$K_1(A, B, I)$

By *Susan C. Geller* at College Station and *Charles A. Weibel**) at New Brunswick

---

In performing calculations in algebraic K-theory, the following setup frequently occurs:

(0.1) The extension situation. $f: A \to B$ is a map of (possibly noncommutative) rings with unit, and I is an ideal of A mapped isomorphically by f onto an ideal of B.

We say that "excision holds for K_i" if the map $K_i(A, I) \to K_i(B, I)$ is an isomorphism. A natural way to study the excision problem is to introduce the double relative groups $K_i(A, B, I)$. These groups fit into a long exact sequence

$$\cdots K_{i+1}(B, I) \to K_i(A, B, I) \to K_i(A, I) \to K_i(B, I) \to \cdots,$$

and are the "obstructions to excision" in the sense that if $K_i(A, B, I) = K_{i-1}(A, B, I) = 0$, then there is an exact "Mayer-Vietoris" sequence

$$\begin{aligned} K_{i+1}(A) &\to K_{i+1}(A/I) \oplus K_{i+1}(B) \to K_{i+1}(B/I) \\ &\xrightarrow{\partial} K_i(A) \to K_i(A/I) \oplus K_i(B) \to K_i(B/I). \end{aligned}$$

To formally define the groups $K_*(A, B, I)$, we proceed as follows. Let $\mathbb{K}(A)$ denote a topological space, natural in A, with $\pi_*(\mathbb{K}(A)) = K_*(A)$. The space $\mathbb{K}(A, I)$ is the homotopy fiber of the map $\mathbb{K}(A) \to \mathbb{K}(A/I)$, and its homotopy groups are the relative K-groups: $\pi_* \mathbb{K}(A, I) = K_*(A, I)$. Finally, the space $\mathbb{K}(A, B, I)$ is the homotopy fiber of the map $\mathbb{K}(A, I) \to \mathbb{K}(B, I)$, and its homotopy groups are the double relative groups: $K_*(A, B, I) = \pi_*(\mathbb{K}(A, B, I))$.

It is classically known that $K_0(A, B, I) = 0$ ([B], p. 482ff). About 1970, Swan [Sw] discovered that $K_1(A, B, I)$ might be nonzero. In 1977, Guin-Loday [GL] and Keune [K1] showed that $K_1(A, B, I)$ is zero when the map $f: A \to B$ is onto. In addition, Guin-Loday [GL] and Keune [K2] showed that, when f is onto, the group $K_2(A, B, I)$ is the 2-sided B-module

$$I/I^2 \underset{A \otimes A^{op}}{\otimes} (\ker f)/(\ker f)^2.$$

---

*) Supported by NSF grant MCS 81-02753

In 1977, Vorst [Vo], (2. 5) extended Swan's results when A and B are commutative, obtaining an exact sequence

$$I/I^2 \otimes \Omega_{B/A} \xrightarrow{\phi_1} K_1(A, I) \longrightarrow K_1(B, I) \longrightarrow 0 .$$

(ϕ_1 is called the Swan-Vorst map, and is sometimes denoted ε.) In [GW], the image of ϕ_1 was identified with a subquotient of $K_2(B/I^2)$, but $K_1(A, B, I)$ itself was not identified. We will show that $K_1(A, B, I)$ actually equals $I/I^2 \otimes \Omega_{B/A}$ in this paper.

Little is known about the higher double relative K-groups. When A is an $S^{-1}\mathbb{Z}$-algebra (e.g., $S^{-1}\mathbb{Z} = \mathbb{Q}$), the groups $K_*(A, B, I)$ are $S^{-1}\mathbb{Z}$-modules (see [W3]). When A is a $\mathbb{Z}/p^n$-algebra, the $K_*(A, B, I)$ are p-groups (see [W2]). It seems reasonable to conjecture that the groups $K_*(A, B, I)$ are modules over the ring $W(R)$ of Witt vectors over the center R of B (see [W2]).

The purpose of this paper is to prove the following theorem.

Theorem (0. 2) (see Theorems (1. 1), (2. 7), (4. 1), and (4. 2)). *Assume the notation of the Excision Situation* (0. 1) (*i.e., I is an ideal of A mapped isomorphically onto an ideal of B by $f: A \to B$*). *There are isomorphisms*:

$$K_1(A, B, I) \cong St(B, I)/\text{image of } St(A, I)$$
$$\cong B \underset{A^e}{\otimes} (I/I^2)/\{b \otimes cx + c \otimes xb - bc \otimes x : b, c \in B,\ x \in I/I^2\} .$$

This abelian group is a module over the center of B/I. There is also an exact sequence in Hochschild homology:

$$0 \to H_1^A(B; I/I^2) \to K_1(A, B, I) \to [B, I/I^2]/[A, I/I^2] \to 0 ,$$

where the right-hand map is the Dieudonné determinant.

When A/I and B/I are commutative, there is a natural isomorphism of B/I-modules:

$$K_1(A, B, I) \cong I/I^2 \underset{B/I}{\otimes} \Omega_{(B/I)/(A/I)} .$$

(The notation employed in the statement of (0. 2) is standard for the subject matter, and is explained further in the text below.)

This paper is arranged as follows. Sections 1 and 2 are mutually independent, and both are independent of the rest of this paper. In § 1, we shall prove that

$$K_1(A, B, I) \cong I/I^2 \otimes \Omega_{B/A}$$

when A and B are commutative. In § 2, we shall use Swan's simplicial definition of higher K-theory in [Sw2] and [K] to show that the topologically defined group $K_1(A, B, I)$ is the cokernel of the group homomorphism $St(A, I) \to St(B, I)$.

§ 3 is a careful exposition of the relative Steinberg group $St(B, I)$. For example, we show that it is generated by the symbols $y_\alpha(b; x)$. We develop enough information about $St(B, I)$ in § 3 to enable us to finish the proof of Theorem (0. 2) in § 4. The previously disconnected threads of § 1, 2 are connected in § 4 as well.

§ 5 is devoted to a few examples and applications of Theorem (0. 2).

Throughout the paper, we have written $\bar{M}$ to denote the image of a group M in another group N under an implicitly understood map $M \to N$. For example, the main part of Theorem (0. 2) reads: $K_1(A, B, I) \cong St(B, I)/\overline{St(A, I)}$.

Acknowledgements. The authors would like to thank W. van der Kallen for explaining the symbols $X(v, x, w)$ to us, and K. Dennis for providing us with Theorem (4. 1), and for several useful discussions about Hochschild homology.

§ 1. The commutative case

In this section, we will give a nontopological proof of our main theorem for the case that A and B are commutative. For this entire section, we will assume the notation of the "Excision Situation" (0. 1), with A and B commutative. That is, $f: A \to B$ maps the ideal I of A isomorphically onto an ideal of B. Our main result is this:

Theorem (1. 1). *When A and B are commutative, there are isomorphisms*

$$\begin{aligned} K_1(A, B, I) &\cong K_1(A/I^2, B/I^2, I/I^2) \\ &\cong I/I^2 \otimes_B \Omega_{B/A} \\ &\cong I \otimes_B \Omega_{B/A}. \end{aligned}$$

Here $\Omega_{B/A}$ is the B-module of relative Kähler differentials, which is generated by symbols db ($b \in B$) (see [H], p. 172).

In Theorem (4. 3) we will connect this computation with the topological computation of § 2 by giving the explicit isomorphism of $I/I^2 \otimes \Omega_{B/A}$ with $St(B, I)/\overline{St(A, I)}$.

This section is independent of the rest of the paper, except that we postpone the proof of the following elementary result until (3. 7): the image of $K_2(B, I^2) \to K_2(B, I)$ is contained in the image of $K_2(A, I) \to K_2(B, I)$. This postponement is only to avoid duplication, and to keep all calculations with the Steinberg groups in the same place.

Theorem (1. 1) is easy to prove when $I^2 = 0$, since then $SK_1(A, I) = 0$ by [B], p. 469. The group $K_1(A, B, I)$ is then the cokernel of the map $K_2(A, I) \to K_2(B, I)$. We know from [GW], (2. 1), though, that, if $I^2 = 0$, the cokernel of this map is isomorphic to $I \otimes \Omega_{B/A}$. This finishes the proof of Theorem (1. 1) when $I^2 = 0$, and establishes the isomorphism $K_1(A/I^2, B/I^2, I/I^2) \cong I/I^2 \otimes \Omega_{B/A}$ in the general setting, since $\Omega_{B/A} \cong \Omega_{(B/I)/(A/I)}$.

In order to prove Theorem (1. 1), then, we only have to show that the map $K_1(A, B, I) \to K_1(A/I^2, B/I^2, I/I^2)$ is an isomorphism. This will follow from an analysis of the following commutative lattice with exact rows and columns:

$$(1.2)\qquad \begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & K_*(A, B, I^2) & \to & K_*(A, B, I) & \to & K_*(A/I^2, B/I^2, I/I^2) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & K_*(A, I^2) & \to & K_*(A, I) & \to & K_*(A/I^2, I/I^2) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & K_*(B, I^2) & \to & K_*(B, I) & \to & K_*(B/I^2, I/I^2) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

The columns are exact by definition; the "sequence for 2 ideals" provides exactness of two rows. The top row is exact by the usual fiber diagram argument. (The squares with four ∂'s commute with a minus sign.)

