MAYER-VIETORIS SEQUENCES AND MOD P K-THEORY

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In this paper we prove that excision holds and that Mayer-Vietoris sequences exist for K-theory with mod p coefficients, as long as we restrict ourselves to $\mathbb{Z}[\frac{1}{p}]$-algebras. Since this theory is related to the usual K-theory by a Universal Coefficient Theorem, this provides a method of recovering at least some of the structure of the usual K-groups.

To show the potential and imperfections in this method, we work through an example in which $\mathbb{Q}/\mathbb{Z}$ appears in the kernel of excision.

Our idea is that massaged K-groups will have Mayer-Vietoris sequences for a wide class of rings. For example, it was proven in [K-V, Appendix A] that the Karoubi-Villamayor Theory KV_* has Mayer-Vietoris sequences under a "Gl-fibration" hypothesis. In [We], it was proven that the groups $K_*(A) \otimes \mathbb{Z}[\frac{1}{p}] = K_*(A;\mathbb{Z}[\frac{1}{p}])$ have Mayer-Vietoris sequences when restricted to the class of rings in which p is nilpotent.

Having introduced K-theory with coefficients $\mathbb{Z}[\frac{1}{p}]$, it seems natural to consider K-theory with coefficients $\mathbb{Z}/p$. The theory $K_*(\ ;\mathbb{Z}/p)$ was introduced in Browder's paper [Br], and we recite the main features of this theory in §2 below. In §3 we construct a theory $KV_*(\ ;\mathbb{Z}/p)$ and provide a spectral sequence relating it to $K_*(\ ;\mathbb{Z}/p)$. We will reap the benefits of this construction in §1, where the applications-oriented reader may access the results without having to read the details of the constructions involved.

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§1. Main Results

Our study of mod p K-theory is motivated by the following fact, observed in [We]: if A is a $\mathbb{Z}[\frac{1}{p}]$-algebra then the groups $NK_*(A)$ are $\mathbb{Z}[\frac{1}{p}]$-modules. This is also true of the relative groups: if I is an ideal in a $\mathbb{Z}[\frac{1}{p}]$-algebra A, then $NK_*(A,I)$ is a $\mathbb{Z}[\frac{1}{p}]$-module by [We, (3.5)].

The mod p K-theory $K_*(\;;\mathbb{Z}/p)$ is related to the usual theory K_* by a Universal Coefficient Theorem (see (2.1) below). Since $K_*(A[x];\mathbb{Z}/p) = K_*(A;\mathbb{Z}/p) \oplus NK_*(A;\mathbb{Z}/p)$, we deduce the following

Consequence 1.1. If A is a $\mathbb{Z}[\frac{1}{p}]$-algebra, then $K_*(A[x];\mathbb{Z}/p) = K_*(A;\mathbb{Z}/p)$ and $K_*(A[x],I[x];\mathbb{Z}/p) = K_*(A,I;\mathbb{Z}/p)$.

This homotopy-like property suggests a comparison with the Karoubi-Villamayer theory KV_* of [K-V]. In §3 below, we construct a theory $KV_*(A;\mathbb{Z}/p)$ for $* \geq 1$ which is related to the KV_*-theory by a Universal Coefficient Theorem. By (3.3) below, there is a spectral sequence converging to $KV_*(A;\mathbb{Z}/p)$ whose E^1 terms are

$$E^1_{st} = \begin{cases} N^s K_t(A;\mathbb{Z}/p), & t > 1,\ s \geq 0 \\ N^s K_1(A) \otimes \mathbb{Z}/p, & t = 1,\ s \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows immediately that (if A is a $\mathbb{Z}[\frac{1}{p}]$-algebra) the spectral sequence collapses to give $K_*(A;\mathbb{Z}/p) = KV_*(A;\mathbb{Z}/p)$ for $* \geq 1$.

More importantly, by (3.4) below, there is a similar spectral sequence obtained by replacing A by (A,I). This converges to groups $KV_*(I;\mathbb{Z}/p)$ which do not depend on A. Again, the spectral sequence collapses to give $K_*(A,I;\mathbb{Z}/p) = KV_*(I;\mathbb{Z}/p)$. We have proven:

Theorem 1.2. Let A be a $\mathbb{Z}[\frac{1}{p}]$-algebra with unit, and let I be an ideal of A. Then for $* \geq 1$ we have

(a) $K_*(A;\mathbb{Z}/p) = KV_*(A;\mathbb{Z}/p)$

(b) $K_*(A,I;\mathbb{Z}/p) = KV_*(I;\mathbb{Z}/p)$

(c) ("Excision") If $f\colon A \to B$ is a ring map with $I \cong f(I)$, and if $f(I)$ is an ideal of B, then $K_*(A,I;\mathbb{Z}/p) = K_*(B,I;\mathbb{Z}/p)$.

Since $K_0(A,I;\mathbb{Z}/p) = K_0(A,I) \otimes \mathbb{Z}/p$, and excision holds for K_0, excision also holds for $K_0(\ ;\mathbb{Z}/p)$. We can then prove the following result by splicing together the long exact ideal sequences for $K_*(\ ;\mathbb{Z}/p)$-theory:

Corollary 1.3 ("Mayer-Vietoris"). Let

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_3 \arrow[r] & A_4 \end{tikzcd}$$

be a pullback square of $\mathbb{Z}[\frac{1}{p}]$-algebras with $A_2 \to A_4$ onto. Then there is a long exact sequence

$$\ldots K_{*+1}(A_4;\mathbb{Z}/p) \to K_*(A_1;\mathbb{Z}/p) \to K_*(A_2;\mathbb{Z}/p) \oplus K_*(A_3;\mathbb{Z}/p) \to K_*(A_4;\mathbb{Z}/p)\ldots$$

valid for all integers $*$.

In many cases, we can use mod p K-theory to gain information about ordinary K-theory. Here are two examples of this philosophy:

Consequence 1.4. If I is a nilpotent ideal in a $\mathbb{Z}[\frac{1}{p}]$-algebra A, then $K_*(A,I)$ is a $\mathbb{Z}[\frac{1}{p}]$-module. In particular, if $\mathbb{Q} \subseteq A$ then $K_*(A,I)$ is a $\mathbb{Q}$-vector space.

