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$K(A, B, I)$: II

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Abstract. The 'KABI' conjecture states that double relative K-theory and cyclic homology agree, at least in characteristic zero. We show that $K_2(A, B, I)$ maps onto $HC_1(A, B, I)$ whenever $A \to B$ is a map of $\mathbf{Q}$-algebras and $I \cong BIB$. We also reinterpret the KABI conjecture in terms of the injectivity of the inverse limit of the map from $NK(A, B, I)$ to the inverse limit of the truncated polynomial versions of $NK(A, B, I)$.

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0. Introduction

A standard scenario for computing the algebraic K-theory of a ring A is the 'Excision situation', in which there is an ideal I of A and a ring map $A \to B$ so that $I \cong BIB$. In this situation, the double relative groups $K_i(A, B, I)$ are the obstructions to the existence of a Mayer–Vietoris sequence for the K-theory of the square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A/I & \longrightarrow & B/I. \end{array}$$

In [4], it was conjectured that, when $A \to B$ is a map of $\mathbf{Q}$-algebras, then there is an isomorphism

$$v\colon K_i(A, B, I) \to HC_{i-1}(A, B, I)$$

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for all i, where the right-hand side is double relative cyclic homology. This 'KABI conjecture' is known to be true for $i \leqslant 1$ [5], if $A \to B$ is onto [8], or if I is nilpotent [3]. In this paper, we establish the following partial result for $i = 2$.

THEOREM A. *If $A \to B$ is a map of $\mathbf{Q}$-algebras and I is an ideal of A with $I \cong BIB$, then there is a surjection*

$$K_2(A,B,I) \to HC_1(A,B,I).$$

The proof goes as follows. From [6] there is an exact diagram

$$\begin{array}{ccccccc} N^2K_2(A,B,I) & \longrightarrow & NK_2(A,B,I) & \longrightarrow & K_2(A,B,I) & \longrightarrow & 0 \\ \downarrow {\scriptstyle N^2\nu} & & \downarrow {\scriptstyle N\nu} & & \downarrow {\scriptstyle \nu} & & \\ N^2HC_1(A,B,I) & \longrightarrow & NHC_1(A,B,I) & \longrightarrow & HC_1(A,B,I) & \longrightarrow & 0, \end{array}$$

where NF is Bass' nil functor of [1], XII. Since $N^2F(A,B,I)$ is a natural summand of $NF(A[t],B[t],I[t])$, $N^2\nu$ will be a surjection whenever $N\nu$ is a surjection for all (A,B,I). This reduces the proof to showing that $N\nu$ is a natural surjection. To this end, we devote section 1 to setting up our notation and establishing an exact diagram

$$\begin{array}{ccccc} NK_2(A,B,I) & \longrightarrow & K_2(A[t]/t^n, B[t]/t^n, I, t) & \to & K_1(A[t], B[t], I, t^n) \\ \downarrow {\scriptstyle N\nu} & & \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle \cong} \\ NHC_1(A,B,I) & \xrightarrow{f_n} & HC_1(A[t]/t^n, B[t]/t^n, I, t) & \to & HC_0(A[t], B[t], I, t^n) \end{array} \qquad (\dagger)$$

for each n (see 1.2, 1.3 and 1.5). The map $N\nu$ is a map of weighted vector spaces (see 2.4 or [4], §9), i.e., it maps the weight s subspace of NK_2 (A, B, I) to the weight s subspace of $NHC_1(A,B,I)$. Therefore the observation that f_n is an injection on the subspaces of weights $s < n$ (see 2.2) is enough to finish the proof of Theorem A.

Our second result, which we prove in Section 2, is a reinterpretation of the 'KABI conjecture'.

THEOREM B. *The following are equivalent for each m:*

(a) *For all $i \leqslant m$ and all $\mathbf{Q}$-algebra maps $A \to B$ with $I \cong BIB$,*

$$\nu: K_i(A,B,I) \cong HC_{i-1}(A,B,I).$$

(b) *For all $i \leqslant m$ and all $\mathbf{Q}$-algebra maps $A \to B$ with $I \cong BIB$,*

$$NK_i(A,B,I) \cong NHC_{i-1}(A,B,I).$$

(c) *For all $i \leqslant m$ and all $\mathbf{Q}$-algebra maps $A \to B$ with $I \cong BIB$,*

$$NK_i(A,B,I) \longrightarrow \varprojlim K_i(A[t]/t^n, B[t]/t^n, I, t)$$

is an injection.