Part of this lattice is the exact diagram

$$(1.3)\qquad \begin{array}{ccccc}
K_2(A/I^2, B/I^2, I/I^2) & \xrightarrow{\alpha} & K_1(A, B, I^2) & \to & K_1(A, B, I) \\
\downarrow & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} \\
K_2(A/I^2, I/I^2) & \xrightarrow{\partial} & SK_1(A, I^2) & \to & SK_1(A, I).
\end{array}$$

Let Φ_i denote the image of ϕ_i $(i=1, 2)$. By [Sw], p. 238, we know that the map $\Phi_2 \to \Phi_1$ is zero. We can now collapse (1. 3) inside (1. 2) to form the following commutative diagram:

$$(1.4)\qquad \begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & K_2(A/I^2, B/I^2, I/I^2) & \xrightarrow{\phi_2\alpha} & \Phi_2 & \xrightarrow{0} & \Phi_1 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K_2(A, I) & \to & K_2(A/I^2, I/I^2) & \xrightarrow{\partial} & SK_1(A, I^2) & \to & SK_1(A, I) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
K_2(B, I^2) & \to & K_2(B, I) & \to & K_2(B/I^2, I/I^2) & \to & SK_1(B, I^2) & \to & SK_1(B, I) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
K_1(A, B, I^2) & \to & K_1(A, B, I) & \to & I/I^2 \otimes \Omega_{B/A} & \to & 0 & & 0 & & \\
\downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow & & & & & & \\
\Phi_2 & \xrightarrow{0} & \Phi_1 & & 0. & & & & & & \\
\downarrow & & \downarrow & & & & & & & & \\
0 & & 0 & & & & & & & &
\end{array}$$

(compare with [GW], (2. 2)). The top right comes from (1. 3), and the bottom right zeros are the $K_0(A, B, I^i)=0$. We have written $I/I^2 \otimes \Omega_{B/A}$ for $K_1(A/I^2, B/I^2, I/I^2)$. Therefore, the exactness of (1. 2) implies exactness of all rows and columns of (1. 4), except for the row

$$K_2(A/I^2, B/I^2, I/I^2) \xrightarrow{\phi_2\alpha} \Phi_2 \xrightarrow{0} \Phi_1 .$$

We will show below in Lemma (1. 6) that the map $\phi_2\alpha$ is onto, so that this row is also exact. Using this fact, a diagram chase on (1. 4) shows that there is a commutative diagram with exact rows:

$$(1.5)\qquad \begin{array}{ccccccccc} K_2(A, I) & \longrightarrow & K_2(B, I) & \longrightarrow & K_1(A, B, I) & \longrightarrow & \Phi_1 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \| & & \\ K_2(B, I^2) \oplus K_2(A, I) & \longrightarrow & K_2(B, I) & \longrightarrow & I/I^2 \otimes \Omega_{B/A} & \longrightarrow & \Phi_1 & \longrightarrow & 0. \end{array}$$

We can eliminate the summand $K_2(B, I^2)$ from the lower left of (1. 5) and retain exactness, because (see Lemma (3. 7) below) the image of $K_2(B, I^2)$ in $K_2(B, I)$ is contained in the image of $K_2(A, I)$ in $K_2(B, I)$. The 5-Lemma then implies that the natural map

$$K_1(A, B, I) \longrightarrow K_1(A/I^2, B/I^2, I/I^2) \cong I/I^2 \otimes \Omega_{B/A}$$

is an isomorphism. This completes the proof of Theorem (1. 1), modulo Lemmas (1. 6) and (3. 7).

Lemma (1. 6). *The map $K_2(A/I^2, B/I^2, I/I^2) \xrightarrow{\phi_2\alpha} \Phi_2$ is onto.*

Proof. It is well-known that $\langle x, by\rangle \langle y, bx\rangle = 1$ in $K_2(B/I^2, I/I^2)$ when $x, y \in I$, $b \in B$. Hence there is an element $q = q(x, y) \in K_2(A/I^2, B/I^2, I/I^2)$ mapping to

$$\langle x, by\rangle \langle y, bx\rangle \in K_2(A/I^2, I/I^2).$$

From Proposition (1. 7) below, we have

$$\phi_2\alpha(q) = \partial\langle x, by\rangle \langle y, bx\rangle = \begin{bmatrix} bx^2y \\ 1-bxy \end{bmatrix} \begin{bmatrix} bxy^2 \\ 1-bxy \end{bmatrix}$$

$$= \begin{bmatrix} xy \\ 1-bxy \end{bmatrix} \begin{bmatrix} bxy \\ 1-bxy \end{bmatrix}^2_* = \begin{bmatrix} xy \\ 1-bxy \end{bmatrix}.$$

These symbols generate Φ_2 by [Sw], (4. 6). We have used an asterisk to indicate that a Mennicke symbol is 1.

Proposition (1.7) (cf. [GW], (2. 3)). *In the ideal sequence for (A, I, I^2), the map $\partial: K_2(A/I^2, I/I^2) \longrightarrow SK_1(A, I^2)$ has*

$$\partial\langle \bar{x}, \bar{a}\rangle = \begin{bmatrix} ax^2(1-ax) \\ 1+ax^2(1-a) \end{bmatrix} \qquad (x \in I,\ a \in A),$$

$$\partial\langle \bar{x}, \bar{y}\rangle = \begin{bmatrix} x^2y \\ 1-xy \end{bmatrix} \qquad (x, y \in I).$$

Proof. We will use Stienstra's version of $\langle x, a\rangle$, which differs by a minus sign from the version used in [GW]. The image of $\langle \bar{x}, \bar{a}\rangle$ in $K_2(A/I^2, I/I^2) \to SK_1(A, I^2)$ is the image of the following element of $St(A)$:

$$X_{21}(-a(1+ax))\,X_{12}(-x)\,X_{21}(a)\,X_{12}(x)$$
$$\cdot X_{12}(+1)\,X_{21}(-1)\,X_{12}(+1)\,X_{12}(ax-1)\,X_{21}(1+ax)\,X_{12}(ax-1).$$

Mapping this element to $Gl(A)$, and multiplying the first four and last six matrices together yields

$$\begin{pmatrix} 1-ax & -ax^2 \\ a^3x^2 & 1+ax+a^2x^2+a^3x^3 \end{pmatrix} \begin{pmatrix} 1+ax & a^2x^2 \\ -a^2x^2 & 1-ax+a^2x^2-a^3x^3 \end{pmatrix}$$
$$= \begin{pmatrix} 1-a^2x^2+a^3x^4 & ax^2(a-1)-a^2x^3(a-1)-a^3x^4+a^4x^5 \\ a^2x^2(a-1)+a^3x^3(a-1)-a^4x^4-a^5x^5 & 1+a^2x^2+a^4x^4(a-1)-a^6x^6 \end{pmatrix}.$$

This matrix is in $Sl(I^2)$; in $SK_1(A, I^2)$ it is $\partial\langle \bar{x}, \bar{a}\rangle$, and is represented by the Mennicke symbol

$$\begin{bmatrix} ax^2(a-1)-a^2x^3(a-1)-a^3x^4+a^4x^5 \\ 1-a^2x^2+a^3x^4 \end{bmatrix}$$
$$= \begin{bmatrix} (ax^2-a^2x^3)\,((a-1)-a^2x^2) \\ 1-a^2x^2+a^3x^4 \end{bmatrix} \begin{bmatrix} ax^2 \\ 1-a^2x^2+a^3x^4 \end{bmatrix}_*$$
$$= \begin{bmatrix} ax^2-a^2x^3 \\ 1-a^2x^2+a^3x^4 \end{bmatrix} \begin{bmatrix} ax^2(a-1)-a^3x^4 \\ 1-a^2x^2+a^3x^4 \end{bmatrix}$$
$$= \begin{bmatrix} ax^2-a^2x^3 \\ 1-a^2x^2+a^2x^3 \end{bmatrix} \begin{bmatrix} ax^2(a-1)-a^3x^4 \\ 1-ax^2 \end{bmatrix}$$
$$= \begin{bmatrix} ax^2-a^2x^3 \\ 1+ax^2(1-a) \end{bmatrix} \begin{bmatrix} -ax^2 \\ 1-ax^2 \end{bmatrix}_* = \begin{bmatrix} ax^2-a^2x^3 \\ 1+ax^2(1-a) \end{bmatrix}.$$

We have used an asterisk to indicate that a Mennicke symbol is 1. This proves the first part of (1. 7). To see the rest, we set $a = y \in I$, and compute:

$$\partial\langle \bar{x}, \bar{y}\rangle = \begin{bmatrix} x^2y(1-xy) \\ 1+x^2y(1-y) \end{bmatrix} \begin{bmatrix} xy(1-y) \\ 1+x^2y(1-y) \end{bmatrix}_*$$
$$= \begin{bmatrix} x^2y(1-y)\,(1-xy) \\ 1+x^2y(1-y) \end{bmatrix} \begin{bmatrix} xy \\ 1+x^2y(1-y) \end{bmatrix}_*$$
$$= \begin{bmatrix} xy+x^2y(1-y) \\ 1+x^2y(1-y) \end{bmatrix} = \begin{bmatrix} x^2y+xy(1-xy) \\ 1-xy \end{bmatrix} = \begin{bmatrix} x^2y \\ 1-xy \end{bmatrix}.$$

§ 2. The topological proof

The purpose of this section is to prove that the topologically defined group $K_1(A, B, I)$ is isomorphic to $St(B, I)/\overline{St(A, I)}$. By definition, $K_i(A, B, I) = \pi_i \mathbb{K}(A, B, I)$ for an appropriate space $\mathbb{K}(A, B, I)$. However, we will use its loop space $\Omega \mathbb{K}(A, B, I)$, or rather a homotopy equivalent model $C(A, B, I)$. In order to obtain these spaces, we will first construct spaces $\Omega \mathbb{K}(A, B)$ and $C(A, B)$ which are not quite homotopy equivalent to each other (their 1-connected covers are). The homotopy groups

$$K_i(A, B) = \pi_{i-1} \Omega \mathbb{K}(A, B)$$

merit study in their own right, since they are the relative groups in the long exact sequence

$$\cdots \to K_i(A, B) \to K_i(A) \to K_i(B) \to K_{i-1}(A, B) \to \cdots.$$

We shall use Swan's definition of higher K-theory [Sw 2], which is also the version used by Keune in [K]. By [K], there is a functor $F_.$ which associates to every associative ring A a simplicial ring $F_.A$ such that each F_iA is a free ring. There is an augmentation $F_.A \to A$, and we define the space $\Omega \mathbb{K}(A)$ to be the fiber of

$$BGl(F_.A) \to BGl(A).$$

By [And], $\Omega \mathbb{K}(A)$ is homotopy equivalent to $\Omega BGl^+(A)$, so that $K_i(A) = \pi_{i-1}(\Omega \mathbb{K}(A))$ for $i \geqq 1$.