To see this, recall that $KV_*(I) = 0$ by [We 2, (2.3)]. By Theorem (1.2) we have $K_*(A,I;\mathbb{Z}/p) = 0$ for $* > 1$. The same is true for $* = 0$ since $K_0(A,I) = 0$ is well known. By the Universal Coefficient Theorem (see [N,(2.4)]), $K_*(A,I;\mathbb{Z}/p) = 0$ for $* = t$ and $t+1$ implies that $K_t(A,I)$ is a $\mathbb{Z}[\frac{1}{p}]$-module, as claimed.

We turn now to consideration of the excision map $\eta: K_*(A,I) \to K_*(B,I)$. In [We, (5.7)], we proved that the kernel of η is a p-divisible abelian group, and that the torsion subgroup of $\mathrm{coker}(\eta)$ is p-divisible. In addition, there is no p-torsion in the cokernel of $K_*(A,I) \to KV_*(I)$. One would like the kernel and cokernel of η to be $\mathbb{Z}[\frac{1}{p}]$-modules, but Example (1.6) below shows that this is not the case.

Swan has pointed out that these results can be improved in the following way. There are doubly relative groups $K_*(A,B,I)$ fitting into a long exact sequence

$$\ldots \xrightarrow{\eta} K_{t+1}(B,I) \to K_t(A,B,I) \to K_t(A,I) \xrightarrow{\eta} K_t(B,I) \ldots$$

Consequence 1.5. If $f: A \to B$ is a map of $\mathbb{Z}[\frac{1}{p}]$-algebras inducing an isomorphism of ideals $I \cong f(I)$, then the doubly relative groups $K_*(A,B,I)$ are $\mathbb{Z}[\frac{1}{p}]$-modules for $* > 0$. Moreover, the p-torsion subgroup of $\ker(\eta_t : K_t(A,I) \longrightarrow K_t(B,I))$ is naturally isomorphic to $\mathrm{coker}(\eta_{t+1}) \otimes \mathbb{Z}/p^\infty$, where $\mathbb{Z}/p^\infty = \mathrm{colim}(\mathbb{Z}/p^n)$.

To see this, we use groups $K_*(A,B,I;\mathbb{Z}/p)$ constructed in §2 below. These al fit into a long exact sequence for mod p K-theory. From Theorem (1.2c) above we se that $K_*(A,B,I;\mathbb{Z}/p) = 0$ for $* > 0$. By Universal Coefficients, the groups $K_*(A,B,I)$ must be $\mathbb{Z}[\frac{1}{p}]$-modules for $* > 0$, and $K_0(A,B,I) = 0$. Finally, apply

$\otimes\, \mathbb{Z}/p^\infty$ to

$$0 \longrightarrow \mathrm{coker}(\eta_{t+1}) \longrightarrow K_t(A,B,I) \longrightarrow \ker(\eta_t) \longrightarrow 0.$$

This yields an isomorphism between $\mathrm{coker}(\eta_{t+1}) \otimes \mathbb{Z}/p^\infty$ and $\mathrm{Tor}\,(\ker(\eta_t), \mathbb{Z}/p^\infty)$ = the p-torsion subgroup of $\ker(\eta_t)$.

EXAMPLE (1.6). Let R be a regular commutative domain. Set $B = R[s,s^{-1},t,t^{-1}]$, $I = (s-1)B$, $A = R \oplus I$. Then $K_1(B,I) = KV_1(I) = K_0(R)$ is well-known. The map $\eta: K_1(A,I) \longrightarrow K_1(B,I)$ is a surjection with kernel $\ker(\eta) \cong R[t,t^{-1}]/\mathbb{Z}\cdot t^{-1} = (\coprod R) \oplus (R/\mathbb{Z}\cdot 1)$. In particular, if $\mathbb{Q} \subseteq R$, $\ker(\eta) = (\mathbb{Q}$-vector spac $\oplus (\mathbb{Q}/\mathbb{Z})$.

The doubly relative group $K_1(A,B,I)$ is naturally isomorphic to $R[t,t^{-1}]$. The cokernel of $K_2(A,I) \longrightarrow K_2(B,I)$ is $\mathbb{Z}$ if $\mathbb{Z} \subseteq R$ and $\mathbb{Z}/n$ if $\mathbb{Z}/n \subseteq R$.

Proof. Since $K_2(B,I) \cong K_1(R[t,t^{-1}])$ and $\Omega_{B/A} \otimes I/I^2 \cong R[t,t^{-1}]$, the exact sequence of [GW, (2.4)] is

$$K_1(R[t,t^{-1}]) \xrightarrow{d} R[t,t^{-1}] \longrightarrow K_1(A,I) \xrightarrow{\eta} K_1(B,I) \longrightarrow 0.$$

We have to analyze d. To α in $K_1(R[t,t^{-1}])$ we first associate $\{\alpha,s\}$ in $K_2(B,I$ and then $\{\alpha,1+\varepsilon\} = \langle\det(\alpha),\varepsilon/\det(\alpha)\rangle$ in $K_2(B/I^2,I/I^2)$, where $s-1$ in B maps to ε in $B/I^2 = R[t,t^{-1}][\varepsilon]$. Consulting [GW], we see that

$$d(\alpha) = d\log(\det(\alpha)) = \det(\alpha)^{-1} \frac{d}{dt} \det(\alpha).$$

Now $K_1(R[t,t^{-1}]) = SK_1(R[t,t^{-1}]) \oplus \mathrm{Units}(R) \oplus \{t^n\}$ by the Fundamental Theorem. The first summand is the kernel of the det map, and $\frac{d}{dt}$ is zero on the second summand. Hence the image of d is the cyclic abelian subgroup of $R[t,t^{-1}]$ generated by $t^{-1} = d\log(t)$. This establishes the formula for $\ker(\eta)$.

It is well-known that $K_*(B,I) = K_{*-1}(R[t,t^{-1}]) = K_{*-1}(R) \oplus K_{*-2}(R)$. Since (A,I) contains $(R[s,s^{-1}],s-1)$, the first summand $K_{*-1}(R)$ splits off as a summand of $K_*(A,I)$. This is not true of the second summand, as we have seen.

