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K_2, K_3 AND NILPOTENT IDEALS

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Introduction

The purpose of this paper is to study the relationship between $K_*(R)$ and $K_*(R/I)$, where $*=2, 3$ and I is a nilpotent ideal of the commutative ring R. The tools that allow us to do so are the relative term $K_2(R, I)$ described by Keune [10] and the Tate map $K_2R \to \Omega_R^2$ as extended by Gersten [7].

Now $K_*(R) \to K_*(R/I)$ is classically an isomorphism for $-\infty < * \leq 1$ (except for units in K_1) and onto for $* = 2$. Van der Kallen's computation [17] shows that it is not generally injective for $* = 2$. The other known results along these lines [14, 19] are that this map is an isomorphism if we replace K_* by its approximations K'_* (for R noetherian) or KV_*.

The main results of this paper are:

(1) $K_3(R) \to K_3(R/I)$ is neither onto nor into and maps exotic elements nontrivially.

(2) $K_3(R) \to K_3(R/I)$ is onto and $K_2(R) \to K_2(R/I)$ is a split surjection when R/I is an integrally closed complete intersection, under various restrictions on I and the characteristic.

(3) If $\frac{1}{2} \in R$, $I^2 = 0$, then there is an exact sequence of R-modules

$$\operatorname{Ker}(\delta) \xrightarrow{\wedge} \Lambda^2 I \to K_2(R, I) \to I \otimes \Omega_{R/I} \to 0,$$

where $\delta: I \otimes I \to I \otimes \Omega_R$. The R-module structure on $K_2(R, I)$ is given by $s\langle a, r\rangle = \langle sa, r\rangle$.

Thus (3) gives a reasonably complete interpretation of $K_2(R, I)$ in terms of differentials.

In Section 1 we study $K_2(R, I)$ in the case $I^2 = 0$. This lays the groundwork for later sections and establishes (3). We give two applications: for $\frac{1}{2} \in R$, $\mathfrak{p}$ a prime ideal of R we have $K_2(R_\mathfrak{p}, I_\mathfrak{p}) = R_\mathfrak{p} \otimes_R K_2(R, I)$. Secondly, if $R = k[x_1, \ldots, x_n]/P^2$, $\operatorname{char}(k) \neq 2$ and $S = R/I$ is a complete intersection, $I = P/P^2$, then $K_2(R, I) = \Lambda^2 I \oplus I \otimes \Omega_S$.

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In Section 2 we consider $R = k[x_1, \ldots, x_n]/P^2$, $I = P/P^2$, where k is a field, $\frac{1}{2} \in k$, and study the map $K_2(R, I) \to K_2(R)$. We show that this map is a split injection if $S = R/I$ is an integrally closed complete intersection. This is one instance of (2).

In Section 3 we consider $R = k[x_1, \ldots, x_n]/P^m$, k a field containing $1/m!$, where $S = R/P$ is normal, and $P = (t)$. We establish (2) in this case and illustrate (1) for $I = (t^j)$, $m > j > m/2$. We obtain similar results in case R is the truncated polynomial ring $k[t]/(t^m)$, k any commutative ring containing $1/m!$

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1. The case $I^2 = 0$

Let R be a commutative ring, I a radical ideal of R. Then [10, Theorem 15] the relative group $K_2(R, I)$ is presented as an abelian group by generators $\langle a, r\rangle$ ($a, b \in I$, $r, s \in R$) and relations

(D1) $\langle a, b\rangle \langle -b, -a\rangle = 1$,

(D2) $\langle a, r\rangle \langle a, s\rangle = \langle a, r + s + ars\rangle$,

(D3) $\langle a, rs\rangle = \langle ar, s\rangle \langle as, r\rangle$,

(D2′) $\langle a, r\rangle \langle b, r\rangle = \langle a + b + abr, r\rangle$,

(D3′) $\langle ab, r\rangle = \langle a, br\rangle \langle b, ar\rangle \langle -abr, -1\rangle$.

Remark. The asymmetry between (D3) and (D3′) follows from the fact that the symbol $\langle 1, a\rangle = \langle -a, -1\rangle$ is not always zero, as the example $\langle 1, -2\rangle = \langle 2, -1\rangle = \{-1, -1\} \neq 0$ in $K_2(\mathbf{Z}/4\mathbf{Z}, 2)$ shows. However, if $\frac{1}{2} \in R$ and $a^N = 0$, then we can choose $b \in aR$ so that $1 + a = (1 + b)^2$, and then $1 = \langle b, 1\rangle = \langle -b, -1\rangle^2 = \langle -a, -1\rangle$, by (D2′).

Notation 1.1. We set $S = R/I$, and write $I \tilde{\wedge} I$ for the quotient of $I \otimes_S I$ by the submodule (or subgroup) generated by $\{a \otimes b + b \otimes a\}$. The quotient map $\tilde{\wedge} : I \otimes I \to I \tilde{\wedge} I$ is the universal skew-symmetric form; $I \tilde{\wedge} I$ is not to be confused with its quotient, the second exterior power $\Lambda^2 I = I \wedge I$ (which has $ar \wedge a = 0$). We shall write the image of $a \otimes b$ in $I \tilde{\wedge} I$ as $a \tilde{\wedge} b$. When $\frac{1}{2} \in R$ we have $I \tilde{\wedge} I = \Lambda^2 I$ and shall drop the tildes. When $I = tR$, we shall write $K_2(R, t)$ for $K_2(R, I)$.