(2. 1). Let $A \to B$ be a map of rings with unit. Define $\Omega \mathbb{K}(A, B)$ to be the fiber of $\Omega \mathbb{K}(A) \to \Omega \mathbb{K}(B)$, and define the space $C(A, B)$ to be the fiber of

$$BGl(F_.A) \to BGl(F_.B).$$

Since $\Omega \mathbb{K}(A, B)$ is Ω of the fiber of $BGl^+(A) \to BGl^+(B)$, we have

$$K_i(A, B) = \pi_{i-1}(\Omega \mathbb{K}(A, B))$$

for $i \geqq 1$. With these definitions, the rows and columns of the following diagram are fibrations up to homotopy:

$$\text{(2.1.1)} \qquad \begin{array}{ccccc} \Omega \mathbb{K}(A, B) & \to & C(A, B) & \longrightarrow & BGl(J) \times Gl(B)/\overline{Gl(A)} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega \mathbb{K}(A) & \longrightarrow & BGl(F_.A) & \to & BGl(A) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega \mathbb{K}(B) & \longrightarrow & BGl(F_.B) & \to & BGl(B) \end{array}$$

(J denotes the kernel of $A \to B$). Note that for discrete groups G such as $Gl(A)$ we have $\pi_1(BG) = G$ and $\pi_i(BG) = 0$ for $i \neq 1$. However, $Gl(F_.A)$ is not a discrete group, so its homotopy groups are more involved. The homotopy sequence of the top row yields $K_i(A, B) \cong \pi_{i-1}(C(A, B))$ for $i \geqq 3$, and also yields the exact sequence of pointed sets:

$$\text{(2.1.2.)} \quad 1 \to K_2(A, B) \to \pi_1(C) \to Gl(J) \to K_1(A, B) \to \pi_0(C) \to Gl(B)/\overline{Gl(A)}.$$

Keune proved in [K 1], p. 168 that $\pi_1 BGl(F_. A) = \pi_0 Gl(F_. A) = St(A)$. Since $BGl(F_. A)$ is connected, the homotopy sequence of the middle column of (2.1.1) ends in

$$St(A) \to St(B) \to \pi_0(C) \to \text{point}.$$

This establishes

Lemma (2. 2) (Keune). *$\pi_0(C(A, B))$ is the right coset space $St(B)/\overline{St(A)}$.*

Here is another of Keune's results, translated into this notation:

Lemma (2. 3) (Keune). *If $B = A/J$, then $\pi_1(C(A, B)) = St(A, J)$.*

Proof. Each $F_i(A) \to F_i(A/J)$ is onto, so that each $St(F_i A) \to St(F_i(A/J))$ is onto with kernel G_i. The sequence $BG_. \to BSt(F_. A) \to BSt(F_. B)$ is a fibration and thus $C(A, B) \cong BG$. The homotopy groups of $G_.$ are $\pi_{i+1}(C(A, B)) = \pi_i(G_.) = L_i St(A, J)$ by [K 1], p. 167. In particular, $\pi_1 C(A, B) = \pi_0(G) = L_0 St(A, J)$, and this is $St(A, J)$ by [K 1], p. 168.

Remark. Plugging (2. 2, 2. 3) into (2.1. 2) yields the exact sequence

$$1 \to K_2(A, J) \to St(A, J) \to Gl(J) \to K_1(A, J) \to 1.$$

We now define $St(A, B)$ to be the subgroup of $St(B)$ which maps under $St(B) \to Gl(B)$ into the subgroup $\overline{Gl(A)}$ of $Gl(B)$. Note that $St(A, B)$ contains both $K_2(B)$ and the image of $St(A)$ in $St(B)$.

Lemma (2. 4). *Let $A \to B$ be an injection of rings with unit. There is an isomorphism of abelian groups*

$$K_1(A, B) \cong St(A, B)/\overline{St(A)}.$$

Proof. The sequence (2.1. 2) and Lemma (2. 2) yield the exact sequence of pointed sets

$$1 \to K_1(A, B) \to St(B)/\overline{St(A)} \to Gl(B)/\overline{Gl(A)}.$$

The image of $K_1(A, B)$ is $St(A, B)/\overline{St(A)}$, whence the lemma.

Proposition (2. 5). *Let $f: A \to B$ be a map of rings with unit, and let J denote* ker (f). *Then $\overline{St(A)} \subset St(A, B)$, and $St(A, B)/\overline{St(A)}$ is an abelian group. There is an exact sequence of groups*

$$1 \to K_2(A, B) \to \pi_1 C(A, B) \to Gl(J) \to K_1(A, B) \to St(A, B)/\overline{St(A)} \to 0.$$

Proof. We have the following chain of subgroups of $St(B)$:

$$\overline{St(A)} = \overline{St(A/J)} \subset St(A, B) \subset St(A/J, B).$$

The quotient $St(A/J, B)/\overline{St(A/J)}$ is abelian by (2. 4), so the subgroup $St(A, B)/\overline{St(A)}$ is also abelian. The exact sequence is just (2.1. 2), rewritten using Lemma (2. 2) and the above analysis of the map $K_1(A, B, I) \to \pi_0(C) = St(B)/\overline{St(A)}$.

We shall now apply the above general remarks to the Excision Situation (0. 1). Define $\Omega\mathbb{K}(A, B, I)$ to be the fiber of the map $\Omega\mathbb{K}(A, A/I) \to \Omega\mathbb{K}(B, B/I)$. By definition, then, $K_i(A, B, I) = \pi_{i-1}\Omega\mathbb{K}(A, B, I)$ for $i \geqq 1$. In particular, $K_1(A, B, I) = \pi_0\Omega\mathbb{K}(A, B, I)$. Define $C(A, B, I)$ to be the fiber of $C(A, A/I) \to C(B, B/I)$. Then the columns of

$$(2.6.1)\qquad \begin{array}{ccccc} \Omega\mathbb{K}(A,B,I) & \to & C(A,B,I) & \to & * \\ \downarrow & & \downarrow & & \downarrow \\ \Omega\mathbb{K}(A,A/I) & \to & C(A,A/I) & \to & BGl(I) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega\mathbb{K}(B,B/I) & \to & C(B,B/I) & \to & BGl(I) \end{array}$$

are fibrations. The bottom two rows of (2. 6. 1) are fibrations by (2. 1. 1). Consequently, the top row of (2. 6. 1) is a fibration, i.e., $\Omega\mathbb{K}(A, B, I) \to C(A, B, I)$ is a homotopy equivalence.

Theorem (2. 7). *Assume the notation of the Excision Situation* (0. 1) (*i.e., I is an ideal of A mapped isomorphically onto an ideal of B by $f: A \to B$*). *Then*

$$K_1(A, B, I) \cong St(B, I)/\overline{St(A, I)}\,.$$

Proof. The homotopy sequence of the middle column of (2. 6. 1) gives

$$\pi_1 C(A, A/I) \to \pi_1 C(B, B/I) \to \pi_0\Omega\mathbb{K}(A, B, I) \to \pi_0 C(A, A/I)\,.$$

By (2. 2) and (2. 3), we may rewrite this sequence as

$$St(A, I) \to St(B, I) \to K_1(A, B, I) \to 1\,.$$

Remark (2. 8). Another approach to this section is to obtain the algebraic excision sequence by applying the snake lemma to

$$\begin{array}{ccccccccc} & & St(A,I) & \to & St(B,I) & \to & St(B,I)/\overline{St(A,I)} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & Gl(I) & \longrightarrow & Gl(I) & \longrightarrow & 1 & \longrightarrow & 1\,. \end{array}$$

One must then prove that the algebraic and topological sequences agree. This approach is not faster than the one presented here.

§ 3. Relative Steinberg groups

Since $K_1(A, B, I) \cong St(B, I)/\overline{St(A, I)}$, we need to know more about $St(B, I)$ in order to identify the quotient group. In this section, we develop the necessary relative Steinberg theory. We also prove Lemma (3. 7) in full generality, completing the proof of Theorem (1. 1).

There are two ways to view the relative Steinberg group $St(B, I)$. We will use a presentation due to van der Kallen [vdK], because it more closely resembles Mennicke symbols and matrix transvections. The second way to view $St(B, I)$ is the original viewpoint of Keune [K 1] and Loday [L], to which we will devote only the paragraph (3. 1). We will, however, take advantage of the convenient notation for the elements $y_\alpha(x)$ and $y_\alpha(b; x)$ that their presentation affords.

(3. 1) Keune and Loday define $St(B, I)$ to be the $St(B)$-group generated by symbols $y_{ij}(x)$, $x \in I$, subject to relations we will not write down. Swan's proof of [Sw], (4. 1) shows that $St(B, I)$ is generated as a group by the elements $y_{ij}(b_1, \ldots, b_n; x)$, defined for $b_i \in B$ and $x \in I$. These are defined inductively from

$$y_{ij}(b; x) = x_{ij}(b) \cdot y_{ji}(x)$$
$$y_{ij}(b_1, \ldots, b_n; x) = x_{ij}(b_1) \cdot y_{ji}(b_2, \ldots, b_n; x).$$

We will see below in (3. 5) that in fact the symbols $y_{ij}(b; x)$ suffice to generate $St(B, I)$.

