MEYER VIETORIS SEQUENCES AND MODULE STRUCTURES ON NK_*

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The purpose of this essay is to point out a number of consequences of some module structures on the nilgroups NK_*, particularly in characteristic p. For example, if we ignore p-torsion and only consider rings in which p is nilpotent, then Quillen K-theory satisfies excision and has Mayer-Vietoris exact sequences. We can deduce similar, but weaker results in characteristic 0 .

In §1 we introduce the notation we need about Witt vectors. In §2 we remind the reader of the $End_0(R)$-module structure on $NK_*(\Lambda)$. It is Stienstra's observation that this extends to a W(R)-module structure, and we explore some elementary consequences of this in §3. In §4 we compute this module structure on $NK_2(k[\varepsilon],\varepsilon)$ and show that it is <u>not</u> the "usual" k-module structure. In §5 we prove the above mentioned results about $K_*(\Lambda) \otimes \mathbb{Z}[\frac{1}{p}]$ using known facts about Karoubi-Villamayor theory. Finally, in §6 we discuss the problem of localization of the ring, and show for example that for a finite group π the groups $NK_*(\mathbb{Z}\pi)$ are torsion groups, the only torsion being at primes dividing the order of π.

I would like to thank J. Stienstra for showing me that I was talking about Witt vectors, and for explaining his point of view to me. I would also like to thank R. G. Swan for motivating this essay with his work on localization in [Sw].

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§1. Witt Vectors

In this section, R denotes any commutative ring with 1. By W(R) we will mean the ring of all (big) Witt vectors. That is, the underlying additive group of W(R) is the multiplicative group $1 + tR[[t]]$ of power series. The multiplication on W(R) is the unique continuous functorial operation $*$ for which $(1-at) * (1-bt) = (1-abt)$; for example, if $d = \text{g.c.d.}(m,n)$ then

$$(1.1)\quad (1-at^m) * (1-bt^n) = (1-a^{n/d}b^{m/d}t^{mn/d})^d .$$

A very quick and readable introduction to this point of view is [Bl 1,§I.1]; other points of view are discussed in [Cl], [Gr], and [SGA6]. Note that the "zero" and "one" of the ring W(R) are 1 and 1-t in our convention, which differs from that of [Bl 1] by a minus sign but agrees with [Stl].

A quick way to check multiplicative formulas in W(R) is to use the ghost map $gh: W(R) \to \prod_1^\infty R$. It is obtained from the abelian group homomorphism

$$-t\frac{d}{dt}(\log) : (1 + tR[[t]])^\times \to (tR[[t]])^+$$

$$\alpha(t) \mapsto \frac{-t}{\alpha(t)}\ \frac{d\alpha}{dt}$$

by identifying the left side with W(R) and the right side with $\prod R$ (via $\Sigma a_n t^n \leftrightarrow (a_1, a_2, \ldots)$). For example, $gh(1-t)^n = (n,n,\ldots)$.

The map gh is a ring homomorphism (for the product structure on ΠR), and is an injection if R has no $\mathbb{Z}$-torsion (q.v. [Bl 1,p.195]). If $\mathbb{Q} \subsetneq R$, gh is a ring isomorphism, so that $W(R) \cong \Pi R$.

Now suppose R is one of the following rings: $S^{-1}\mathbb{Z}$ for some set S of primes, $\hat{\mathbb{Z}}_p$, or a (commutative) Q-algebra. Then for $r \in R$ the coefficients of the power series

$$\lambda_t(r) = (1-t)^r = 1-rt + \binom{r}{2} t^2 - \binom{r}{3} t^3 + \dots$$

all belong to R . This is obvious for $\mathbb{Q}$-algebras, and may be proven for $\hat{\mathbb{Z}}_p$ by an easy convergence argument. Here is a proof for $S^{-1}\mathbb{Z}$ that I learned from R. G. Swan: let $r=k/s$, and suppose by induction that $\binom{r}{n} \in R$ for $n < N$. The coefficients of t^N in the equation

$$[(1-t)^r]^s = [\sum_{i=0}^{\infty} \binom{r}{i} (-t)^i]^s = \sum_{i=0}^{\infty} \binom{k}{i} (-t)^i = (1-t)^k$$

are all in R , with the possible exception of $s\binom{r}{N}$. Hence $\binom{r}{N} \in R$ as well.

<u>Proposition 1.2</u> If R is $S^{-1}\mathbb{Z}$, $\hat{\mathbb{Z}}_p$, or a Q-algebra, the map $\lambda_t : r \mapsto (1-t)^r$ defines a ring injection from R to W(R).

Proof. Composition with the ghost map, which is a ring injection, gives the set map $gh(\lambda_t(r)) = (r,r,\ldots)$. Since $gh(\lambda_t)$ is a ring injection, λ_t must also be a ring injection.

Remark 1.3 We can formulate Proposition 1.2 in the language of λ-rings (see [SGA6]). A binomial ring is a commutative ring R with no $\mathbb{Z}$-torsion, such that $\binom{r}{n} \in R$ for every $r \in R$, $n \in \mathbb{N}$. The above discussion shows that $S^{-1}\mathbb{Z}$, $\hat{\mathbb{Z}}_p$, and $\mathbb{Q}$-algebras are binomial rings. A λ-ring is a commutative ring R with a given ring homomorphism $\lambda_t : R \to W(R)$. Proposition 1.2 shows that every binomial ring is a λ-ring in such a way that λ_t is an injection, and in this guise is proven on p.322 of [SGA6].

(1.4) In order to perform brute force computations, it is useful to introduce the following three abelian group endomorphisms of $W(R)$: the homothety $[r] : \alpha(t) \mapsto \alpha(rt)$, the Verschiebung $V_m : \alpha(t) \mapsto \alpha(t^m)$, and the Frobenius transfer

$$F_m : \alpha(t) \mapsto \sum_{\zeta^m=1} \alpha\,(\zeta t^{1/m}) \ .$$

F_m is a ring endomorphism, and $F_m(\lambda_t(r)) = \lambda_t(r)$ for the ring map λ_t of Proposition 1.2. Since every Witt vector can be written uniquely as a product $\alpha(t) = \Pi(1-r_m t^m)$, we can think of multiplication by $\alpha(t)$ as the endomorphism $\Sigma\ V_m[r_m]\ F_m$. The theoretical foundations of this viewpoint are developed in [C1] and [C2]. This viewpoint is applied to K-theory computations in [Bl 1], [LR], and [St1].

