establishes the commutativity of the right-hand face in the following cube:

$$\begin{tikzcd}
K_i(A, A_+) \arrow[rr, "\phi^*"] \arrow[dr, Rightarrow] \arrow[dd, "F_m\omega *"'] & & K_i(A[y], y) \arrow[dr, "V_m"] \arrow[dd, "F_m\omega *"', near start] & \\
& K_i(B, B_+) \arrow[rr, "\phi^*", near start] \arrow[dd, "w *'"', near end] & & K_i(B[x], x) \arrow[dd, "\omega *"] \\
K_i(A, A_+) \arrow[rr, "\phi^*"', near end] \arrow[dr, Rightarrow] & & K_i(A[x], x) \arrow[dr, "V_m"'] & \\
& K_i(B, B_+) \arrow[rr, "\phi^*"] & & K_i(B[x], x).
\end{tikzcd}$$

The front and back faces commute by naturality. The top and bottom faces commute by the above discussion, the ϕ^* being split injections. Thus the left-hand face also commutes, which was to be shown.

Remark 4.2.1. If R contains the rational numbers, the injection $\lambda_t\colon R \to W(R)$ is invariant under the Frobenius, i.e., $F_m(\lambda_t(r)) = \lambda_t(r)$. Hence regrading A does not affect the R-module structure on $K_i(A, A_+)$. It does change the filtration on $K_i(A, A_+)$, however, as can be seen in Example 0.3.

PROPOSITION 4.3 (Morita invariance). *If A is Morita equivalent over R to B, then the natural isomorphism $NK_i(A) \cong NK_i(B)$ is an isomorphism of $W(R)$-modules.*

Proof. If L is an $R[x]$-bimodule and M is an $A-B$ bimodule, then there is an isomorphism $L \otimes_R M \cong M \otimes_R L$ of $A[x]-B[x]$ bimodules. For example, the Morita equivalence $A \approx B$ of R-algebras is induced by a functor $\otimes N\colon \textbf{mod-}A \to \textbf{mod-}B$, where N is an $A-B$ bimodule, and the $A[x]-B[x]$ bimodule $R[x] \otimes_R N \cong N \otimes_R R[x]$ induces a Morita equivalence $A[x] \approx B[x]$. On the other hand, if P is the $R[x]$-bimodule such that $\otimes_{R[x]} P$ induces $(1-rt^m)*$ on $NK_i(R)$, then $P \otimes_R A \approx A \otimes_R P$ induces $(1-rt^m)*$ on $NK_i(A)$, and similarly for $P \otimes_R B$. There is an $A[x]-B[x]$ bimodule isomorphism

$$\begin{aligned}&(P \otimes_R A) \otimes_{A[x]} (R[x] \otimes_R N)\\ &\qquad \approx P \otimes_R N \cong N \otimes_R P \cong (N \otimes_R R[x]) \otimes_{B[x]} (B \otimes_R P).\end{aligned}$$

This establishes commutativity up to natural transformation of

$$\begin{tikzcd}
\mathbf{P}(A[x]) \arrow[r, "N \otimes R[x]"] \arrow[d, "P \otimes A"] & \mathbf{P}(B[x]) \arrow[d, "B \otimes P"'] \\
\mathbf{P}(A[x]) \arrow[r, "R[x] \otimes N"] & \mathbf{P}(B[x]).
\end{tikzcd}$$

On the K-theory level, this implies that the Morita isomorphism $NK_i(A) \cong NK_i(B)$ commutes with $(1-rt^m)*$, whence the result.

If we want to discuss Morita invariance of graded R-algebras, we have to discuss graded Morita equivalences. Rather than pursue this tangential issue, we content ourselves with a special case. If A is a graded R-algebra, so is the matrix ring $M_n(A) = M_n(A_0) \oplus M_n(A_1) \oplus \cdots$, and the corresponding isomorphisms $K_i(A) \cong K_i(M_n(A))$ and $K_i(A_0) \cong K_i(M_n(A_0))$ induce a natural isomorphism $K_i(A, A_+) \cong K_i(M_n(A), M_n(A_+))$.