The cokernel of $K_2(A,I) \to K_2(B,I) = KV_2(I)$ is the E^∞_{11} term of the Gersten-Anderson spectral sequence $E^1_{st} = N^sK_t(A,I) \Longrightarrow K_{s+t}(B,I) = KV_{s+t}(I)$, described for example in [We 1,(2.6)]. It is not hard to work out from the above that $E^1_{st} = N^sK_1(A,I) = N^s\ker(K_1(A,I) \longrightarrow K_1(B,I)) \cong x_1 \ldots x_s R[t,t^{-1},x_1,\ldots,x_s]$ in the more or less obvious notation for $s \neq 0$, and from this that $E^\infty_{11} = E^2_{11}$ is the image of $\underset{\sim}{Z}$ in R. (The $\underset{\sim}{Z} \cdot t^{-1}$ factored out in $R[t,t^{-1}]/\underset{\sim}{Z} \cdot t^{-1} \subsetneq E^1_{01}$ gives rise to the E^2_{11} term.) This establishes the formula for $\mathrm{coker}(\eta_2)$.

To determine the doubly relative groups, we use the analogous spectral sequence

$$E^1_{st}(\mathrm{rel}) = N^sK_t(A,B,I) \Longrightarrow 0$$

constructed in (3.5) below. Since B is regular, we have $E^1_{st}(\mathrm{rel}) = N^sK_t(A,I)$ for $s \neq 0$. From the above description of $N^sK_1(A,I)$, we obtain the exact sequence

$$0 \longrightarrow E^2_{11}(\mathrm{rel}) \longrightarrow R[t,t^{-1}] \longrightarrow K_1(A,B,I) \longrightarrow E^2_{01}(\mathrm{rel}) \longrightarrow 0.$$

Since the outer terms are zero, the middle two must be isomorphic. This finishes the proof of (1.6).

The ultimate point in introducing a new theory such as $K_*(;Z/p)$ is in order to say more about the structure of the usual groups K_*. We can be somewhat successful at this, but not completely. In order to illustrate this point, we analyze the above example.

It is convenient to use coefficients mod p^∞. These are defined as $K_t(\ ;\mathbb{Z}/p^\infty) = \underset{\overrightarrow{n}}{\operatorname{colim}}\, K_t(\ ;\mathbb{Z}/p^n)$, and behave exactly as coefficients mod p^n. If $\frac{1}{p} \in R$ in Example (1.6) we have

$$K_*(A,I;\mathbb{Z}/p^\infty) = K_*(B,I;\mathbb{Z}/p^\infty) = K_{*-1}(R;\mathbb{Z}/p^\infty) \oplus K_{*-2}(R;\mathbb{Z}/p^\infty).$$

By Universal Coefficients, there is a noncanonical isomorphism

$$K_*(A,I;\mathbb{Z}/p^\infty) \cong K_*(A,I) \otimes \mathbb{Z}/p^\infty \oplus (\text{p-torsion in } K_{*-1}(A,I)).$$

Since the kernel of η is p-divisible, it follows that every p-torsion element in the kernel of $K_{*-1}(A,I) \to K_{*-1}(B,I)$ is detected by a $\mathbb{Z}/p^\infty$-summand in $K_{*-2}(R;\mathbb{Z}/p^\infty)$. By (1.5), these summands also detect elements in the cokernel of $K_*(A,I) \longrightarrow K_*(B,I)$, a p-torsion free group. The case $* = 2$ is described in (1.6) above.

For concreteness, consider the case $R = \mathbb{Q}$. By [Bo], [Q1], each group $K_t(\mathbb{Q})$ is a nondivisible torsion group unless $t = 0$ or $t \equiv 1 (\mathrm{mod}\ 4)$, a $K_t(\mathbb{Q})$ is $\mathbb{Z} \oplus$ (finite group) for $t = 0,5,9,13,\ldots$. By Universal Coefficients we se that $K_t(\mathbb{Q};\mathbb{Z}/p^\infty)$ contains no $(\mathbb{Z}/p^\infty)$-summands if $t \not\equiv 1 (\mathrm{mod}\ 4)$, $t \neq 0$, one $(\mathbb{Z}/p^\infty$ summand if $t = 0$ or if $t \equiv 1 (\mathrm{mod}\ 4)$, $t \neq 1$, and a countably infinite number of $(\mathbb{Z}/p^\infty)$-summands if $t = 1$. We summarize:

<u>Proposition 1.7.</u> When $R = \mathbb{Q}$ in Example (1.6), the kernels of the $\eta_t\colon K_t(A,I) \longrightarrow K_t(B,I)$ are $\mathbb{Q}$-vector spaces, except for $t = 1$ and possibly $t \equiv 2 (\mathrm{mod}\ 4)$. Each η_t is onto, except for $t = 2$ and possibly $t \equiv 3 (\mathrm{mod}\ 4)$. Fc $t = 6,10,14,\ldots$ either a) η_{t+1} is onto and $\ker(\eta_t)$ is a $\mathbb{Q}$-vector space, or b) $\operatorname{coker}(\eta_{t+1}) = \mathbb{Z}$ and $\ker(\eta_t) = (\mathbb{Q}\text{-vector space}) \oplus \mathbb{Q}/\mathbb{Z}$.

In the final case, either a) η_3 is onto and $\ker(\eta_2)$ is a $\mathbb{Q}$-vector space, or b) $\operatorname{coker}(\eta_3)$ is a countably generated free abelian group and $\ker(\eta_2) = (\mathbb{Q}\text{- vector space}) \oplus (\operatorname{coker} \eta_3 \otimes \mathbb{Q}/\mathbb{Z})$.

§2. Mod p K-Theory

This section consists of an expansion of Browder's observations in [Br]. We will write $\mathcal{A}$ for an exact category, so that the topological space $BQ\mathcal{A}$ gives the K-theory of $\mathcal{A}$ via $K_m(\mathcal{A}) = \pi_{m+1}(BQ\mathcal{A})$ as in [Q]. The primary example is $\mathcal{A} = \underline{P}(A)$, the category of finitely generated projective A-modules, and we write $K_n(A)$ for $K_n(\underline{P}(A))$.

