

## Homotopy Algebraic $K$ -Theory

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**Abstract.** We describe a homotopy-invariant algebraic  $K$ -theory  $KH_*$  for both associative rings and schemes. It agrees with Quillen  $K$ -theory  $K_*$  for regular rings and schemes, or if coefficients mod  $\ell$  are taken and  $\ell$  is a unit. The Fundamental Theorem holds for  $KH$ . This theory also satisfies excision and Zariski cohomological descent. Many known "Chern characters" on  $K_*$  factor through  $KH_*$ .

In this paper I construct a homotopy invariant version  $KH_*$  of algebraic  $K$ -theory for both associative rings and schemes. Here "homotopy invariant" means that  $KH_*(A) \cong KH_*(A[t])$ . There is a natural map  $K_*(A) \rightarrow KH_*(A)$ , which is an isomorphism if the ring (or scheme)  $A$  is regular. There is a corresponding theory with coefficients mod  $\ell$ , and if  $A$  is a  $\mathbf{Z}[1/\ell]$ -algebra then

$$K_*(A; \mathbf{Z}/\ell) \cong KH_*(A; \mathbf{Z}/\ell).$$

The Fundamental Theorem of  $K$ -theory holds for rings and schemes:

$$KH_n(A[t, t^{-1}]) \cong KH_n(A) \oplus KH_{n-1}(A),$$

and there is a strongly convergent spectral sequence

$$E_2^{pq} = \check{H}^p(\mathcal{U}; KH_{-q}) \Rightarrow KH_{-p-q}(X)$$

for every cover  $\mathcal{U}$  of a scheme  $X$ . (If  $X$  is quasiprojective, we can get the usual Brown-Gersten spectral sequence, replacing Čech cohomology with the cohomology of the sheaf associated to the presheaf  $U \mapsto KH_{-q}(U)$ .)

The  $KH$ -theory we present is very closely related to Karoubi-Villamayor  $K$ -theory  $KV_*$ . For example,  $KV_n(A) = KV_n(A[t])$  for  $n \geq 1$ , but not for  $n = 0$  or  $n$  negative. Had Karoubi or Gersten insisted on homotopy invariance for  $n \leq 0$ , they could have come up with  $KH$ -theory in the early 1970's. Indeed, the series of papers [K], [KV], [KVCR], [R], [GMV], [GLP], [GR], [GL] and [Gsch] skirted the issue, introducing hypotheses like  $K_0$ -regularity instead. (In the presence of  $K_0$ -regularity,  $KV(A) \cong KH(A)$ .) We shall show that

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every theorem known for  $KV$ -theory holds for  $KH$ -theory without the need for hypotheses such as ‘ $GL$ -fibration’ or ‘ $K_0$ -regularity.’

Had I fully accepted homotopy invariance, I could have discovered  $KH$ -theory in the late 1970’s. I knew of the  $KH$  construction (for rings), but did not think it was interesting because it changed  $K_0(A)$ . In retrospect, many of the technicalities in my early papers [WH], [WN], [WKV], [DW] were due to the fact that I used  $KV$ -theory instead of  $KH$ -theory; my positive results were always in cases when  $KV(A) = KH(A)$ . In many ways, this paper is a rewrite of my 1977 Ph.D. thesis [Wth].

Here is an outline of this paper. In Section 1 we define a spectrum  $KH(A)$  for each associative ring  $A$  and define  $KH_n(A)$  to be  $\pi_n KH(A)$  for  $n \in \mathbf{Z}$ . This section is written in the style of Karoubi and Gersten in the early 1970’s.

In Section 2 we establish excision for ideals, prove that  $KH(A) \cong KH(A_{\text{red}})$ , and show that  $KH$ -theory has localization sequences. (We do not need the usual non-zero divisor hypothesis!) Here, and elsewhere in this paper, we write  $\cong$  for a homotopy equivalence of spectra. In Section 3 we restrict to commutative rings. We compare  $KH$ -theory to  $KV$ -theory for low dimensional rings and show that  $KH(A) \cong KH({}^+A)$ , where  ${}^+A$  is the seminormalization of  $A$ . We also show that  $KH$ -theory has a long exact sequence for localizing at invertible ideals, a ring-theoretic special case of a more general result proven later. These sections are written in the style of my papers in the late 1970’s.

In Section 4 we show that Jouanolou’s device [J] enables us to extend  $KH$ -theory to quasiprojective schemes. In Section 5 we prove that  $KH$  has the Mayer-Vietoris property (i.e., excision for open subschemes) and deduce the Brown-Gersten spectral sequence for quasiprojective schemes. This implies that  $KH$ -theory satisfies Čech cohomological descent. In Section 6 we use this descent property to define  $KH$ -theory for all schemes. This requires a homotopy limit process, and we use Thomason’s  $\tilde{H}$  construction [T]. The discussion here is in the spirit of [WBG], presenting a streamlined version of the proof sketched by Thomason (in disguise) in the widely distributed letters [TL] and at the 1986 Berkeley ICM.

My original intention was to include Thomason as a joint author of this paper, and it was only after his insistence that our styles were too disparate that I was dissuaded from doing so. Nevertheless, the observation that one could use Jouanolou’s trick to extend  $KH$ -theory to schemes, prove that this  $KH$ -theory satisfied Čech cohomological descent and establish a Brown-Gersten spectral sequence is entirely due to him. At the least, I owe a profound debt of thanks to Bob Thomason for his permission to include his results in this paper.

### §1. Associative Rings.

In this section we shall use “ring” to mean an associative ring, possibly without unit, and “ring with unit” to mean associative ring with unit. For any ring with unit  $A$ , the Quillen  $K$ -groups  $K_n(A)$  are the homotopy groups of the infinite loop space  $BGL^+(A)$  for  $n \geq 1$ , while  $K_0(A)$  and  $K_n(A)$  for

negative  $n$  are defined in terms of projective  $A$ -modules [Bass]. Alternatively, the spectrum  $K^{GL}(A)$  associated to  $BGL^+(A)$  is the connected cover of the spectrum  $K(A)$  associated to  $K_0(A) \times BGL^+(A)$ , which in turn is the connective cover of a nonconnective spectrum  $K^B(A)$ , and  $K_n(A) = \pi_n K^B(A)$  for all  $n \in \mathbf{Z}$ . The spectrum  $K^B(A)$  is constructed in [Gsp], [Wag], [PW] and implicitly in [KV]; these different constructions give homotopy equivalent spectra [PW1], so there is no ambiguity.

It will be convenient to let  $K^P(A)$  denote the cofiber of  $K^{GL}(A) \rightarrow K^B(A)$ , so that  $\pi_n K^P(A)$  is zero for  $n \geq 1$ , and equals  $K_n(A)$  for  $n \leq 0$ . (The  $P$  stands for projective modules.)

If  $A$  is any ring,  $\mathbf{Z} \oplus A$  is a ring with unit under the product  $(m, a) \times (n, b) = (mn, mb + an + ab)$ . If  $F$  is a product-preserving functor from rings with unit to spectra, we define  $F(A)$  to be the fiber of the evident map  $F(\mathbf{Z} \oplus A) \rightarrow F(\mathbf{Z})$ ; if  $A$  has a unit this definition is consistent because then  $\mathbf{Z} \oplus A \cong \mathbf{Z} \times A$ . In this way, we can extend  $K^B, K^{GL}$ , etc. to all rings.

Next, we introduce the simplicial ring  $\Delta A$ , following [R]. For each  $n$ ,  $\Delta_n A$  is the coordinate ring  $A[t_0, \dots, t_n]/(\sum t_i - 1)A$  of the “standard  $n$ -simplex” over  $A$ . All face and degeneracy maps in  $\Delta A$  are determined by their obvious geometric counterparts; for example,  $d_i(t_i) = 0$  and  $s_i(t_i) = t_i + t_{i+1}$ .

Let  $KV(A)$  denote the (fibrant) geometric realization of the simplicial spectrum  $K^{GL}(\Delta A)$ , and set  $KV_n(A) = \pi_n KV(A)$  for  $n \geq 1$ . By [R], [GR], these are the Karoubi-Villamayor groups of [KVCR]. It is traditional to set  $KV_n(A) = K_n(A)$  for  $n \leq 0$ , and we shall follow this tradition.

**Definition 1.1.** Let  $KH(A)$  denote the (fibrant) geometric realization of the simplicial spectrum  $K^B(\Delta A)$ . For  $n \in \mathbf{Z}$ , we shall write  $KH_n(A)$  for  $\pi_n KH(A)$ .

It is clear from the definition that  $KH(A)$  commutes with filtered colimits of rings, and that there are natural transformations  $K_n(A) \rightarrow KV_n(A) \rightarrow KH_n(A)$  for all  $n \in \mathbf{Z}$ .

**THEOREM 1.2.** *Let  $A$  be an associative ring.*

(i) *(Homotopy Invariance) For every set  $X$ , let  $A[X]$  denote the polynomial ring in the commuting variables  $X$ . Then*

$$KH(A) \cong KH(A[X]).$$

(ii) *(Free Rings) For every set  $X$ , let  $A\{X\}$  denote the free associative  $A$ -algebra generated by the noncommuting variables  $X$  and containing  $A$ . Then*

$$KH(A) \cong KH(A\{X\}).$$

(iii) *(Fundamental Theorem) For all  $n \in \mathbf{Z}$*

$$KH_n(A[x, x^{-1}]) \cong KH_n(A) \oplus KH_{n-1}(A).$$

*On the level of spectra, the corresponding result is*

$$KH(A[x, x^{-1}]) \cong KH(A) \times \Omega^{-1}KH(A).$$

(iv) (Graded Rings) If  $A = A_0 \oplus A_1 \oplus \cdots$  is a graded ring, then

$$KH(A_0) \cong KH(A).$$

PROOF: For (i),  $A[X]$  is the filtered colimit of finite polynomial extensions, so it is enough to prove that  $KH(A) \cong KH(A[x])$ . This follows from the fact that  $\Delta A \rightarrow \Delta(A[x])$  is a simplicial homotopy equivalence of rings [WKV, 2.4]. Part (ii) follows from part (iv) since  $A\{X\}$  is graded. If  $A$  is a graded ring, define  $\varphi : A \rightarrow A[t]$  by  $\varphi(a) = at^n$  for  $a \in A_n$ . Since  $\varphi$  is a ring homomorphism split by the map ' $t = 1$ ,'  $KH(\varphi)$  is a homotopy equivalence by part (i). Part (iv) now follows from

$$\begin{array}{ccc} KH(A) & \xrightarrow[\cong]{\varphi} & KH(A[t]) \\ \downarrow & & \cong \downarrow_{t=0} \\ KH(A_0) & \longrightarrow & KH(A). \end{array}$$

Finally, to prove part (iii), recall that the Fundamental Theorem for Bass-Quillen  $K$ -theory states that there is a naturally split exact sequence

$$0 \rightarrow K_n(A) \rightarrow K_n(A[x]) \oplus K_n(A[x^{-1}]) \rightarrow K_n(A[x, x^{-1}]) \rightarrow K_{n-1}(A) \rightarrow 0.$$

The proof in [GQ] makes it clear that we can interpret this statement as a (split) homotopy fibration

$$\frac{K(A[x])}{K(A)} \times \frac{K(A[x^{-1}])}{K(A)} \rightarrow \frac{K(A[x, x^{-1}])}{K(A)} \rightarrow \Omega^{-1}K(A).$$

Applying this to  $\Delta A$  and taking realizations yields a homotopy fibration whose first term is contractible by part (i). That is, the map

$$KH(A[x, x^{-1}])/KH(A) \rightarrow \Omega^{-1}KH(A)$$

is a weak homotopy equivalence, hence a homotopy equivalence by [Adams, p. 150], proving part (iii).