(1.5) The subgroups $I_N = (1+t^N R[[t]])$ of $W(R)$ are actually ideals, as is clear from the formula (1.1). We will call the resulting topology on $W(R)$ the <u>t-adic topology</u>. $W(R)$ is separated and complete in this topology, and the quotient rings $W_N(R) = W(R)/I_{N+1}$ are the rings of <u>truncated Witt vectors.</u> In particular, $W_1(R) \cong R$, so the ring maps λ_t of Proposition 1.2 are split injections.

When $p^m = 0$ in R and N is fixed, $(1-t)^{p^n} \in I_N$ for large n . Thus "p" is nilpotent in each $W_N(R)$, but "$p^n \neq 0$" in $W(R)$. From this, we see that the composite $\hat{\mathbb{Z}}_p \to W(\hat{\mathbb{Z}}_p) \to W(R)$ is a ring injection, which is continuous with respect to the p-adic and t-adic topologies.

§2. $End_0(R)$

In this section, we recall some facts about the ring $End_0(R)$. A very readable survey of the relation of $End_0(R)$ to $W(R)$ may be found in [Gr].

We can define $End_0(\Lambda)$ for any ring Λ with 1. Let $\underline{\underline{End}}(\Lambda)$ denote the exact category of endomorphisms of finitely generated projective right Λ-modules. This is the category denoted $\underline{\underline{P}}(\Lambda)^{\mathbb{N}}$ on p.5 of [Ba]: objects are pairs (M,f) with $f \in \text{End}(M)$, and morphisms $(M_1,f_1) \to (M_2,f_2)$ are maps $\alpha: M_1 \to M_2$ with $f_2\alpha = \alpha f_1$.

There are two interesting subcategories of $\underline{\underline{\text{End}}}(\Lambda)$. One is the full exact subcategory $\underline{\underline{\text{Nil}}}(\Lambda)$ of nilpotent endomorphisms, and the other is the reflective subcategory of zero endomorphims, which is naturally equivalent to $\underline{\underline{P}}(\Lambda)$, the category of f.g. Λ-projectives. We define $\text{End}_n(\Lambda)$ and $\text{Nil}_n(\Lambda)$ by the splittings

$$K_n \underline{\underline{\text{End}}}(\Lambda) = K_n(\Lambda) \oplus \text{End}_n(\Lambda)$$
$$K_n \underline{\underline{\text{Nil}}}(\Lambda) = K_n(\Lambda) \oplus \text{Nil}_n(\Lambda) \quad .$$

Now suppose that Λ is an R-algebra for some commutative ring R. Then there are exact pairings

$$\otimes : \underline{\underline{\text{End}}}(R) \times \underline{\underline{\text{End}}}(\Lambda) \to \underline{\underline{\text{End}}}(\Lambda)$$
$$\otimes : \underline{\underline{\text{End}}}(R) \times \underline{\underline{\text{Nil}}}(\Lambda) \to \underline{\underline{\text{Nil}}}(\Lambda)$$
$$(M,f) \otimes (N,g) = (M \otimes_R N, f \otimes g) \ .$$

These induce maps $K_0 \underline{\underline{\text{End}}}(R) \otimes K_* \underline{\underline{\text{End}}}(\Lambda) \to K_* \underline{\underline{\text{End}}}(\Lambda)$, $K_0 \underline{\underline{\text{End}}}(R) \otimes K_* \underline{\underline{\text{Nil}}}(\Lambda) \to K_* \underline{\underline{\text{Nil}}}(\Lambda)$ by the usual generators-and-relations tricks on K_0 . It is easy to see that $(0,0)$ and $(R,1) \in K_0(\underline{\underline{\text{End}}}(R)$ act as the zero and identity maps. If we take $R = \Lambda$, we see that K_0 End (R) is a commutative ring with 1. $K_0(R)$ is an ideal, generated by the idempotent (R,0), and the quotient ring is $End_0(R)$. Since $(R,0)\otimes$ reflects End (Λ) into P (Λ), $K_0(R)$ acts as zero on End_* (Λ) and Nil_* (Λ). The following is immediate (and well-known):

Proposition 2.1 If Λ is an R-algebra with 1, End_* (Λ) and Nil_* (Λ) are graded modules over the ring End_0 (R) .

Remark 2.2 Of course, we can use the construction in [Wa, §9] to see that End_* (Λ) and Nil_* (Λ) are graded modules over the graded ring End_* (R) . We will not use this, except to make the following observation: there is another embedding of P(R) into End(R), namely as the subcategory of identity endomorphisms. It is not hard to see that this induces a ring homomorphism $K_*(R) \to End_*(R)$ preserving "one". The resulting $K_*(R)$-module structure on Nil_* (Λ) agrees with the "usual" one, obtained by identifying Nil_* (Λ) with a $K_*(R)$-submodule of K_{*+1} ($\Lambda[y]$). We will return to this point in §3.

There is a well-defined map $\chi: End_0(R) \to W(R)$ given by taking characteristic polynomials : $\chi(M,f) = \det(1-tf)$. Note that $\chi(R,0)=1$ and $\chi(R,1)=1-t$. It is easy to see that χ is a ring homomorphism, and that the image of χ is the set of all rational functions in W(R), i.e., quotients of polynomials in $1+tR[t]$. The induced t-adic topology on $End_0(R)$ is defined by the ideals $I_N = \{f \in \text{End}_0(R) \mid \chi(f) \equiv 1 \pmod{t^N}\}$, and $\text{End}_0(R)$ is separated in this topology. The key fact is:

<u>Theorem 2.3</u> (Almkvist [A]). The map $\chi : \text{End}_0(R) \to W(R)$ is a ring <u>injection</u>, and $W(R)$ is the t-adic completion of $\text{End}_0(R)$.