If $m > 2$ and $p > 1$, the mod p Moore space $P^m(\underset{\sim}{Z}/p)$ is $S^{m-1} \cup_p e^m$ (an m-cell attached to S^{m-1} by a degree p map). Basic properties of $P^m(\underset{\sim}{Z}/p)$ may be found in [N]. The notation $\pi_m(X;\underset{\sim}{Z}/p)$ denotes $[P^m(\underset{\sim}{Z}/p),X]$ for $m > 2$, and a space X.

We now define the mod p K-theory of $\mathcal{A}$ to be $K_m(\mathcal{A};\underset{\sim}{Z}/p) = \pi_{m+1}(BQ\mathcal{A};\underset{\sim}{Z}/p)$ for $m > 1$, $K_0(\mathcal{A};\underset{\sim}{Z}/p) = K_0(\mathcal{A}) \otimes \underset{\sim}{Z}/p$ for $m = 0$, and $Km(\mathcal{A};\underset{\sim}{Z}/p) = 0$ for $m < 0$ (cf. [Br, p. 45]). Mod p K-theory is related to the usual K-theory by the

Universal Coefficient Theorem (2.1): For all m there is a short exact sequence of abelian groups

$$0 \longrightarrow K_m(\mathcal{A}) \otimes \underset{\sim}{Z}/p \longrightarrow K_m(\mathcal{A};\underset{\sim}{Z}/p) \longrightarrow \mathrm{Tor}(K_{m-1}(\mathcal{A}),\underset{\sim}{Z}/p) \longrightarrow 0.$$

If $p \not\equiv 2 \pmod 4$ this sequence splits (not naturally), so that $K_m(\mathcal{A};\underset{\sim}{Z}/p)$ is a $\underset{\sim}{Z}/p$-module. If $p \equiv 2 \pmod 4$, $K_m(\mathcal{A};\underset{\sim}{Z}/p)$ is a $\underset{\sim}{Z}/2p$-module.

The proof of Theorem (2.1) may be found in [N, pp. 3,37] or [AT, p. 78]. Note that from [AT, p. 79] it follows that $K_2(\underset{\sim}{Z};\underset{\sim}{Z}/2) = \underset{\sim}{Z}/4$, while $K_m(A;\underset{\sim}{Z}/2)$ is a $\underset{\sim}{Z}/2$-module whenever multiplication by $[-1] \in K_1(A)$ is the zero map from $K_{m-1}(A)$ to $K_m(A)$. For example, this is the case for finite fields, an observation made in [Br].

In order to proceed further, we are going to have to introduce a method of computing $\pi_0(;\underset{\sim}{Z}/p)$ and $\pi_1(;\underset{\sim}{Z}/p)$ for infinite loop spaces. It seems simplest to work with spectra instead of topological spaces. For an introduction to spectra, the reader is encouraged to read Chapter 1 of [A], and to consult [A1][Sw] for details. One good reason for preferring spectra is that spectra form an additive category (see [A1, p. 156]). Another is that $\pi_*(;\underset{\sim}{Z}/p)$ is a homology theory on spectra, but not on spaces.

A <u>CW-spectrum</u> $\underline{E}$ is a sequence $(E_0,E_1,\dots)$ of based CW-complexes togethe with maps $\varepsilon:\Sigma E_i \to E_{i+1}$ (or by adjointness, maps $E_i \to \Omega E_{i+1}$). For technical reasons, each ε must embed ΣE_i as a subcomplex of E_{i+1}. We call $\underline{E}$ an <u>Ω-spectrum</u> if the $E_i \to \Omega E_{i+1}$ are weak equivalences. Every infinite loop space the initial space of an Ω-spectrum, and $\Omega BQ\mathcal{A}$ is no exception. For example, could take the Ω-spectrum

$$\underline{BQ}\mathcal{A} = (\Omega BQ\mathcal{A}, BQ\mathcal{A}, BQ^2\mathcal{A},\dots),$$

where $Q^i\mathcal{A}$ is the multicategory defined on [Wa, p. 194] (ε comes from the multicategory map $Q^i\mathcal{A} \otimes (\cdot \Longrightarrow \cdot) \to Q^{i+1}\mathcal{A}$ described on p. 197 of [Wa]), or els could create $\underline{BQ}\mathcal{A}$ with an infinite loop space machine. A technical point: $\underline{BQ}\mathcal{A}$ should be (-1)-connected to avoid $\lim^1$ difficulties.

There is an adjoint pair $(\Sigma^\infty,\Omega^\infty)$ of functors between spaces and spectra. The spectrum $\Sigma^\infty X$ is $(X,\Sigma X,\Sigma^2 X,\dots)$. If $\underline{E}$ is an Ω-spectrum then $\Omega^\infty\underline{E}$ is E in general $\Omega^\infty\underline{E}$ is an infinite loop space. The homotopy groups of a spectrum a $\pi_m(\underline{E}) = [\Sigma^\infty S^m,\underline{E}] = \pi_m(\Omega^\infty\underline{E})$, where $[\underline{D},\underline{E}]$ denotes homotopy classes of maps in t category of spectra. For example, $\pi_m(\underline{BQ}\mathcal{A}) = K_m(\mathcal{A})$.

We define the mod p homotopy groups $\pi_n(\underline{E};\underset{\sim}{Z}/p)$ to be $[\Sigma^\infty P^n(\underset{\sim}{Z}/p),\underline{E}]$, agreein that when $n < 2$ we write $\Sigma^\infty P^n(\underset{\sim}{Z}/p)$ for the spectrum $(\text{point},\dots,\text{point},P^2(\underset{\sim}{Z}/p)$ $P^3(\underset{\sim}{Z}/p),\dots)$. By adjointness we have that $\pi_m(\underline{BQ}\mathcal{A};\underset{\sim}{Z}/p) = \pi_m(\Omega BQ\mathcal{A};\underset{\sim}{Z}/p) = K_m(\mathcal{A};\underset{\sim}{Z}$ for $m > 2$, while $\pi_1(\underline{BQ}\mathcal{A};\underset{\sim}{Z}/p) = \pi_2(BQ\mathcal{A};\underset{\sim}{Z}/p) = K_1(\mathcal{A};\underset{\sim}{Z}/p)$. Also $\pi_0(\underline{BQ}\mathcal{A};\underset{\sim}{Z}/p) = \pi_2(BQQ\mathcal{A};\underset{\sim}{Z}/p) = K_0(\mathcal{A}) \otimes \underset{\sim}{Z}/p$ by the Universal Coefficient Theore Thus we could have defined $K_*(\mathcal{A};\underset{\sim}{Z}/p)$ as $\pi_*(\underline{BQ}\mathcal{A};\underset{\sim}{Z}/p)$ in the first place.