**Remark 1.2.1.** The  $KV$ -analogue of part (iv) was proven in [Wth] and [Pv]. Gersten proved the  $K$ -theoretic analogue of part (ii) for regular noetherian rings in [GF]. The  $KV$ -analogue of part (iii) was proven using a  $K_0$ -regularity assumption in [K] and [GLP].

For our next result, we need to recall some terminology. Let  $F$  be a functor from rings to abelian groups. Then  $NF(A)$  is the kernel of ' $t = 0$ ':  $F(A[t]) \rightarrow F(A)$ , so that  $N^p F(A)$  is the intersection of the kernels of the maps

$$'t_i = 0' : F(A[t_1, \dots, t_p]) \rightarrow F(A[t_1, \dots, \hat{t}_i, \dots, t_p]), \quad i = 1, \dots, p.$$

(cf. [Bass, XII].) As in [GH], the homotopization  $[F]A$  of  $F(A)$  is the coequalizer of

$$F(A[t]) \begin{array}{c} \xrightarrow{t=0} \\ \xrightarrow{t=1} \end{array} F(A).$$

**THEOREM 1.3.** *There is a right half-plane homology spectral sequence*

$$E_{pq}^1 = N^p K_q(A) \Rightarrow KH_{p+q}(A)$$

and  $E_{0q}^2 = [K_q]A$  for all  $q \in \mathbf{Z}$ .

**PROOF:** This is the standard spectral sequence of a simplicial spectrum; the  $q^{\text{th}}$  row is the Moore complex used to compute the simplicial homotopy groups of  $E_{pq}^2 = \pi_p K_q(\Delta A)$ .

**Remark 1.3.1.** The spectral sequence is convergent in the sense that each group  $KH_n(A)$  has a filtration

$$0 = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \dots, \quad \cup F_i = KH_n(A),$$

and  $F_p/F_{p-1} \cong E_{p,n-p}^\infty$ . The image of  $K_n(A) \rightarrow KH_n(A)$  is  $F_0$ , and the image of  $KV_n(A) \rightarrow KH_n(A)$  is  $F_{n-1}$  for  $n \geq 1$ . Since  $K_2(A) \rightarrow KV_2(A)$  is not onto by [SS], this spectral sequence does not always collapse. The corresponding 1st quadrant spectral sequence for  $KV_*(A)$  was discovered by Gersten [G341] and Anderson [And]; there is also a 4th quadrant spectral sequence for the cofiber  $KH^p$  of  $KV \rightarrow KH$ :

$$E_{pq}^1 = N^p K_q(A) \Rightarrow KH_{p+q}^p(A), \quad p \geq 0 \text{ and } q \leq 0.$$

**Example 1.4.** When  $A$  is regular noetherian (say on the left), then  $K_n(A) = KV_n(A) = KH_n(A)$  for all  $n \in \mathbf{Z}$ , and  $K_n(A) = 0$  for  $n < 0$ . This follows as in [G341] using the spectral sequence 1.3.

More generally, recall that  $A$  is *coherent* if the category of finitely presented  $A$ -modules is abelian, and *coherent regular* if in addition every finitely presented  $A$ -module has finite projective dimension. If  $A[X]$  is coherent regular for every set  $X$ , then the usual proofs [GF, 4.6] go through to show that again  $K_n(A) = KV_n(A) = KH_n(A)$  for all  $n \in \mathbf{Z}$ , and  $K_n(A) = 0$  for  $n < 0$ . On the level of spectra, this implies [Adams, p. 150] that

$$K(A) \cong K^B(A) \cong KV(A) \times K_0(A) \cong KH(A).$$

A weaker condition than regularity is  $K_n$ -regularity. Recall that a ring  $A$  is  $K_n$ -regular if  $K_n(A) = K_n(A[X])$  for every set  $X$ . Vorst has shown that if  $A$  is  $K_n$ -regular then  $A$  is also  $K_{n-1}$ -regular. (See [V] for  $n \geq 1$  and [DW, 4.4] for  $n \leq 0$ .) Therefore, from the spectral sequence 1.3 we can also deduce

**PROPOSITION 1.5.** *If  $A$  is  $K_0$ -regular, then  $KV_i(A) \cong KH_i(A)$  for all  $i \in \mathbf{Z}$ . More generally, if  $A$  is  $K_n$ -regular, then*

- (i)  $KH_i(A) \cong K_i(A)$  for all  $i \leq n$
- (ii)  $KH_{n+1}(A) \cong [K_{n+1}]A$ , the homotopization of  $K_{n+1}(A)$ .

**Remark 1.5.1.** The hypothesis that  $A$  be  $K_0$ -regular was introduced by Karoubi [K] and Gersten [GLP] in 1970, in order to make assertions like  $KV_1(A[x, x^{-1}]) \cong KV_1(A) \oplus K_0(A)$ .  $K_1$ -regularity was studied by several people in the late 1970's, including Vorst, Geller, Roberts, Dayton and Weibel.

PROPOSITION 1.6. *Let  $A$  be a ring in which the integer  $\ell$  is invertible. Then*

$$K_*(A; \mathbf{Z}/\ell) \cong KH_*(A; \mathbf{Z}/\ell).$$

Here  $K_*(A; \mathbf{Z}/\ell)$  denotes the mod  $\ell$  homotopy groups  $\pi_*(K^B(A); \mathbf{Z}/\ell)$  of the spectrum  $K^B(A)$ , and similarly for  $KH_*(A; \mathbf{Z}/\ell)$ .

PROOF: As in [Wp, 3.4], there is a right half-plane spectral sequence

$$E_{pq}^1 = N^p K_q(A; \mathbf{Z}/\ell) \Rightarrow KH_{p+q}(A; \mathbf{Z}/\ell).$$

Since  $N^p K_q(A; \mathbf{Z}/\ell) = 0$  for  $p \neq 0$  [Wp], this yields the result.

**Remark 1.6.1.** By the Universal Coefficient Theorem, there are short exact sequences (split if  $\ell \not\equiv 2 \pmod{4}$ )

$$0 \rightarrow K_n(A) \otimes \mathbf{Z}/\ell \rightarrow K_n(A; \mathbf{Z}/\ell) \rightarrow \text{Tor}(K_{n-1}(A), \mathbf{Z}/\ell) \rightarrow 0.$$

In particular,  $K_1(A; \mathbf{Z}/\ell)$  includes the  $\ell$ -torsion in  $K_0(A)$ , and  $K_0(A; \mathbf{Z}/\ell)$  includes the  $\ell$ -torsion in  $K_{-1}(A)$ . This differs slightly from Browder's meaning of  $K_1(A; \mathbf{Z}/\ell)$  in [Br].

We conclude this section by factoring some maps through  $K \rightarrow KH$ . If  $A$  is a noetherian ring with unit, the finitely generated  $A$ -modules form an abelian category  $\mathbf{M}(A)$  whose  $K$ -theory is denoted  $G_*(A)$ . In fact, there is a connective spectrum  $G(A)$  associated to  $\Omega BQM(A)$  and  $G_*(A)$  is  $\pi_* G(A)$  [Q], [Gillet]. The spectrum  $K(A)$  is associated to  $\Omega BQP(A)$ , where  $\mathbf{P}(A)$  is the subcategory of projective modules in  $\mathbf{M}(A)$ , and the inclusion of exact categories induces a map  $K(A) \rightarrow G(A)$  called the *Cartan map*.

THEOREM 1.7. *Let  $A$  be a noetherian ring with unit. Then the Cartan map factors:*

$$\begin{array}{ccc} K(A) & \longrightarrow & G(A) \\ \downarrow & & \uparrow \\ K^B(A) & \longrightarrow & KH(A) \end{array}$$

PROOF: (cf. [WKV, 3.1]) We first factor the Cartan map through  $K^B$ . Let  $K^B(A)(-n)$  denote the  $(-n-1)$  connected cover of  $K^B(A)$ , with  $K^B(A)(-0) = K(A)$ . We will use a similar notation for  $G(A)$ , although  $G(A)(-n) \cong G(A)$  for all  $n \geq 0$ . The natural map  $K^B(A) \rightarrow \Omega K^B(A[t, t^{-1}])$  discussed in the proof of (1.2) induces a map

$$K^B(A)(-n) \rightarrow (\Omega K^B(A[t, t^{-1}])(-n)) = \Omega(K^B(A[t, t^{-1}])(-n+1))$$

for each  $n \geq 0$ . Similarly, the fibration

$$G(A) \rightarrow G(A[t]) \rightarrow G(A[t, t^{-1}])$$

of [Q] induces a map from  $\Omega G(A[t, t^{-1}])$  to  $G(A)$ . By induction on  $n$ , we have a natural map  $K(A)(-n+1) \rightarrow G(A)$  and a diagram

$$\begin{array}{ccc} K(A) & \longrightarrow & K^B(A)(-n) \\ \downarrow & & \downarrow \\ \Omega K(A[t, t^{-1}]) & \longrightarrow & \Omega(K^B(A[t, t^{-1}])(-n+1)) \\ \downarrow & & \downarrow \\ \Omega G(A[t, t^{-1}]) & \xrightarrow{\cong} & \Omega(G(A[t, t^{-1}])(-n+1)) \\ \downarrow & & \downarrow \\ G(A) & \xrightarrow{\cong} & G(A)(-n), \end{array}$$

and the left composite is the Cartan map. This factors the Cartan map through  $K^B(A)(-n)$  in a way compatible with the factorization through  $K^B(A)(-n+1)$ . In the colimit, we obtain a natural map  $K^B(A) \rightarrow G(A)$  factoring the Cartan map.