(d) *For all $i \leqslant m$, all $\mathbf{Q}$-algebra maps $A \to B$ with $I \cong BIB$, and all n, the weighted vector space $K_i(A[t], B[t], t^nI[t])$ has no nonzero subspaces of weight less than n.*

1. Triple Relative Groups

We need to extend Goodwillie's theorem, that the relative groups $K_n(A,I)$ and $HC_{n-1}(A,I)$ are isomorphic when $\mathbf{Q} \subset A$ and I is nilpotent, to multiple relative situations (see 1.3 below). This is straightforward, and largely an exercise in notation.

Let us begin by setting up the notation. If X is a subset of a ring A, we write A/X for A/AXA. Suppose we are given a functor F from rings to spectra. We write $F(A,X)$ for the homotopy fiber of the map $F(A) \to F(A/X)$. If $f\colon A \to B$ is a ring map, we write $F(A,B,X)$ for the homotopy fiber of the map $F(A,X) \to F(B,f(X))$. If Y is also a subset of A, we write $F(A,B,X,Y)$ for the homotopy fiber of the map $F(A,B,X) \to F(A/Y,B/Y,X)$.It is easy to see that (up to homotopy equivalence) the definition of $F(A,B,X,Y)$ is symmetric in X and Y. The homotopy groups $\pi_n F(?)$ are traditionally written as $F_n(?)$.

If F is a functor from rings to chain complexes, we can repeat the above definitions, replacing the 'homotopy fiber' of a map $u\colon C' \to C$ of chain complexes with the 'shifted mapping cone' $\mathrm{Con}(u)[+1]$ of [2], pp. 37–8, and replacing 'homotopy equivalence' with 'homologism' ($=$ isomorphism on homology). When F is the chain complex for cyclic homology, we shall write $HC_n(?)$ for $H_n(\mathrm{F}(?))$.

There is a Dold–Kan functor from chain complexes to spectra; if $X(C)$ is the topological space of [10], section 4, the spectrum associated to a chain complex C is the sequence $X(C), X(C[-1]), \ldots$ of spaces with identifications $\Omega X(C[-n-1]) \cong X(C[-n])$. The conventions we have made for chain complex-valued functors are compatible with the conventions for spectrum-valued functors under the Dold–Kan map.

Here is an application of our notation. First note that if $X \subset Y$, then $F(A,B,X,Y) = F(A,B,X)$ almost by definition. For convenience, we phrase the next result in the language of spectra; the result for chain-complexes is the same, except that 'homotopy fibration' needs to be interpreted as a 'distinguished triangle'.

LEMMA 1.1. *If $J \subseteq AXA \cap AYA$, then there is a homotopy fibration*

$$F(A,B,J) \longrightarrow F(A,B,X,Y) \longrightarrow F(A/J,B/J,X,Y).$$

Proof. Consider the following diagram

$$\begin{array}{ccccc}
F(A,B,X,Y,J) & \longrightarrow & F(A,B,X,J) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
F(A,B,X,Y) & \longrightarrow & F(A,B,X) & \longrightarrow & F(A/Y,B/Y,X) \\
\downarrow & & \downarrow & & \downarrow \\
F(A/J,B/J,X,Y) & \longrightarrow & F(A/J,B/J,X) & \longrightarrow & F(A/Y,B/Y,X)
\end{array}$$

By definition, the three columns and bottom two rows are homotopy fibrations. Hence the top row is also a homotopy fibration, and therefore

$$F(A,B,X,Y,J) \cong F(A,B,X,J) \cong F(A,B,J).$$ □

Here is another application of our notation. If I is an ideal of A and $A \to B$ is a ring map, the definition of NF yields

$$NF(A,I) = F(A[t],I,t),$$

$$NF(A,B,I) = F(A[t],B[t],I,t).$$

Mapping $A[t]$ to $A_n = A[t]/t^n$ and $B[t]$ to $B_n = B[t]/t^n$, we see that $F(A[t],B[t],I,t^n)$ is the homotopy fiber of the map

$$NF(A,B,I) \to F(A_n,B_n,I,t).$$

By naturality of the map v of [10, 6.9], we obtain the following proposition.