We now assume $I^2 = 0$. Then (D2′) is additivity in I. Using additivity (D1) is equivalent to $\langle a, b\rangle \langle b, a\rangle = 1$. From this (D3′) is equivalent to bilinearity in $I \times I$.

Thus there is a map (of abelian groups) $\psi: I \tilde{\wedge} I \to K_2(R, I)$ given by $\psi(a \tilde{\wedge} b) = \langle a, b\rangle$. We also note that (D2) can be replaced by the following "twisted additivity" in R:

(D4) $\langle a, r\rangle \langle a, s\rangle = \langle a, r+s\rangle \langle a, rsa\rangle$.

Relation (D4) shows that we cannot put a module structure on $\mathrm{Im}(\psi)$ to make ψ a module map. For example, let $F = \mathbf{F}_2(t)$, $R = F[\varepsilon]$. By [17, Example 3], $K_2(F[\varepsilon], \varepsilon) \cong F/F^2 \oplus F$, the isomorphism being $\langle \varepsilon x, \varepsilon\rangle\langle \varepsilon y, t\rangle \leftrightarrow (x, y)$. Identifying F with $I \tilde{\wedge} I$ on generator $\varepsilon \tilde{\wedge} \varepsilon$, we have $\psi(1) = \langle \varepsilon, \varepsilon\rangle = 0$, yet $\psi(t) = \langle \varepsilon t, \varepsilon\rangle \neq 0$.

In order to circumvent this difficulty, we consider $\bar{K}_2(R, I)$, the quotient of $K_2(R, I)$ by the subgroup generated by the $\langle a, ra\rangle$, $a \in I$, $r \in R$. Since $2\langle a, ra\rangle = \langle a^2, ra\rangle = 0$, the kernel of $K_2(R, I) \to \bar{K}_2(R, I)$ is an elementary 2-group.

Lemma 1.2. *$\bar{K}_2(R, I)$ is an $S(=R/I)$-module under the action $s\langle a, r\rangle = \langle as, r\rangle$, and $\bar{\psi}$: $\Lambda^2 I \to \bar{K}_2(R, I)$ is an S-module homomorphism.*

We have $K_2(R, I) = \bar{K}_2(R, I)$ in the following situations:

(a) $\frac{1}{2} \in R$.

(b) *$R/\mathrm{ann}(I)$ is perfect of characteristic 2.*

(c) *$I \subseteq (\mathrm{ann}\, I)^2$, in which case $\psi = 0$.*

Proof. Since $\langle a, rb\rangle = \langle ar, b\rangle$, the action sends all the relations of $K_2(R, I)$ except (D2) into (sums of) relations, as well as fixing the set $\{\langle a, ra\rangle\}$. The S-module structure now follows from (D4).

When $\frac{1}{2} \in R$, (D2) yields $1 = \langle a, ra/2\rangle^2 = \langle a, ra\rangle\langle a, r^2a^2/4\rangle = \langle a, ra\rangle$. In case (b) we have $\langle a, ra\rangle = \langle a, s^2a\rangle = \langle as, sa\rangle^2 = 1$. Finally, case (c) follows from the observation that $\langle I, (\mathrm{ann}\, I)^2\rangle = 1$. To see this, we note that for $x_i, y_i \in \mathrm{ann}\,(I)$ we have

$$\left\langle a, \sum x_i y_i\right\rangle = \prod \langle a x_i, y_i\rangle \langle a y_i, x_i\rangle = 1.$$

In the special case $R = S[\varepsilon]$, $I = \varepsilon R$, the map $\bar{\psi}$ is zero and $\bar{K}_2(R, \varepsilon) \cong \Omega_S$ as an S-module by results of van der Kallen [17].

Vorst [18] has shown that there exists a well-defined group map $\varphi: K_2(R, I) \to I \otimes_S \Omega_S$ given by $\varphi\langle a, r\rangle = a \otimes \mathrm{d}\bar{r}$. This fits together with ψ to give the following description of $K_2(R, I)$.

Theorem 1.3. *When $I^2 = 0$ the following sequence of abelian groups is exact and natural in R and I:*

$$I \tilde{\wedge} I \xrightarrow{\psi} K_2(R, I) \xrightarrow{\varphi} I \otimes_S \Omega_S \to 0.$$

This induces an exact sequence of S-modules:

$$\Lambda^2 I \xrightarrow{\bar{\psi}} \bar{K}_2(R, I) \xrightarrow{\varphi} I \otimes \Omega_S \to 0.$$

Proof. As $\varphi\psi = 0$, $\varphi\langle as, r\rangle = s\varphi\langle a, r\rangle$, and φ is onto, we have only to show that $\varphi: \mathrm{coker}(\psi) \to I \otimes \Omega_S$ is an isomorphism. For each $a \in I$ define

$$D_a: S \to \mathrm{coker}(\psi)$$

by the formula $D_a(\bar{r}) = \langle a, r\rangle$. Using (D4) we see that D_a is a well-defined additive map and that $\mathrm{coker}(\psi)$ is an S-module. By (D3) it is a derivation, and so induces a map $\delta_a : \Omega_S \to \mathrm{coker}(\psi)$ of S-modules, $\delta_a(s\,\mathrm{d}\bar{r}) = \langle as, r\rangle$. The δ_a induce a map $\delta : I \otimes \Omega_S \to \mathrm{coker}(\psi)$ in the standard way. The formula for δ_a shows that δ is onto and that $\varphi\delta = \mathrm{id}(I \otimes \Omega_S)$, whence $\varphi : \mathrm{coker}(\psi) \cong I \otimes \Omega_S$, as desired.