(3. 2) Let B be an associative ring with unit, and J a 2-sided ideal of B. Let V be a right B-module, W a left B-module. Every bilinear form $W \otimes V \to B$ gives rise to a group $St(V, J, W)$, defined by the following presentation.

Generators: $X(v, x, w)$, where $(v, x, w) \in V \times J \times W$ and $w \cdot v = 0$.

Relations: (a) $X(v, bx, w) = X(vb, x, w)$, $b \in B$,
(b) $X(v, xb, w) = X(v, x, bw)$, $b \in B$,
(c) $X(v, x_1 + x_2, w) = X(v, x_1, w)\, X(v, x_2, w)$
(d) $X(v_1 + v_2, x, w) = X(v_1, x, w)\, X(v_2, x, w)$ if $w \cdot v_i = 0$,
(e) $X(v, x, w_1 + w_2) = X(v, x, w_1)\, X(v, x, w_2)$ if $w_i \cdot v = 0$,
(f) $X(v', x', w')\, X(v, x, w)\, X(v', x', w')^{-1}$
$= X\big(v + v'x'(w' \cdot v), x, w - (w \cdot v')\, x'w'\big)$.

The primary examples use the free B-module B^∞, which has basis $\{\hat{e}_1, \hat{e}_2, \ldots\}$. We view B^∞ as both a left and right B-module. The bilinear form $B^\infty \otimes B^\infty \to B$ is given by the familiar rule $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ ($=1$ if $i = j$, and $=0$ if $i \neq j$). In [vdK], (A 6), [vdK 2], van der Kallen shows that $St(B)$ is isomorphic to $St(B^\infty, B, B^\infty)$, with $x_{ij}(b)$ corresponding to the symbol $X(\hat{e}_i, b, \hat{e}_j)$.

Van der Kallen also shows in [vdK], (A 8) that $St(B, I)$ is isomorphic to $St(B^\infty, I, B^\infty)$. The symbol $y_{ij}(x)$ of the presentation (3. 1) corresponds to the symbol $X(\hat{e}_i, x, \hat{e}_j)$ of the presentation (3. 2). If $g \in St(B)$ acts on B^∞ via the map $St(B) \to Gl(B)$, then the element $g \cdot y_{ij}(x)$ of $St(B, I)$ corresponds to $X(g\hat{e}_i, x, \hat{e}_j g^{-1}) \in St(B^\infty, I, B^\infty)$. In fact:

Lemma (3. 3). *The $St(B)$-group structure on $St(B, I)$ is induced from the $Gl(B)$-group structure on $St(B^\infty, I, B^\infty)$ given by*

$$g \cdot X(v, x, w) = X(gv, x, wg^{-1}).$$

Proof. It is easily checked that the above formula yields a $Gl(B)$-group structure on $St(B^\infty, I, B^\infty)$. The fact that the induced $St(B)$-group structure agrees with Keune's is pointed out in (A 8) of [vdK].

Remark (3. 3. 1). It is convenient to think of v as a column vector, and of w as a row vector. With this convention, the matrix product vxw makes sense for $x \in I$, and vxw is a matrix whose square is zero. The map $St(B, I) \to Gl(B, I)$ sends $X(v, x, w)$ to $1 + vxw$, and this map is a map of $Gl(B)$-groups. In particular, the image of

$$St(B^n, I, B^n) \to Gl_n(I) \to Gl_n(B)$$

is a normal subgroup of $Gl_n(B)$ containing $E_n(B, I)$ and contained in $E_{n+1}(B, I) \cap Gl_n(B)$. (This last observation goes back in spirit to [V].)

We can create variant Steinberg groups using the 2-sided submodule $I^\infty = IB^\infty = B^\infty I$ of B^∞. The following result connects some of these variants:

Theorem (3. 4) (van der Kallen). *There are isomorphisms of $Gl(B)$-groups*:

(a) $St(B, I) \to St(B^\infty, I, B^\infty)$ *via* $y_{ij}(x) \to X(\hat{e}_i, x, \hat{e}_j)$,
(b) $St(B^\infty, I, B^\infty) \to St(B^\infty, B, I^\infty)$ *via* $X(v, x, w) \to X(v, 1, xw)$,
(c) $St(B^\infty, I, B^\infty) \to St(I^\infty, B, B^\infty)$ *via* $X(v, x, w) \to X(vx, 1, w)$.

Proof. This is (A 8) of [vdK].

We now illustrate the power of van der Kallen's presentation by showing that $St(B, I)$ is generated by the elements $y_\alpha(b; x)$. This generalizes Suslin and Vaserstein's result that $E(B, I)$ is generated by the elements $e_{ij}(b; x) = e_{ij}(b)\, e_{ji}(x)\, e_{ij}(-b)$. (See [SV].) Our proof is in fact a transcription of their proof.

Theorem (3. 5). *The group $St(B, I)$ is generated as a group (not just as an $St(B)$-group) by the elements*

$$y_{ij}(b; x) = e_{ij}(b) \cdot y_{ji}(x) = X(\hat{e}_j + \hat{e}_i b, x, \hat{e}_i - b\hat{e}_j)$$

for $b \in B$, $x \in I$, and $i \neq j$.

Remark (3. 5. 1). The proof below yields the following identity (for $x \in I$; $b, c \in B$; $i \neq j \neq k$):

$$\begin{aligned} x_{ij}(c) \cdot y_{ji}(b; x) = {} & y_{ij}\big((1+cb)\, x(1+bc)\big)\, y_{ik}\big(-(1+cb)\, xb(1+cb)\big) \\ & y_{jk}\big(b; x(1+bc)\big)\, y_{ji}(-bxb)\, y_{jk}\big(bxb(1+cb)\big) \\ & y_{ik}(1+cb;\ -xb)\, y_{ki}(xb)\, y_{kj}\big(-x(1+bc)\big). \end{aligned}$$

Let $St_0(B, I)$ denote the subgroup of $St(B, I)$ generated by the symbols $y_{ij}(x) = X(\hat{e}_i, x, \hat{e}_j)$, and let $St_1(B, I)$ denote the subgroup of $St(B, I)$ generated by the symbols $y_{ij}(b; x)$. Thus $St_0(B, I) \subset St_1(B, I) \subset St(B, I)$, and we need to prove that $St_1 = St$.

(3. 5. 2) Whenever the coordinates of v and w are disjoint, the symbol $X(v, x, w)$ belongs to $St_0(B, I)$. Indeed, the bilinearity relations yield

$$X(\Sigma\, \hat{e}_i v_i, x, \Sigma\, w_j \hat{e}_j) = \prod_{i \neq j} X(\hat{e}_i, v_i x w_j, \hat{e}_j).$$

Note that if $\beta \in St_0(B, I)$, then $e_{ij}(b) \cdot \beta \in St_1(B, I)$. This is a consequence of the fact that $e_{ij}(b) \cdot y_{kl}(x) \in St_0(B, I)$ whenever $(ij) \neq (lk)$.

Proof of (3. 5). When $w \in B^n$, we abbreviate

$$X\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, x, (w\ \ 0)\right] \quad \text{for} \quad X\left(\hat{e}_{n+1}, x, \sum_{i=1}^{n} w_i \hat{e}_i\right), \quad \text{etc.}$$

These elements are in $St_0(B, I)$. We will also abbreviate

$$\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad \left(\begin{array}{cc|c} 1 & 0 & v_1 \\ 0 & 1 & v_n \\ \hline 0 \ldots & 0 & 1 \end{array}\right).$$

When $v, w \in B^n$, we then have

$$\left\{\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix} \cdot X\left[\begin{pmatrix}0\\ 1\end{pmatrix}, x, (w\ 0)\right]\right\} X\left[\begin{pmatrix}0\\ 1\end{pmatrix}, x, (w\ 0)\right]^{-1}$$

$$= X\left[\begin{pmatrix}v\\ 1\end{pmatrix}, x, (w\ 0)\right] X\left[\begin{pmatrix}0\\ 1\end{pmatrix}, x, (w\ 0)\right]^{-1} \tag{3.5.3}$$

$$= X(v, x, w).$$

By Lemma (3. 6), $X(v, x, w) \in St_1(B, I)$, and hence $St_1 = St$, as desired.

Lemma (3. 6) (Suslin-Vaserstein [SV], Lemma (2. 1)). *Let $x \in I$, $v \in B^s$, $w \in B^r$. Then for every $r \times s$ matrix u over B*:

$$\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} \cdot X\left[\begin{pmatrix}0\\ v\end{pmatrix}, x, (w\ 0)\right] \in St_1(B, I).$$

Proof. By bilinearity, we can assume that $(0\ v) = \hat{e}_{k+r}$ and $(w\ 0) = \hat{e}_l$ for $1 \leqq k \leqq s$, $1 \leqq l \leqq r$. We first assume that $u_{lk} = 0$. Then

$$\beta = \begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} \cdot X\left[\begin{pmatrix}0\\ v\end{pmatrix}, x, (w\ 0)\right]$$

$$= X\left[\begin{pmatrix}uv\\ v\end{pmatrix}, x, (w, -wu)\right].$$

The l^{th} entry of (uv, v) and the $(k+r)^{\text{th}}$ entry of $(w, -wu)$ are zero when $u_{lk} = 0$. In this case, $\beta \in St_0(B, I)$ by (3. 5. 2), the coordinates of (uv, v) and $(w, -wu)$ being distinct.