The operations on $\text{End}_0(R)$ inducing the homothety, Verschiebung, and Frobenius of (1.4) are discussed in [Gr] . Grayson also points out in [Gr] that $\text{End}_0(R)$ is a λ-ring via $\lambda_t(M,f) = \Sigma\,(\Lambda^n M, \Lambda^n f)t^n$, and that χ is a λ-ring homomorphism. The λ-ring structure on $W(R)$ is given in [SGA6,p.319].

<u>Exercise 2.4</u> Show that $\chi(f) = 1 + a_1 t + \ldots + a_m t^m$ for the endomorphism

$$f = \begin{bmatrix} 0 & & & & -a_m \\ 1 & 0 & & 0 & \vdots \\ & 1 & \ddots & & \vdots \\ & 0 & \ddots & 0 & -a_2 \\ & & & 1 & -a_1 \end{bmatrix}$$

of R^m . Then show that if $\nu^N = 0$ then $f \otimes \nu$ represents 0 in $\text{Nil}_0(\Lambda)$ whenever $\chi(f) \equiv 1$ modulo t^N .

§3. <u>The W(R)-modules $NK_*(\Lambda)$.</u>

We keep the notation that R is a commutative ring with 1 and that Λ is an R-algebra with 1. We take $NK_n(\Lambda)$ to be the kernel of "y=0" : $K_n(\Lambda[y]) \to K_n(\Lambda)$. $NK_n(\Lambda)$ is isomorphic to $Nil_{n-1}(\Lambda)$ via the composite

$$NK_n(\Lambda) \subset K_n(\Lambda[y]) \subset K_n(\Lambda[x,y]/(xy=1)) \xrightarrow{\partial} K_{n-1}\ \underline{\underline{Nil}}\ (\Lambda)$$

(this is proven on p.237 of [GQ]). Thus the groups $NK_n(\Lambda)$ are $End_0(R)$ modules. For $n \geq 1$, this is just Proposition 2.1; for $n = 0$ (and $n < 0$) this follows from the functoriality of the module structure and the fact that $NK_0(\Lambda)$ is the "contracted functor" of $NK_1(\Lambda)$, q.v. [Ba, XII §7].

<u>Theorem 3.1</u> (Stienstra [St2]). For every $\gamma \in Nil_*(\Lambda)$ there is an N so that γ is annihilated by the ideal $I_N = \{f|\chi(f) \equiv 1 \bmod t^N\}$ of $End_0(R)$. Consequently, $NK_*(\Lambda)$ is a module over the t-adic completion $W(R)$ of $End_0(R)$.

The second sentence follows from the first by Theorem 2.3. The first sentence may be proven in the spirit of Exercise 2.4, but we will refer the reader to [St2] for a careful proof.

<u>Exercise 3.2</u> Use the sign convention that $[1-\nu y] \in NK_1(\Lambda)$ corresponds to $(N,\nu) - (N,0) \in Nil_0(\Lambda)$ and Exercise 2.4 to show that the $W(R)$-module structure on $NK_1(\Lambda)$ is completely determined by the formula $\alpha(t) * [1-\nu y] = [\alpha(\nu y).]$

Show that the ring map $K_*(R) \to End_*(R)$ induces the "usual" $K_*(R)$-module structure on $NK_*(\Lambda)$, i.e., that coming from the $K_*(R)$-module structure on $K_*(\Lambda[y])$. Use this to show that for $\gamma \in K_{n-1}(R)$, $(N,\nu) \in \underline{\underline{Nil}}(\Lambda)$, and $\alpha(t) \in W(R)$ we have the formula

$$\alpha(t) * \{\gamma, 1-\nu y\} = \{\gamma, \alpha(\nu y)\} \in NK_n(\Lambda).$$

This formula was first proven by Bloch on p.238 of [Bl 1], and is especially useful in determining the W(R)-module structure on $NK_2(\Lambda)$. (See Example (4.4).)

<u>Corollary 3.3</u> Fix an integer p and a ring Λ with 1.

(a) If Λ is an $S^{-1}\mathbb{Z}$- algebra, $NK_*(\Lambda)$ is an $S^{-1}\mathbb{Z}$-module.

(b) If Λ is a $\mathbb{Q}$-algebra, $NK_*(\Lambda)$ is a center(Λ) -module.

(c) If Λ is a $\hat{\mathbb{Z}}_p$-algebra, $NK_*(\Lambda)$ is a $\hat{\mathbb{Z}}_p$-module.

(d) If $p^m=0$ in Λ, $NK_*(\Lambda)$ is a p-group.

<u>Proof.</u> The first three parts follow from (1.2) and (3.1). In case (d), note by (1.5) and (3.1) that every element of $NK_*(\Lambda)$ is annihilated by some $p^n \in \hat{\mathbb{Z}}_p$, i.e., that $NK_*(\Lambda)$ is a p-group.

<u>Historical Remark (3.4)</u> The observation that NK_0, NK_1 are p-groups for $\mathbb{Z}/p^m\mathbb{Z}$-algebras is due to Chase, and may be found

on p.646 of [Ba]. Chase asked in [Ge, Problem 18] if the same were true for all NK_*. The affirmative answer of (3.3) is implicit in [Bl 1] (as well as [Bl 2], [vdK, p.310], and [St1]). For $\mathbb{Q}$-algebras, it was remarked on pp.13,51 of [Ba2] that NK_0, NK_1 are divisible groups. For $S^{-1}\mathbb{Z}$-algebras, Swan proved in [Sw] that NU, NPic are $S^{-1}\mathbb{Z}$-modules, and observed that the same was true for NK_0, NK_1. The $End_0(R)$ approach is due to Stienstra, mentioned on p.68 of [St1], and will appear in [St2].