PROPOSITION 1.2. *If $A \to B$ is a map of $\mathbf{Q}$-algebras, and I is any ideal of A, there is a map of long exact sequences*

$$\begin{array}{ccccccc}
\cdots \to & K_i(A[t],B[t],I,t^n) & \to & NK_i(A,B,I) & \to & K_i(A_n,B_n,I,t) & \to \cdots \\
& \downarrow v & & \downarrow v & & \downarrow v & \\
\cdots \to & HC_{i-1}(A[t],B[t],I,t^n) & \to & NHC_{i-1}(A,B,I) & \to & HC_{i-1}(A_n,B_n,I,t) & \to \cdots
\end{array}$$

THEOREM 1.3. (Goodwillie). *Let $A \to B$ be a map of $\mathbf{Q}$-algebras, and let I be a nilpotent ideal of A such that BIB is also nilpotent. Then for all i*

$$v\colon K_i(A,B,I) \cong HC_{i-1}(A,B,I),$$

and for all $X \subset A$,

$$v\colon K_i(A,B,X,I) \cong HC_{i-1}(A,B,X,I).$$

Proof. By Goodwillie's theorem, $K_i(R,I) \cong HC_{i-1}(R,I)$ for $R = A, B, A/X$ or B/X, and all i. By the 5-lemma, this yields $K_i(A,B,I) \cong HC_{i-1}(A,B,I)$ and a similar isomorphism mod X. Again using the 5-lemma, we find that

$$K_i(A,B,I,X) \cong HC_{i-1}(A,B,I,X).$$

As $F(A,B,I,X) \cong F(A,B,X,I)$ for any F, this proves 1.3. □

THEOREM 1.4. *If $A \to B$ is a map of $\mathbf{Q}$-algebras, $J = AXA \cap AYA$, and $BJB = BXB \cap BYB$, then for all i*

$$v\colon K_i(A/J,B/J,X,Y) \cong HC_{i-1}(A/J,B/J,X,Y).$$

Proof. Consider the diagram for $F = K$ (resp., HC):

$$\begin{array}{ccccc}
F(A/J,B/J,X,Y) & \to & F(A/J,B/J,X) & \to & F(A/Y,B/Y,X) \\
\downarrow & & \downarrow & & \downarrow \\
F(A/J,A/Y,X) & \to & F(A/J,X) & \to & F(A/Y,X) \\
\downarrow & & \downarrow & & \downarrow \\
F(B/J,B/Y,X) & \to & F(B/J,X) & \to & F(B/Y,X)
\end{array}$$

By definition, the right two columns and all three rows are homotopy fibrations. Hence the left column is also a homotopy fibration. By naturality of v, there is a map of long exact sequences

$$\begin{array}{ccccccc}
\cdot \to K_{i+1}(B/J, B/Y, X) & \longrightarrow & K_i(A/J, B/J, X, Y) & \longrightarrow & K_i(A/J, A/Y, X) \to \cdot \\
\downarrow v_A \quad \downarrow v_B & & \downarrow v & & \downarrow v_A \quad \downarrow v_B \\
\cdot \to HC_i(B/J, B/Y, X) & \longrightarrow & HC_{i-1}(A/J, B/J, X, Y) & \longrightarrow & HC_{i-1}(A/J, A/Y, X) \to \cdot
\end{array}$$

By [8], the maps v_A and v_B are isomorphisms, so the 5-lemma yields the desired result that v is an isomorphism. □