In the general case, write $u = u' + u''$, where u'' has only one nonzero entry, namely u_{lk}, and the $(l, k)^{\text{th}}$ entry of u' is zero. Then $\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} = e_{l,k+r}(u_{lk}) \begin{pmatrix}1 & u'\\ 0 & 1\end{pmatrix}$, and $St_0(B, I)$ contains the element $\beta = \begin{pmatrix}1 & u'\\ 0 & 1\end{pmatrix} \cdot X\left[\begin{pmatrix}0\\ v\end{pmatrix}, x, (w\ 0)\right]$. By (3. 5. 2) we then have

$$\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} \cdot X\left[\begin{pmatrix}0\\ v\end{pmatrix}, x, (w\ 0)\right] = e_{l,k+r}(u_{lk}) \cdot \beta \in St_1(B, I).$$

Corollary (3. 6. 1). *Let N be the normal subgroup of $St(B, I)$ generated by the $y_{ij}(x)$. Then*

$$X(v, x, w) \equiv \prod_i y_{i,n+1}(v_i; xw_i) \text{ modulo } N.$$

Proof. This follows from (3. 5. 3) and a careful perusal of the proof of Lemma (3. 6).

We now complete the proof of Theorem (1. 1) by proving the following result.

Lemma (3. 7). *The image of $St(B, I^2) \to St(B, I)$ is contained in the image of $St(A, I) \to St(B, I)$. The image of $K_2(B, I^2) \to K_2(B, I)$ is contained in the image of $K_2(A, I) \to K_2(B, I)$.*

Proof. By Theorem (3. 5), $St(B, I^2)$ is generated by symbols $y_{ij}(b; z)$, $b \in B$, $z \in I^2$. By relation (3. 2. c), these symbols are additive in z, so it suffices to consider the case $z = xy$ $(x, y \in I)$. Fix $k \neq i, j$. Then in $St(B, I)$

$$\begin{aligned} y_{ij}(b; xy) &= x_{ij}(b) \cdot [y_{jk}(x), y_{ki}(y)] \\ &= [y_{ik}(bx)\, y_{jk}(x), y_{kj}(-yb)\, y_{ki}(y)] \\ &\in St_0(B, I) \subset \overline{St(A, I)}. \end{aligned}$$

This establishes the first sentence. The second sentence follows from a chase on the following diagram with exact rows:

$$\begin{array}{ccccccc} & & K_2(B, I^2) & \to & St(B, I^2) & & \\ & & \downarrow & & \downarrow & & \\ 1 & \to & K_2(B, I) & \longrightarrow & St(B, I) & \longrightarrow & Gl(I) \\ & & \uparrow & & \uparrow & & \| \\ & & K_2(A, I) & \longrightarrow & St(A, I) & \longrightarrow & Gl(I) . \end{array}$$

We now need to know the relationship between $St(A, I)$ and $St(B, I)$. We will give a direct proof of the fact that $St(B, I)/\overline{St(A, I)}$ is abelian, noting that this fact is implicit in § 2 (it is the fundamental group of a loop space).

Theorem (3. 8). *Assume the notation of the Excision Situation* (0. 1) (*i.e., I is an ideal of A mapped isomorphically onto an ideal of B by $f: A \to B$*). *Then the image of $St(A, I)$ in $St(B, I)$ is a normal subgroup, and the action of $St(A)$ on the quotient group*

$$St(B, I)/\overline{St(A, I)}$$

is trivial.

Proof. First note that $St(A, I) \xrightarrow{\iota} St(B, I) \xrightarrow{\pi} Gl(I)$ are $Gl(A)$-group maps by Lemma (3. 3). If $\alpha \in St(A, I)$, $\beta \in St(B, I)$ then $\beta(\iota(\alpha))\beta^{-1} = \pi(\beta) \cdot \iota(\alpha) = \iota(\pi(\beta) \cdot \alpha)$ by relation (3. 2. f). This shows that $\iota St(A, I)$ is a normal subgroup of $St(B, I)$. In fact, $\iota St(A, I)$ is the smallest $St(A)$-subgroup of $St(B, I)$ containing $St_0(B, I)$.

Choose $x \in I$ and $v, w \in B^n$ with $w \cdot v = 0$. We will show that $\alpha \cdot X(v, x, w) \equiv X(v, x, w)$ in $St(B, I)$ modulo $St(A) \cdot St_0(B, I)$ for all $\alpha \in St(A)$. It is enough to check $\alpha = x_{n+i,j}(a)$ and $\alpha = x_{i,n+j}(a)$ for $i, j \geqq 1$, since these α generate $St(A)$. In $St(B, I)$ we have

$$\begin{aligned} x_{n+i,j}(a) \cdot X(v, x, w) &= X[(e_{n+i,j}(a)\, v, x, w e_{n+i,j}(-a)] \\ &= X(v + \hat{e}_{n+i} a v_j, x, w) \\ &= X(v, x, w)\, X(\hat{e}_{n+i} a v_j, x, w) \\ &\equiv X(v, x, w) \quad \text{modulo} \quad St_0(B, I) \quad \text{by} \quad (3. 5. 2). \end{aligned}$$

Similarly, $\{x_{i,n+j}(a) \cdot X(v, x, w)\}\, X(v, x, w)^{-1} = X\left[\begin{pmatrix} v \\ 0 \end{pmatrix}, x, (0, -w_i a \hat{e}_{n+j})\right]$. By (3. 5. 2) this too is in $St_0(B, I)$, proving the theorem.

Corollary (3. 9). *If the coordinates of v and w are distinct, then* $X(v, x, w)=0$ *in* $St(B, I)/\overline{St(A, I)}$.

Proof. This is (3. 5. 2), since $St_0(B, I)\subset \overline{St(A, I)}$.

Corollary (3. 10). *The group* $St(B, I)/\overline{St(\mathbb{Z}\oplus I, I)}$ *is abelian. In this group we have* $X(v, x, w)\, X(v', x, w')=X\left[\begin{pmatrix} v \\ v' \end{pmatrix}, x, (w\ w')\right]$.

Proof. First note that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the image of an element of $St(\mathbb{Z}\oplus I)$, so that by Theorem (3. 8)

$$X(v', x, w')=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\cdot X\left[\begin{pmatrix} v' \\ 0 \end{pmatrix}, x, (w'\ 0)\right]=X\left[\begin{pmatrix} 0 \\ v' \end{pmatrix}, x, (0\ w')\right],$$

and

$$X\left[\begin{pmatrix} v \\ v' \end{pmatrix}, x, (w\ w')\right]=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\cdot X\left[\begin{pmatrix} v \\ v' \end{pmatrix}, x, (w\ w')\right]=X\left[\begin{pmatrix} v' \\ v \end{pmatrix}, x, (w'\ w)\right].$$

Finally we have that $X(v, x, w)\, X(v', x', w')$ equals

$$X\left[\begin{pmatrix} v \\ 0 \end{pmatrix}, x, (w\ 0)\right]X\left[\begin{pmatrix} 0 \\ v' \end{pmatrix}, x, (w\ 0)\right]_* X\left[\begin{pmatrix} 0 \\ v' \end{pmatrix}, x, (0\ w')\right]X\left[\begin{pmatrix} v \\ 0 \end{pmatrix}, x, (0\ w')\right]_*$$

$$\equiv X\left[\begin{pmatrix} v \\ v' \end{pmatrix}, x, (w\ 0)\right]X\left[\begin{pmatrix} v \\ v' \end{pmatrix}, x, (0\ w')\right]=X\left[\begin{pmatrix} v \\ v' \end{pmatrix}, x, (w\ w')\right].$$

Here the $*$ is affixed to symbols that are trivial by (3. 9).

Corollary (3. 11). *Assume the notation of the Excision Situation* (0.1) (*i.e., I is an ideal of A mapped isomorphically onto an ideal of B by* $f\colon A\to B$). *Then the group* $St(B, I)/\overline{St(A, I)}$ *is abelian. It is generated by the symbols* $y_{ij}(b; x)$, *where* $b\in B$ *and* $x\in I$. *These symbols are independent of the choice of i and j, and*

$$y_{21}(b; x)=X\left[\begin{pmatrix} 1 \\ b \end{pmatrix}, x, (-b\ \ 1)\right].$$

Proof. For an appropriate permutation matrix $g\in St(\mathbb{Z})$, we have

$$y_{21}(b; x)=g\cdot y_{ij}(b; x).$$

The theorem now follows from (3. 5), (3. 8), and (3. 10).

Remark (3. 12). The map $St(B, I)/\overline{St(A, I)}\to K_1(A, I)$ induced by $St(B, I)\to Gl(I)$ sends $y_{21}(b; x)$ to the class of the matrix

$$e_{21}(b; x)=\begin{pmatrix} 1-xb & x \\ -bxb & 1+bx \end{pmatrix}.$$

In particular, when A and B are commutative, this is the image of $x \otimes db$ under the usual "Swan-Vorst" map $I/I^2 \otimes \Omega_{B/A} \to K_1(A, I)$ of [Sw], p. 235, [Vo], (2. 5). This is hardly surprising, since our method is based on Swan's work.

Corollary (3. 13). *Assume the notation of the Excision Situation* (0. 1) (*i.e., I is an ideal of A mapped isomorphically onto an ideal of B by $f: A \to B$). Then the action of $St(B, I)$ on $St(B, I)/\overline{St(A, I)}$ is trivial. Consequently, the $Gl(I)$-group structure on $St(B, I)$ induces the structure of a $K_1(B, I)$-group on $St(B, I)/\overline{St(A, I)}$.*

Proof. The first statement follows from relation (3. 2. f) and the fact that

$$St(B, I)/\overline{St(A, I)}$$

is abelian. The second follows from the fact that $K_1(B, I)$ is the cokernel of $St(B, I) \to Gl(I)$.

§ 4. The group $St(B, I)/\overline{St(A, I)}$

For this section we assume that we are in the Excision Situation (0. 1), with A and B noncommutative rings. That is, $f: A \to B$ maps the ideal I of A isomorphically onto an ideal of B. In order to emphasize the main points, we have relegated the technical details to a series of propositions at the end of the section. We first state and prove the Theorems of this section.