Remark. When $R = S \oplus I$, $I^2 = 0$, we recover results of van der Kallen [17] and Dennis–Krusemeyer [3, (6.7)]. From [3, (6.3)] the map ψ is an isomorphism when $R = \mathbf{Z} \oplus I$. The example $R = S[\varepsilon]$, S a perfect field of characteristic 2, shows that ψ can be zero when $I \tilde{\wedge} I \neq 0$. However, ψ is an isomorphism when $R = \mathbf{Z}/4$, even though $S = \mathbf{Z}/2$. The following result shows that ψ is "almost" a split injection if $R = S \oplus I$. It was proven in the case $\frac{1}{2} \in S$ using different methods in [3].

Corollary 1.4. *If $R = S \oplus I$, the second sequence of Theorem 1.3 is split exact, and*

$$\bar{K}_2(R, I) \cong \Lambda^2 I \oplus (I \otimes \Omega_S).$$

If $\frac{1}{2} \in S$, this gives $K_2(R, I)$ as well. If S is perfect of characteristic 2, ψ induces $\Lambda^2 I \cong K_2(R, I)$.

Proof. We define $\rho : \bar{K}_2(R, I) \to \Lambda^2 I$ by $\langle a, s \oplus b\rangle \to a \wedge b$. All relations are preserved, so ρ is a well-defined group map. Since ρ is a left inverse for $\bar{\psi}$ in Theorem 1.3, the decomposition of $\bar{K}_2(R, I)$ follows. The two special cases follow from Lemma 1.2.

Our goal is to understand the kernel of ψ in the non-split case. One simple but useful case follows immediately from 1.2(c):

Corollary 1.5. *When $I \subseteq \mathrm{ann}(I)^2$, $\varphi : K_2(R, I) \cong I \otimes \Omega_S$.*

To handle the general case, we set $T = R \oplus I$, $J = 0 \oplus I$, and $S = R/I$. By [11, Theorem 58] there is an exact sequence

$$J \otimes_S I \xrightarrow{\delta} J \otimes_R \Omega_R \to J \otimes_S \Omega_S \to 0,$$

where $\delta(a \otimes b) = a \otimes \mathrm{d}b$. We identify T with Keune's $R(I)_1$ by associating $r \oplus a \in T$ with $(r, r + a) \in R(I)_1$. By [10, Theorem 14] there is an exact sequence

$$J \otimes_R I \xrightarrow{\eta} K_2(T, J) \to K_2(R, I) \to 0,$$

where $\eta(a \otimes b) = [x_{12}(-b \oplus b), y_{21}(0 \oplus a)]^{-1} = \langle a, b \oplus -b\rangle$ (see p. 175 of [10]). By naturality, the map $(T, J) \to (R, I)$ induces a map of exact sequences in Theorem 1.3.

All this data fits together to form a commutative diagram:

$$\begin{array}{ccccccc}
 & & J \tilde{\wedge} J & = & I \tilde{\wedge} I & & \\
 & & \downarrow{\scriptstyle \psi_J} & & \downarrow{\scriptstyle \psi_I} & & \\
J \otimes I & \xrightarrow{\eta} & K_2(T, J) & \longrightarrow & K_2(R, I) & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \varphi_J} & & \downarrow{\scriptstyle \varphi_I} & & \\
J \otimes I & \xrightarrow{\delta} & J \otimes \Omega_R & \longrightarrow & J \otimes \Omega_S & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

We now state our main result on $\operatorname{Ker}(\psi)$, again in terms of its quotient $\operatorname{Ker}(\bar{\psi})$.

Theorem 1.6. *Let* $\delta : I \otimes I \to I \otimes \Omega_R$ *be* $\delta(a \otimes b) = a \otimes \mathrm{d}b$. *Then* $\operatorname{Ker}(\bar{\psi}_I) = \wedge(\operatorname{Ker} \delta)$, *and we have the exact sequence of S-modules*

$$\operatorname{Ker}(\delta) \xrightarrow{\wedge} \Lambda^2 I \xrightarrow{\bar{\psi}} \bar{K}_2(R, I) \xrightarrow{\varphi} I \otimes \Omega_{R/I} \to 0.$$

Proof. We first pass from $I \tilde{\wedge} I$ to $\Lambda^2 I$ and note that

$$\wedge : J \otimes I \cong I \otimes I \to \Lambda^2 I$$

is the same map as $-\rho_J \eta$, where ρ_J is the map of Corollary 1.4. Explicitly, $\rho_J \eta(a \otimes b) = \rho_J \langle a, b \oplus -b \rangle = -a \wedge b$.

Let $\beta \in \operatorname{Ker}(\psi_I)$. A diagram chase shows that $\psi_J(\beta) = \eta(\alpha)$ for some $\alpha \in J \otimes I$, and that $\delta(\alpha) = 0$. From Corollary 1.4 we see that $\wedge(\alpha) = -\rho_J \psi_J(\beta) = -\bar{\beta}$ in $\Lambda^2 I$.

Conversely, suppose $\delta(\alpha) = 0$ for $\alpha \in J \otimes I$. Then $\varphi_J \eta(\alpha) = 0$, so $\eta(\alpha) = \psi_J(\beta)$ for some $\beta \in I \tilde{\wedge} I$. It follows that $\beta \in \operatorname{Ker}(\psi_I)$ and $\bar{\beta} = \rho_J \eta(\alpha) = -\wedge(\alpha)$. Done.