We now use the construction in [WKV, §3] of a simplicial exact category  $\mathbf{M}_\bullet$  with  $BQM_p \cong BQM(\Delta_p A)$  for each  $p$ .  $\mathbf{M}_p$  is the full subcategory of  $\mathbf{M}(\Delta_p A)$  of all modules which are Tor-independent of the maps  $\Delta_p A \rightarrow \Delta_q A$ . We have constructed a map of simplicial spectra from  $K^B(\Delta A)$  to  $K(\mathbf{M}_\bullet)$ , yielding a diagram of spectra

$$\begin{array}{ccc} K^B(A) & \longrightarrow & G(A) \\ \downarrow & & \downarrow \cong \\ KH(A) = |K^B(\Delta A)| & \longrightarrow & |K(\mathbf{M}_\bullet)| \end{array}$$

If  $A$  is a Banach algebra over  $\mathbf{R}$  or  $\mathbf{C}$ , it is possible to define the topological  $K$ -theory of  $A$  by

$$K_n^{\text{top}}(A) = \pi_n BGL^{\text{top}}(A), \quad n \geq 1.$$

These agree with the groups defined in [KV], and are extendable to negative  $K$ -groups by the method of [KV], which implicitly constructs a spectrum  $K^{\text{top}}(A)$  with  $K_n^{\text{top}}(A) = \pi_n K^{\text{top}}(A)$ ,  $n \in \mathbf{Z}$ . (This spectrum is explicitly constructed in [PW1, §5].) The natural map  $BGL(A)^+ \rightarrow BGL^{\text{top}}(A)$  induces a natural map  $K^B(A) \rightarrow K^{\text{top}}(A)$ .

PROPOSITION 1.8. *If  $A$  is a Banach algebra, the map from  $K^B(A)$  to  $K^{\text{top}}(A)$  factors:*

$$\begin{array}{ccc} K^B(A) & \longrightarrow & K^{\text{top}}(A) \\ \downarrow & \nearrow & \\ KH(A) & & \end{array}$$

PROOF: Let  $C_n$  denote the ring of continuous maps from the  $n$ -simplex into  $A$ , so that  $\Delta A$  is a simplicial subalgebra of  $C_\bullet$ . This gives a diagram of maps

$$\begin{array}{ccccc} K^B(A) & \longrightarrow & & \longrightarrow & K^{\text{top}}(A) \\ \downarrow & & & & \downarrow \\ K^B(\Delta A) & \longrightarrow & K^B(C_\bullet) & \longrightarrow & K^{\text{top}}(C_\bullet) \end{array}$$

of simplicial spectra. Since  $A \rightarrow C_n$  is a homotopy equivalence of Banach algebras,  $K^{\text{top}}(A) \cong K^{\text{top}}(C_n)$  for each  $n$ . Hence the geometric realization of  $K^{\text{top}}(C_\bullet)$  is homotopy equivalent to  $K^{\text{top}}(A)$ , and the result follows.

§2. **Excision and Localization.**

In this section we develop some tools to help us compute the  $KH$ -theory of rings. The first result was proven for  $KV$ -theory in [KV][GMV] under the added hypothesis that  $A \rightarrow A/I$  is a ‘ $GL$ -fibration.’

THEOREM 2.1. (Excision for Ideals.) *Let  $I$  be an ideal in a ring  $A$ . Then*

$$KH(I) \rightarrow KH(A) \rightarrow KH(A/I)$$

*is a homotopy fibration, i.e., there is a long exact sequence*

$$\cdots KH_{n+1}(A/I) \rightarrow KH_n(I) \rightarrow KH_n(A) \rightarrow KH_n(A/I) \cdots \quad (n \in \mathbf{Z}).$$

PROOF: Let us write  $K^B(A, I)$  for the homotopy fiber of  $K^B(A) \rightarrow K^B(A/I)$ . The sequence of simplicial spectra

$$K^B(\Delta A, \Delta I) \rightarrow K^B(\Delta A) \rightarrow K^B(\Delta A/I)$$

is degreewise a fibration, hence a fibration on the geometric realizations. (Here it is important that we work with spectra instead of spaces, because geometric realization does not preserve homotopy fibrations of spaces. However, a homotopy fibration sequence of spectra is the same thing as a homotopy cofibration sequence of spectra, so it is preserved by geometric realization.) Therefore  $|K^B(\Delta A, \Delta I)|$  is the homotopy fiber of  $KH(A) \rightarrow KH(A/I)$ .

Now write  $K^{GL}(A, I)$  for the connected cover of the fiber of  $K^{GL}(A) \rightarrow K^{GL}(A/I)$ , so that  $KV(I) \cong |K^{GL}(\Delta A, \Delta I)|$  by [WKV, 2.6]. If  $K^P(A, I)$  denotes the cofiber of  $K^{GL}(A, I) \rightarrow K^B(A, I)$ , then we know from [Bass, IX

(5.4)] that  $K^P(I) \cong K^P(A, I)$  because  $K_n(I) \cong K_n(A, I)$  for all  $n \leq 0$ . The result now follows from the 5-lemma applied to

$$\begin{array}{ccccc} KV(I) & \longrightarrow & KH(I) & \longrightarrow & |K^P(\Delta I)| \\ \downarrow \cong & & \downarrow & & \downarrow \cong \\ |K^{GL}(\Delta A, \Delta I)| & \longrightarrow & |K^B(\Delta A, \Delta I)| & \longrightarrow & |K^P(\Delta A, \Delta I)|. \end{array}$$

COROLLARY 2.2. (Mayer-Vietoris sequence of an ideal.) *Let  $I$  be an ideal of  $A$ , and let  $A \rightarrow B$  map  $I$  isomorphically onto an ideal of  $B$ . Then the square*

$$\begin{array}{ccc} KH(A) & \longrightarrow & KH(B) \\ \downarrow & & \downarrow \\ KH(A/I) & \longrightarrow & KH(B/I) \end{array}$$

is homotopy cartesian, and there is a long exact sequence

$$\cdots KH_{n+1}(B/I) \rightarrow KH_n(A) \rightarrow KH_n(B) \oplus KH_n(A/I) \rightarrow KH_n(B/I) \cdots (n \in \mathbf{Z}).$$

PROOF:  $KH(I)$  is the homotopy fiber of both vertical arrows.

THEOREM 2.3. *If  $I$  is a nilpotent ring, then  $KH(I)$  is contractible. If  $I$  is a nilpotent ideal in a ring  $A$ , then  $KH(A) \cong KH(A/I)$ .*

PROOF: Every  $\Delta_p I \cong I[t_1, \dots, t_p]$  is nilpotent, and by [Bass, IX (1.3)]  $K_0(\Delta_p I) = 0$ . Hence  $I$  is  $K_0$ -regular, so  $KV(I) \cong KH(I)$  by 1.5. But  $KV_*(I) = 0$  by [WN, 2.2] or [Pv], so we have  $KV(I) \cong *$  by [Adams, p. 150]. The second sentence follows from this and Excision for  $I$  (2.1).

**Remark 2.3.1.** A ring (or ideal) is called *locally nilpotent* if every finitely generated subring is nilpotent. If  $I$  is locally nilpotent, then it is the union of nilpotent rings, so  $KH(I) \cong *$  by 2.3. In particular, in any ring  $A$  the lower nilradical of  $A$ ,

$$\text{nil}(A) = \sum \{\text{nilpotent ideals of } A\}$$

is locally nilpotent. Consequently,  $KH(A) \cong KH(A/\text{nil}(A))$ .

There are other situations for which one can define  $KH$  groups. For example, let  $A$  be a ring with unit,  $S$  a multiplicative set of central nonzerodivisors of  $A$ , and consider the category  $\mathbf{H}_S(A)$  of  $S$ -torsion  $A$ -modules having finite resolutions by finitely generated projective  $A$ -modules. The Quillen  $K$ -groups  $K_n \mathbf{H}_S(A)$  for positive  $n$  are the homotopy groups of a connective spectrum  $K\mathbf{H}_S(A)$  with  $\Omega^\infty K\mathbf{H}_S(A) = \Omega BQ\mathbf{H}_S(A)$ . Using Bass' contracted functors [Bass, XII], Carter showed in [Carter] that this was the connective cover of a spectrum  $K^B \mathbf{H}_S(A)$ , and that

$$K^B \mathbf{H}_S(A) \rightarrow K^B(A) \rightarrow K^B(S^{-1}A)$$

is a fibration. Let  $H_n \subset \mathbf{H}_S(\Delta_n A)$  be the full exact subcategory of all modules  $M$  Tor-independent of the maps  $\Delta_n A \rightarrow \Delta_q A$ .  $H_\bullet$  is a simplicial exact category with  $K^B(H_n) = K^B(\mathbf{H}_S(\Delta_n A))$  as in [W3, §3]. Define  $KH(\mathbf{H}_S(A))$  to be the geometric realization of the simplicial spectrum  $K^B(H_\bullet)$ . Clearly,

$$(2.4) \quad KH(\mathbf{H}_S(A)) \rightarrow KH(A) \rightarrow KH(S^{-1}A)$$

is a fibration, yielding a long exact sequence in homotopy.

Here is an application of the localization fibration (2.4). Let  $S$  be a multiplicatively closed set of central nonzerodivisors of  $A$ . A map  $i : A \rightarrow B$  of rings with unit is called an *analytic isomorphism along nonzerodivisors*  $S$  if  $i(S)$  consists of central nonzerodivisors of  $B$  and  $A/sA \cong B/sB$  for every  $s \in S$ . This last condition is equivalent to requiring the  $S$ -adic completions  $\hat{A} = \lim A/sA$  and  $\hat{B} = \lim B/sB$  to be isomorphic, whence the name. For example, if  $s \in A$  is a central nonzerodivisor, then  $A \rightarrow \hat{A}$  is an analytic isomorphism along  $\{s^n\}$ .

**THEOREM 2.5.** ('Analytic Isomorphisms') *If  $A \rightarrow B$  is an analytic isomorphism along nonzerodivisors  $S$ , then the square*

$$\begin{array}{ccc} KH(A) & \longrightarrow & KH(B) \\ \downarrow & & \downarrow \\ KH(S^{-1}A) & \longrightarrow & KH(S^{-1}B) \end{array}$$

*is homotopy cartesian, and there is a long exact sequence*

$$\cdots KH_n(A) \rightarrow KH_n(B) \oplus KH_n(S^{-1}A) \rightarrow KH_n(S^{-1}B) \cdots \quad (n \in \mathbf{Z}).$$

**PROOF:** (cf. [WA, 1.3]). We know that  $\mathbf{H}_S(A) \cong \mathbf{H}_S(B)$  by [WA, 1.1], a result due to Karoubi. Similarly,  $\mathbf{H}_S(\Delta_n A) \cong \mathbf{H}_S(\Delta_n B)$  for all  $n \geq 0$ . Hence  $KH(\mathbf{H}_S A) \cong KH(\mathbf{H}_S B)$ . By (2.4), these are the homotopy fibers of the vertical arrows in 2.5, proving that the square 2.5 is homotopy cartesian.

It is possible to remove the nonzerodivisor hypothesis from Theorem 2.5 using Excision for Ideals.