Theorem (4. 1) (R. K. Dennis). *There is an isomorphism of $St(B, I)/\overline{St(A, I)}$ with the group*

$$H = B \underset{A^e}{\otimes} (I/I^2)/\{b \otimes cx + c \otimes xb - bc \otimes x : b, c \in B,\ x \in I/I^2\}.$$

Under this isomorphism the element $y_\alpha(b; x)$ of $St(B, I)/\overline{St(A, I)}$ corresponds to $b \otimes x$. The $\otimes$ is taken over the ring $A^e = A \underset{Z}{\otimes} A^{op}$.

Proof. Consider the set map $y_\alpha(;): B \times I \to St(B, I)/\overline{St(A, I)}$ which is given by $y_\alpha(;)\,(b; x) = y_\alpha(b; x)$. By (4. 4) this map is additive; by (4. 7), it induces an onto homomorphism $y_\alpha: H \to St(B, I)/\overline{St(A, I)}$. In (4. 9), we construct a homomorphism π going in the other direction, and taking $y_\alpha(b; x)$ to $b \otimes x$. Hence y_α is an isomorphism with inverse π.

Remark (4. 1. 1). It is immediate that $K_1(A, B, I) \cong St(B, I)/\overline{St(A, I)}$ carries the structure of a module over the center of B/I. If $r \in B$ has image $\bar{r}$ in the center of B/I, then the module structure is given by $\bar{r} \cdot y_\alpha(b; x) = y_\alpha(b; rx) = y_\alpha(b; xr)$.

Remark (4. 1. 2). A similar argument shows that $K_1(A, B, I)$ also equals

$$(B/A \underset{Z}{\otimes} I/I^2)/\{b \otimes cx + c \otimes xb - bc \otimes x\}.$$

This shows what was missing in Swan's map in [Sw].

Theorem (4. 2). *There is an exact sequence*

$$0 \to H_1^A(B; I/I^2) \to St(B, I)/\overline{St(A, I)} \to [B, I/I^2]/[A, I/I^2] \to 0,$$

where $H_*^A(B; M)$ denotes the Hochschild homology of the A-algebra B with coefficients in the 2-sided B-module M, and $[B, M]$ denotes the subgroup of M generated by all $bm - mb$ with $b \in B$, $m \in M$.

Proof. To compute Hochschild homology, we follow [Mac], p. 288. One considers the chain complex

$$(B \underset{A}{\otimes} B) \underset{A^e}{\otimes} M \xrightarrow{\partial_2} B \underset{A^e}{\otimes} M \xrightarrow{\partial_1} M/[A, M],$$

$$\partial_2 : b \otimes c \otimes m \mapsto (c \otimes mb - bc \otimes m + b \otimes cm),$$

$$\partial_1 : b \otimes m \mapsto (mb - bm).$$

By (4. 1), $St(B, I)/\overline{St(A, I)}$ is the cokernel of ∂_2 for $M = I/I^2$. The result is immediate.

Remark (4. 2. 1). When A and B are commutative, $H_1^A(B; M) \cong M \underset{B}{\otimes} \Omega_{B/A}$. This is easily seen from the proof of (4. 2) and the presentation of $\Omega_{B/A}$ given in [H], p. 172. If we take $M = I/I^2$, we obtain the isomorphism $St(B, I)/\overline{St(A, I)} \cong I/I^2 \otimes \Omega_{B/A}$.

Remark (4. 2. 2). We always have $H_*^A(B, I/I^2) = H_*^{A/I}(B/I, I/I^2)$. When B/I is commutative (even if B is not), we therefore have

$$St(B, I)/\overline{St(A, I)} \cong H_1^{A/I}(B/I, I/I^2)$$
$$\cong I/I^2 \underset{B/I}{\otimes} \Omega_{(B/I)/(A/I)}.$$

Remark (4. 2. 3). The map $K_1(A, B, I) \to [B, I/I^2]/[A, I/I^2]$ sends $y_\alpha(b; x)$ to $xb - bx$. It is therefore the Dieudonné determinant map used by Swan in [Sw], (2. 6).

Theorem (4. 3). *In the Excision Situation* (0. 1) *with A and B commutative rings, the isomorphism*

$$I/I^2 \otimes \Omega_{B/A} \cong K_1(A, B, I) \cong H_1^A(B; I/I^2) \cong St(B, I)/\overline{St(A, I)}$$

is given by sending the generator $x \otimes db$ of $I/I^2 \otimes \Omega_{B/A}$ to the element

$$y_\alpha(b; x) = X\left[\begin{pmatrix} 1 \\ b \end{pmatrix}, x, (-b \;\; 1)\right].$$

The inverse map sends $X(v, x, w)$ to $\sum x w_i \otimes d(v_i)$.

Proof. The isomorphism $I/I^2 \otimes \Omega_{B/A} \cong H_1^A(B; I/I^2)$ associates the element $x \otimes db$ of $I/I^2 \otimes \Omega_{B/A}$ and the homology class of the cycle $b \otimes x$ in $B \otimes I/I^2$. The result follows from (4. 2), given the description of the maps y_α and π.

Remark. The map $I/I^2 \otimes \Omega_{B/A} \cong K_1(A, B, I) \to SK_1(A, I)$ is the Swan-Vorst map of [Sw] and [Vo], (2. 5). This follows from (4. 3) and Remark (3. 12). Compare with [GW], (2. 3).

We conclude this section with the promised parade of technical details needed to prove Theorem (4. 1).

Proposition (4. 4). *The symbols $y_\alpha(b; x)$ in $St(B, I)/\overline{St(A, I)}$ are additive in both b and x.*

Proof. Let $b, c \in B$, $x \in I$. By (3. 5) and (3. 2. c), the symbol $y_\alpha(b; x)$ is additive in x in the group $St(B, I)$. We have to show that $y_{21}(b; x)\, y_{21}(c; x) \equiv y_{21}(b + c; x)$ modulo the image of $St(A, I)$. In $St(B^\infty, I, B^\infty)/\overline{St(A^\infty, I, A^\infty)}$ we have

$$y_{21}(b; x)\, y_{21}(c; x) = X\left[\begin{pmatrix}1\\ b\end{pmatrix}, x, (-b \;\; 1)\right] X\left[\begin{pmatrix}1\\ c\end{pmatrix}, x, (-c \;\; 1)\right]$$

$$\equiv X\left[\begin{pmatrix}1\\ b\\ 1\\ c\end{pmatrix}, x, (-b, 1, -c, 1)\right] \qquad \text{by (3. 10)}$$

$$\equiv X\left[\begin{pmatrix}1\\ b+c\\ 1\\ c\end{pmatrix}, x, (-b, 1, -c, 0)\right] \qquad \text{using } x_{24}(1) \in St(A) \text{ in (3. 8)}$$

$$\equiv X\left[\begin{pmatrix}1\\ b+c\\ 0\\ c\end{pmatrix}, x, (-(b+c), 1, c, 0)\right] \qquad \text{using } x_{31}(-1) \text{ in (3. 8)}$$

$$\equiv X\left[\begin{pmatrix}1\\ b+c\end{pmatrix}, x, (-(b+c), 1)\right] X\left[\begin{pmatrix}0\\ c\end{pmatrix}, x, (c \;\; 0)\right]_* \qquad \text{by (3. 10)}$$

$$\equiv y(b+c; x) \qquad \text{by (3. 9)}.$$

Corollary (4. 5). *y_α induces an abelian group map from $B \underset{A^e}{\otimes} I/I^2$ onto $St(B, I)/\overline{St(A, I)}$.*

Proof. $y_\alpha(B; I^2) = 0$ by (3. 7). The relations

$$y_\alpha(ba; x) = y_\alpha(b; ax) \qquad \text{and} \qquad y_\alpha(ab; x) = y_\alpha(b; xa)$$

are immediate from (4. 6) below. The map is onto because, by (3. 5), the $y_\alpha(b; x)$ generate $St(B, I)/\overline{St(A, I)}$.

Proposition (4. 6). *In* $St(B, I)/\overline{St(A, I)}$

$$y_\alpha(bc; x) = y_\alpha(b; cx)\, y_\alpha(c; xb).$$

Proof. Take $\alpha = 21$. Then

$$y_\alpha(b; cx)\, y_\alpha(c; xb) = X\left[\begin{pmatrix}1\\ b\end{pmatrix}, cx, (-b, 1)\right] X\left[\begin{pmatrix}1\\ c\end{pmatrix}, xb, (-c\ 1)\right]$$

$$= X\left[\begin{pmatrix}c\\ bc\\ 1\\ c\end{pmatrix}, x, (-b, 1, -bc, b)\right] \qquad \text{by (3. 2. a, b) and (3. 10)}$$

$$= X\left[\begin{pmatrix}1\\ bc\\ 1\\ c\end{pmatrix}, x, (-b, 1, b-bc, 0)\right] \qquad \text{using } e_{13}(1)\, e_{14}(-1) \text{ in (3. 8)}$$

$$= X\left[\begin{pmatrix}1\\ bc\\ 0\\ c\end{pmatrix}, x, (-bc, 1, b-bc, 0)\right] \qquad \text{using } e_{31}(-1) \text{ in (3. 8)}$$

$$= X\left[\begin{pmatrix}1\\ bc\end{pmatrix}, x, (-bc, 1)\right] = y_{21}(bc; x) \qquad \text{by (3. 9)}.$$

Corollary (4. 7). y_α *induces an onto map*

$$y_\alpha : H = B \underset{A^e}{\otimes} (I/I^2)/\{b \otimes cx + c \otimes xb - bc \otimes x\} \to St(B, I)/\overline{St(A, I)}.$$

Proof. By (4. 5) with $a \in A$, $b \in B$ we have $y_\alpha(ba; x) = y_\alpha(b; ax)$ and $y_\alpha(ab; x) = y_\alpha(b; xa)$ in $St(B, I)/\overline{St(A, I)}$. Therefore, y_α induces an onto map from $B \underset{A^e}{\otimes} I/I^2$ to $St(B, I)/\overline{St(A, I)}$. The elements $b \otimes cx + c \otimes xb - bc \otimes x$ are in the kernel of y_α by (4. 6).