We can apply this result in the following geometric situation. Let k be a field of characteristic $\neq 2$, and let P be a prime ideal of $A = k[x_1, \ldots, x_n]$ generated by an A-sequence. Set $R = A/P^2$, $S = A/P$; $\operatorname{Spec}(R)$ is the "first neighbourhood of infinitesimals" [5, 16.1.2] of the complete intersection $\operatorname{Spec}(S)$ in $\mathbf{A}^n$. The ideal $I = P/P^2$ is a free S-module.

In this case we have only to consider the map $d : I \to S \otimes \Omega_R = S \otimes \Omega_A$. By Theorem 1.6 we see that ψ is an injection iff d is.

Corollary 1.7. *If* $S = A/P$ *is a complete intersection in* $A = k[x_1, \ldots, x_n]$, $\operatorname{char}(k) \neq 2$, *then the map* ψ_I *is a split injection, so that as abelian groups*

$$K_2(R, I) \cong \Lambda^2 I \oplus (I \otimes \Omega_S).$$

Here $R = A/P^2$, $I = P/P^2$.

Proof. Let F be the subfield $\mathbf{Q}$ or $\mathbf{Z}/p\mathbf{Z}$ of k. The quotient field L of S is separably generated over F [5]. (If L/k were separable, we could have used $F = k$ instead.) It is well-known [11] that the separability condition implies that $d: I \to S \otimes \Omega_{A/F}$ is an injection. By the above remarks ψ is an injection. Thus the sequence of Theorem 1.3 is a s.e.s. of vector spaces over F, and splits as such. Done.

Warning. The splitting will not respect the S-module structure on $K_2(R, I)$ in general.

We conclude the section by noting the following simple local-global principle, and remarking that nil ideals in commutative rings are never K_2-regular.

Proposition 1.8. *When $\frac{1}{2} \in R$ and $I^2 = 0$, for every multiplicatively closed set U of R we have $K_2(R_U, I_U) = K_2(R, I)_U$ and $\mathrm{Ker}(\psi)_U = \mathrm{Ker}(\psi_U)$, where $\psi_U : \Lambda^2 I_U \to K_2(R_U, I_U)$. In particular, ψ (resp. $K_2(R, I)$) is zero iff ψ_m (resp. $K_2(R_m, I_m)$) is zero for every $m \in \mathrm{Max}(R)$.*

Proof. By Lemma 1.2, $K_2(R, I)$ is an R-module, so $K_2(R, I)_U$ makes sense. The sequence of Theorem 1.6 is natural in R, and the outside terms localize properly. The result now follows from the five-lemma.

Proposition 1.9. *$NK_2(R, I) \neq 0$ whenever I is a nil ideal in a commutative ring R.*

Proof. A standard argument of Algebraic Topology (see the weaving argument of [12]) gives the exact

$$NK_2(R, I^2) \to NK_2(R, I) \to NK_2(R/I^2, I/I^2) \to NSK_1(R, I^2).$$

The hypotheses imply that $NSK_1(R, I^2) = 0$, so we start all over with $I^2 = 0$. Now the summand $I[t] \otimes dt$ of $I[t] \otimes \Omega_S$ is never zero, so it follows from Theorem 1.3 that the subgroup $\langle I[t], t\rangle$ is a nonzero summand of $NK_2(R, I)$.

2. Image in $K_2(R)$

We now apply the computations of Section 1 to compute $K_2(A/P^2)$, where $A = k[x_1, \ldots, x_n]$, k a field of characteristic $\neq 2$. We restrict to the complete intersection case in order to expedite computation. Set $R = A/P^2$, $S = A/P$, $I = P/P^2$. The sequence

(2.1) $$K_3(R) \to K_3(S) \to K_2(R, I) \xrightarrow{\iota_*} K_2(R) \to K_2(S) \to 0$$

shows that $K_2(R)$ is determined (up to extension) by $K_2(S)$ and $K_2(R, I)$ when ι_* is an injection. In this case we will also have $K_3(R) \to K_3(S)$ onto. We will also see that this gives $K_2(R) = K_2(S) \oplus \Lambda^2 I \oplus (I \otimes \Omega_S)$.

Our primary tool will be the "Tate map" from $K_2(R)$ to Ω^2_R, taking $\{u, v\}$ to $(uv)^{-1}\,\mathrm{d}u \wedge \mathrm{d}v$. This map was defined in [7] up to sign; R. K. Dennis has shown [2] that it takes $\langle a, b\rangle$ to $(1+ab)^{-1}\,\mathrm{d}a \wedge \mathrm{d}b$. We will compose with $\Omega^2_R \to S\otimes\Omega^2_R$ in this section. The advantage gained is that

$$S\otimes\Omega^2_R \cong S\otimes\Omega^2_A$$

is a free S-module, and $\langle a, b\rangle$ maps to $\mathrm{d}a \wedge \mathrm{d}b$. Call the map $K_2(R)\to S\otimes\Omega^2_R$ "$\overline{\text{Tate}}$".

We first pause to acquire the following useful results from exterior algebra. We forget our restriction on k, S until (2.6).

Proposition 2.2. *Let $V\subset W$ be an inclusion of free S-modules, S any commutative ring. The kernel of $V\otimes W\to \Lambda^2 W/\Lambda^2 V$ is contained in $V\otimes(V_L\cap W)$, where $V_L\cap W=\{w\in W \mid Sw\cap V\neq 0\}$.*

Proof. Let L be the total quotient ring of S, and write $V_L = V\otimes L$, etc. Now L is a semilocal ring of dimension 0, so [8, § 20.4] W_L/V_L must be free. Thus we can choose a complementary L-module Z so that $W_L = V_L\oplus Z$. We have $\Lambda^2 W_L = \Lambda^2 V_L\oplus \Lambda^2 Z\oplus(V_L\otimes Z)$ and $V\otimes(W/V)_L$ embeds in the third summand. The result follows from the fact that the kernel of $W\to(W/V)_L = W_L/V_L$ is $V_L\cap W$. Done.