Corollary 3.5 If I is an ideal in Λ, the relative groups $NK_*(\Lambda,I)$ are $W(R)$-modules, and there is an exact sequence of $W(R)$-modules

$$(*)\quad NK_{*+1}(\Lambda) \to NK_{*+1}(\Lambda/I) \to NK_*(\Lambda,I) \to NK_*(\Lambda) \to NK_*(\Lambda/I).$$

In particular, if Λ is an R-algebra for R one of $S^{-1}\mathbb{Z}$, $\hat{\mathbb{Z}}_p$, $\mathbb{Q}$-algebra, then $NK_*(\Lambda,I)$ is an R-module. If $1 \in \Lambda$ and $p^m = 0$, then $NK_*(\Lambda,I)$ is a p-group.

Proof. The commutative diagram

$$\begin{array}{ccc} \underline{\underline{Nil}}(\Lambda) \times \underline{\underline{End}}(R) & \longrightarrow & \underline{\underline{Nil}}(\Lambda) \\ \downarrow & & \downarrow \\ \underline{\underline{Nil}}(\Lambda/I) \times \underline{\underline{End}}(R) & \longrightarrow & \underline{\underline{Nil}}(\Lambda/I) \end{array}$$

is a special case of diagram (5.5) of [We3], with $A = C = \underline{\underline{Nil}}(\Lambda)$, $B = \underline{\underline{End}}(R)$, etc. The discussion following (5.5) - especially the

the penultimate paragraph of Section 5 - applies here to prove that (after discarding the summand $K_*(R)$ of $K_*\underline{\underline{End}}(R)$) the sequence (*) is an exact sequence of $End_0(R)$-modules. By Stienstra's theorem (3.1), (*) is actually a sequence of $W(R)$-modules. The rest of the Corollary follows as in (3.3) above.

Corollary 3.6 (Murthy-Pedrini [MP]) Let Λ be an algebra over a field $k \supseteq \mathbb{Q}$. Then $NK_n(\Lambda)$ is either zero or a torsionfree divisible group of rank at least $[k:\mathbb{Q}]$. In particular, if k is uncountable, then $NK_n(\Lambda)$ is either zero or of uncountable rank. The same is true of $NK_n(\Lambda,I)$.

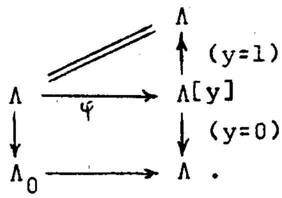
Proof Each group $NK_n(\Lambda)$, $NK_n(\Lambda,I)$ is a k-vector space. We remark that the rank of these groups is **always** infinite - see Proposition 4.1 below.

Proposition 3.7 Let $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \ldots$ be a graded ring, and write $K_*(\Lambda) = K_*(\Lambda_0) \oplus \tilde{K}_*(\Lambda)$. Then:

(a) if $\mathbb{Q} \subset \Lambda$ (resp. Λ is an $S^{-1}\mathbb{Z}$-module), $\tilde{K}_*(\Lambda)$ is a $\mathbb{Q}$-vector space (resp. an $S^{-1}\mathbb{Z}$-module)

(b) if Λ is a $\mathbb{Z}/p^m\mathbb{Z}$-algebra, $\tilde{K}_*(\Lambda)$ is a p-group.

<u>Proof</u> There is a ring homomorphism $\varphi : \Lambda \to \Lambda[y]$ defined on Λ_n by $\varphi(\lambda) = \lambda y^n$. Apply K_* to the following commutative diagram:



We see that $K_*(\varphi)$ maps $\tilde{K}_*(\Lambda)$ into $NK_*(\Lambda)$ as a summand. Now apply (3.3).

§4. <u>Other methods</u>

There are two other ways to obtain a W(R)-module structure on $NK_*(\Lambda)$. For completeness, we mention them here. Proof that they agree with the $End_0(R)$ pairing will appear in [St2].

The first is to use the Dieudonné ring D = End(W). Cartier proved in [C1] that every endomorphism of the functor W :(R-algebras)→ (abelian groups) can be written uniquely as $\Sigma\, V_m\, [r_{mn}]\, F_n$, where the operations [r], V_m, F_n are those of (1.5). There are three pertinent types of endomorphism of $NK_*(\Lambda)$: The homothety $[r] = \rho_r^*$, the Verschiebung $V_m = (\iota_m)^*$, and the Frobenius transfer $(\iota_m)_*$. These are induced by the endomorphisms $\rho_r(y) = ry$ and $\iota_m(y) = y^m$ of $\Lambda[y]$. It is proven on p.317 of [Bl 2] that F_m corresponds to the endofunctor

$\theta(N,\nu) = (N,\nu^m)$ of $\underline{\underline{\text{Nil}}}$ (Λ), so for every γ in $NK_*(\Lambda)$ we have $F_m(\gamma) = 0$ for large m. Thus the expression $\Sigma V_m[r_{mn}]F_n(\gamma)$ makes sense. Cartier has given in [C2] the necessary relations for this to define a D-module structure on NK_*, hence a W(R)-module structure using the map $W(R) \to D$ of (1.5). This brute force approach is the one taken in [Bl 1], [LR], and [Stl]. A nice application of this method is the following, which van der Kallen attributes to Farrell:

Proposition 4.1([vdK],p.310) For every i the group $NK_i(\Lambda)$ is either zero or is not finitely generated.

Proof If not, then there is an M such that $F_m \equiv 0$ on $NK_i(\Lambda)$ for all $m > M$. Pick $\alpha \neq 0$ in $NK_i(\Lambda)$ and choose an integer $m > M$ with $m\alpha \neq 0$, and note that $F_m(V_m\alpha) = m\alpha$, a contradiction.

The second method is given on p.315 of [Bl 2]. Bloch considers the biexact functor

$$\underline{\underline{P}}\,(R[[t]]) \times \underline{\underline{\text{Nil}}}\,(\Lambda) \to \underline{\underline{P}}\,(\Lambda),$$
$$M \otimes (N,\nu) = M \underset{R[[t]]}{\otimes} N$$

where N is considered to be an R-module with t acting via ν. Waldhausen's machinery (in §9.2 of [Wa]) produces a map

$W(R) \otimes NK_*(\Lambda) \subseteq K_1(R[[t]]) \otimes K_{*-1}$ $\underline{\underline{Nil}}$ $(\Lambda) \to K_*(\Lambda)$.