**THEOREM 2.6.** *Let  $S$  be a central multiplicative set in  $A$ , and  $i : A \rightarrow B$  a ring map so that  $i(S)$  is in the center of  $B$  and  $A/sA \cong B/sB$  for every  $s \in S$ . Then the square*

$$\begin{array}{ccc} KH(A) & \longrightarrow & KH(B) \\ \downarrow & & \downarrow \\ KH(S^{-1}A) & \longrightarrow & KH(S^{-1}B) \end{array}$$

*is homotopy cartesian.*

**PROOF:** Replacing  $A, B$  by  $\mathbf{Z} \times A, \mathbf{Z} \times B$ , we may assume that  $i$  is a map of rings with unit. By a direct limit argument, we may assume  $S = \{s^n\}$  for

some  $s$ . Set  $I = \cup \text{ann}_A(s^n) = \ker(A \rightarrow S^{-1}A)$  and  $J = \cup \text{ann}_B(s^n)$ . We may also assume, using 2.3, that the locally nilpotent ideal  $I \cap \ker(i)$  is zero. It is easy to see that  $s$  is not a zerodivisor of  $A/I$  or  $B/J$ , and that  $I$  maps isomorphically onto an ideal of  $B$ . By 2.2 and 2.5 the two squares of

$$\begin{array}{ccccc}
 KH(A) & \longrightarrow & KH(B) & & \\
 \downarrow & & \downarrow & & \\
 KH(A/I) & \longrightarrow & KH(B/I) & \longrightarrow & KH(B/J) \\
 \downarrow & & & & \downarrow \\
 KH(S^{-1}A) & \longrightarrow & & \longrightarrow & KH(S^{-1}B)
 \end{array}$$

are homotopy cartesian. It is therefore enough to see that  $KH(B/I) \cong KH(B/J)$ , which is a consequence of 2.3.1 and the following elementary lemma:

LEMMA 2.7. *The ideal  $J/I$  of  $B/I$  is a union of nilpotent ideals.*

PROOF: Every  $x \in B$  can be written as  $x = i(a) - sb$  for some  $a \in A, b \in B$ . If  $x \in \text{ann}_B(s^n)$ , then

$$0 = xs^n = i(as^n) - s^{n+1}b.$$

Since  $A/s^{n+1} \cong B/s^{n+1}B$ , this implies that  $as^n \in s^{n+1}A$ . But then  $a \in sA + I$ . This proves that  $\text{ann}_B(s^n) \subset sB + I$ . Hence the image of  $\text{ann}_B(s^n)$  in  $B/I$  is a nilpotent ideal. But  $J/I$  is the union of these ideals.

COROLLARY 2.8. *Let  $s, t$  be central elements of a ring  $A$  so that  $sA + tA \cong A$ . Then the square*

$$\begin{array}{ccc}
 KH(A) & \longrightarrow & KH(A[t^{-1}]) \\
 \downarrow & & \downarrow \\
 KH(A[s^{-1}]) & \longrightarrow & KH(A[s^{-1}, t^{-1}])
 \end{array}$$

*is homotopy cartesian.*

### §3. Commutative Rings.

In this section we shall use ‘commutative ring’ to mean a commutative ring with unit, and ‘dimension’ to mean Krull dimension. Since the nilradical  $\text{nil}(A)$  of a commutative ring  $A$  is a locally nilpotent ideal,  $KH(\text{nil } A) \cong 0$  and  $KH(A) \cong KH(A_{\text{red}})$  by (2.3.1), where  $A_{\text{red}} = A/\text{nil}(A)$ . Thus we concentrate our attention on reduced rings. Recall from [Bass, IX (3.2)] that  $H^0(A) = H^0(\text{Spec}(A); \mathbf{Z})$  is the direct summand of  $K_0(A)$  which measures the rank of a finitely generated projective  $A$ -module.

PROPOSITION 3.1. *If  $A$  is a 0-dimensional commutative ring, then*

- i)  $KH(A) \cong KH(A_{\text{red}}) \cong K(A_{\text{red}})$
- ii)  $KH_0(A) = K_0(A) = H^0(A)$
- iii)  $KH_n(A) = K_n(A) = 0$  for  $n$  negative.

PROOF: If  $A$  is Artinian, this is easy. More generally, let  $A$  be a reduced 0-dimensional ring. Then  $A$  is absolutely flat, i.e., every  $A$ -module is flat. In particular, every finitely presented  $A$ -module is projective, proving that  $A$  is regular coherent. Moreover,  $A$  is "supercoherent": each  $A[x_1, \dots, x_n]$  is coherent by [Sab], hence regular coherent [GF, 4.1]. The result now follows from 1.4, except for the equality  $K_0 = H^0$ , which is a consequence of the classical fact that every finitely presented module has the form  $Ae_1 \oplus \dots \oplus Ae_n$  for appropriate idempotents  $e_i \in A$ . (See [Bour, I.2, Ex. 18].)

If  $A$  is any commutative ring, the *seminormalization*  ${}^+A$  of  $A$  is the smallest extension of  $A_{\text{red}}$  which is Pic-regular [Swan].  ${}^+A$  is the filtered union of all subrings obtained from  $A_{\text{red}}$  by a finite number of extensions of the form  $C \subset C[x]$ , where  $x^2, x^3 \in C$  [Swan, 2.8]. The following result is implicit in [WN, 3.14].

PROPOSITION 3.2.  $KH(A) \cong KH({}^+A)$ .

PROOF: It is enough to prove that if  $C = A[x]$  with  $x^2, x^3 \in A$  then  $KH(A) \cong KH(C)$ . Setting  $I = Cx^2$ , the cartesian square

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ A/I & \longrightarrow & C/I \end{array}$$

gives rise to a homotopy-cartesian square of  $KH$ -spectra by 2.2. But  $(A/I)_{\text{red}} = (C/I)_{\text{red}}$ , so  $KH(A/I) \cong KH(C/I)$  by 2.3, and we deduce that  $KH(A) \cong KH(C)$ , as desired.

THEOREM 3.3. *Let  $A$  be a 1-dimensional commutative noetherian ring, and  ${}^+A$  the seminormalization of  $A_{\text{red}}$ . Then  $KH(A) \cong KH({}^+A) \cong KV({}^+A)$ , and we have:*

$$\begin{aligned} KH_n(A) &= KV_n(A) && \text{if } n \geq +2; \\ KH_1(A) &= ({}^+A)^* \oplus SKV_1(A), && \text{where } SKV_1 = [SK_1]; \\ KH_0(A) &= H^0(A) \oplus [Pic]A, && \text{a quotient of } K_0(A); \\ KH_{-1}(A) &= K_{-1}(A); \\ KH_n(A) &= 0 && \text{if } n \leq -2. \end{aligned}$$

PROOF: By [WA, 2.8],  $A$  is  $K_{-1}$ -regular and  $K_n(A) = 0$  for  $n \leq -2$ . We also know that  $K_0(A) = H^0(A) \oplus \text{Pic}(A)$  by Serre's Theorem [Bass], and that  $K_0(A[X]) = H^0(A) \oplus \text{Pic}(A[X])$  by [VW, 1.5] (this is easy if  $A$  has finite

normalization), so  $A$  is  $K_0$ -regular iff  $A_{\text{red}}$  is seminormal [Swan]. The result for  $n \leq 0$  follows from 1.5 and the above remarks. This transfers attention to  ${}^+A$ , which may not be noetherian. However, since  ${}^+A$  is the filtered colimit of noetherian 1-dimensional rings, we have

$$K_0({}^+A[X]) = H^0({}^+A) \oplus \text{Pic}({}^+A[X]) = K_0({}^+A),$$

so that  ${}^+A$  is  $K_0$ -regular. Hence  $KV({}^+A) \cong KH({}^+A)$  by 1.5. Finally, we observe that by  $KV_n(A) = KV_n({}^+A)$  for  $n \geq 2$  and  $SKV_1(A) = SKV_1({}^+A)$  by [WN, 3.14]. (The hypothesis in *op. cit.* that  ${}^+A$  be finite over  $A$  is easily removed.)

**Remark 3.3.1.** This difference between  $KV_1$  and  $KH_1$  is the explanation for the phenomena discussed in [WH]. In terms of the spectral sequence (1.3) for  $KH$ ,  $({}^+A)^*/A^*$  is the homology of

$$\text{Pic}(A) \leftarrow N \text{Pic}(A) \leftarrow N^2 \text{Pic}(A).$$

**Scholium 3.3.2.** If  $A$  is a 1-dimensional commutative ring, then every finitely generated projective  $A$ -module of constant rank is of the form  $F \oplus L$ , where  $F$  is free and  $L$  is a rank 1 projective  $A$ -module. This follows from Heitmann’s generalization [Heit, 2.6] of Serre’s Theorem. A non-noetherian version of an Eisenbud-Evans conjecture asks if every finitely generated projective  $A[t_1, \dots, t_n]$ -module of constant rank has the form  $F \oplus L$ , where  $F$  is free and  $L$  has rank 1. The stable version of this conjecture asks if  $K_0(A[X]) = H^0(A) \oplus \text{Pic}(A[X])$ . If this were true, we could erase the hypothesis ‘noetherian’ from the statement of Theorem 3.3.

For commutative rings, there is a slightly stronger localization theorem than in §1: we can “invert” fin. gen. locally principal ideals. Here is what this means.

**LEMMA 3.4.** *Let  $I$  be a fin. gen. locally principal ideal in  $A$ . Then there is a ring  $I^{-1}A$  so that*

$$\text{Spec}(I^{-1}A) = \text{Spec}(A) - V(I).$$

**PROOF:** Set  $U = \text{Spec}(A) - V(I)$ , and note that  $J = \cup \text{ann}(I^n)$  is zero on  $U$ , i.e.,  $U \subset \text{Spec}(A/J)$ . The image of  $I$  in  $A/J$  is a fin. gen. locally invertible ideal, and we set  $I^{-1}A$  equal to  $\cup (IA/J)^{-n}$ . The map  $\text{Spec}(I^{-1}A) \rightarrow \text{Spec}(A)$  induces an isomorphism of  $\text{Spec}(I^{-1}A)$  with  $U$  since it does so locally.

**THEOREM 3.5.** (Analytic Isomorphisms for Cartier Divisors) *Let  $I$  be a fin. gen. locally principal ideal in  $A$ , and let  $A \rightarrow B$  be a commutative ring map such that  $A/I^n \cong B/I^n B$  for all  $n$ . Assume that  $I$  and  $IB$  are locally generated by a nonzerodivisor. Then there is a homotopy cartesian square*

$$\begin{array}{ccc} K^B(A) & \longrightarrow & K^B(B) \\ \downarrow & & \downarrow \\ K^B(I^{-1}A) & \longrightarrow & K^B(I^{-1}B). \end{array}$$

PROOF: Let  $\mathbf{H}_I A$  be the category of all finitely presented  $A$ -modules  $M$  with  $pd_A(M) \leq 1$  and  $M \otimes I^{-1}A = 0$ . Using the connective spectrum  $K(\mathbf{H}_I A)$  to define the  $K$ -groups of  $\mathbf{H}_I A$ , the localization theorem of [GQ, p. 229] states that there is a long exact sequence

$$(3.5.1) \quad \cdots K_{n+1}(B) \rightarrow K_n(\mathbf{H}_I(A)) \rightarrow K_n(A) \rightarrow K_n(B) \cdots$$

ending in  $K_0(B)$ .