Corollary 2.3. *Let W be a free S-module, $v\in W$ a faithful element. Then the kernel of $(v\wedge -): W\to\Lambda^2 W$ is the submodule $I^{-1}v$, where I is the trace ideal of v in W.*

Proof. Fix a basis $\{x_\alpha\}$ for W and write $v=\sum v_\alpha x_\alpha$, $V=Sv$. Note that I is generated by the v_α so $V_L\cap W = I^{-1}v$. The proposition now implies that $\ker(v\wedge -)\subseteq I^{-1}v$. Conversely, given $s\in I^{-1}$ we have

$$v\wedge sv = \sum_{\alpha<\beta}[v_\alpha(sv_\beta)-v_\beta(sv_\alpha)]x_\alpha\wedge x_\beta = 0.$$

Remarks. If W is not free we have the following counterexample. Let $S=k[v, w]$ and let W be the ideal (v, w). Then $\Lambda^2 W\cong k$ on generator $v\wedge w$ and the kernel of $(v\wedge -)$ is (v, w^2).

If v is not faithful, then $v\wedge \operatorname{ann}(v)W = 0$, while $\operatorname{ann}(v)W\not\subset I^{-1}v$.

The following result is essentially a nonseparable version of [15] and [16]. I am indebted to W. Vasconcelos for the main ideas in the proof.

Proposition 2.4. *Let $S=A/P$ be a complete intersection in $A=k[x_1,\ldots,x_n]$. Then Ω_S is torsionfree if and only if S is integrally closed.*

Proof. Let L be the quotient field of S. Since L is separably generated over $\mathbf{Q}$ or $\mathbf{Z}/p\mathbf{Z}$, the sequence

$$0\to P/P^2 \xrightarrow{\delta} S\otimes\Omega_A\to\Omega_s\to 0$$

is exact. Restricting attention to a finitely generated summand of $S\otimes\Omega_A$, we argue as in [16]: as $\mathrm{pd}(\Omega_S)\leq 1$, the associated primes of Ω_S (hence all its primes, as S is C-M) have height ≤ 1. Thus S is normal iff $\Omega_{S_\mathfrak{p}}$ is free for all height 1 primes $\mathfrak{p}$. In particular, this proves the "if" part.

Conversely, suppose some $R=S_\mathfrak{p}$ is not a DVR, height $(\mathfrak{p})=1$. Then some $e\in L-R$ has $e\mathfrak{p}\subset R$. As δ is not invertible, some $f\in P-\mathfrak{p}P$ has $\delta(f)\in\mathfrak{p}(R\otimes\Omega_A)$. Let ω denote the image of $e\delta(f)$ in Ω_R; ω is a torsion element. Moreover, $\omega\neq 0$ since $\omega=0$ would imply that $e\delta(f)=\delta(g)$ for some $g\in P$, i.e., that $ef\in R\otimes P/P^2$. But $e\notin R$ and some coordinate of f is a unit of R, so this is absurd. Done.

Corollary 2.5. *Let $I=P/P^2$, $S=k[x_1,\ldots,x_n]/P$ a complete intersection. Then the map*

$$\wedge(\mathrm{d}\otimes 1): I\otimes\Omega_S\to(S\otimes\Omega_A^2)/\mathrm{d}(\Lambda^2 I)$$

given by $\bar{x}\otimes\mathrm{d}\bar{y}\mapsto\mathrm{d}x\wedge\mathrm{d}y$ (mod $\mathrm{d}\Lambda^2 I$) is an injection if S is integrally closed. If $P=(x)$ is principal this map ($\mathrm{d}x\wedge -:\Omega_S\to S\otimes\Omega_A^2$) is an injection if and only if S is a normal hypersurface in $\mathbf{A}^n$.

Proof. We apply Proposition 2.2 with $V=\mathrm{d}I$, $W=S\otimes\Omega_A$. Since $\Omega_S=W/V$, $V_L\cap W=V$ iff Ω_S is torsionfree. The last statement follows from Corollary 2.3.

Theorem 2.6. *Let $S=A/P$ be an integrally closed complete intersection in $A=k[x_1,\ldots,x_n]$, $\mathrm{char}(k)\neq 2$. Set $R=A/P^2$, $I=P/P^2$. Then:*

(i) $K_2(R)=K_2(S)\oplus\Lambda^2 I\oplus(I\otimes\Omega_S)$.

(ii) *The map $K_3(R)\to K_3(S)$ is onto.*

Proof. By the introductory remarks of this section, it is enough to show that the composite

$$\overline{\mathrm{Tate}}(\iota_*): K_2(R,I)\to K_2(R)\to S\otimes\Omega_R^2$$

is an injection. This is a vector space map over $\mathbf{Q}$ or $\mathbf{Z}/p\mathbf{Z}$ and splits as such, so that $\overline{\mathrm{Tate}}$ splits ι_*.

The map $\overline{\mathrm{Tate}}(\iota_*\psi):\Lambda^2 I\to S\otimes\Omega_R^2$ is just $\Lambda^2\mathrm{d}: f\wedge g\to \mathrm{d}f\wedge\mathrm{d}g$, so it is enough to show that the induced map $I\otimes\Omega_S\to S\otimes\Omega_A^2/\mathrm{d}(\Lambda^2 I)$ is an injection. This situation is covered by Corollary 2.5 since the induced map is easily seen to be $\wedge(\mathrm{d}\otimes 1)$. Done.