Bloch then injects $NK_*(\Lambda)$ as a summand in $NK_*(\Lambda[x])$ via $y \mapsto xy$ and obtains a pairing $W(R) \otimes NK_*(\Lambda) \to NK_*(\Lambda)$. Work is then needed to show that this defines a module structure. Note that the original map $W(R) \otimes NK_*(\Lambda) \to K_*(\Lambda)$ is the composite of the module map and the projection "y=1" : $NK_*(\Lambda) \to K_*(\Lambda)$. Bloch also shows in [Bl 1,p.224] that

(4.2) The relative groups $K_*(R[\varepsilon]/(\varepsilon^{m+1}), \varepsilon) = C_m K_*(R)$ have the structure of W(R)-modules.

We warn the reader that there are two <u>different</u> W(R)-module structures on $NK_*(R[\varepsilon]/(\varepsilon^{m+1}),\varepsilon)$: one from the $End_0(R)$ pairing, and one induced from the pairing (4.2) on $K_*(R[\varepsilon,y],\varepsilon)$. We will not consider Bloch's pairing (4.2), except to give the following two examples of the difference in module structure:

<u>Example 4.3</u> Let k be a Q-algebra, and set $R = k[\varepsilon]/(\varepsilon^{n+1}=0)$. If we give $K_1(R,\varepsilon)$ the R-module structure induced from the W(R)-module structure of (4.2), then there is an R-module isomorphism $\varepsilon R \cong K_1(R,\varepsilon)$, $f \mapsto \exp(f)$. As a sample of the W(R)-module structure, we note that $(1-rt) * \exp(f(\varepsilon)) = \exp(f(r\varepsilon))$. The R-module structure on $NK_1(R,\varepsilon)$ induced from the $End_0(R)$ pairing agrees with this : $\lambda_t(r) * \exp(f) = \exp(rf)$ for $f = f(\varepsilon,y)$ in $\varepsilon yR[y]$. The W(R)-structure on $NK_1(R,\varepsilon)$ gives $(1-rt) * \exp(f(\varepsilon,y)) = \exp(f(\varepsilon,ry))$ however, which is different. The map

"$y = 1$": $NK_1(R,\varepsilon) \to K_1(R,\varepsilon)$ is an R-module map but not a W(R)-module map.

Example (4.4) We describe the R-module structure on $NK_2(R,\varepsilon)$ for $R = k[\varepsilon]$, $\varepsilon^2 = 0$, k a Q-algebra. It was a surprise to me that we cannot replace "Q-algebra" by "field of characteristic $\neq 2,3$", since the formulas require Ω_k to be a Q-module. Recall that there is a well-known group isomorphism $\Omega_k \cong K_2(R,\varepsilon)$ given by $a\,db \mapsto \langle a\varepsilon, b\rangle$. (We are using Stienstra's $\langle , \rangle$ notation in order to make use of the computations in [St1].) Using Bloch's pairing in (4.2), $K_2(R,\varepsilon)$ is a $k = W_1(k)$-module and $\Omega_k \cong K_2(R,\varepsilon)$ is a k-module isomorphism: this observation is stated on p.62 of [St1].

If we use the k[y]-module structure on $NK_2(R,\varepsilon)$ coming from the (4.2) pairing on $K_2(R[y],\varepsilon)$, we obtain the k[y]-module isomorphism $(y\Omega_k[y]) \oplus k[y] \cong NK_2(R,\varepsilon)$ which (for b in k and f_i in k[y]) associates $yf_1\,db \oplus f_2$ and $\langle \varepsilon y f_1, b\rangle\langle \varepsilon f_2, y\rangle$. On the other hand, we will show that under the k-module structure induced from the $End_0(R)$-pairing we have the formulas:

(4.4.1) $\lambda_t(r) * \langle \varepsilon y f_1, b\rangle = \langle r\varepsilon y f_1, b\rangle$ (b in k,

(4.4.2) $\lambda_t(r) * \langle \varepsilon, y f_2\rangle = \langle \varepsilon, r y f_2\rangle$. $f_i(y)$ in k[y])

Thus (for example) the two actions of r on $\langle \varepsilon, y^i\rangle$ are different if

$dr \neq 0$, so that "y=1" : $NK_2(R,\varepsilon) \to K_2(R,\varepsilon)$ is not a k-module homomorphism. Using these formulas, it is easy to see that there is a k-module isomorphism

$$(y\Omega_k[y]) \oplus k[y] \cong NK_2(R,\varepsilon)$$
$$yf_1 db \oplus f_2 \mapsto \langle\varepsilon y f_1, b\rangle \langle\varepsilon, y f_2\rangle \ .$$

We now derive the formulas (4.4.1)and (4.4.2). By functoriality of the W(R)-module structure, we can assume that b is a unit of k. Formula (4.4.1) then follows from Exercise 3.2 and Example 4.3, given the identification $\langle f\varepsilon, b\rangle = \{1-bf\varepsilon, b\}$. To establish (4.4.2), we first consider the case $f_2(y) = b$. Using the formulas on p.62 of [St1], which are also valid for the module structure on $NK_2(R)$, we find that

$$(1-rt^m) * \langle\varepsilon, by\rangle = \begin{cases} \langle\varepsilon, rby\rangle, & m = 1 \\ 0, & m \neq 1 \ . \end{cases}$$

The formula (4.4.2) for $\langle\varepsilon, by\rangle$ is immediate. For $\langle\varepsilon, by^i\rangle$ we use the following formula of Bloch, found on p.316 of [Bl2] (note the missing V_m in [Bl2]):

(4.4.3) $\quad V_m[F_m(\omega) * \gamma] = \omega * V_m(\gamma) \quad (\omega \text{ in } W(R), \gamma \text{ in } NK_*(\Lambda)).$

We can now compute:

$$\lambda_t(r)*\langle\varepsilon,by^m\rangle = \lambda_t(r)*V_m\langle\varepsilon,by\rangle = V_m[F_m\lambda_t(r)*\langle\varepsilon,by\rangle]$$
$$= V_m[\lambda_t(r)*\langle\varepsilon,by\rangle] = V_m\langle\varepsilon,rby\rangle = \langle\varepsilon,rby^m\rangle .$$

The formula (4.4.2) now follows from the fact that $\langle\varepsilon,yf_2\rangle$ is additive in f_2 .