COROLLARY (4.4). *The natural isomorphism* $K_i(A, A_+) \cong K_i(M_n(A), M_n(A_+))$ *is an isomorphism of* $W(R)$*-modules.*

Proof. The maps $A \to^{\phi} A[x] \to M_n(A[x])$ and $A \to M_n(A) \to^{\phi} M_n(A[x])$ agree, so the Morita isomorphism $K_i(A[x]) \cong K_i(M_n(A[x]))$ sends the summands $\phi^* K_i(A)$ and $\phi^* K_i(A_0)$ to $\phi^* K_i(M_n A)$ and $\phi^* K_i(M_n A_0)$. Hence it sends $\phi^* K_i(A, A_+)$ to $\phi^* K_i(M_n A, M_n A_+)$, so we can deduce this result from 4.3.

5. EXAMPLES

In this section, we give some formulas to illustrate the $W(R)$-module structure. The action of $W(R)$ on $K_i(A, A_+)$ is completely determined by the action of the Witt vectors $(1 - rt^m)$, so we concentrate on their effect.

The action on $K_0(A, A_+)$ is clear from the construction in Section 1: if M is a projective A-module with $M/A_+M \cong (A_0)^n$, then $[M] - n$ is an element of $K_0(A, A_+)$ and $(1 - rt^m) * ([M] - n) = [M \otimes_A P] - mn$.

The action on $K_1(A, A_+)$ is more complicated, but can be written down directly from the left action of $GL_n(A)$ on $A^n \otimes_A P \cong A^{mn}$. Here is one special case:

LEMMA 5.1. *If* v *is a nilpotent* $n \times n$ *matrix with entries in* A_i $(i \neq 0)$, *the action of* $W(R)$ *on the corresponding element* $(1 - v)$ *of* $K_1(A, A_+)$ *is given by* $(1 - rt^m) * (1 - v) = (1 - r^{i/d} v^{m/d})^d$, $d = \gcd(m, i)$.

Proof. Using the embedding ϕ^* of $K_1(A, A_+)$ in $K_1(A[x], x)$, we can assume that $A = A_0[x]$ and $v = \alpha x^i$ for α a nilpotent matrix with entries in A_0. By Morita invariance, we can replace A_0 by $M_n(A_0)$ to assume α is in A_0. Replacing R and A_0 by $R[\alpha]$, we can assume that $A = R[x]$. But the formula is well-known in this case (see, e.g., [B1, II.2.3]).

Next, consider the case in which R contains the rational numbers. In this case, there is a ring map $\lambda_t \colon R \to W(R)$ sending r to $(1 - t)^r = \sum \binom{r}{i}(-t)^i$. The abelian groups $K_i(A, A_+)$ become R-modules in this way. Our next result describes the R-module structure on $K_1(A, A_+)$ when A_+ is nilpotent.

PROPOSITION 5.2. *Suppose that* A *contains the rational numbers and that* A_+ *is nilpotent. Then every element of* $K_1(A, A_+)$ *is represented by a unit* $(1 - f)$, $f \in A_+$, *and the* R*-module structure is given by* $r * (1 - f) = (1 - f)^r$. *If we let* $[A, A_+]$ *denote the subgroup of* A *generated by all* $af - fa$, $a \in A$ *and* $f \in A_+$, *then there is an* R*-module isomorphism*

$$\exp\colon \quad A_+/[A, A_+] \to K_1(A, A_+).$$

Proof. It is well known that every element of $K_1(A, A_+)$ is represented by a unit $1-f$ of A with f in A_+. In the Appendix, we show that ln and exp induce an isomorphism of $A_+/[A, A_+]$ with $K_1(A, A_+)$, so if $(1-f)=\Pi(1-f_i)$ then $(1-f)^r$ and $\Pi(1-f_i)^r$ represent the same element of $K_1(A, A_+)$. Therefore, to see that the module structure is given by $r*(1-f)=(1-f)^r$, we can factor $1-f$ into terms $(1-f_i)$ with f_i in A_i. Replacing A by $R[f_i]$, we can assume A commutative. Via the R-module injection $\phi^*\colon K_1(A, A_+)\to K_1(A[x], x)$ we can assume $A=R[x]$. The result in this special case is well-known (see, e.g., [S0, II.5.10; B1, II.3.5; W1, p. 480]).