The arguments in [Carter] for  $\mathbf{H}_S A$  go through verbatim to show that  $K(\mathbf{H}_I A)$  is the connective cover of a spectrum  $K^B(\mathbf{H}_I A)$ , and that

$$(3.5.2) \quad K^B(\mathbf{H}_I A) \rightarrow K^B(A) \rightarrow K^B(I^{-1}A)$$

is a homotopy fibration. Thus the long exact sequence 3.5.1 continues with negative  $K$ -groups.

Now the categories  $\mathbf{H}_I A$  and  $\mathbf{H}_I B$  are equivalent, since they are locally equivalent (see the proof of 2.5). Hence the fibers  $K^B(\mathbf{H}_I A)$  and  $K^B(\mathbf{H}_I B)$  of the vertical arrows in the square 3.5 are homotopy equivalent, proving that the square is homotopy cartesian.

COROLLARY 3.6. *Let  $I$  be a fin. gen. locally principal ideal in  $A$ , and let  $A \rightarrow B$  be a commutative ring map such that  $A/I^n \cong B/I^n B$  for all  $n$ . Then there is a homotopy cartesian square*

$$\begin{array}{ccc} KH(A) & \longrightarrow & KH(B) \\ \downarrow & & \downarrow \\ KH(I^{-1}A) & \longrightarrow & KH(I^{-1}B) \end{array}$$

PROOF: First suppose that  $I$  and  $IB$  are locally generated by a nonzerodivisor. Applying 3.5 to  $\Delta_n A \rightarrow \Delta_n B$ , we get a square of simplicial spectra

$$\begin{array}{ccc} K^B(\Delta A) & \longrightarrow & K^B(\Delta B) \\ \downarrow & & \downarrow \\ K^B(\Delta I^{-1}A) & \longrightarrow & K^B(\Delta I^{-1}B) \end{array}$$

which is homotopy cartesian in each degree. Hence its geometric realization, the square of 3.6, is homotopy cartesian. (Alternatively, we could mimic the proof of 2.5, constructing  $KH(\mathbf{H}_I A)$  and using the analogue of (2.4).)

The general case follows from this special case in exactly the same way that 2.6 follows from 2.5.

**§4. Jouanolou's Device.**

In this section, we shall show that Jouanolou's Device enables us to extend  $KH$  to quasiprojective schemes. First recall [Hart, Ex. II (5.18)] that a (rank  $n$ ) vector bundle  $E$  over a scheme  $X$  is a scheme that is locally  $X \times \mathbf{A}^n$ , with linear patching maps.

LEMMA 4.1. *If  $X = \text{Spec}(A)$  and  $E \rightarrow X$  is a vector bundle, then  $E = \text{Spec}(B)$  for some  $B$  and  $KH(A) \cong KH(B)$ .*

PROOF: We know [Hart, *ibid.*] that  $B$  is the symmetric algebra of some projective  $A$ -module, so in particular  $B$  is graded with  $A$  in degree 0. We now cite Theorem 1.2 (iv).

**Definition 4.2.** An *affine vector bundle torsor* over a scheme  $X$  is an affine scheme  $W$  and an affine map  $W \rightarrow X$  such that, for some vector bundle  $E$  over  $X$  (thought of as a group scheme over  $X$ ),  $W$  is a torsor for  $E$ . That is,  $E$  acts on  $W$  and  $W$  is locally isomorphic to  $E$ , but the patching maps are only affine and need not be linear.

For example, if  $X = \text{Spec}(A)$  then  $W \cong E = \text{Spec}(B)$  because  $H_{\text{et}}^1(X, E) = 0$ . In fact, when  $X$  is affine, affine vector bundle torsors over  $X$  are the same thing as vector bundles over  $X$ . If  $X$  is a quasiprojective scheme (over some affine base), Jouanolou proved in [J, 1.5] that some affine vector bundle torsor over  $X$  exists. Thomason has observed that Jouanolou's device works in a slightly more base-free setting:

PROPOSITION 4.3. (Jouanolou-Thomason) *Let  $X$  be a quasicompact separated scheme with an ample line bundle. Then an affine vector bundle torsor over  $X$  exists.*

PROOF: (Thomason) Let  $\mathcal{L}$  be an ample line bundle over  $X$ . Replacing  $\mathcal{L}$  by  $\mathcal{L}^{\otimes n}$  if need be, we may assume that there are global sections  $s_0, \dots, s_N$  of  $\mathcal{L}$  such that the subschemes

$$X_{s_i} = \{x \in X : s_i(x) \neq 0\}$$

are affine and form a cover of  $X$ . (See [EGA 2, 4.5.2] or the proof of [Hart, II (7.7)]; Hartshorne assumes  $X$  to be noetherian.) These  $s_i$  determine an affine map  $s : X \rightarrow \mathbf{P}_{\mathbf{Z}}^N$  [Hart, p. 128]. Jouanolou noted [J, 1.5] that the Stiefel scheme  $W_0$  of rank 1 idempotent matrices in  $M_{N+1}(\mathbf{Z})$  forms an affine vector bundle torsor over  $\mathbf{P}_{\mathbf{Z}}^N$ . Pulling back  $W_0$  to  $X$ , we get the affine vector bundle torsor  $W = s^*W_0$  over  $X$ .

**Remark 4.3.1.** This construction is not functorial. For example, if  $U$  is open in  $X$  but  $i : U \rightarrow X$  is not affine, then  $i^*W$  need not be an affine scheme, even though it is a vector bundle torsor over  $U$ . To get an affine vector bundle torsor over  $U$  which maps to  $W$ , pull back the ample bundle to  $i^*W$  and find an affine vector bundle torsor  $W'$  over  $i^*W$ , hence over  $U$ .

There is another case in which we can assert the existence of affine vector bundle torsors. Let  $X$  be a quasicompact quasiseparated scheme.  $X$  has an *ample family* of line bundles if there are global sections  $s_i$  of line bundles  $\mathcal{L}_i$  ( $i = 0, \dots, N$ ) such that the non-vanishing loci  $X_{s_i}$  are all affine and cover  $X$ . (Cf. [SGA6] II 2.2.3 and 2.2.4) For example, separated regular noetherian schemes have an ample family of line bundles by [SGA6] II 2.2.7.1.

**PROPOSITION 4.4.** (Thomason) *Let  $X$  be a quasicompact quasiseparated scheme with an ample family of line bundles. Then an affine vector bundle torsor over  $X$  exists.*

**PROOF:** We adapt the above notation, letting  $\mathcal{E}$  denote the direct sum of the line bundles  $\mathcal{L}_i$ . The map

$$s = (s_0, \dots, s_N) : \mathcal{O}_X \rightarrow \mathcal{E}$$

is a split monomorphism on each  $X_{s_i}$ , so the cokernel  $\mathcal{F}$  of  $s$  is a rank  $N$  vector bundle on  $X$ . Let  $W$  denote the difference  $\mathbf{P}\mathcal{E} - \mathbf{P}\mathcal{F}$  of the corresponding projective space bundles over  $X$ , and  $\pi : W \rightarrow X$  the projection.

We claim that  $W$  is an affine vector bundle torsor over  $X$ . Let  $S(\mathcal{E})$  be the symmetric algebra on  $\mathcal{E}$ , so that  $\mathbf{P}\mathcal{E} = \mathbf{Proj}(S(\mathcal{E}))$ . It is not hard to verify that  $W = \mathbf{Spec}(S(\mathcal{E})/(s - 1))$  is a torsor for the vector bundle  $\mathbf{V}\mathcal{F} = \mathbf{Spec}(S(\mathcal{E})/(s - 0))$ , where  $\mathbf{Spec}$  denotes relative spec over  $X$ . This implies that  $\pi$  is an affine map. To see that  $W$  is an affine scheme we use the local criterion of [EGA2, 5.2.1] [Hart, Ex. II.2.17]. The  $s_i$  induce elements  $f_i$  of  $\Gamma(W, \mathcal{O}_W)$  such that  $f_0 + \dots + f_N = 1$  and each  $W_{f_i} = \pi^{-1}(X_{s_i})_{f_i}$  is affine, being a localization of the affine  $\pi^{-1}(X_{s_i})$ .

**LEMMA 4.5.** (Jouanolou [J, 1.6]) *If  $V = \mathbf{Spec}(A)$  and  $W = \mathbf{Spec}(B)$  are both affine vector bundle torsors over  $X$ , then  $KH(A) \cong KH(B)$ .*

**PROOF:** Form the cartesian square:

$$\begin{array}{ccc} V \times_X W & \longrightarrow & W \\ \downarrow & & \downarrow \\ V & \longrightarrow & X. \end{array}$$

$V \times_X W$  is an affine vector bundle torsor over the affine scheme  $V$ , so in particular  $V \times_X W \rightarrow V$  (and by symmetry  $V \times_X W \rightarrow W$ ) is a vector bundle. By 4.1, we have

$$KH(V) \xrightarrow{\cong} KH(V \times_X W) \xrightarrow{\cong} KH(W).$$

**Definition 4.6.** (Jouanolou's Device) Let  $X$  be a scheme with an affine vector bundle torsor. Define  $KH(X)$  to be  $KH(B)$ , where  $W = \mathbf{Spec}(B)$  is some given affine vector bundle torsor over  $X$ . If  $X$  is affine, we may take  $W = X$  to ensure that the spectrum  $KH(\mathbf{Spec}(A))$  is  $KH(A)$ , and not just homotopy equivalent to it. By 4.5,  $KH(X)$  is well-defined up to homotopy equivalence of spectra, and the groups  $KH_i(X) = \pi_i KH(X)$  are well-defined up to isomorphism.

If  $f : X \rightarrow Y$  is a morphism of schemes, with given affine vector bundle torsors  $W_X$  and  $W_Y$ , there may not be any map  $W_X \rightarrow W_Y$  over  $f$ . However,  $(f^* W_Y) \times_X W_X$  is an affine vector bundle torsor over  $X$  which maps to  $W_Y$ . Thus there is a map  $KH(X) \rightarrow KH(Y)$ , but it is only defined up to homotopy equivalence.