Lemma (4. 8). *In the group* H, $a \otimes y = 0$ *for* $a \in A$, $y \in I/I^2$.

Proof. First note that $0 = 1 \otimes (1 \cdot y) + 1 \otimes (y \cdot 1) - (1 \cdot 1) \otimes y = 1 \otimes y$. We then have $a \otimes y = 1 \otimes ay = 0$.

Proposition (4. 9). *The formula* $\pi(X(v, x, w)) = \sum v_i \otimes xw_i$ *defines a well-defined homomorphism*

$$\pi : St(B, I)/\overline{St(A, I)} \to H.$$

In addition, $\pi(y_\alpha(b; x)) = b \otimes x$.

Proof. We have to check the relations (3. 2):

(a) $\sum v_i \otimes (bx) w_i - \sum (v_i b) \otimes x w_i = \sum b \otimes (x w_i) v_i$
$= b \otimes x(\sum w_i v_i) = 0;$

(b) $\sum v_i \otimes (xb) w_i = \sum v_i \otimes x(b w_i);$

(c), (d) and (e) are immediate;

(f) $\sum_i (v_i + v_i' x'(w' \cdot v)) \otimes x (w_i - (w \cdot v') x' w_i')$

$$= \sum (v_i + v_i' x'(w' \cdot v)) \otimes x w_i \quad \text{because } x(w \cdot v') x' \in I^2$$
$$= \sum (v_i \otimes x w_i + \sum v_i' x'(w' \cdot v)) \otimes x w_i$$
$$= \sum v_i \otimes x w_i .$$

The last equality holds because $a_i = v_i' x'(w' \cdot v) \in A$, and $a \otimes y = 0$ by lemma (4. 8). Finally, we use (3. 5) and (4. 8) to see that

$$\pi(y_{ij}(b; x)) = \pi(X(\hat{e}_j + \hat{e}_i b, x, \hat{e}_i - b\hat{e}_j)) = b \otimes x - 1 \otimes xb = b \otimes x .$$

§ 5. Applications

In this section we give some examples to link our new results with known ones, and to show some of the potential of our approach. We first point out exactly how the groups $K_i(A, B, I)$ affect the Mayer-Vietoris sequence. Next, we analyze some classical commutative cases. Finally, we scratch the surface of the noncommutative world, showing that even the cases $I^2 = 0$ and $I^3 = 0$ are markedly different.

We begin by pointing out how much can be obtained from the vanishing of just one double relative group. Write 'quo-$K_i(A)$' for the quotient of $K_i(A)$ by the image of $K_i(A, B, I)$, and write 'sub-$K_i(B/I)$' for the subgroup of $K_i(B/I)$ of elements mapping to zero under $K_i(B/I) \to K_{i-1}(B, I) \to K_{i-2}(A, B, I)$.

Proposition (5. 1) (cf. [Mil], [Stein], [GW]). *There is always an exact sequence*

$$K_{i+1}(A/I) \oplus K_{i+1}(B) \to \text{sub-}K_{i+1}(B/I) \to \text{quo-}K_i(A) \to K_i(A/I) \oplus K_i(B) .$$

If in addition we have $K_{i-1}(A, B, I) = 0$, *then there is an exact sequence*

$$K_{i+1}(A/I) \oplus K_{i+1}(B) \to K_{i+1}(B/I) \to \text{quo-}K_i(A) \to K_i(A/I) \oplus K_i(B)$$
$$\to \text{sub-}K_i(B/I) \to K_{i-1}(A) \to K_{i-1}(A/I) \oplus K_{i-1}(B) .$$

Proof. Chase the diagram

$$\begin{array}{ccccccccccccccc}
 & & & & K_i(A, B, I) & & & & & & K_{i-1}(A, B, I) & & & & \\
 & & & & \downarrow & & & & & & \downarrow & & & & \\
\bullet & \to & \bullet & \to & K_i(A, I) & \to & K_i(A) & \to & K_i(A/I) & \to & K_{i-1}(A, I) & \to & \bullet & \to & \bullet \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bullet & \to & \bullet & \to & K_i(B, I) & \to & K_i(B) & \to & K_i(B/I) & \to & K_{i-1}(B, I) & \to & \bullet & \to & \bullet \\
 & & & & \downarrow & & & & & & \downarrow & & & & \\
 & & & & K_{i-1}(A, B, I) & & & & & & K_{i-2}(A, B, I) & & & &
\end{array} .$$

Corollary (5. 2). *Because $K_0(A, B, I)=0$, the usual Mayer-Vietoris sequence in algebraic K-theory begins*

$$K_2(A/I) \oplus K_2(B) \to K_2(B/I) \to \text{quo-}K_1(A) \to K_1(A/I) \oplus K_1(B) \to \cdots .$$

If $K_1(A, B, I)=0$, we can extend the sequence to the left as far as this:

$$K_3(A/I) \oplus K_3(B) \to K_3(B/I) \to \text{quo-}K_2(A) \to K_2(A/I) \oplus K_2(B)$$
$$\to K_2(B/I) \to K_1(A) \to K_1(A/I) \oplus K_1(B) \to \cdots .$$

In order to incorporate $K_1(A, B, I)$ into known results, we turn to the familiar node and cusp examples.

Example (5. 3) ("node"). Let k be a field. Set $B=k[t]$, $I=(t^2-t)\,B$, and $A=k\oplus I$. We know that $K_1(A, B, I)=I/I^2 \otimes \Omega_{B/A}=0$ because $\Omega_{B/A}=0$ (see [GR]). From (5. 2) we obtain the exact sequence

$$K_3(k) \oplus K_3(k) \xrightarrow{\Delta} K_3(k) \oplus K_3(k) \to \text{quo-}K_2(A) \to K_2(k) \to 0 .$$

On the other hand, we know from [W4] that $K_2(A)=K_2(k) \oplus K_3(k) \oplus k^+$. It follows that $K_2(A, B, I)$ maps onto k^+. This example shows that $K_2(A, B, I)$ is not

$$I/I^2 \otimes \ker(A \to B)/\ker(A \to B)^2$$

in general.

In fact, $K_2(A, B, I) \cong k^+$. To see this, we use the exact sequence

$$K_3(A, I) \to K_3(B, I) \to K_2(A, B, I) \to k^+ \to 0 .$$

The group $K_3(B, I)$ is $K_4(k)$, and may be identified with $KV_3(I)$. The spectral sequence $E^1_{pq}=N^p K_q(A, I) \Rightarrow KV_{p+q}(I)$ of [W3], (3. 4) degenerates enough to yield the exact sequence

$$NK_3(A, I) \to K_3(A, I) \to KV_3(I) \to 0 .$$

Hence $K_3(A, I)$ maps onto $K_3(B, I)$, leaving the isomorphism of $K_2(A, B, I)$ with k^+.

Example (5. 4) ("cusp"). Let k be a field of characteristic p. Set $B=k[t]$, $I=t^p B$ and $A=k\oplus I$. Then $K_1(A, B, I^2) \cong I^2/I^4 \otimes \Omega_{B/A} \cong B/I$. From

$$K_2(A/I^2, B/I^2, I/I^2) \to K_1(A, B, I^2) \to K_1(A, B, I) \xrightarrow{\cong} K_1(A/I^2, B/I^2, I/I^2)$$

we see that $K_2(A/I^2, B/I^2, I/I^2)$ maps onto B/I. (This calculation could also be made for $p \neq 2$ by showing that the kernel of $K_2(A/I^2, I/I^2) \to K_2(B/I^2, I/I^2)$ is $\Lambda^2(I/I^2) \cong B/I$. This method uses [W1], (1. 4).)

Example (5. 5). One might hope to obtain information about $K_3(B/I)$ by analyzing the chain of maps

$$K_3(B/I) \to K_2(B, I) \to K_1(A, B, I) \to K_1(A, I) .$$

For example, let $B = k[t]$, $I = t^n B$, and set $A = k \oplus I$. Then

$$K_1(A, B, I) = I/I^2 \otimes \Omega_{B/A} \cong B/(nt^{n-1}, t^n)\,B,$$

and the image of $K_1(A, B, I) \to K_1(A, I)$ is well understood (see [GW]). The group $K_3(B/I) \cong K_2(B/I) \oplus K_3(k)$ maps onto the kernel of $K_1(A, B, I) \to K_1(A, I)$, giving a bound for $K_3(B/I)$.

Unfortunately, these techniques will rarely yield more than the methods exploited by Stienstra [St]. To see this, recall that $K_1(A, B, I) = K_1(A/I^2, B/I^2, I/I^2)$. Therefore the map $K_3(B/I) \to K_2(B, I) \to K_1(A, B, I)$ factors through the image of $K_3(B/I)$ in $K_2(B/I^2, I/I^2)$, which is the kernel of $K_2(B/I^2, I/I^2) \to K_2(B/I^2)$. But the map

$$K_2(B/I^2, I/I^2) \to K_2(B/I^2)$$

is exactly what Stienstra considers.

We now turn from commutative examples to noncommutative examples. We first give a simple application that compares the double relative K_1 and K_2 groups.