<u>Exercise 4.5</u> Show that for $R = k[\varepsilon]/(\varepsilon^{n+1})$, k a **Q-algebra**, there is an isomorphism of R-modules

$$(\Omega_k \otimes \varepsilon yR[y]) \oplus \varepsilon yR[y] \cong NK_2(R,\varepsilon)$$

$$\varepsilon yf_1 db \oplus \varepsilon yf_2 \mapsto \langle\varepsilon yf_1,b\rangle \langle\varepsilon, \frac{\exp(\varepsilon yf_2)-1}{\varepsilon}\rangle .$$

§5. <u>Main Results</u>

In order to pass from the NK_*-groups to K_*-groups, it is necessary to consider the Karoubi-Villamayor groups, whose key property is that $KV_*(\Lambda) = KV_*(\Lambda[y])$. Recall that the groups $N^pF(\Lambda)$ are defined by iteration from $N^0F = F$ using the formula $NF(\Lambda)$ = kernel of $F(\Lambda[y]) \to F(\Lambda)$. If $F = K_*$ and Λ is an R-algebra, we see that the $N^pK_*(\Lambda)$ are W(R)-modules for $p \neq 0$ by functoriality of the W(R)-module structure on NK_* . The result we need is this:

Theorem 5.1 If Λ is a ring with unit, there is a first quadrant spectral sequence (defined for $p \geq 0$, $q \geq 1$)

$$E^1_{pq} = N^pK_q(\Lambda) \Rightarrow KV_{p+q}(\Lambda) .$$

If I is an ideal in Λ, there is a spectral sequence

$$E^1_{pq} = N^pK_q(\Lambda,I) \Rightarrow KV_{p+q}(I) .$$

Moreover, there is a long exact sequence

$$\cdots \to KV_{i+1}(\Lambda/I) \to H_i(C_*) \oplus KV_i(I) \to KV_i(\Lambda) \to \cdots$$
$$\cdots \to KV_1(\Lambda/I) \to K_0(I)/\mathrm{im}NK_1(\Lambda/I) \to K_0(\Lambda) \to K_0(\Lambda/I) .$$

Here C_* is a chain complex with C_p the cokernel of $N^pK_1(\Lambda) \to N^pK_1(\Lambda/I)$.

Proof See [We2], theorems 2.5, 2.6, and 3.2. The spectral sequence was originally discovered by Gersten and Anderson.

Since p-groups form a Serre subcategory of all abelian groups (and since $K_o(\Lambda) = KV_o(\Lambda)$ by definition), we immediately deduce the following result:

Theorem 5.2 Let Λ be a $\mathbb{Z}/p^m\mathbb{Z}$-algebra with unit, and let I be an ideal in Λ. Then $K_*(\Lambda) \otimes \mathbb{Z}[\frac{1}{p}] \stackrel{\cong}{\to} KV_*(\Lambda) \otimes \mathbb{Z}[\frac{1}{p}]$ and $K_*(\Lambda,I) \otimes \mathbb{Z}[\frac{1}{p}] \stackrel{\cong}{\to} KV_*(I) \otimes \mathbb{Z}[\frac{1}{p}]$.

Corollary 5.3 ("Excision") If $f = \Lambda_1 \to \Lambda_2$ is a map of $\mathbb{Z}/p^m\mathbb{Z}$-algebras with unit, and I is an ideal of Λ_1 with $I \cong f(I)$, then

$$K_*(\Lambda_1, I) \otimes Z[\tfrac{1}{p}] \overset{\cong}{\to} K_*(\Lambda_2, I) \otimes Z[\tfrac{1}{p}] .$$

Corollary 5.4 If I is a nilpotent ideal in a $\mathbb{Z}/p^m\mathbb{Z}$-algebra Λ with unit, then $K_*(\Lambda,I)$ is a p-group.

Proof: We have $KV_*(I) = 0$ by [We 1, Theorem 2.3].

Theorem 5.5 ("Mayer-Vietoris"). Let

$$\begin{array}{ccc} \Lambda_1 & \to & \Lambda_2 \\ \downarrow & & \downarrow \\ \Lambda_3 & \to & \Lambda_4 \end{array}$$

be a pullback square of $\mathbb{Z}/p^m\mathbb{Z}$-algebras with $\Lambda_2 \to \Lambda_4$ onto. Then there is a long exact sequence

$$\cdots K_{*+1}(\Lambda_4)\otimes Z[\tfrac{1}{p}] \to K_*(\Lambda_1)\otimes Z[\tfrac{1}{p}] \to [K_*(\Lambda_2)\oplus K_*(\Lambda_3)]\otimes Z[\tfrac{1}{p}] \to K_*(\Lambda_4)\otimes Z[\tfrac{1}{p}] \cdots$$

valid for all integers $*$. The same is true if we replace K_* by KV_* .

Proof: We splice together the long exact ideal sequences for $\Lambda_1 \to \Lambda_3$ and $\Lambda_2 \to \Lambda_4$ in the familar way.

<u>Example 5.6</u> Let k be a finite field of characteristic p. Then $K_*(k[x,y]/(x^3=y^2)) = K_*(k) \oplus$ (p-group) follows either from the fact that the ring is graded and (3.7), or from the conductor pullback square. For the node $\Lambda_1 = k[x,y]/(y^2=x^2+x^3)$ it is well-known that $KV_*(\Lambda_1) = K_*(k) \oplus K_{*+1}(k)$ from the conductor pullback square. Since $K_*(k)$ is a group of order prime to p (for $* \neq 0$), we must have $K_*(\Lambda_1) = K_*(k) \oplus K_{*+1}(k) \oplus$ (p-group).