**Example 4.7.** (Jouanolou [J, 3.1]) Fix a commutative ring  $k$ . Then

$$KH(\mathbf{P}_k^1) \cong KH(k) \times KH(k)$$

and there is a homotopy cartesian square

$$(4.8) \quad \begin{array}{ccc} KH(\mathbf{P}_k^1) & \longrightarrow & KH(k[t]) \\ \downarrow & & \downarrow \\ KH(k[t^{-1}]) & \longrightarrow & KH(k[t, t^{-1}]). \end{array}$$

PROOF: By [J, 3.1]  $\text{Spec}(A)$  is an affine vector bundle torsor over  $\mathbf{P}^1$ , where  $A = k[x, y, z]/(x^2 - x + yz)$ ; the map  $\text{Spec}(A) \rightarrow \mathbf{P}^1$  sends  $(x, y, z)$  to  $(x : y) = (z : 1 - x)$ . The open set  $V$  of  $\text{Spec}(A)$  lying over  $\text{Spec}(k[t]) \subset \mathbf{P}^1$  is the complement of the divisor given by  $I = (x, z)A$ , and the open set  $U$  of  $\text{Spec}(A)$  lying over  $\text{Spec}(k[t^{-1}])$  is the complement of the divisor given by  $J = (y, 1 - x)A$ . In particular,  $U, V$  and  $U \cap V$  are affine, so the square 4.8 is homotopy equivalent to the square

$$\begin{array}{ccc} KH(A) & \longrightarrow & KH(I^{-1}A) \\ \downarrow & & \downarrow \\ KH(J^{-1}A) & \longrightarrow & KH(I^{-1}(J^{-1}A)). \end{array}$$

This square is homotopy cartesian by 3.6. Finally, the Fundamental Theorem states that  $KH(k[t, t^{-1}]) \cong KH(k) \times \Omega^{-1}KH(k)$ , so from the square we deduce the split fibration

$$KH(k) \rightarrow KH(\mathbf{P}_k^1) \xrightarrow{\simeq} KH(k[t]),$$

whence the calculation of  $KH(\mathbf{P}_k^1)$ .

Many of the properties of  $KH$ -theory which hold for rings can be immediately extended to schemes with an affine vector bundle torsor via Jouanolou's device. For example, there is a map

$$K(X) \rightarrow K(W) = K(B) \rightarrow KH(B) \cong KH(X),$$

where  $K(X)$  is the connected spectrum with  $\Omega^\infty K(X) = \Omega BQ\mathbf{P}(X)$ . If  $X$  is regular and quasiprojective, so is any vector bundle torsor over  $X$ , so that  $KH(X) \cong K(X)$  by [Q, p. 128]. Clearly  $KH(\mathbf{A}_X^n) \cong KH(X)$ ; more generally,  $KH(E) \cong KH(X)$  if  $E \rightarrow X$  is a vector bundle, or even a torsor under a vector bundle. (Because an affine vector bundle torsor over  $E$  is also one over  $X$ .)

Another property, immediate from 2.3 and 3.2, is that

$$KH(X) \cong KH(X_{\text{red}}) \cong KH(^+X)$$

where  $X_{\text{red}}$  is the reduced scheme of  $X$  and  $^+X$  is the seminormalization of  $X_{\text{red}}$ . The analogue of the Mayer-Vietoris sequence of an ideal (1.6) is not quite as immediate. We offer two versions here:

PROPOSITION 4.9. (Conductor Square) *Let  $p : \tilde{X} \rightarrow X$  be a finite map of schemes, and suppose there is a closed subscheme  $Y$  of  $X$  so that, setting  $\tilde{Y} = p^{-1}(Y)$ ,*

$$\tilde{X} - \tilde{Y} \cong X - Y.$$

*Assuming  $X$  is quasiprojective (or has an affine vector bundle torsor), then the square*

$$\begin{array}{ccc} KH(X) & \longrightarrow & KH(\tilde{X}) \\ \downarrow & & \downarrow \\ KH(Y) & \longrightarrow & KH(\tilde{Y}) \end{array}$$

*is homotopy cartesian, and there is a long exact sequence*

$$\cdots KH_{n+1}(\tilde{Y}) \rightarrow KH_n(X) \rightarrow KH_n(\tilde{X}) \oplus KH_n(Y) \rightarrow KH_n(\tilde{Y}) \cdots$$

PROOF: Using Jouanolou's device, we may assume that  $X = \text{Spec}(A)$  and  $Y = \text{Spec}(A/J)$  for some radical ideal  $J$  of  $A$ . We may also assume  $A$  is reduced by 2.3.1. Since  $p$  is affine, there is a fin. gen.  $A$ -algebra  $B$  so that  $\tilde{X} = \text{Spec}(B)$ . The hypothesis implies that  $A[s^{-1}] \cong B[s^{-1}]$  for each  $s \in J$ . Letting  $p$  denote the map  $A \rightarrow B$ , this implies that  $p : J \rightarrow B$  is an injection and that for each  $s \in J$  some  $Bs^n$  lies in  $p(A)$ . Consequently,

$$I = \{s \in J : Bs \subseteq p(A)\}$$

is an ideal of  $A$  with radical  $J$ , and  $p(I)$  is an ideal of  $B$ . The square we want is equivalent to the square 2.2, since evidently  $KH(Y) \cong KH(A/I)$  and  $KH(\tilde{Y}) \cong KH(B/I)$ .

COROLLARY 4.10. (Mayer-Vietoris for closed covers) *Let  $Y_1, Y_2$  be closed subschemes of  $X$  with  $X = Y_1 \cup Y_2$ . Assume that  $X$  is quasiprojective (or more generally, that  $X$  has an affine vector bundle torsor). Then the square*

$$\begin{array}{ccc} KH(X) & \longrightarrow & KH(Y_1) \\ \downarrow & & \downarrow \\ KH(Y_2) & \longrightarrow & KH(Y_1 \cap Y_2) \end{array}$$

*is homotopy cartesian.*

PROOF: Take  $\tilde{X} = Y_1$  and  $Y = Y_2$  in 4.9.

## §5. Brown-Gersten Theorem.

The goal of this section is to prove a result of Thomason:

**THEOREM 5.1.** (Mayer-Vietoris property) *Let  $X$  be a quasicompact separated scheme with an ample line bundle (or with an ample family of line bundles). Then for every Zariski open cover  $X = U \cup V$ , the following square is homotopy cartesian:*

$$\begin{array}{ccc} KH(X) & \longrightarrow & KH(V) \\ \downarrow & & \downarrow \\ KH(U) & \longrightarrow & KH(U \cap V). \end{array}$$

The proof here follows the idea sketched by Bob Thomason in [TL], but is machine-free in the spirit of [WBG]. I am grateful to him for pointing this result out to me, and for explaining the details of [TL]. One important consequence of this result is

**COROLLARY 5.2.** (Brown-Gersten Spectral Sequence) *Let  $X$  be a quasiprojective scheme over an affine base, which is noetherian of finite Krull dimension. Let  $\tilde{K}\tilde{H}_q$  be the sheaf associated to the presheaf  $U \mapsto KH_q(U)$  on  $X$ . Then there is a convergent right half-plane spectral sequence*

$$E_2^{p,q} = H^p(X; \tilde{K}\tilde{H}_{-q}) \Rightarrow KH_{-p-q}(X).$$

**PROOF:** We cite the usual Brown-Gersten Theorem of [BG, p. 283], [T, 2.5].

We will prove Theorem 5.1 by a bootstrapping process, starting with 2.8. Using it, we establish an intermediate version of 5.1:

**PROPOSITION 5.3.** *Let  $i : U \rightarrow X$  be an open immersion with  $X = \text{Spec}(A)$  affine. Let  $s \in A$  be such that  $X = U \cup V$ , where  $V = D(s) = \text{Spec}(A_s)$ . Then there is a homotopy cartesian square*

$$\begin{array}{ccc} KH(X) & \longrightarrow & KH(V) \\ \downarrow & & \downarrow \\ KH(U) & \longrightarrow & KH(U \cap V) \end{array}$$

**PROOF:** Since  $\mathcal{O}_X$  is an ample line bundle on  $U$ , there is an affine vector bundle torsor  $p : \text{Spec}(B) \rightarrow U$ , and  $B$  is an  $A$ -algebra. By Jouanolou's device, we have to prove that the square

$$\begin{array}{ccc} KH(A) & \longrightarrow & KH(A_s) \\ \downarrow & & \downarrow \\ KH(B) & \longrightarrow & KH(B_s) \end{array}$$

is homotopy cartesian. Let  $J$  be such that  $V(J) = X - U$ ; since  $sA + J = A$ , some  $t \in J$  has  $sA + tA = A$ . By 2.8 we have homotopy cartesian squares:

$$\begin{array}{ccc} KH(A) & \longrightarrow & KH(A_s) & & KH(B) & \longrightarrow & KH(B_s) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ KH(A_t) & \longrightarrow & KH(A_{st}) & & KH(B_t) & \longrightarrow & KH(B_{st}). \end{array}$$

Since  $\text{Spec}(A_t) \subset U$  we see that  $\text{Spec}(B_t) \rightarrow \text{Spec}(A_t)$  is a vector bundle. Hence  $KH(A_t) \cong KH(B_t)$  and  $KH(A_{st}) \cong KH(B_{st})$ . The result follows from a diagram chase of the relevant homotopy fibers.

PROOF OF 5.1: We can assume  $X = \text{Spec}(A)$  by Jouanolou's device. Let  $I$  and  $J$  be ideals of  $A$  defining the complements of  $U$  and  $V$ , respectively. Because  $I + J = A$ , some  $t \in J$  has  $I + tA = A$ . Hence  $D(t) \subset V$  and  $\{U, D(t)\}$  also cover  $X$ . By 5.3, the outer square in the following diagram is homotopy cartesian:

$$\begin{array}{ccccc} KH(X) & \longrightarrow & KH(V) & \longrightarrow & KH(D(t)) \\ \downarrow & & \downarrow & & \downarrow \\ KH(U) & \longrightarrow & KH(U \cap V) & \longrightarrow & KH(U \cap D(t)). \end{array}$$

We will show that the right-hand square is also homotopy cartesian; a standard diagram chase shows that the left-hand square is homotopy cartesian, proving the theorem.

Let  $W = \text{Spec}(B)$  be an affine vector bundle torsor over  $V$ . Then  $B$  is an  $A$ -algebra, and the open set  $D_B(t)$  of  $\text{Spec}(B)$  lies over the open set  $D_A(t)$  of  $\text{Spec}(A)$ . Pulling the right-hand square back to  $W = \text{Spec}(B)$  yields a homotopy equivalent square which is homotopy cartesian by 5.3, proving the claim.

## §6. Čech Descent.

The Mayer-Vietoris property for  $KH$ -theory allows us to extend  $KH$ -theory from quasiprojective schemes to all schemes. The idea is very simple: choose a cover  $\{U_i\}$  of  $X$  by affine open subschemes, and define  $KH(X)$  to be the homotopy limit of the  $KH(U_1 \cap \cdots \cap U_n)$ . Then prove that  $KH(X)$  is independent of the choice of cover, and show it agrees with the previous definition of  $KH(X)$  if  $X$  happens to be quasiprojective (or supports an ample line bundle). The goal of this section is to make this idea work, using Thomason's Čech hypercohomology construction.