COROLLARY 1.5. *Let $A \to B$ be a map of $\mathbf{Q}$-algebras, and suppose I is an ideal of A such that $I \cong BIB$. Then for all n*

$$v: K_1(A[t], B[t], I, t^n) \cong HC_0(A[t], B[t], I, t^n).$$

Proof. We apply 1.1 with $X = I$, $Y = \{t^n\}$ and $J = t^n I[t]$ to get the exact diagram

$$\begin{array}{ccc}
K_2(A[t]/J, B[t]/J, I, t^n) & \xrightarrow{\cong} & HC_1(A[t]/J, B[t]/J, I, t^n) \\
\downarrow & & \downarrow \\
K_1(A[t], B[t], t^n I[t]) & \xrightarrow{\cong} & HC_0(A[t], B[t], t^n I[t]) \\
\downarrow & & \downarrow \\
K_1(A[t], B[t], I, t^n) & \longrightarrow & HC_0(A[t], B[t], I, t^n) \\
\downarrow & & \downarrow \\
K_1(A[t]/J, B[t]/J, I, t^n) & \xrightarrow{\cong} & HC_0(A[t]/J, B[t]/J, t^n) \\
\downarrow & & \downarrow \\
K_0(A[t], B[t], t^n I[t]) & \xrightarrow{\cong} & HC_{-1}(A[t], B[t], t^n I[t])
\end{array}$$

The maps labeled '$\cong$' are isomorphisms by 1.4, [5], 1.4 and [5], respectively. (The bottom groups are knows to be zero.) The result we want follows from the 5-lemma. □

REMARK 1.5.1. Extending the above diagram, we obtain surjections from $K_2(A[t], B[t], t^n I[t])$ and $K_2(A[t], B[t], I, t^n)$ onto the corresponding HC_1 terms. We will not need this result.

2. The Weight Structure of NHC

One chain complex whose homology is HC_* of a k-algebra A is the total complex $C(A)$ of the double chain complex $\mathbf{C}(A)$ of [7], section 1. In degree d it is a direct sum of groups $A^{\otimes n} = A \otimes \cdots \otimes A$ (n *terms*), $1 \leqslant n \leqslant d + 1$.

Now suppose that A is graded. As in [4], the complex $C(A)$, and thus the groups $HC_*(A)$, are graded by *weight*. The weight s subcomplex of $C(A)$ is that part generated by terms $a_1 \otimes \cdots \otimes a_n$ with $a_i \in A_{s_i}$ and $\Sigma s_i = s$. If I is a graded ideal of A, the map $C(A) \to C(A/I)$ is onto, so its kernel $C'(A, I)$ has the same homology as $C(A, I)$. The complex $C'(A, I)$ is also graded by weight; its weight s summand is generated by terms $a_1 \otimes \cdots \otimes a_n$ with $a_i \in A_{s_i}$, $\Sigma s_i = s$, and some a_i in I. In particular, the weight s subcomplex of $C'(A, I)$ is zero whenever $I_0 = I_1 = \cdots = I_s = 0$. This proves

LEMMA 2.1. (Cf. [4, 9.5]) *If $I = I_n \oplus I_{n+1} \oplus \dots$ is a graded ideal of a graded k-algebra $A = A_0 \oplus A_1 \oplus \dots$, then $HC_*(A, I)$ is graded by weight, and the weight s summands are zero for all $s < n$.*

It is easy to formulate multiple relative versions of 2.1, most of which we shall leave to the reader. We shall concentrate on the situations needed for Theorems A and B.

PROPOSITION 2.2. *Let $A \to B$ be a map of (ungraded) k-algebras. If $I \subset A$ and $J = t^n I[t]$, then for all n $HC_*(A[t], B[t], I, t^n)$, $HC_*(A[t]/J, B[t]/J, I, t^n)$ and $HC_*(A[t], B[t], J)$ are graded by weight, and their weight s summands are zero for all $s < n$. Consequently, the map*

$$f_n \colon NHC_*(A, B, I) \to HC_*(A[t]/t^n, B[t]/t^n, I, t)$$

is a map of weighted vector spaces which is an isomorphism on the subspaces of weight less than n.

Proof. Let X denote either $\{t^n\}$ or $t^n I$. The shifted mapping cone $C'(A[t], B[t], X)$ of $C'(A[t], X) \to C'(B[t], X)$ has the same homology as $C(A[t], B[t], X)$, yet has nothing of weight s when $s < n$. Hence the shifted mapping cone $C'(A[t], B[t], t^n, I)$ of

$$C'(A[t], B[t], t^n) \to C'((A/I)[t], (B/I)[t], t^n)$$

has the same homology as $C(A[t], B[t], I, t^n)$, and has nothing of weight s when $s < n$. The same remarks apply modulo J. This proves the first assertion. The second assertion follows from this, given the long exact sequence of 1.2. □

COROLLARY 2.3. *If $A \to B$ is a map of k-algebras and $I \subset A$, then*

$$\varprojlim f_n \colon NHC_*(A, B, I) \to \varprojlim HC_*(A[t]/t^n, B[t]/t^n, I, t)$$

is an injection.