EXAMPLE 5.3. Let $A=R[\varepsilon, x]/(\varepsilon^N)$, where for convenience R is a field. There are several ways to grade A, and each gives a different $W(R)$-module structure on $NK_*(R[\varepsilon], \varepsilon)$. To illustrate this, consider the product $(1-rt^m)*(1-\varepsilon^i x^j)$. Set $v=\varepsilon^i x^j$, $d=\gcd(m, i)$ and $e=\gcd(m, j)$. When $A_0=R[x]$ and $\deg(\varepsilon)=1$, the product is $(1-r^{i/d}v^{m/d})^d$; when $A_0=R[\varepsilon]$ and $\deg(x)=1$, the product is $(1-r^{j/e}v^{m/e})^e$. This clarifies the remark on p. 480 of [W1] that there are different $W(R)$-module structures on $NK_*(R[\varepsilon], \varepsilon)$. In fact, they arise from different gradings of $A=R[\varepsilon, x]$.

We now turn to the action of $W(R)$ on the relative K_2 group. We will assume that A_+ is a nilpotent ideal and that A is commutative, so that we know that $K_2(A, A_+)$ is additively generated by symbols $\langle a, s\rangle$ and $\langle a, b\rangle$, where $s\in A_0$, $a\in A_i$ and $b\in A_j$ $(i, j\neq 0)$. First, we describe the R-module structure in characteristic zero. To do this, we shall adopt the convention that the expression $(1-(1-ax)^r)/x$ means the polynomial

$$a\sum_{k=0}^{\infty}\binom{r}{k+1}(-ax)^k=ra-\binom{r}{2}a^2x+\binom{r}{3}a^3x^2-\cdots.$$

PROPOSITION 5.4. *Suppose that A is commutative, that R contains the rational numbers, and that A_+ is nilpotent. Then the R-module structure on $K_2(A, A_+)$ is given by the formulas ($s\in A_0$, $a\in A_i$ and $b\in A_j$ $(i, j\neq 0)$),*

$$r*\langle a, s\rangle=\left\langle\frac{1-(1-as)^r}{s}, s\right\rangle$$

$$r*\langle a, b\rangle=\left(\frac{i}{i+j}\right)\left\langle\frac{1-(1-ab)^r}{b}, b\right\rangle+\left(\frac{j}{i+j}\right)\left\langle a, \frac{1-(1-ab)^r}{a}\right\rangle$$

$$=\left\langle\frac{1-(1-ab)^r}{b}, b\right\rangle+\left(\frac{j}{i+j}\right)\left\langle ab, \frac{1-(1-ab)^r}{ab}\right\rangle.$$

Proof. To compute $r*\langle a, s\rangle$ we can reduce to the generic case $R=\mathbf{Q}[r, s]$, $A=R[a]/(a^n)$. For this A, $K_2(A, A_+)$ embeds in $K_2(A[s^{-1}],$

$A_+[s^{-1}]$), so we can assume that s is a unit of R. But then the product formula yields

$$r * \langle a, s\rangle = r * \{1-as, s\} = \{r * (1-as), s\} = \{(1-as)^r, s\}$$
$$= \left\langle \frac{1-(1-as)^r}{s}, s\right\rangle.$$

To compute $r * \langle a, b\rangle$, we apply ϕ^* to get

$$\begin{aligned}
r * \langle ax^i, bx^j\rangle &= r * (\langle abx^i, x^j\rangle + \langle ax^{i+j}, b\rangle)\\
&= \left(\frac{j}{i+j}\right) r * \langle ab, x^{i+j}\rangle + r * \langle ax^{i+j}, b\rangle\\
&= \left(\frac{j}{i+j}\right)\left\langle ab, \frac{(1-abx^{i+j})^r}{ab}\right\rangle + \left\langle \frac{(1-abx^{i+j})^r}{b}, b\right\rangle\\
&= \left(\frac{j}{i+j}\right)\left\langle a, \frac{(1-abx^{i+j})^r}{a}\right\rangle + \left(\frac{i}{i+j}\right)\left\langle \frac{(1-abx^{i+j})^r}{b}, b\right\rangle\\
&= \left(\frac{j}{i+j}\right)\left\langle ax^i, \frac{(1-abx^{i+j})^r}{ax^i}\right\rangle + \left(\frac{i}{i+j}\right)\left\langle \frac{(1-abx^{i+j})^r}{bx^j}, bx^j\right\rangle\\
&\quad -\left(\frac{j}{i+j}\right)\left\langle x^i, \frac{(1-abx^{i+j})^r}{x^i}\right\rangle - \left(\frac{i}{i+j}\right)\left\langle \frac{(1-abx^{i+j})^r}{x^j}, x^j\right\rangle\\
&= \left(\frac{j}{i+j}\right)\phi^*\left(\left\langle a, \frac{(1-ab)^r}{a}\right\rangle\right) + \left(\frac{i}{i+j}\right)\phi^*\left(\left\langle \frac{(1-ab)^r}{b}, b\right\rangle\right).
\end{aligned}$$