<u>Remark</u>. All the results of "Higher algebraic K-theory: I and II" ([Q] and [GQ] hold for mod p K-theory. This applies specifically to: additivity for character exact sequences, reduction by resolution, devissage, localization sequences for abelian categories, localization theorems (for projective modules and for fin. g modules), and the Fundamental Theorem. This is because only very elementary properties of homotopy groups are used. A subtle point is that exact functors $\mathcal{A} \to \mathcal{B}$ induce maps $BQ\mathcal{A} \to BQ\mathcal{B}$ of H-spaces, which follows from Theorem 2 of [Q In fact, $\underline{BQ}\mathcal{A} \to \underline{BQ}\mathcal{B}$ is a map of spectra.

We can now construct the relative groups $K_*(A,I;\mathbb{Z}/p)$. For an ideal I in a ring A, write $\underline{BQ}\,\underline{P}(A,I)$ for the homotopy fiber of the map $\underline{BQ}\,\underline{P}(A) \rightarrow \underline{BQ}\,\underline{P}(A/I)$ of Ω-spectra. The homotopy groups are the usual Quillen relative K-groups: $K_*(A,I) = \pi_*(\underline{BQ}\,\underline{P}(A,I))$. We then define $K_*(A,I;\mathbb{Z}/p) = \pi_*(\underline{BQ}\,\underline{P}(A,I);\mathbb{Z}/p)$. As in [N, p.4], there is a functorial exact sequence ending in

$$\ldots K_1(A/I;\mathbb{Z}/p) \rightarrow K_0(A,I;\mathbb{Z}/p) \rightarrow K_0(A;\mathbb{Z}/p) \rightarrow K_0(A/I;\mathbb{Z}/p) \rightarrow K_{-1}(A,I;\mathbb{Z}/p) \rightarrow 0.$$

Similarly, if $(A,I) \rightarrow (B,J)$ is a ring map, we can define $K_*((A,I),(B,J))$ to be the homotopy groups of the fiber $\underline{BQ}\,\underline{P}((A,I),(B,J))$ of $\underline{BQ}\,\underline{P}(A,I) \rightarrow \underline{BQ}\,\underline{P}(B,J)$, and define $K_*((A,I),(B,J);\ \mathbb{Z}/p)$ as $\pi_*(\underline{BQ}\,\underline{P}((A,I),(B,I));\ \mathbb{Z}/p)$. When $I \cong J$ we simplify the notation, writing (A,B,I) for $((A,I),(B,I))$. This yields the groups $K_*(A,B,I)$ used in (1.5) above.

<u>Warning</u>. We have $K_{-1}(A,I) = K_0(A/I)/\text{im}\,K_0(A)$ in the exact sequence

$$0 \rightarrow K_0(A,I) \otimes \mathbb{Z}/p \rightarrow K_0(A,I;\mathbb{Z}/p) \rightarrow \text{Tor}\,(K_{-1}(A,I),\mathbb{Z}/p) \rightarrow 0.$$

This $K_{-1}(A,I)$ depends on the choice of A, and is not the usual negative K-theory of [Ba, ch. XII]. As a consequence, excision need not hold for $K_0(A,I;\mathbb{Z}/p)$. Similarly, although $K_0(A,B,I) = 0$, the groups $K_{-1}(A,B,I)$ and $K_{-2}(A,B,I)$ may be nontrivial, and the module structure on $K_*(A,B,I)$ described in Consequence (1.5) does not extend to the cases $* < 0$. These caveats may be illustrated by the subring $R \oplus I$, $I = (t^2-1)B$, of $B = R[t]$, $R = \mathbb{C}[x,y]/(y^2=x^3-x)$. Here $K_{-1}(A,I) = K_{-1}(A,B,I) = 0$ but $K_{-1}(B,I) = K_{-2}(A,B,I) = K_0(R) = \mathbb{Z} \oplus (\mathbb{C}/\mathbb{Z}\times\mathbb{Z})$. It follows that $K_0(B,I;\mathbb{Z}/p) = K_0(A,I;\mathbb{Z}/p) \oplus (\mathbb{Z}/p)^2$.

<u>Construction 2.2</u>. If $\underline{E}$ is (n-1)-connected, let $\underline{E}^{(n)}$ denote the fiber of the map $\underline{E} \rightarrow K(\pi_n(\underline{E}),n)$. We call $\underline{E}^{(n)}$ the n-connected cover of $\underline{E}$. We will write $\underline{K}(A)$, $\underline{K}(A,I)$, and $\underline{K}(A,B,I)$ for the 0-connected covers of $\underline{BQ}\,\underline{P}(A)$, $\underline{BQ}\,\underline{P}(A,I)$, and $\underline{BQ}\,\underline{P}(A,B,I)$. The space $\Omega^\infty\underline{K}(A)$ is $BG\ell^+(A)$ by the "+ = Q" theorem, and Browder wrote $K_*(A;\mathbb{Z}/p)$ for $\pi_*(\underline{K}(A);\mathbb{Z}/p)$ in [Br]. By Universal Coefficients, Browder's groups and ours agree except possibly when $* = 0,1$.

If we take $\mathcal{A}$ to be the category $\underline{\mathrm{Nil}}(A)$ or $\underline{\mathrm{End}}(A)$, we obtain groups $\mathrm{Nil}_*(A;\mathbb{Z}/p) = K_*(\underline{\mathrm{Nil}}(A))/K_*(A)$ and $\mathrm{End}_*(A;\mathbb{Z}/p) = K_*(\underline{\mathrm{End}}(A))/K_*(A)$, and have Universal Coefficient theorems for Nil_* and End_*. Note that we have $\mathrm{End}_0(A;\mathbb{Z}/p)$ $\mathrm{End}_0(A) \otimes \mathbb{Z}/p$, $\mathrm{Nil}_0(A;\mathbb{Z}/p) = \mathrm{Nil}_0(A) \otimes \mathbb{Z}/p = NK_1(A) \otimes \mathbb{Z}/p$. For $* > 0$ $\mathrm{Nil}_*(A;\mathbb{Z}/p) \cong NK_{*+1}(A;\mathbb{Z}/p)$ holds by the Fundamental Theorem in [GQ].