If we try to use the same technique for our other λ-rings, the best we can do is this:

<u>Theorem 5.7</u> Let Λ be an R-algebra with unit, and let I be an ideal in Λ, where $R = S^{-1}\mathbb{Z}$ or $\mathbb{Q}$. Let η be either $K_*(\Lambda) \to KV_*(\Lambda)$ or $K_*(\Lambda,I) \to KV_*(I)$. Then:

(a) If $R = S^{-1}\mathbb{Z}$, ker (η) is divisible by the primes in S and coker (η) has no S-torsion.

(b) If $R = \mathbb{Q}$, ker (η) is divisible and coker (η) is torsionfree.

<u>Remark</u> From Example (4.4) we know that $d_1: E^1_{12} \to E^1_{02}$ is not a k-module homomorphism for a $\mathbb{Q}$-algebra k, so it is not reasonable to expect a k-module version. In [We4], I will give an example to show that ker (η) and coker (η) need not be $S^{-1}\mathbb{Z}$-modules.

Proof We prove (a); (b) is a special case. In the spectral sequence of (5.1), the E^1_{pq} are R-modules for $p \geq 1$. By induction, we observe that E^n_{pq} is an R-module for $p \geq n$, S-torsionfree for $0 < p < n$, and $K_q(\Lambda)$/(S-divisible group) for $p = 0$. The key observation here is that, if h is any map from an $S^{-1}\mathbb{Z}$-module to an abelian group, im (h) is S-divisible and ker (h) is S-torsionfree. For $n = \infty$ we see that $K_q(\Lambda)$/(S-divisible group) is a subgroup of $KV_q(\Lambda)$, and that the quotient is filtered by S-torsionfree groups, whence the result.

Corollary 5.8 Let I be a nilpotent ideal in a ring Λ. Then:

(a) If Λ is an $S^{-1}\mathbb{Z}$-algebra, $K_*(\Lambda,I)$ is S-divisible.

(b) If $\mathbb{Q} \subset \Lambda$, $K_*(\Lambda,I)$ is divisible.

Proof Again, $KV_*(I) = 0$ by [We 1, Theorem 2.3].

In a subsequent paper [We4], I expect to show that $K_*(,I)$ is an R-module in Corollary 5.8 by the method of K-theory with mod p coefficients.

§6. Localization of R

In this section, we consider the effect of localizing Λ on the resulting map $NK_*(\Lambda) \to NK_*(S^{-1}\Lambda)$.

<u>Lemma 6.1</u> Let R be one of $S^{-1}\mathbb{Z}$, $\hat{\mathbb{Z}}_p$, or a $\mathbb{Q}$-algebra. Then (for each nonnilpotent s in R and integer i) $\binom{1/s}{i}$ is in $R[1/s]$ and $\binom{s^n}{i}$ is in sR for large n.

<u>Proof</u> The first part follows from (1.1). For the second part, choose n large enough so that g.c.d. $(i!, s^{n-1})$ = g.c.d.$(i!s^n)$.

<u>Proposition 6.2</u> Let R be one of $S^{-1}\mathbb{Z}$, $\hat{\mathbb{Z}}_p$, or a $\mathbb{Q}$-algebra. Then for each nonnilpotent s in R and integer N there is an isomorphism

$$W_N(R)\left[\frac{1}{\lambda_t(s)}\right] \approx W_N(R)\left[\frac{1}{1-st}\right] \approx W_N(R\left[\frac{1}{s}\right]) .$$

<u>Proof</u> For a,b in any commutative ring A we have $A[\frac{1}{a}] = A[\frac{1}{b}]$ just in case there are α,β in A and integers m,n so that $a\alpha = b^m$, $b\beta = a^n$. We take $a = \lambda_t(s) = (1-t)^s$, $b = (1-st)$ and choose m,n so that

$$\alpha(t) = (1-s^m t)^{1/s} \quad , \quad \beta(t) = (1-t/s)^{s^n}$$

belong to $W_N(R)$; this is possible by lemma 6.1. We then have $a*\alpha = 1-s^m t = b^{*m}$ and $b*\beta = (1-t)^{s^n} = a^{*n}$, proving the first isomorphism. The second isomorphism is easy to establish from the equation $(1-rs^{-i}t^m)*(1-st) = (1-rs^{m-i}t^m)$.

Remark For the (big) Witt vectors, $W(R)[\lambda_t(s)^{-1}]$ and $W(R)[(1-st)^{-1}]$ are distinct subrings of $W(R[s^{-1}])$, at least for $R = S^{-1}\mathbb{Z}$ or $\hat{\mathbb{Z}}_p$. This is because $\alpha(t)$, $\beta(t) \notin W(R)$ when s is a prime of R .

Theorem 6.3 (Vorst) Let R be a commutative ring with unit, and let Λ be an R-algebra with unit. Then for every multiplicative set $S \subset R$ of nonzerodivisors on Λ there is an isomorphism

$$\{(1-st)\,|\,s\in S\}^{-1}W(R) \underset{W(R)}{\otimes} NK_*(\Lambda) \to NK_*(S^{-1}\Lambda) .$$

Proof This is (1.4) of [V], as interpreted in Remark (1.8) of [V].

Corollary 6.4 Let R be one of $T^{-1}\mathbb{Z}, \hat{\mathbb{Z}}_p$, or a $\mathbb{Q}$-algebra, and let Λ be an R-algebra. Then for every multiplicative set $S \subset R$ of nonzerodivisors on Λ there is an isomorphism of $S^{-1}R$-modules:

$$S^{-1}R \underset{R}{\otimes} NK_*(\Lambda) \to NK_*(S^{-1}\Lambda) .$$

Proof $NK_*(\Lambda)$ is the direct colimit over the family of finitely generated W(R)-submodules M . By Stienstra's Theorem (3.1) we have

$$S^{-1}R \underset{R}{\otimes} M = \{\lambda(s)\}^{-1}W_N(R) \underset{W(R)}{\otimes} M = \{(1-st)\}^{-1}W(R) \underset{W(R)}{\otimes} M .$$

The corollary now follows from (6.3) by taking colimits of both sides.