Proposition (5. 6) (cf. Dennis-Krusemeyer [DK], (3. 1)). *Let J be an ideal of B such that $B = A \oplus J$, and suppose that I is an ideal of A with $IJ = JI = 0$. Then*

$$K_1(A, B, I) = H_1^A(B; I/I^2) = I/I^2 \underset{A^e}{\otimes} J/J^2,$$
$$K_2(B, I) = K_2(A, I) \oplus K_1(A, B, I).$$

Proof. This is a simple calculation using (4. 1) and (4. 2). Alternatively, note that $K_1(A, B, I) = K_2(B, A, I)$, and we know the latter group is $I/I^2 \underset{A^e}{\otimes} J/J^2$ from [K 2], [GL].

Next, we explore the map $K_2(B/I) \to$ quo-$K_1(A)$ given in (5. 2). If A and B are both commutative, it is well-known that we can write $K_1(A) = U(A) \oplus SK_1(A)$, and that the image of $K_2(B/I)$ is in $SK_1(A)$, or rather in the cokernel of $\phi_1 : K_1(A, B, I) \to SK_1(A)$. This is not true if B is noncommutative, even if A is commutative, as we shall see in Example (5. 7) below.

When I is a radical ideal of A, let $W(A, I)$ denote the subgroup of $(1+I)^*$ generated by $\{(1+ax)(1+xa)^{-1} : a \in A,\ x \in I\}$. Then by [Sw], (2. 1) we have

$$K_1(A, I) \cong (1+I)^*/W(A, I).$$

Now consider the map $\phi_1 : K_1(A, B, I) \to K_1(A, I) \cong (1+I)^*/W(A, I)$ in the excision situation. Clearly, we have exact sequences

$$K_2(A, I) \to K_2(B, I) \to K_1(A, B, I) \xrightarrow{\phi_1} W(B, I)/W(A, I) \to 0.$$
$$0 \to W(B, I)/W(A, I) \to K_1(A, I) \to K_1(B, I) \to 0.$$

In fact, the map $\phi_1 : K_1(A, B, I) \to W(B, I)/W(A, I)$ sends $y_\alpha(b; x)$ to $(1-by)^{-1}(1-yb)$, where $y=x(1+bx)^{-1}$. This calculation is given on p. 227, line 21 of [Sw]. For a radical ideal, then, there is an exact sequence

$$K_2(A/I) \oplus K_2(B) \to K_2(B/I) \to K_1(A)/\overline{W(B, I)} \to K_1(A/I) \oplus K_1(B) \to \cdots.$$

Example (5. 7). Let $B=k\{b, c\}/(b^2=c^2=cb=0)$, where k is any commutative ring. B has k-basis $\{1, b, c, bc\}$. Set $J=(b, c)B$, $I=bcB$, and $A=k\oplus I$. Since $[B, I]=0$, we have $W(B, I)=0$ and quo-$K_1(A)=K_1(A)=K_1(k)\oplus(1+I)^*$. On the other hand, $W(B, J)=(1+I)^*$ and $K_1(B)=K_1(k)\oplus(1+J)^*/(1+I)^* \cong K_1(B/I)$. It follows that the map $K_2(B/I) \to K_1(A)$ is onto $(1+I)^* \cong k^+$, and hence nonzero. In fact, B/I is commutative, and the element $\langle b, \alpha c\rangle$ of $K_2(B/I)$ $(\alpha \in k)$ maps to $(1-\alpha bc) \in (1+I)^*$. This may be seen by inspecting the proof of Proposition (1. 7) above.

In this example it is easy to calculate that

$$K_1(A, B, I) = H_1^A(B; k) \cong J/I.$$

It follows that there is an exact sequence

$$K_2(A, I) \to K_2(B, I) \to J/I \to 0.$$

This is an instance of the general result (5. 8) for ideals I with $I^2=0$.

Theorem (5. 8) (Kassel [Kas]). *If $I^2=0$, there are exact sequences*

$$K_2(A, I) \to K_2(B, I) \to H_1^A(B; I) \to 0,$$
$$0 \to [B, I]/[A, I] \to K_1(A, I) \to K_1(B, I) \to 0.$$

Proof. By [Sw], (2. 6), we have $K_1(A, I) \cong I/[A, I]$ and $K_1(B, I) \cong I/[B, I]$. The second sequence is immediate. By the above remark, the map $K_1(A, B, I) \to I/[A, I]$ sends $y_\alpha(b; x)$ to $(bx-xb)$, and is therefore the Dieudonné determinant map of (4. 2. 3). The first sequence now follows from the extension of $K_1(A, B, I)$ given in (4. 2).

Example (5. 9). Let $A=k[x]/(x^3)$, where k is any commutative ring. Let B be the noncommutative k-algebra $A\{b\}/(bx=x^2,\ xb=0)$, i.e., B is a twisted product of A and $k[b]$ over k. Let $I=xA$. It is a simple matter to see that

$$K_1(A, B, I) \cong H_1^A(B; k) \cong k^+, \qquad (k^+ \text{ means } k \text{ as an additive group})$$
$$K_1(A, I) = (1+xA)^*,$$
$$W(B, I) = (1+x^2A)^* \cong k^+,$$
$$K_1(B, I) = (1+xA)^*/(1+x^2A)^* \cong k^+.$$

In this case, $K_1(A, B, I) \cong W(B, I)$ with $y_\alpha(b; ax)$ mapping to $1+ax^2$. Hence

$$K_2(A, xA) \to K_2(B, xA)$$

is onto, and there is an exact sequence

$$0 \to H_1^A(B; I/I^2) \to K_1(A, I) \to K_1(B, I) \to 0.$$

This shows that the extensions of (5. 8) fail if $I^2 \neq 0$, even though $I^3=0$.

References

[And] *D. W. Anderson*, Relationship among *K*-theories, Lecture Notes in Math. **341**, Berlin-Heidelberg-New York 1973.

[B] *H. Bass*, Algebraic *K*-theory, New York 1968.

[DK] *R. K. Dennis* and *M. Krusemeyer*, $K_2(A[X, Y]/XY)$, a problem of Swan and related computations, J. Pure Appl. Alg. **15** (1979), 125—148.

[GL] *D. Guin-Waléry* and *J.-L. Loday*, Obstruction à l'excision en *K*-théorie algébrique, Lecture Notes in Math. **854**, Berlin-Heidelberg-New York 1981.

[GR] *S. Geller* and *L. Roberts*, Kähler differentials and excision for curves, J. Pure Appl. Alg. **17** (1980), 85—112.

[GW] *S. Geller* and *C. Weibel*, K_2 measures excision for K_1, Proc. AMS **80** (1980), 1—9.

[H] *R. Hartshorne*, Algebraic Geometry, Berlin-Heidelberg-New York 1977.

[K] *F. Keune*, Nonabelian derived functors and algebraic *K*-theory, Lecture Notes in Math. **341**, Berlin-Heidelberg-New York 1973.

[K1] *F. Keune*, The relativization of K_2, J. Algebra **54** (1978), 159—177.

[K2] *F. Keune*, Doubly relative *K*-theory and the relative K_3, J. Pure Appl. Alg. **20** (1981), 39—53.

[Kas] *C. Kassel*, *K*-théorie relatif d'un idéal bilatère de carré nul: étude homologique en basse dimension, Lecture Notes in Math. **854**, Berlin-Heidelberg-New York 1981.

[L] *J.-L. Loday*, Cohomologie et groupe de Steinberg relatifs, J. Algebra **54** (1978), 178—202.

[Mac] *S. Maclane*, Homology, Berlin-Heidelberg-New York 1963.

[Mil] *J. Milnor*, Introduction to algebraic *K*-theory, Annals of Math Studies **72**, Princeton 1971.

[Stein] *M. Stein*, Excision and K_2 of group rings, J. Alg. **18** (1980), 213—224.

[St] *J. Stienstra*, On K_2 and K_3 of truncated polynomial rings, Lecture Notes in Math. **854**, Berlin-Heidelberg-New York 1981.

[SV] *A. Suslin* and *L. Vaserstein*, Serre's problem on projective modules over polynomial rings and algebraic *K*-theory, Math. USSR Izv. **10** (1976), 937—1001 (= Izv. Akad. Nauk SSSR **40** (1976), 993—1054).

[Sw] *R. Swan*, Excision in algebraic *K*-theory, J. Pure Appl. Alg. **1** (1977), 221—252.

[Sw2] *R. Swan*, Nonabelian homological algebra and algebraic *K*-theory, Proc. Symp. Pure Math. **17** (1970), 88—123.

[vdK] *W. van der Kallen*, Appendix to Localization of the *K*-theory of polynomial extensions (by T. Vorst), Math. Annalen **244** (1979), 51—53.

[vdK2] *W. van der Kallen*, Another presentation for Steinberg groups, Indag. Math **39** (1977), 304—312.

[V] *L. Vaserstein*, On the stabilization of the general linear group, Math USSR Sbornik **8** (1969), 383—400 (= Mat. Sbornik **79** (1969), 405—424).

[Vo] *T. Vorst*, Polynomial extensions and excision for K_1, Math. Annalen **244** (1979), 193—204.

[W1] *C. Weibel*, K_2, K_3 and nilpotent ideals, J. Pure Appl. Alg. **18** (1980), 333—345.

[W2] *C. Weibel*, Mayer-Vietoris sequences and module structures on NK_*, Lecture Notes in Math. **854**, Berlin-Heidelberg-New York 1981.

[W3] *C. Weibel*, Mayer-Vietoris sequences and mod p *K*-theory, Lecture Notes in Math. **966**, Berlin-Heidelberg-New York 1982 (to appear).

[W4] *C. Weibel*, *K*-theory and analytic isomorphisms, Inv. Math. **61** (1980), 177—197.

Mathematics Department, Texas A&M University, College Station, Texas 77843, USA

Mathematics Department, Rutgers University, New Brunswick, NJ 08903, USA

Eingegangen 9. Juni 1982