We would like to say that the $\mathrm{End}_0(A)$-module structure on $NK_*(A)$ induces natural $\mathrm{End}_0(A) \otimes \mathbb{Z}/p$-module structure on $NK_*(A;\mathbb{Z}/p)$, but this is not always s we have to avoid $p \equiv 2 (\mathrm{mod}\ 4)$. The module structure arises from the biexact pai ing $\underline{\mathrm{End}} \times \underline{\mathrm{Nil}} \to \underline{\mathrm{Nil}}$, and we generalize Browder's result [Br, (1.7)] accordingly

<u>Theorem 2.3</u>. Let $\mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ be a biexact pairing of exact categories in the sense of Waldhausen [Wa, §9]. Then for $p \not\equiv 2\ (\mathrm{mod}\ 4)$ there is an induced pairin

$$K_*(\mathcal{A};\mathbb{Z}/p) \otimes K_*(\mathcal{B};\mathbb{Z}/p) \longrightarrow K_*(\mathcal{C};\mathbb{Z}/p).$$

<u>Proof</u>. By [Wa, (9.2)], the pairing induces a map of topological spaces $BQ\mathcal{A} \wedge BQ\mathcal{B} \to BQQ\mathcal{C}$; the cited proof shows that a map $\underline{BQ}\mathcal{A} \wedge \underline{BQ}\mathcal{B} \to \underline{BQ}\mathcal{C}$ is induced from the multicategory maps $BQ^i\mathcal{A} \otimes BQ^j\mathcal{B} \to BQ^{i+j}\mathcal{C}$. (To make this map unique at the spectrum level, however, we must eliminate $\lim^1$ ambiguities; the applications we have in mind are insensitive to this ambiguity.) The result we want now follows from the following remark (or from [Br, (1.6)]).

<u>Remark</u>. A pairing $\underline{D} \wedge \underline{E} \to \underline{F}$ of spectra induces a map $\pi_*(\underline{D};\mathbb{Z}/p) \otimes \pi_*(\underline{E};\mathbb{Z}/p) \to \pi_*(\underline{F};\mathbb{Z}/p)$ in the following way. For $p \not\equiv 2\ (\mathrm{mod}\ 4)$ there is an isomorphism (cf. [Br, (1.4)]) $\underline{M} \wedge \underline{M} \cong \underline{M} \vee \Sigma^{-1}\underline{M}$ in the category of spectra, where $\underline{M} = \Sigma^\infty P^0(\mathbb{Z}/p)$. The map now comes from

$$[\underline{M},\underline{D}] \otimes [\underline{M},\underline{E}] \longrightarrow [\underline{M}\wedge\underline{M},\underline{D}\wedge\underline{E}] \longrightarrow [\underline{M}\wedge\underline{M},\underline{F}] \longrightarrow [\underline{M},\underline{F}],$$

where the last map is induced by a splitting $\rho\colon \underline{M} \longrightarrow \underline{M}\wedge\underline{M}$.

Corollary 2.4. Let R be a commutative ring, A an R-algebra, and $p \not\equiv 2 \pmod 4$. Then the groups $NK_*(A;\mathbb{Z}/p)$ are modules over the ring $W(R) \otimes \mathbb{Z}/p$ of Witt vectors mod p.

Proof. This follows from (2.3) above and [We, (3.1)], to wit: $NK_*(A;\mathbb{Z}/p)$ is a module over the ring $\mathrm{End}_0(R) \otimes \mathbb{Z}/p$ and every element is annihilated by some ideal $I_N \otimes \mathbb{Z}/p$, so $NK_*(A;\mathbb{Z}/p)$ is a module over the t-adic completion $W(R) \otimes \mathbb{Z}/p$ of $\mathrm{End}_0(R) \otimes \mathbb{Z}/p$.

Remark. If $\frac{1}{p} \in R$, $W(R) \otimes \mathbb{Z}/p = 0$. At the opposite extreme, if R is a perfect field of characteristic p (p any prime $\neq 0$), the ghost map (composed with a projection) induces an isomorphism

$$W(R) \otimes \mathbb{Z}/p \cong \prod_{i=1}^{\infty} R.$$

We conclude this section with a related result we will not need, which I learned from J. P. May and J. Neisendorfer.

Theorem 2.5. A commutative associative biexact pairing $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ makes $K_*(\mathcal{A};\mathbb{Z}/p)$ a graded commutative, associative ring under the following restriction on p: if $2|p$ then $16|p$, and if $3|p$ then $9|p$.

In particular, when R is a commutative ring and p is as above, $K_*(R;\mathbb{Z}/p)$ and $\mathrm{End}_*(R;\mathbb{Z}/p)$ are commutative associative rings. Moreover, $K_*(A;\mathbb{Z}/p)$ is a $K_*(R;\mathbb{Z}/p)$-module and $NK_*(A;\mathbb{Z}/p)$ is an $\mathrm{End}_*(R;\mathbb{Z}/p)$-module for every R-algebra A.

Proof. The pairing induces a commutative, associative map of spectra $\underline{BQ}\mathcal{A} \wedge \underline{BQ}\mathcal{A} \to \underline{BQ}\mathcal{A}$. Choosing $\rho: \underline{M} \to \underline{M}\wedge\underline{M}$ gives the product on $K_*(\mathcal{A};Z/p)$ for $p \not\equiv 2 \pmod 4$. By [O], ρ is cocommutative when $p \not\equiv 4 \pmod 8$, and ρ is coassociative under the stated restriction on p (the case $p \equiv 0 \pmod 9$ is pro in [O1, Theorem (3.3)]). Category theory now shows that $K_*(\mathcal{A};\underset{\sim}{Z}/p)$ is commutat and associative.