The astute reader may wonder what happens if Ω_R^2 is used instead of its quotient $S\otimes\Omega_R^2$. The answer seems complicated in general. To illustrate the point, we consider the special case of a plane curve given by $t(x,y)=0$. The map from $K_2(R,t)\cong\Omega_S$ to Ω_R^2 is $\omega\to\mathrm{d}t\wedge\omega+t\,\mathrm{d}\omega$. Using the ideas of Corollaries 2.3 and 2.5 we see that the kernel of this map lies inside the subgroup $I^{-1}\,\mathrm{d}t$ of Ω_S. To compute the image in Ω_R^2, we project to the summand $\Omega_{R/k}^2=(R/tJ)\,\mathrm{d}x\wedge\mathrm{d}y$, where $J=(\partial t/\partial x,\partial t/\partial y)S$.

Lemma 2.7. *Let* $R = k[x, y]/(t^2)$, $S = k[x, y]/(t)$, *and pick* $e \in J^{-1}$ *so that* $\omega = e\,\mathrm{d}t \in \Omega_S$. *Write* $\omega = \omega_0 + \omega_x\,\mathrm{d}x + \omega_y\,\mathrm{d}y$ *and let* $s \in S$ *be such that* $\omega_y(\partial t/\partial x) - \omega_x(\partial t/\partial y) \equiv st \pmod{t^2}$ *in* $k[x, y]$. *Then the map* $I^{-1}\mathrm{d}t \to S/J \cong t\Omega^2_{R/k}$ *induced by* $\mathrm{Tate}(\iota_*): \Omega_S \to \Omega^2_R$ *sends* ω *to* $s + (\partial\omega_y/\partial x - \partial\omega_x/\partial y)$.

As an application, set $t = x^3 - y^2$. A typical element of $I^{-1}\,\mathrm{d}t$ is $\omega = (a + bx)(3y\,\mathrm{d}x - 2x\,\mathrm{d}y)$, $a, b \in k$, when $\mathrm{char}(k) \neq 2, 3$. The element s is $-6(a + bx)$ and the image of ω in $S/J = k[x]/(x^2)$ is $-(11a + 13bx)$. We thus distinguish four cases. In the first three, $a \in k$ is such that $\mathrm{d}a = 0$ in Ω_k.

$\mathrm{char}(k) = 3$: The kernel of $\mathrm{Tate}(\iota_*)$ is all $\langle at, y\rangle$.

$\mathrm{char}(k) = 11$: The kernel of $\mathrm{Tate}(\iota_*)$ is all $\langle 3ayt, x\rangle\langle -2axt, y\rangle$.

$\mathrm{char}(k) = 13$: The kernel of $\mathrm{Tate}(\iota_*)$ is all $\langle 3axyt, x\rangle\langle -2ax^2t, y\rangle$.

$\mathrm{char}(k) = 0, 5, 7$ or $p > 13$: The map $\mathrm{Tate}(\iota_*)$ is an injection. Hence $K_3(R) \to K_3(S)$ is onto and $K_2(R) = K_2(S) \oplus \Omega_S$.

3. Higher order nilpotence

Consider the following two situations. For convenience we fix an integer $N \geqslant 3$.

(A) k is a commutative ring containing $1/N!$. Set $A = k[t]$, $S = k$, and let $R_m = A/(t^m)$ be the truncated polynomial ring for $m \geqslant 1$.

(B) k is a field containing $1/N!$. Set $A = k[x_1, \ldots, x_n]$, and let $0 \neq t \in A$ be such that $S = A/(t)$ is an integrally closed domain. Let R_m denote $A/(t^m)$, the $(m-1)$th neighbourhood of infinitesimals of S for $m \geqslant 1$.

Note that if S is regular in (B) we recover case (A) with $k = S$. For then S is formally smooth over the ground field ($\mathbf{Q}$ or $\mathbf{Z}/p\mathbf{Z}$) and $R_m \to S$ splits.

We now demonstrate the main results (1), (2) mentioned in the introduction.

Theorem 3.1. *Assume either case* (A) *or* (B), *and let* $2 \leqslant m \leqslant N$.

(a) *For* $m > j > m/2$ *the following sequence is exact*:

$$K_3(R_m) \to K_3(R_j) \to R_{m-j} \to 0.$$

The map $K_3(R_m) \to K_3(S)$ *is onto.*

(b) *For* $m \geqslant 3$ *the maps* $K_3(R_m) \to K_3(R_{m-1})$ *are not onto. For* $N > m \geqslant 4$ *every element of* $K_3(R_m)$ *not coming from* $K_3(R_{m+1})$ *maps to a nonzero element of* $K_3(R_{m-1})$. *Thus* $K_3(R_m) \to K_3(R_{m-1})$ *is a nontrivial map on the "exotic" elements, i.e., those elements mapping to* 0 *in* $K_3(S)$ *but not coming from* $K_3(R_{m+1})$.

(c) *We have* $K_2(R_m) = K_2(S) \oplus K_2(R_m, t)$. *The abelian group* $K_2(R_m, t)$ *is isomorphic to* $S^{m-1} \otimes_S \Omega_S$ *and is filtered by the images* F_j *of* $K_2(R_m, t^j)$ *with* $F_j/F_{j+1} \cong \Omega_S$. *Moreover, the composite*

$$K_2(R_m, t) \longrightarrow K_2(R_m) \xrightarrow{\mathrm{Tate}} R_{m-1} \otimes \Omega^2_A$$

is an injection. In case (A), *we can identify* $K_2(R_m, t)$ *with the summand* $R_{m-1}\,\mathrm{d}t \otimes_k \Omega_k$ *of* $R_{m-1} \otimes \Omega^2_A$.