Consequence (6.5) Let π be a finite group of order n, and let p be a prime dividing n. Then $NK_*(\Lambda)$ is a p-group for $\Lambda = \mathbb{Z}_{(p)}[\pi]$ and $\hat{\mathbb{Z}}_p[\pi]$. $NK_*(\mathbb{Z}[\pi])$ is an n-torsion group, and $\mathbb{Z}_{(p)} \otimes NK_*(\mathbb{Z}[\pi]) = NK_*(\mathbb{Z}_{(p)}\pi)$. Similar statements hold for $NK_*(R[\pi])$, where R is a finite extension of $S^{-1}\mathbb{Z}$ or $\hat{\mathbb{Z}}_p$.

Proof $\mathbb{Z}[\frac{1}{n}] \otimes NK_*(\Lambda) = NK_*(\Lambda[\frac{1}{n}]) = 0$, since $\Lambda[\frac{1}{n}]$ is a regular ring. (See [Ba], pp.648, 695 for previous results.)

Consequence (6.6) Let Λ be a ring with no $\mathbb{Z}$-torsion. Then the rank of the abelian group $NK_*(\Lambda)$ is the dimension of the Q-vector space $NK_*(Q\Lambda)$, and the p-torsion subgroup of $NK_*(\Lambda)$ is the torsion subgroup of $NK_*(\mathbb{Z}_{(p)}\Lambda)$.

In order to apply these results when S contains zerodivisors, we can use the following result of Vorst:

Theorem 6.7 (Vorst [V],(1.7)) Let A be a reduced commutative ring. Then for every multiplicative set $S \subset A$ there is an isomorphism

$$\{(1-st) | s \in S\}^{-1} W(A) \otimes NK_*(A) \to NK_*(S^{-1}A) .$$

Corollary 6.8 Let R be one of $T^{-1}\mathbb{Z}$, $\hat{\mathbb{Z}}_p$, or a $\mathbb{Q}$-algebra, and let A be a reduced commutative R-algebra. Then for every multiplicative set $S \subset R$ there is an $S^{-1}R$-module isomorphism $S^{-1}R \otimes NK_*(A) \to NK_*(S^{-1}A)$.

Consequence (6.9) Let R be the coordinate ring of an affine variety over a field of characteristic zero. Then $NK_*(R)$ is a torsion R-module supported only at the singular locus of the variety.

Consequence (6.10) ([Ba],p.648) Let R be a reduced commutative ring, and n an integer with $R[\frac{1}{n}]$ regular. Then $NK_*(R)$ is an n-torsion abelian group.

Consequence (6.11) Vorst's Theorem 1.9 of [V] in characteristic zero merely states that the R-module $NK_i(R)$ is zero iff it is locally zero.

Remark In [Sw], Swan proved that for every commutative ring A and for every multiplicative set $S \subset \mathbb{Z}$ there are isomorphisms $S^{-1}NU(A) \to NU(S^{-1}A)$, $S^{-1}NPic(A) \to NPic(S^{-1}A)$. The possibility of generalizing this result to the higher nilgroups was the original motivation for this paper.

References

[A] G. Almkvist, The Grothendieck group of the category of endomorphisms, J. Alg. 28 (1974), 375-388.

[Ba] H. Bass, Algebraic K-theory, Benjamin, New York, 1968.

[Ba2] H. Bass, Introduction to some methods of algebraic K-theory, CBMS Regional Conf. series, No. 20, AMS, Providence, 1974.

[Bl1] S. Bloch, Algebraic K-theory and Crystalline Cohomology, Publ. Math. I.H.E.S. 47 (1978), 188-268.

[Bl2] S. Bloch, Some formulas pertaining to the K-theory of commutative groupschemes, J. Alg. 53 (1978), 304-326.

[C1] P. Cartier, Groupes formels associés aux anneaux de Witt généralisés, C.R. Acad. Sci. Paris 265 (1967), 49-52.

[C2] P. Cartier, Modules associés à un groupe formel commutatif, courbes typiques, C.R. Acad. Sci. Paris 265 (1967), 129-132.

[Ge] S. Gersten, Problems about higher K-functors, Lecture Notes in Math. 341, Springer-Verlag, New York, 1973.

[GQ] D. Grayson, Higher algebraic K-theory: II (after Daniel Quillen), Lecture Notes in Math. 551, Springer-Verlag, New York, 1976.

[Gr] D. Grayson, Grothendieck rings and Witt vectors, Comm. in Alg. 6 (1978), 249-255.

[LR] J. Labute and P. Russell, On K_2 of truncated polynomial rings, J.P.A.A. 6 (1975), 239-251.

[MP] M. P. Murthy and C. Pedrini, K_0 and K_1 of polynomial rings, Lecture Notes in Math. 342, Springer-Verlag, New York, 1973.

[SGA6] P. Berthelot, S.G.A. VI, Exposé V, Lecture Notes in Math. 225, Springer-Verlag, New York, 1971.

[St1] J. Stienstra, Deformations of the second Chow group, a K-theoretic approach, thesis, Utrecht, 1978.

[St2] J. Stienstra, to appear.

[Sw] R. Swan, On seminormality, preprint (1979).

[V] T. Vorst, Localization of the K-theory of polynomial equations, Math. Ann. 244 (1979), 33-53.

[vdK] W. van der Kallen, Generators and relations in algebraic K-theory, Proc. I.C.M. (Helsinki,1978), 305-310.

[Wa] F. Waldhausen, Algebraic K-theory of generalized free products, Ann. Math. 108 (1978), 135-256.

[We1] C. Weibel, Nilpotence in K-theory, J. Alg. 61 (1979), 298-307.

[We2] C. Weibel, KV-theory of categories, Trans. Amer. Math. Soc., to appear.

[We3] C. Weibel, A survey of products in algebraic K-theory, these Proceedings.

[We4] C. Weibel, Meyer-Vietoris Sequences and mod p K-theory, Oberwolfach (1980), Lect. Notes in Math., to appear.