If $\mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{B}$ is a biexact pairing, associative with respect to the pairing on $\mathcal{A}$, then the resulting module structure $\underline{BQ}\mathcal{A} \wedge \underline{BQ}\mathcal{B} \longrightarrow \underline{BQ}\mathcal{B}$ mak $K_*(\mathcal{B};\underset{\sim}{Z}/p)$ a $K_*(\mathcal{A};\underset{\sim}{Z}/p)$-module (under the restriction on p). The pairings $\underline{P}(R) \times \underline{P}(A) \longrightarrow \underline{P}(A)$, $\underline{End}(R) \times \underline{Nil}(A) \longrightarrow \underline{Nil}(A)$ have been used to establish the last sentence in the theorem.

§3. Mod p KV-Theory

In this section we construct functors $KV_*(A;\underset{\sim}{Z}/p)$, prove an excision result, and construct the spectral sequences which we used in section 1 above. The results of this section parallel those of [We 1], and many remain valid if we replace $\underline{P}$ by an exact functor $\mathcal{A}$.

By the phase "simplicial CW-spectrum" we will mean a simplicial object $E_\bullet$ in the category $\mathcal{S}$ of CW-spectra before passage to homotopy (this category is described on pages 139-144 of [A1]). We want to construct the "total spectrum" $|E|$ by geometric realization of the underlying topological spaces (see [M, p. 101]), so we will in fact insist that the face and degeneracy maps be honest geometric maps, i.e., "functions" in the sense of [A1, p. 140].

If A is a ring with unit, we can form a simplicial ring $A_\bullet$ which in degree n is the coordinate ring $A[x_0,\ldots,x_n]/(\sum x_i = 1)$ of the "standard n-simplex," the face and degeneracy maps being dictated by the geometry. Constructing $\underline{K}(A_\bullet)$ as in (2.2) produces a simplicial CW-spectrum whose initial spaces form the simplicial topological space $BGl^+(A_\bullet)$. (We warn the reader that deep spectrum work requires a functorial version of (2.2), an issue we shall neglect, as we are only interested in the homotopy groups involved.) Thus $\pi_0(|\underline{K}(A_\bullet)|) = 0$, and for $m > 1$ we have $KV_m(A) = \pi_m(|\underline{K}(A_\bullet)|)$.

We define $KV_m(A;\underset{\sim}{Z}/p)$ to be $\pi_m(|\underline{K}(A_\bullet)|;\underset{\sim}{Z}/p)$ for $m > 1$ and ignore KV_0. Thus $KV_1(A;\underset{\sim}{Z}/p) = KV_1(A) \otimes \underset{\sim}{Z}/p$, and for $m > 2$ there is a Universal Coefficient Theorem as in (2.1) above:

$$0 \longrightarrow KV_m(A) \otimes \underset{\sim}{Z}/p \longrightarrow KV_m(A;\underset{\sim}{Z}/p) \longrightarrow \mathrm{Tor}(KV_{m-1}(A),\underset{\sim}{Z}/p) \longrightarrow 0.$$

We are going to need some spectral sequences arising from simplicial spectra such as $\underline{K}(A_\bullet)$. it seems best to do this in the following generality. Recall from [A] that the homotopy category of CW-spectra $h\mathcal{S}$ is an additive category with the property that every split epi $E \to E_1$ has a kernel E_2, i.e., $E \cong E_1 \vee E_2$ for some E_2 in E.

<u>Definition 3.1.</u> Let $E_\bullet$ be a simplicial object in an additive category $\mathcal{E}$, and assume that every split epi in $\mathcal{E}$ has a kernel in $\mathcal{E}$. Define NE_t to be the kernel

of the split epi $d_0 : E_{t+1} \longrightarrow E_t$. By shifting the face and degeneracy indices d one, $NE_\bullet$ becomes a simplicial object as well. We have $E_{t+1} \cong E_t \oplus NE_t$ by construction. We can iterate this construction to obtain $N^sE_\bullet = N(N^{s-1}E_\bullet)$, sett $N^0E_\bullet = E_\bullet$ by convention. By abuse, we will write N^sE for N^sE_0. It is an ea exercise to see that

$$E_n \cong (1+N)^nE = E \oplus \binom{n}{1}NE \oplus \dots \oplus \binom{n}{i}N^iE \oplus \dots \oplus N^nE.$$

Using this formula, it follows that the cokernel of the (split) map

$$(\sigma_0,\dots,\sigma_{s-1}) : \bigoplus_{i=0}^{s-1} E_{s-1} \longrightarrow E_s$$

is naturally isomorphic to N^sE.

If F is an additive functor on $\mathcal{E}$ we have $N^sF(E_\bullet) = F(N^sE_\bullet)$. For exampl if $\underline{E}_\bullet$ is a simplicial spectrum then

$$N^*\underline{E}:\ * \longleftarrow \underline{E} \xleftarrow[d_1]{} N\underline{E} \xleftarrow[d_2]{} N^2\underline{E} \xleftarrow[d_3]{} \cdots$$

is a chain complex in the additive category $h\mathcal{S}$. The homotopy groups of the simplicial abelian group $[D,\underline{E}_\bullet]$ may be computed as

$$\pi_s[D,\underline{E}_\bullet] = H_s(N^*[D,\underline{E}]) = H_s([D,N^*\underline{E}]).$$

For this reason, we may think of $N^*\underline{E}$ as the Moore complex associated to $\underline{E}$.

When E is a functor from rings to $\mathcal{E}$, we can form $E(A_\bullet)$. In this case $NE(A)$ is the kernel of $E(t=0): E(A[t]) \longrightarrow E(A)$, and we recover the original definition of the functor NE in [Ba, p. 658]. In particular,
$N^sK_t(A) = \pi_t(N^s\underline{BQ}\,\underline{P}(A))$.