We begin by making some general remarks. Since $K_1(R_m, t)=0$, the long exact ideal sequence ends in

$$(3.2)\qquad K_3(R_m)\to K_3(R_j)\to K_2(R_m, t^j)\xrightarrow{\iota_*} K_2(R_m)\to K_2(R_j)\to 0.$$

If $m>j>m/2$, Theorem 1.3 gives an isomorphism $\varphi: K_2(R_m, t^j)= t^jR_m\otimes\Omega_{R_j}=R_{m-j}\otimes_A\Omega_A$. Thus the proof is effectively reduced to a computation of ι_*.

The following lemma gives general conditions under which ι_* is not an injection.

Lemma 3.3. *Let R be a commutative ring containing both $1/m$ and t, where $t^m=0$. Then $\langle t^j, t^{m-j}\rangle=0$ in $K_2(R, t)$ and $K_2(R)$ for $0<j<m$.*

Since $K_2(R, t^j)=R/\operatorname{ann}(t^j)\otimes\Omega_R$ whenever $m>j>m/2$ and $\langle t^j, t^{m-j}\rangle$ corresponds to $d(t^{m-j})$, ι_* cannot be an injection whenever $d(t^{m-j})\neq 0$. In our cases this is so, since $dt\neq 0$in $S\otimes_A\Omega_A$ implies that $d(t^{m-j})\neq 0$ in $R_{m-j}\otimes\Omega_R$.

Proof. We first recall the general fact that if $1+xy^n$ is a unit of R we have $\langle x, y^n\rangle=n\langle xy^{n-1}, y\rangle$. This follows from relation (D3). We are now reduced to the case $j=m-1$, since $\langle at^j, t^{m-j}\rangle=(m-j)\langle at^{m-1}, t\rangle$. We have $-\langle at^{m-1}, t\rangle=\langle t, at^{m-1}\rangle=\langle at, t^{m-1}\rangle=(m-1)\langle at^{m-1}, t\rangle$, so $m\langle at^{m-1}, t\rangle=0$. In particular, $\langle at^{m-1}, t\rangle=m\langle (a/m)t^{m-1}, t\rangle=0$.

While we are on the subject, we note that the following sharpened version of (3.3) can be extracted from part (c) of Theorem 3.1. We will not use this result.

Corollary 3.4. *Let R be a commutative ring containing both $1/m!$ and t, where $m\geqslant 2$ and $t^m=0$. Then for all $i, j\geqslant 1$ we have $\langle t^i, t^j\rangle=0$ in $K_2(R, t)$ and $K_2(R)$. In particular $\langle t, t\rangle=0$.*

Proof. By naturality of K_2 it is enough to consider the case $R=k[t]/(t^m)$, $k=\mathbf{Z}[1/m!]$. Since $\langle t^i, t^j\rangle$ is killed by the Tate map, part (c) of (3.1) will imply that it is zero in $K_2(R, t)$.

We now prove part (c). Note that case (A) is a slightly modified version of J. Graham's result [9]. For in this case we have the isomorphism $\Omega_{R_m}=R_m\otimes\Omega_k\oplus R_{m-1}\,dt$, $df=Df\oplus f'\,dt$. The element $df\wedge dg$ projects to $f'Dg-g'Df$ in the summand $R_{m-1}\otimes\Omega_k$ of $\Omega^2_{R_m}$.

Proposition 3.5. *Assume the notation of case* (A), *$m\leqslant N$. The Tate map induces an isomorphism of $K_2(R_m, t)$ with $R_{m-1}\otimes\Omega_k$, sending $\langle tf, g\rangle$ to $(1+tfg)^{-1}\times[fDg+t(f'Dg-g'Df)]$.*

Proof. If $m=2$ the Tate map sends $\varphi^{-1}(f\,dg)$ to $f\,dg$ for $f, g\in R_1=k$. Inductively, we take $j=m-1$ in (3.2), noting that $\varphi: K_2(R_m, t^{m-1})\cong\Omega_k\oplus k\,dt$. By Lemma 3.3

the summand $\varphi^{-1}(k\ \mathrm{d}t) = k\langle t^{m-1}, t\rangle$ lies in the kernel of ι_*. Thus both rows are exact in the diagram

$$\begin{array}{ccccccccc} & & \Omega_k & \xrightarrow{\iota_*} & K_2(R_m, t) & \longrightarrow & K_2(R_{m-1}, t) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \overline{\text{Tate}}} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & \Omega_k & \xrightarrow{(m-1)t^{m-2}} & R_{m-1}\otimes\Omega_k & \longrightarrow & R_{m-2}\otimes\Omega_k & \longrightarrow & 0. \end{array}$$

The displayed formula for the Tate map shows that the diagram commutes. The proposition follows from a diagram chase.

In case (B) we have to be more careful, since we have no splitting of Ω_R, and must be content with showing that the Tate map is an injection on $K_2(R_m, t)$. Since we have an underlying field, the image of $K_2(R_m, t)$ is a summand of $R_{m-1}\otimes\Omega^2_A$, whence the splitting $K_2(R_m) = K_2(S)\oplus K_2(R_m, t)$.