Since ϕ^* is an injection, we deduce the formula for $r * \langle a, b\rangle$.

Here are the general formulas for the module structure on K_2 when A_+ is nilpotent:

PROPOSITION 5.5. *Let A be a commutative graded R-algebra with $A_0 = R$ and A_+ nilpotent. The $W(R)$-module structure on $K_2(A, A_+)$ is completely determined by the formulas*:

(a) $(1-rt^m) * \langle a, s\rangle = d\langle a^{m/d}r^{i/d}s^{m/d-1}, s\rangle$, *where $r, s \in R$, $a \in A_i$ and $d = \gcd(m, i)$*

(b) $(1-rt^m) * \langle a, b\rangle = (um+iv)\langle a^k b^{k-1} r^n, b\rangle - jv\langle a^{k-1}b^k r^n, a\rangle + ju\langle (ab)^k r^{n-1}, r\rangle + j(d-1)\langle -(ab)^k r^n, -1\rangle$, *where $a \in A_i$, $b \in A_j$, $r \in R$, $d = \gcd(i+j, m)$, $k = m/d$, $n = (i+j)/d$ and u and v are integers such that $d = um + v(i+j)$.*

Proof. We compute as in [W1, (4.4)], using the formulas (and symbols) on p. 62 of [S0] (which may be derived from Sect. 2 of the published version [S1]). These formulas give the $W(R)$-module structure on $K_2(A[x], x)$. One of these formulas is

$$(1-rt^m) * \langle ax^i, s\rangle = d\langle a^{m/d} r^{i/d} s^{m/d-1} x^{im/d}, s\rangle, \qquad d=\gcd(m, i).$$

Now this is just ϕ^* applied to formula (a), so 5.5(a) holds in $K_2(A, A_+)$. The second formula from [S0] is (in the notation of (b)),

$$\begin{aligned}(1-rt^m) &* \langle cx^{i+j-1}, x\rangle\\ &= um\langle c^k r^n x^{mn-1}, x\rangle + u\langle c^k r^{n-1} x^{mn}, r\rangle\\ &\quad - v\langle c^{k-1} r^n x^{mn}, c\rangle + (d-1)\langle -c^k r^n x^{mn}, -1\rangle\\ &= um\langle czx^{nd-1}, x\rangle + u\langle c^k r^{n-1} x^{mn}, r\rangle\\ &\quad - v\langle zx^{nd}, c\rangle + (d-1)\langle -czx^{nd}, -1\rangle,\end{aligned}$$

where $z=r^n(cx^{nd})^{k-1}$. Now $\phi^*(\langle a, b\rangle)=\langle ax^i, bx^j\rangle = j\langle abx^{i+j-1}, x\rangle + \langle ax^{i+j}, b\rangle$. Set $c=ab$, so that $c^k x^{mn}=\phi(c^k)$ and $z=\phi(r^n c^{k-1})$. The two cited formulas yield in $K_2(A[x], x)$ that

$$\begin{aligned}(1-rt^m) * \langle ax^{i+j}, b\rangle &= d\langle a^k r^n b^{k-1} x^{mn}, b\rangle = d\langle azx^{nd}, b\rangle;\\ (1-rt^m) * \langle ax^i, bx^j\rangle &= jum\langle czx^{nd-1}, x\rangle + \phi^*(\beta)\\ &\quad - jv\langle zx^{nd}, c\rangle + d\langle azx^{nd}, b\rangle,\end{aligned}$$