Suppose that  $X$  is a quasiprojective scheme, or more generally that  $X$  is a quasicompact separated scheme with an ample line bundle. Then  $KH(U)$  is defined for every open  $U$  in  $X$ , and we can think of  $KH$  as a presheaf of fibrant spectra on  $X$ . Actually, since  $KH$  is not functorial (see Remark

4.3.1) we need to replace  $KH(U)$  by a homotopy equivalent spectrum to get a presheaf, but we shall relegate this technicality to the Appendix.

The presheaf  $KH$  is suitable fodder for the machinery of Brown-Gersten [BG] and Thomason [T], and the theory of *op. cit.* applies. In order to allow a reader who is unfamiliar with that machinery to follow along, we shall pause for some definitions before proceeding.

Let  $F$  be a presheaf of fibrant spectra on a topological space  $X$ . Given an open cover  $\mathcal{U} = \{U_i\}$  of  $X$ , let  $f\mathcal{U}$  denote the poset of finite intersections  $U_\alpha = U_{i_1} \cap \dots \cap U_{i_n}$  of opens in  $\mathcal{U}$ . As in [T, 1.9] we define

$$\check{H}(\mathcal{U}; F) = \operatorname{holim}_{f\mathcal{U}} F(U_i \cap \dots \cap U_j)$$

or equivalently as

$$\operatorname{holim}_{\Delta} (\Pi F(U_i) \rightrightarrows \Pi F(U_i \cap U_j) \rightrightarrows \dots).$$

For example, when  $\mathcal{U}$  has only two elements the square

$$(6.1) \quad \begin{array}{ccc} \check{H}(\{U_1, U_2\}; F) & \longrightarrow & F(U_1) \\ \downarrow & & \downarrow \\ F(U_2) & \longrightarrow & F(U_1 \cap U_2) \end{array}$$

is homotopy cartesian. The functor  $\check{H}(\mathcal{U}; \_)$  from presheaves of fibrant spectra to fibrant spectra preserves homotopy equivalences and homotopy limits [T, 1.5], and by [T, 1.6] there is a Čech cohomology spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{U}; \pi_{-q}F) \Rightarrow \pi_{-p-q}\check{H}(\mathcal{U}; F).$$

For change of covers, we have the following result from [T, 1.12, 1.20, 1.21].

**PROPOSITION 6.2.** *Let  $\mathcal{U}$  and  $\mathcal{V}$  be two Zariski open covers of  $X$ , and let  $F$  be a presheaf of fibrant spectra on  $X$ . If  $\mathcal{V}$  is a refinement of  $\mathcal{U}$ , then there is a map  $\check{H}(\mathcal{U}; F) \rightarrow \check{H}(\mathcal{V}; F)$ . If  $\mathcal{U}$  and  $\mathcal{V}$  are refinements of each other, this is a homotopy equivalence.*

I learned the next result from Thomason, but could locate no appropriate literature reference for it.

**THEOREM 6.3.** *Let  $F$  be a presheaf of fibrant spectra on a topological space  $X$ . Suppose that for every open sets  $U, V$  of  $X$*

$$\begin{array}{ccc} F(U \cup V) & \longrightarrow & F(V) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(U \cap V) \end{array}$$

is homotopy cartesian. Then for every finite cover  $\mathcal{U}$  of  $X$  the map  $F(X) \rightarrow \check{H}(\mathcal{U}; F)$  is a homotopy equivalence. If  $X$  is quasicompact, then  $F(X) \rightarrow \check{H}(\mathcal{U}; F)$  is a homotopy equivalence for every cover  $\mathcal{U}$  of  $X$ .

PROOF: We proceed by induction on the size of  $\mathcal{U}$ , since the case  $\mathcal{U} = \{X\}$  is trivial, and the case  $\mathcal{U} = \{U, V\}$  is given by (6.1). Given  $\mathcal{U} = \{U_0, \dots, U_n\}$ , set  $U = U_0$  and  $V = U_1 \cup \dots \cup U_n$ . Let  $F|U$  denote the presheaf on  $X$  given by cobase-change, i.e., by  $(F|U)(W) = F(U \cap W)$ . Since  $\{U\}$  and  $\{U \cap U_i, i = 0, \dots, n\}$  refine each other as covers of  $U$ , we see by 6.2 that

$$\check{H}(\mathcal{U}; F|U) = \check{H}(\{U \cap U_i\}; F) \cong \check{H}(\{U\}; F) \cong F(U).$$

If we define  $F|(U \cap V)$  and  $F|V$  similarly, we see that the same refinement argument gives  $\check{H}(\mathcal{U}; F|(U \cap V)) \cong F(U \cap V)$ , while our inductive assumption gives

$$\check{H}(\mathcal{U}; F|V) = \check{H}(\{V \cap U_i\}; F) \cong \check{H}(\{U_1, \dots, U_n\}; F) \cong F(V).$$

Now our hypothesis asserts that for every open set  $W$  in  $X$  the square

$$\begin{array}{ccc} F(W) & \longrightarrow & (F|V)(W) \\ \downarrow & & \downarrow \\ (F|U)(W) & \longrightarrow & (F|(U \cap V))(W) \end{array}$$

is homotopy cartesian. Since  $\check{H}(\mathcal{U}; \_)$  preserves homotopy limits, there is a map of homotopy cartesian squares:

$$\begin{array}{ccccc} F(X) & \longrightarrow & F(V) & & \check{H}(\mathcal{U}; F) & \longrightarrow & \check{H}(\mathcal{U}; F|V) \\ \downarrow & & \downarrow & \longrightarrow & \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(U \cap V) & & \check{H}(\mathcal{U}; F|U) & \longrightarrow & \check{H}(\mathcal{U}; F|(U \cap V)). \end{array}$$

Since three of the corners are homotopy equivalent, the fourth corner  $F(X) \rightarrow \check{H}(\mathcal{U}; F)$  is also a homotopy equivalence, completing the inductive step.

Now suppose that  $X$  is quasicompact and that  $\mathcal{U}$  is a cover of  $X$ . By quasicompactness,  $\mathcal{U}$  has a finite subcover  $\mathcal{V}$ . Let  $G$  be the presheaf on  $X$  defined by  $G(W) = \check{H}(W \cap \mathcal{V}; F)$ . Since  $\mathcal{V}$  is finite, we know each  $F(W) \rightarrow \check{H}(W \cap \mathcal{V}; F)$  is a homotopy equivalence, i.e., that  $F \simeq G$  as presheaves on  $X$ . Therefore the vertical arrows in the following diagram are homotopy equivalences.

$$\begin{array}{ccccc} F(X) & \longrightarrow & \check{H}(\mathcal{U}; F) & \longrightarrow & \check{H}(\mathcal{V}; F) \\ \simeq \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ \check{H}(\mathcal{V}; F) = G(X) & \longrightarrow & \check{H}(\mathcal{U}; G) & \longrightarrow & \check{H}(\mathcal{V}; G). \end{array}$$

The top composite is the same map as the left vertical map. It follows that the horizontal arrows are inverse homotopy equivalences, proving the theorem.

**COROLLARY 6.4.** (Čech Cohomological Descent) *Let  $X$  be a quasicompact separated scheme with an ample line bundle (or with an ample family of line bundles). Then for every Zariski open cover  $\mathcal{U}$  of  $X$  there is a homotopy equivalence of spectra*

$$KH(X) \rightarrow \check{H}(\mathcal{U}; KH).$$

**PROOF:** By 5.1,  $KH$  satisfies the Mayer-Vietoris property needed to apply 6.3.

**Definition 6.5.** Let  $X$  be a scheme. Choose a covering  $\mathcal{U}$  of  $X$  by affine open subschemes, and set

$$KH(X) = \check{H}(\mathcal{U}; KH).$$

This definition makes sense, because in the homotopy limit used to define  $\check{H}(\mathcal{U}; KH)$  we only need the spectra  $KH(U_1 \cap \dots \cap U_n)$ , and  $U_1 \cap \dots \cap U_n$  is quasiprojective. (If  $X$  is separated, then  $U_1 \cap \dots \cap U_n$  is affine [Hart, Ex. II (4.3)], but in general this is not so.) By 6.4 this definition is homotopy equivalent to the prior definition of  $KH(X)$  if  $X$  is affine, or quasiprojective, or is a quasicompact separated scheme with an ample line bundle. The next result shows that  $KH(X)$  is independent of the choice of affine cover. Recall that a scheme is quasiaffine if it is an open subscheme of an affine scheme.

**PROPOSITION 6.6.** *If  $\mathcal{U}$  and  $\mathcal{V}$  are coverings of  $X$  by quasiaffine open subschemes, then  $\check{H}(\mathcal{U}; KH) \cong \check{H}(\mathcal{V}; KH)$ .*

**PROOF:** (Thomason) Let  $F$  and  $G$  be the presheaves of fibrant spectra on  $X$  defined by  $F(U) = \check{H}(U \cap \mathcal{U}; KH)$  and  $G(U) = \check{H}(U \cap \mathcal{V}; KH)$ . By 6.4 we see that  $F(U) \cong KH(U) \cong G(U)$  whenever  $U$  is a quasiaffine subscheme of  $X$ . In particular,  $\check{H}(\mathcal{V}; F) \cong \check{H}(\mathcal{V}; KH)$  and  $\check{H}(\mathcal{U}; G) \cong \check{H}(\mathcal{U}; KH)$ . On the other hand, since homotopy limits commute we have

$$\begin{aligned} \check{H}(\mathcal{V}; F) &= \check{H}(\mathcal{V}; \check{H}(- \cap \mathcal{U}; KH)) \\ &\cong \check{H}(\mathcal{U}; \check{H}(- \cap \mathcal{V}; KH)) \\ &\cong \check{H}(\mathcal{U}; G). \end{aligned}$$

Having extended the definition of  $KH(X)$  to all schemes  $X$ , we can easily derive the extension of several earlier results.

**PROPOSITION 6.7.** (Mayer-Vietoris Property) *Let  $U, V$  be open subschemes of  $X$  with  $U \cup V = X$ . Then the following square is homotopy cartesian:*

$$\begin{array}{ccc} KH(X) & \longrightarrow & KH(V) \\ \downarrow & & \downarrow \\ KH(U) & \longrightarrow & KH(U \cap V) \end{array}$$

In particular, for each finite cover  $\mathcal{U}$  of  $X$  there is a Čech spectral sequence

$$E_2^{pq} = \check{H}^p(\mathcal{U}; KH_{-q}) \Rightarrow KH_{-p-q}(X).$$

PROOF: Cover  $U, V$  and  $U \cap V$  by affine open subschemes, so that the union of these forms a cover  $\mathcal{U}$  of  $X$ . Let  $F$  be the presheaf  $\check{H}(- \cap \mathcal{U}; KH)$  on  $X$ , so that  $F(X) = KH(X)$ . Since  $U \cap \mathcal{U}$  is a cover by quasiaffine opens,  $F(U) \cong KH(U)$  by 6.2. Similarly,  $F(V) \cong KH(V)$  and  $F(U \cap V) \cong KH(U \cap V)$ . Now let  $G(W) = \check{H}(\{W \cap U, W \cap V\}; KH)$  for quasiprojective subschemes  $W$  of  $X$ ; by 6.4 we see that the map  $KH(W) \rightarrow G(W)$  is a homotopy equivalence. Since homotopy limits commute, we have

$$\check{H}(\{U, V\}; F) \cong \check{H}(\mathcal{U}; G) \cong \check{H}(\mathcal{U}; KH) = KH(X).$$

The desired result now follows from (6.1) and (6.3).