(2.4) We conclude by interpreting the 'KABI conjecture' for **Q**-algebras. By [9], the $NK_*(A, B, I)$ have the natural structure of a continuous module over the ring $W(\mathbf{Q}) = \Pi_{i=1}^{\infty} \mathbf{Q}$. The coordinate idempotents e_s in $\Pi\mathbf{Q}$ make $NK_*(A, B, I)$ into a weighted vector space, the weight s summand being $e_s * NK_*(A, B, I)$. As observed in [4] section 9, the substitution $t \mapsto 2t$ induces automorphisms on $NK_*(A, B, I)$ and $NHC_*(A, B, I)$, and the weight s subspaces are the eigenspaces for the eigenvalue 2^s in each case. Hence each

$$v_i \colon NK_i(A, B, I) \to NHC_{i-1}(A, B, I)$$

is a map of weighted vector spaces.

The above considerations apply more generally to graded rings, using the continuous module structures of [11]. In particular, all the maps in the diagram of

Proposition 1.2 are maps of weighted vector spaces. If the KABI conjecture is true, all the maps v in 1.2 are isomorphisms whenever $I \cong BIB$.

Proof of Theorem B. Clearly (a) implies (b). If (b) holds, then $N^pK_q(A,B,I) \cong N^pHC_{q-1}(A,B,I)$ for all $p \geq 1$ and $q \leq m$. By [11], 6.4 and 6.9, and [4], 2.1, there are spectral sequences:

$$E^1_{pq} = N^pF_q(A,B,I) \Rightarrow F_{p+q-1}(A,B,I) \quad (p \geq 1, q \geq 0)$$

for $F_* = \mathrm{K}_*$ or $F_* = HC_*$. Thus (a) holds by the comparison theorem. Hence, (a) and (b) are equivalent.

(a) $\Rightarrow$ (d). This follows immediately from 2.2, given the observation that (a) implies that for all $i \leq m$

$$K_i(A[t], B[t], t^nI) \cong \mathrm{HC}_{i-1}(A[t], B[t], t^nI).$$

(d) $\Rightarrow$ (c). By 1.2, it is enough to show that for all n $K_i(A[t], B[t], I, t^n)$ has nothing of weight less than n. By 1.1 with $J = t^nI[t]$, there is an exact sequence

$$K_i(A[t], B[t], J) \to K_i(A[t], B[t], I, t^n) \to K_i(A[t]/J, B[t]/J, I, t^n)$$

of weighted vector spaces. By (d) there is nothing of weight less than n in the first term. By 1.4,

$$K_i(A[t]/J, B[t]/J, I, t^n) \cong HC_{i-1}(A[t]/J, B[t]/J, I, t^n),$$

and this has nothing of weight less than n by 2.2. Hence $K_i(A[t], B[t], I, t^n)$ has nothing of weight less than n, proving (c).

(c) $\Rightarrow$ (b). We proceed by induction on i, assuming that (a) (the KABI Conjecture) holds for all $j \leq i$. The proof of 1.5 may be applied (with different subscripts) to show that for all n

$$K_i(A[t], B[t], I, t^n) \cong HC_{i-1}(A[t], B[t], I, t^n).$$

Now consider the diagram:

$$\begin{array}{ccc} NK_{i+1}(A,B,I) & \rightarrow & \varprojlim K_{i+1}(A[t]/t^n, B[t]/t^n, I, t) \\ \downarrow & & \downarrow \\ NHC_i(A,B,I) & \rightarrow & \varprojlim HC_i(A[t]/t^n, B[t]/t^n, I, t). \end{array}$$

The right vertical map is an isomorphism by 1.3. The horizontal maps are injections by 2.3 and the assumption (c) for $i+1$. Hence Nv is an injection. On the other hand, Nv is a surjection by diagram (†) of the introduction, with i replacing 1. This proves (b) □

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