where $\beta = ju\langle c^k r^{n-1}, r\rangle + j(d-1)\langle -c^k r^n, -1\rangle$. Since $j\langle czx^{nd-1}, x\rangle = \langle czx^i, x^j\rangle$ and $iv = -um + d - jv$, we obtain

$$\begin{aligned}(1-rt^m) &* \langle ax^i, bx^j\rangle - \phi^*(\beta)\\ &= um(\langle azx^i, bx^j\rangle - \langle azx^{i+j}, b\rangle) + d\langle azx^{i+j}, b\rangle\\ &\quad - jv(\langle azx^{i+j}, b\rangle + \langle bzx^{i+j}, a\rangle)\\ &= um\langle azx^i, bx^j\rangle + iv\langle azx^{i+j}, b\rangle - jv\langle bzx^{i+j}, a\rangle.\end{aligned}$$

The result now follows from the observation that

$$\begin{aligned}\phi^*(i\langle a^k b^{k-1} r^n, b\rangle &- j\langle a^{k-1} b^k r^n, a\rangle)\\ &= i\langle azx^i, bx^j\rangle - j\langle bzx^j, ax^i\rangle\\ &= i\langle azx^{i+j}, b\rangle - j\langle bzx^{i+j}, a\rangle\end{aligned}$$

because

$$i\langle abzx^i, x^j\rangle = ij\langle abzx^{i+j-1}, x\rangle = j\langle abzx^j, x^i\rangle.$$

The complicated formula for $(1-rt^m) * \langle a, b\rangle$ simplifies quite a bit when m and $(i+j)$ are units of R, for then we can divide by these elements in $K_2(A, A_+)$. This is because m and $(i+j)$ are units in the ring $W(R)$. First, note that $j(d-1)\langle -(ab)^k r^n, -1\rangle = 0$, because either $(d-1)$ is even or else 1/2 is in A. Second, note that $(1/k)\langle a, b^k\rangle = \langle ab^{k-1}, b\rangle$ for a or b in A_+. Thus formula (5.5)(b) becomes

$$\begin{aligned}(1-rt^m) &* \langle a, b\rangle \\ &= d\langle a^k b^{k-1} r^n, b\rangle - jv\langle (ab)^{k-1} r^n, ab\rangle + ju\langle (ab)^k r^{n-1}, r\rangle \\ &= (d/k)\langle a^k r^n, b^k\rangle - (jv/k)\langle r^n, (ab)^k\rangle + (ju/n)\langle (ab)^k, r^n\rangle \\ &= (d/k)\langle a^k r^n, b^k\rangle + (j/nk)\langle (ab)^k, r^n\rangle \\ &= (d/k - j/nk)\langle a^k r^n, b^k\rangle + (j/nk)\langle a^k, r^n b^k\rangle.\end{aligned}$$

In summary, we have derived the

SIMPLIFICATION 5.6. When m and $i+j$ are units in R we have the simpler formula

$$(1-rt^m) * \langle a, b\rangle = \left(\frac{i}{nk}\right)\langle a^k r^n, b^k\rangle + \left(\frac{j}{nk}\right)\langle a^k, r^n b^k\rangle$$

($a \in A_i$, $b \in A_j$, $r \in R$, $d = \gcd(i+j, m)$, $k = m/d$, and $n = (i+j)/d$).

We conclude with an application of these ideas to the paper [vdk-S] of Stienstra and van der Kallen. Let $A = R[y_1, \ldots, y_r, \ldots, y_s]/I$, where I is an ideal generated by monomials of $R[y_1, \ldots, y_r]$ and containing some power of each of $\{y_1, \ldots, y_r\}$. We grade A by putting $A_0 = R[y_{r+1}, \ldots, y_s]$ and letting $y_1, \ldots, y_r$ belong to A_1. For $\alpha = (\alpha_1, \ldots, \alpha_s)$ an s-tuple of nonnegative integers, write y^α for $\Pi y_i^{\alpha_i}$. If y^α belongs to I and $\alpha_i \neq 0$, Stienstra and van der Kallen define group maps

$$\Gamma_{\alpha,i}\colon\ (1 + xR[\![x]\!])^* \to K_2(A, A_+),$$

$$\Gamma_{\alpha,i}(1 - xf(x)) = \langle f(y^\alpha)(y^\alpha/y_i), y_i\rangle,$$

and use these maps to completely describe $K_2(A, A_+)$ when R is a perfect field of characteristic p. (See [vdK-S, (2.6)].) Our observation is