COROLLARY 6.8. (Brown-Gersten) *If  $X$  is a noetherian scheme of finite Krull dimension, there is a spectral sequence*

$$E_2^{pq} = H^p(X; \check{K}\check{H}_{-q}) \Rightarrow KH_{-p-q}(X).$$

To map  $K_*(X)$  to  $KH_*(X)$ , recall that if  $\mathbf{P}(X)$  denotes the exact category of finite type vector bundles on  $X$ , then  $\Omega BQ\mathbf{P}(X)$  is an infinite loop space. Let  $K(X)$  be a connective  $\Omega$ -spectrum so that  $\Omega^\infty K(X) = \Omega BQ\mathbf{P}(X)$ . Then  $\pi_n K(X) = K_n(X)$  for  $n \geq 0$  and  $\pi_n K(X) = 0$  for  $n < 0$ .

LEMMA 6.9. *There is a natural map  $K(X) \rightarrow KH(X)$  which is well-defined up to homotopy.*

PROOF: If  $X$  is affine, this follows from the definition of  $KH(X)$ . If  $W$  is an affine vector bundle torsor over  $X$ , the map is

$$K(X) \rightarrow K(W) \rightarrow KH(W) = KH(X).$$

In general, if  $\mathcal{U}$  is a cover of  $X$  by affine opens, the maps  $K(U_i \cap \cdots \cap U_j) \rightarrow KH(U_i \cap \cdots \cap U_j)$  assemble to give the map

$$f_{\mathcal{U}} : K(X) \rightarrow \check{H}(\mathcal{U}; K) \rightarrow \check{H}(\mathcal{U}; KH) \cong KH(X).$$

If  $\mathcal{V}$  is another affine cover of  $X$  refining  $\mathcal{U}$ , then the functoriality of  $\check{H}(\mathcal{U};) \rightarrow \check{H}(\mathcal{V};)$  and 6.6 shows that  $f_{\mathcal{U}}$  is homotopic to  $f_{\mathcal{V}}$ .

PROPOSITION 6.10. (Gersten [Gsch]) *Let  $X$  be a separated, regular noetherian scheme. Then  $K(X) \cong KH(X)$ .*

PROOF: By [Q,7.3] the presheaf  $K$  has the Mayer-Vietoris property, so by 6.3 we have  $K(X) \cong \check{H}(\mathcal{U}; K)$ . On the other hand, we know that  $\check{H}(\mathcal{U}; K) \cong \check{H}(\mathcal{U}; KH)$ , because for each  $U = U_1 \cap \cdots \cap U_n$  we have  $K(U) \cong KH(U)$ .

For convenience, let us write  $X[t]$  and  $X[t, t^{-1}]$  for the schemes  $X \times_{\mathbf{Z}} \text{Spec}(\mathbf{Z}[t])$  and  $X \times_{\mathbf{Z}} \text{Spec}(\mathbf{Z}[t, t^{-1}])$ .

**THEOREM 6.11.** *Let  $X$  be any scheme. Then*

- (a) *(Homotopy Invariance)*  $KH(X[t]) \cong KH(X)$ ;
- (b) *(Fundamental Theorem)*  $KH(X[t, t^{-1}]) \cong KH(X) \times \Omega^{-1}KH(X)$ ,  
i.e., for all  $n \in \mathbf{Z}$  we have  
 $KH_n(X[t, t^{-1}]) \cong KH_n(X) \oplus KH_{n-1}(X)$ ;
- (c) *(Projective Line)*  $KH(\mathbf{P}_X^1) \cong KH(X) \times KH(X)$ .

**PROOF:** Fix a cover  $\mathcal{U} = \{U_i\}$  of  $X$  by affine open subschemes. Then  $\mathcal{U}[t] = \{U_i[t]\}$  covers  $X[t]$ . By the affine (and quasiprojective) versions of Homotopy Invariance we have  $KH(U_i) \cong KH(U_i[t])$ . Thus

$$KH(X[t]) = \check{H}(\mathcal{U}[t]; KH) \cong \check{H}(\mathcal{U}; KH) = KH(X).$$

The computation for  $X[t, t^{-1}]$  is similar, using the open cover  $\{U_i[t, t^{-1}]\}$ , and noting that the homotopy fiber  $F(U)$  of  $KH(U) \rightarrow KH(U[t, t^{-1}])$  is naturally isomorphic to  $KH(U)$ . For the projective line, apply this line of argument to the open cover  $\{U_i \times \mathbf{P}^1\}$  by quasiprojective subschemes, using the result 4.7 to see that  $KH(U \times \mathbf{P}^1) \cong KH(U) \times KH(U)$ .

**APPENDIX.**

The  $\check{H}$  construction of Thomason [T] requires strict functoriality of  $KH$  on the category  $\mathbf{Top}(X)$  of open subschemes of a scheme  $X$ , at least up to infinitely coherent homotopy. Unfortunately, Jouanolou’s device only provides us with a homotopy class of spectra  $KH(U)$ , one for each affine vector bundle torsor over  $U$ , so that strict functoriality fails. In this appendix we remedy this (following a suggestion of Thomason) by providing a strict functor  $B(U)$  from  $\mathbf{Top}(X)$  to commutative rings, so that  $U \mapsto KH(B(U))$  is a strict functor, i.e., a presheaf of fibrant spectra on  $X$ .

Let  $X$  be a scheme, and let  $\mathcal{C}_X$  denote the category whose objects are tuples

$$\alpha = (W_\alpha \rightarrow X, \{W_\alpha|U \rightarrow U, U \in \mathbf{Top}(X)\})$$

of torsors  $W_\alpha$  under a finite rank vector bundle over  $X$ , provided with a choice of a pullback  $W_\alpha|U$  over each open  $U$  in  $X$ . A morphism in  $\mathcal{C}_X$  is a map  $W_\alpha \rightarrow W_\beta$  of vector bundle torsors over  $X$ . If  $i : U \rightarrow X$  is the inclusion of an open subscheme, there is a natural functor  $i^* : \mathcal{C}_X \rightarrow \mathcal{C}_U$  given by

$$i^*(\alpha) = (W_\alpha|U \rightarrow X, \{W_\alpha|V \rightarrow V, V \subseteq U\}).$$

The  $\mathcal{C}_U$  with  $U \in \mathbf{Top}(X)$  assemble to give a category  $\mathcal{C}(X)$  which is cofibered over the opposite category of  $\mathbf{Top}(X)$ . Note that  $\mathcal{C}(X)$  has a small skeletal subcategory, i.e., a set of isomorphism classes of objects, because each  $\mathcal{C}_U$  does.

Given  $U$  open in  $X$ , let us define the scheme over  $U$

$$W_\infty(U) = \prod \{W_\alpha, \text{ all iso. classes } \alpha \text{ in } \mathcal{C}_U\}.$$

For  $i : V \rightarrow U$  in  $\mathbf{Top}(X)$  we have a map

$$W_\infty(V) \rightarrow i^*W_\infty(U) \rightarrow W_\infty(U)$$

so  $W_\infty$  is a strict functor from  $\mathbf{Top}(X)$  to schemes.

LEMMA A.1. *If  $X$  has an affine vector bundle torsor  $W$ , then  $W_\infty(X)$  is an affine scheme.*

PROOF: Choose an  $\omega \in \mathcal{C}_X$  representing  $W$ . For each finite set  $I$  of objects of  $\mathcal{C}_X$  including  $\omega$ ,  $W_I = \prod\{W_\alpha : \alpha \in I\}$  is a torsor over  $W$ , hence a vector bundle, hence an affine scheme. Since  $W_\infty(X)$  is the colimit of  $W_I$ 's, it too is affine.

PROPOSITION A.2. *If  $X$  has an affine vector bundle torsor  $W = \text{Spec}(A)$  and  $B$  is such that  $W_\infty(X) = \text{Spec}(B)$ , then  $KH(A) \cong KH(B)$ .*

PROOF: Choose an  $\omega \in \mathcal{C}_X$  representing  $W$  to get  $W_\infty(X) \rightarrow W$ . For each finite set  $I$  of objects of  $\mathcal{C}_X$  including  $\omega$ ,  $W_I = \text{Spec}(B_I)$  for some  $A$ -algebra  $B_I$  and  $KH(A) \cong KH(B_I)$  by (4.1). Since  $B$  is the colimit of  $B_I$ 's as an  $A$ -algebra, we have  $KH(A) \cong KH(B)$ .

If  $U$  is open in  $X$ , let  $B(U)$  denote the ring of global functions on the scheme  $W_\infty(U)$ . Then  $B$  is a strict functor from  $\mathbf{Top}(X)$  to commutative rings, and therefore  $U \mapsto KH(B(U))$  is a presheaf of fibrant spectra on  $X$ .

If  $X$  is a quasicompact separated scheme with an ample line bundle, then every open  $U$  in  $X$  has an affine vector bundle torsor by (4.3). By Proposition A.2,  $KH(B(U))$  is homotopy equivalent to the  $KH(U)$  provided by Jouanolou's device in (4.6). Therefore, the statement of Čech Cohomological Descent (6.4) should be read with  $KH$  denoting the presheaf  $U \mapsto KH(B(U))$ .

In the definition (6.5) of  $KH(X)$  for an arbitrary scheme  $X$ , the presheaf  $KH$  should also be interpreted as  $U \mapsto KH(B(U))$ . As noted there, the only  $U$  for which  $KH$  is needed are quasiprojective, so that  $KH(U) \cong KH(B(U))$ .

If  $Y \rightarrow X$  is a map of schemes, the above construction does not provide a strictly functorial map  $KH(X) \rightarrow KH(Y)$ . One way to do this is to work in a fixed Grothendieck universe and proceed as follows. Modify the above discussion by replacing  $\mathbf{Top}(X)$  with the category of all schemes over  $X$  in the universe. Thus objects  $\alpha$  of  $\mathcal{C}_X$  contain choices of pullbacks  $W_\alpha|Y$  for every  $Y \rightarrow X$  in the universe.  $\mathcal{C}_X$  is no longer small, but has a skeletal subcategory in the next bigger universe. Thus  $W_\infty(X)$  and  $B(X)$  live in the next bigger universe, as does the presheaf  $U \mapsto KH(B(U))$  of fibrant spectra on  $X$ .

This presheaf is strictly functorial within our fixed universe, providing a strictly functorial  $KH(X) \rightarrow KH(Y)$ .

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