Now let $E_\bullet$ be a simplicial CW-spectrum, and write $|E|$ for the total spectrum. Write $F_s|E|$ for the subspectrum of $|E|$ generated by E_s. It is standard (cf. [M, p. 102]) that the cofiber of $F_{s-1}|E| \longrightarrow F_s|E|$ is

$$\Sigma^s(N^sE) = \Sigma^sE_s/(\mathrm{im}(\sigma_0,\dots,\sigma_{s-1}):\bigvee E_{s-1} \longrightarrow E_s).$$

This yields an exact couple in the additive category $h\mathcal{J}$:

$$\begin{tikzcd}
\bigvee_s F_{s-1}|E| \arrow[rr] & & \bigvee_s F_s|E| \arrow[dl] \\
& \bigvee_s \Sigma^s N^s E. \arrow[ul] &
\end{tikzcd}$$

Embedding $h\mathcal{J}$ in an abelian category gives an Atiyah-Hirzebruch type "spectral sequence of spectra" with $E^1_{st} = \Sigma^{-t} N^s E \Rightarrow \Sigma^{-s-t}|E|$. Convergence follows for example from [Sz, pp. 338-9]. The same is true if we apply an exact functor such as $[D,]$:

Theorem 3.2. Let $E_\bullet$ be a simplicial CW-spectrum. For every spectrum D there is a right half-plane homology spectral sequence

$$E^1_{st} = [\Sigma^t D, N^s E] = N^s[\Sigma^t D, E_\bullet] \Rightarrow [\Sigma^{s+t} D, |E|].$$

Applications 3.3. If we take $D = \Sigma^\infty S^0$ we obtain the stable Bousfield-Kan spectral sequence $E^1_{st} = \pi_t(N^s E) \Rightarrow \pi_{s+t}(|E|)$ with $E^2_{st} = \pi_s \pi_t(E_\bullet)$. For $E_\bullet = \underline{K}(A_\bullet)$ this yields the Gersten-Anderson spectral sequence $E^1_{st} = N^s K_t(A) \Rightarrow KV_{s+t}(A)$, defined for $s > 0$, $t > 1$.

If we take $D = \Sigma^\infty P^0(\underset{\sim}{Z}/p)$, we obtain a mod p analogue: $E^1_{st} = \pi_t(N^s E; \underset{\sim}{Z}/p) \Rightarrow \pi_{s+t}(|E|; \underset{\sim}{Z}/p)$. For $E_\bullet = \underline{K}(A_\bullet)$ this yields a first quadrant spectral sequence (defined for $s > 0$, $t > 1$):

$$E^1_{st} = \left\{ \begin{matrix} N^s K_t(A; \underset{\sim}{Z}/p), \ t > 1 \\ \\ N^s K_1(A) \times \underset{\sim}{Z}/p, \ t = 1 \end{matrix} \right\} \Longrightarrow KV_{s+t}(A; \underset{\sim}{Z}/p).$$

We will now construct relative versions of the above spectral sequences. Whe I is an ideal of A we can form the simplicial spectrum $\underline{K}(A_\bullet,I_\bullet)$. By [We 1, (2.6) the homotopy groups of the total spectrum $|\underline{K}(A_\bullet,IA_\bullet)|$ are independent of the choi of the ambient ring A, and for $m > 1$ we have $KV_m(I) = \pi_m(|\underline{K}(A_\bullet,IA_\bullet)|)$. This being said, we define

$$KV_m(I;\underset{\sim}{Z}/p) = \pi_m(|\underline{K}(A_\bullet,IA_\bullet)|;\underset{\sim}{Z}/p)$$

for $m > 1$. Note that $KV_1(I;\underset{\sim}{Z}/p) = KV_1(I) \otimes \underset{\sim}{Z}/p$ by the Universal Coefficient Theorem, since $\pi_0(|\underline{K}(A_\bullet,IA_\bullet)|) = 0$.

Applying Theorem (3.2) to $D = \Sigma^\infty S^0$ and $E_\bullet = \underline{K}(A_\bullet,IA.)$ gives the spectral sequence $E^1_{st} = N^sK_t(A,I) \Rightarrow KV_{s+t}(I)$ of [We 1, Theorem 2.6]. Using $D = \Sigma^\infty P^0(\underset{\sim}{Z}/$ instead yields

Corollary 3.4. There is a first quadrant spectral sequence (defined for $s > 0$, $t > 1$):

$$E^1_{st} = \left\{\begin{matrix} N^sK_t(A,I;\underset{\sim}{Z}/p), & t > 1 \\ N^sK_1(A,I) \otimes \underset{\sim}{Z}/p, & t = 1 \end{matrix}\right\} \Rightarrow KV_{s+t}(I;\underset{\sim}{Z}/p).$$

Application 3.5. Consider the simplicial spectrum $\underline{K}(A_\bullet,B_\bullet,I_\bullet)$ associated with excision. Since $\pi_0\underline{BQ}\,\underline{P}(A,B,I) = 0$ is known, each sequence $\underline{K}(A_t,B_t,I_t) \longrightarrow \underline{K}(A_t,I_t) \longrightarrow \underline{K}(B_t,I_t)$ is a fibration sequence of connected spectra. It follows that $|\underline{K}(A_\bullet,B_\bullet,I_\bullet)| \longrightarrow |\underline{K}(A_\bullet,I_\bullet)| \longrightarrow |\underline{K}(B_\bullet,I_\bullet)|$ is a fibration. Since the latter map is a homotopy equivalence by [We 1,(2.6)], $|\underline{K}(A_\bullet,B_\bullet,I_\bullet)|$ is contractible. By Theorem 3.2, there are spectral sequences

$$E^1_{st} = N^sK_t(A,B,I) \Rightarrow 0,$$

$$E^1_{st} = N^sK_t(A,B,I;A/p) \Rightarrow 0,$$

defined for $s > 0$, $t > 1$.

Remark (M. Karoubi). It would be interesting to have an axiomatic description of $KV_*(\,;\underset{\sim}{Z}/p)$ similar to the axioms in [K-V] for the theory KV_*. It is not clear what the definitions should be for $KV_t(\,;\underset{\sim}{Z}/p)$, $t = 0,1$. For example, if $A \to A/I$ is a "Gl-fibration" in the sense of [K-V], then there is a fibration

$$\pi \times \underline{K}(A_\bullet, IA_\bullet) \longrightarrow \underline{K}(A_\bullet) \longrightarrow \underline{K}(A/I_\bullet),$$

where π is a constant simplicial abelian group. The long exact sequence for mod p homotopy yields a long exact ideal sequence ending in

$$\ldots KV_1(A;\underset{\sim}{Z}/p) \longrightarrow KV_1(A/I;\underset{\sim}{Z}/p) \longrightarrow \pi \otimes \underset{\sim}{Z}/p \longrightarrow 0.$$

In general, π is a subgroup of $K_0(I)$ and $\pi \otimes \underset{\sim}{Z}/p$ need not inject into $K_0(I) \otimes \underset{\sim}{Z}/p$.

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