THEOREM 5.7. *Given* α, *let* $e = \deg(y^\alpha) = \alpha_1 + \cdots + \alpha_r$, *and identify* $(1 + xR[\![x]\!])^*$ *with the ideal* $V_e W(R)$ *of* $W(R)$ *via* $x = t^e$. *Then*

(a) *If* $i > r$, *the map* $\Gamma_{\alpha,i}$ *is a* $W(R)$*-module homomorphism.*

(b) *If* $i \leqslant r$ *and* R *is a perfect field of characteristic* $p \neq 0$, *the map* $\Gamma_{\alpha,i}$ *is a* $W(R)$*-module homomorphism.*

Proof. The ideal $V_e W(R)$ is generated by $1-x$, so it is enough to check that $(1-rt^m) * \Gamma_{\alpha,i}(1-x) = \Gamma_{\alpha,i}((1-rt^m) * (1-x))$. Write $d=\gcd(m, e)$, so that the right-hand side is $d\langle r^{e/d}y^{m\alpha/d}/y_i, y_i\rangle$. If $i>r$ then $y_i \in A_0$, and part (a) follows immediately from formula (5.5)(a).

If $i \leqslant r$ the formula is more complicated. Set $e=nd$, $m=kd$ and choose u, v so that $1=uk+vn$. Formula (5.5)(b) then reads

$$\begin{aligned}(1-rt^m) * \Gamma_{\alpha,i}(1-x) = d\langle r^n y^{k\alpha}/y_i, y_i\rangle + v\langle y^\alpha, r^n y^{(k-1)\alpha}\rangle \\ + u\langle y^{k\alpha}r^{n-1}, r\rangle + (d-1)\langle -r^n y^{k\alpha}, -1\rangle.\end{aligned}$$

Since $\operatorname{char}(R) \neq 0$, the last term is zero. Since R is perfect we can extract pth roots of r, and therefore $\langle y^{k\alpha}r^{n-1}, r\rangle$ is p-divisible. As $K_2(A, A_+)$ is a p-group, this term must also be zero. The theorem will now follow once we show that $\langle y^\alpha, r^n y^{(k-1)\alpha}\rangle = 0$ for all k. If $p \nmid k$, this term equals $(1/k)\langle y^{k\alpha}, r^n\rangle = 0$. We now proceed by induction on k, using [S1, p. 414],

$$\begin{aligned}0 &= p\langle y^\alpha, sy^{(k-1)\alpha}\rangle = \langle y^\alpha, s^p y^{p(k-1)\alpha} y^{(p-1)\alpha}\rangle \\ &= \langle y^\alpha, s^p y^{(pk-1)\alpha}\rangle.\end{aligned}$$

Since $r^n = s^p$ for some s in R, this establishes the result.

APPENDIX

In this Appendix, we give a proof that $K_1(A, I)$ carries a natural module structure in characteristic 0 whenever I is nilpotent.

THEOREM A.1. *Let A be a ring containing $\mathbf{Q}$, and I a nilpotent ideal of A. Then there is a natural isomorphism $K_1(A, I) \cong I/[A, I]$, where $[A, I]$ is the subgroup of I generated by all $[a, x] = ax - xa$ with $a \in A$ and $x \in I$.*

Before giving our proof, we note that Goodwillie has proven that for any nilpotent ideal there is an isomorphism

$$K_i(A, I; \mathbf{Q}) \cong \mathrm{HC}_{i-1}(A \otimes \mathbf{Q}, I \otimes \mathbf{Q}).$$

Here HC denotes cyclic homology over $\mathbf{Q}$, and our indexing convention is such that $\mathrm{HC}_i(A) \to \mathrm{HC}_i(A/I) \to \mathrm{HC}_{i-1}(A, I) \to \mathrm{HC}_{i-1}(A)$ is exact. When A contains $\mathbf{Q}$, we know from [W2, 1.4] that $K_i(A, I; \mathbf{Q}) = K_i(A, I)$. This yields a more general result:

THEOREM A.2. *Let A be a ring containing $\mathbf{Q}$, and I a nilpotent ideal of A. There is a natural isomoprhism $K_i(A, I) \cong HC_{i-1}(A, I)$ for all i. In par-*

ticular, these groups are modules over the center of A. The case $i=1$ yields the isomorphism $K_1(A, I) \cong HC_0(A, I) \cong I/[A, I]$ of Theorem A.1.

To be more explicit about the isomorphism, we use a more explicit discription of $K_1(A, I)$, due to Vaserstein. For any radical ideal I in any ring A, let $W(A, I)$ denote the subgroup of $1+I$ generated by all $(1+ax)(1+xa)^{-1}$ with $a \in A$ and $x \in I$. Then

$$K_1(A, I) \cong (1+I)/W(A, I).$$

(See Theorem 2.1 of [Sw].) We will show that the power series expansions for ln and exp provide the isomorphisms in Theorem A.1.

We shall also need the Campbell–Hausdorff formula, which may be found in [J, pp. 170–174]. It states that for x, y in a complete radical ideal I that there are u, v in I such that

$$\exp(x)\exp(y) = \exp(x+y+[u, x]+[v, y]).$$

The actual formula is explicit enough to see that if $y \in I^n$ then $u \in I^n$. In fact, $u = \frac{1}{4}y + \frac{1}{12}[x, y] + \cdots$ is in the closure of the ideal AyA.

As an application, consider the set map $\ln : (1+I) \to I$, whose inverse is the set map exp. The Campbell–Hausdorff formula shows that ln is not a group homomorphism. In fact, for x, y in I it yields

$$\begin{aligned} \ln((1+x)(1+y)) = \ln(1+x) + \ln(1+y) &+ [u, \ln(1+x)] \\ &+ [v, \ln(1+y)]. \end{aligned}$$

We summarize this computation:

LEMMA A.3. *If I is any complete radical ideal which is also a* **Q**-*vector space, then* ln *induces a group epimorphism*

$$1+I \xrightarrow{\ln} I/[I, I].$$

COROLLARY A.4. *If I is a complete radical ideal in a ring A which contains* **Q**, *then* ln *induces a surjection $K_1(A, I) \to I/[A, I]$.*

Proof. Fix $a \in A$ and $x \in I$, and set

$$y = x + \sum_{i=1}^{\infty} x(-ax)^i/(i+1) = x + \sum_{i=1}^{\infty} (-xa)^i x/(i+1).$$

Then modulo $[A, I]$ we have that

$$\ln((1+ax)(1+xa)^{-1}) = \ln(1+ax) - \ln(1+xa) = ay - ya = 0.$$

In trying to construct an inverse to the map of A.3, we are led to consider $\exp([I, I])$. Note that for every n the set $\exp([I, I^n])$ is a subgroup of $1+I$ by Campbell–Hausdorff.

LEMMA A.5. *If $a \in A$, $x \in I$ and $y \in I^n$, then*

(i) $\exp(x+y)\exp(-x)\exp(-y)$ *is in* $\exp([I, I^n])$.

(ii) $\exp(xa)\exp(-ax)$ *is in* $W(A, I)$.

Proof. There are $u, v, u' \in I$ and $v', v'' \in I^n$ such that

$$\begin{aligned}&\exp(x+y)\exp(-x)\exp(-y)\\&\quad=\exp(y+[u,x]+[v,y])\exp(-y)\\&\quad=\exp([u,x]+[v,y]+[u',y]+[v',[u,x]]+[v'',[v,y]]).\end{aligned}$$

We claim that $[u, x]$ is in $[I, I^n]$. To see this, note that by the Campbell–Hausdorff formula we can write $u=u_1+u_2$, where $u_1 \in I^n$ and $[u_2, x]=0$. This establishes (i). For (ii), note that

$$z=x+\sum_{i=1}^{\infty} x(ax)^i/(i+1)!=x+\sum_{i=1}^{\infty}(xa)^i x/(i+1)!$$

satisfies $1+za=\exp(xa)$ and $(1+az)^{-1}=\exp(ax)^{-1}=\exp(-ax)$.

PROPOSITION A.6. *When I is a nilpotent ideal which is also a* **Q**-*vector space, then*

(i) $\exp([I, I])=W(\mathbf{Q}\oplus I, I)$.

(ii) exp *induces a (well-defined) group epimorphism*:

$$I \xrightarrow{\exp} (1+I)/W(\mathbf{Q}\oplus I, I)=K_1(I).$$

Proof. Since $[\mathbf{Q}\oplus I, I]=[I, I]$, A.4 implies that $W(\mathbf{Q}\oplus I, I)$ lies in $\exp([I, I])$. We show by descending induction on n that $\exp([I, I^n])$ lies in $W(\mathbf{Q}\oplus I, I)$, the case $n \gg 0$ being given. For $x \in I$, $y \in I^n$ we have

$$\begin{aligned}&\exp([x,y])\exp(yx)\exp(-xy)\\&\quad=\exp(xy+[u,[x,y]]+[v,yx])\exp(-xy)\\&\quad=\exp(w),\qquad w\in[I, I^{n+1}].\end{aligned}$$

The result follows from A.5.

THEOREM A.7. *Let I be a nilpotent ideal in a ring A containing* $\mathbf{Q}$.

(i) $W(A, I) = \exp([A, I])$

(ii) exp and ln induce an isomorphism $K_1(A, I) \cong I/[A, I]$.

Proof. We need only show that $\exp([A, I])$ is contained in $W(A, I)$, since we can then cite A.4 and A.6. But modulo $W(\mathbf{Q} \oplus I, I)$, Lemma A.5 shows that $\exp(\Sigma[a_i, x_i]) \equiv \Pi \exp([a_i, x_i])$, which is in $W(A, I)$.

REFERENCES

[Bass] H. BASS, "Algebraic K-theory," Benjamin, New York, 1968.

[B1] S. BLOCH, Algebraic K-theory and crystalline cohomology, *Publ. I. H. E. S. (Paris)* **47** (1978), 188–268.

[B2] S. BLOCH, Some formulas pertaining to the K-theory of commutative groupschemes, *J. Algebra* **53** (1978), 304–326.

[C] P. CARTIER, Modules associés à un groupe formal commutatif, courbes typiques, *C. R. Acad. Sci. Paris* **265** (1967), 129–132.

[J] N. JACOBSON, "Lie Algebras," Interscience, New York, 1962.

[Q] D. QUILLEN, "Higher Algebraic K-Theory: I," Lecture Notes in Math., No. 341, Springer-Verlag, Berlin, 1973.

[Reid] L. REID, "Some Results on the Lower K-Theory of Singular Affine Algebras," *J. Pure Applied Algebra* **42** (1986), 185–198.

[S0] J. STIENSTRA, "Deformations of the Second Chow-group," Ph.D. thesis, Utrecht, 1978.

[S1] J. STIENSTRA, "On K_2 and K_3 of Truncated Polynomial Rings," Lecture Notes in Math., No. 854, Springer-Verlag, Berlin, 1981.

[S2] J. STIENSTRA, "Operations in the Higher K-theory of Endomorphisms," CMS conference proceedings, Vol. 2, Part 2, 1983.

[Sr] V. SRINIVAS, Vector bundles on the cone over a curve, *Compositio Math.* **47** (1982), 249–269.

[Sw] R. SWAN, Excision in algebraic K-theory, *J. Pure Appl. Algebra* **1** (1971), 221–252.

[vdK] W. VAN DER KALLEN, Le K_2 de nombres duaux, *C. R. Acad. Sci. Paris* **273** (1971), 1204–1207.

[vdK-S] W. VAN DER KALLEN AND J. STIENSTRA, The relative K_2 of truncated polynomial rings, *J. Pure Appl. Algebra* **34** (1984), 277–289.

[W1] C. WEIBEL, "Mayer–Vietoris Sequences and Module Structures on NK_*," Lecture Notes in Math., No. 854, Springer-Verlag, Berlin, 1981.

[W2] C. WEIBEL, "Mayer–Vietoris Sequences and mod p K-Theory," Lecture Notes in Math., No. 966, Springer-Verlag, Berlin, 1982.

[W3] C. WEIBEL, Complete intersection points on smooth affine surfaces, unpublished preprint, 1981.