Since the result for $m = 2$ follows from Theorem 2.6, we induct using the diagram

$$\begin{array}{ccccccccc} & & \Omega_S & \xrightarrow{\iota_*} & K_2(R_m, t) & \longrightarrow & K_2(R_{m-1}, t) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \overline{\text{Tate}}} & & \downarrow{\scriptstyle \overline{\text{Tate}}} & & \\ 0 & \longrightarrow & S\otimes\Omega^2_A & \xrightarrow{t^{m-2}} & R_{m-1}\otimes\Omega^2_A & \longrightarrow & R_{m-2}\otimes\Omega^2_A & \longrightarrow & 0. \end{array} \qquad (3.6)$$

The Ω_S appears in (3.6) for the following reason. By Lemma 1.2 the subgroup $\langle Rt^{m-1}, t\rangle$ of $K_2(R, t)$ has an R-module structure (for convenience we use R for the fixed R_m). By (3.3) its generator $\langle t^{m-1}, t\rangle$ is zero. Using Theorem 1.3 the quotient $K_2(R, t^{m-1})/\langle Rt^{m-1}, t\rangle$ is isomorphic to Ω_S under $\langle gt^{m-1}, h\rangle \mapsto g\ \mathrm{d}h$.

A short computation shows that (3.6) commutes if the left vertical arrow is $g\ \mathrm{d}h \mapsto (m-1)\ \mathrm{d}t \wedge g\ \mathrm{d}h$. As S is integrally closed, this map is monic by Corollary 2.5. A diagram chase shows that ι_* and $\overline{\text{Tate}}: K_2(R_m, t) \to R_{m-1}\otimes\Omega^2_A$ are monic. This proves part (c) of Theorem 3.1.

Remark. The situation is more complicated in the case $R_m = A/P^m$, P generated by an A-sequence of length >1, even if A/P is normal (so the Tate map is an injection for $m = 2$ by (2.5)). In the case A/P regular and $\mathrm{char}(k) = 0$, a formula has been given by Bloch [1]. The difficulty in extending the proof of case (B) is exemplified by the fact that the kernel of $K_2(R_3, t^2) \to R_2\otimes\Omega^2_A$ always contains $\langle s^2, s\rangle$ and $\langle t^2, t\rangle$ for $s, t \in P$, but not always $\langle s^2, t\rangle$.

The proof of (a) is now easy. We map R_{m-j} into $K_2(R_m, t^j) = R_{m-j}\otimes\Omega_A$ by sending r to

$$e(r) = \langle rt^j, t\rangle + \langle t^{j+1}/(j+1), r\rangle = r\ \mathrm{d}t + t/(j+1)\ \mathrm{d}r.$$

This lies in the kernel of ι_* as it is killed by the Tate map, so we have two things to establish.

(1) The map e is an injection. Write $r = st^i$ with $s \notin tR_{m-j}$; we have to show that $r\,\mathrm{d}t + t\,\mathrm{d}r/(j+1) = ur\,\mathrm{d}t + t^{i+1}\,\mathrm{d}s/(j+1)$ is nonzero, where u is the unit $(i+j+1/j+1)$. In the subquotient $S \otimes \Omega_A = t^i R_{m-j} \otimes \Omega_A / t^{i+1} R_{m-j} \otimes \Omega_A$ this is $us\,\mathrm{d}t$. But the map $\mathrm{d}t : S \to S \otimes \Omega_A$ is an injection (as in (1.7)), so $us\,\mathrm{d}t \neq 0$.

(2) $\mathrm{Ker}(\iota_*) = e(R_{m-j})$. A typical element of $R_{m-j} \otimes \Omega_A$ is $t^i\omega$, where $\omega \notin tR_{m-j} \otimes \Omega_A$, i.e. $\bar{\omega} \neq 0$ in $S \otimes \Omega_A$. Now

$$\mathrm{Tate}(\iota_* \varphi^{-1}) : t^i\omega \mapsto (i+j)t^{i+j-1}\,\mathrm{d}t \wedge \omega + t^{i+j}\,\mathrm{d}\omega$$

so if $\varphi^{-1}(t^i\omega) \in \mathrm{Ker}(\iota_*)$ we must have $\bar{\omega} = s\,\mathrm{d}t$ in $S \otimes \Omega_A$. Lift s to R_{m-j} and subtract $e(u^{-1}st^i)$ from ω to increase i. Eventually we get $\omega \equiv 0 \bmod e(R_{m-j})$ as desired.

This establishes (a) and the first sentence of part (b) of Theorem 3.1. When $N > m \geq 4$ we consider the commutative diagram

$$\begin{array}{ccccccc}
K_3(R_{m+1}) & \longrightarrow & K_3(R_m) & \longrightarrow & S & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
K_3(R_{m+1}) & \longrightarrow & K_3(R_{m-1}) & \longrightarrow & R_2 & \longrightarrow & 0.
\end{array} \tag{3.7}$$

The S corresponds to the subgroup $\langle St^m, t\rangle$ of $K_2(R_{m+1}, t^m)$, while the R_2 corresponds to the subgroup of all $\langle rt^{m-1}, t\rangle + \langle t^m/m, r\rangle$, $r \in R_2$, of $K_2(R_{m+1}, t^m)$. It follows that the right vertical arrow of (3.7) is the injection $s \mapsto st$. This proves (b) since $K_3(R_{m+1})$ maps onto $K_3(S)$.

Done with Theorem 3.1.

The proof of part (b) actually gives a stronger result:

Corollary 3.8. *Assume* $1/m! \in k$, *and let* $r \in K_3(R_{m-1})$, $r \notin \mathrm{im}\,K_3(R_m)$. *The image of* r *in* $K_3(R_j)/\mathrm{im}\,K_3(R_m)$ *is zero iff* $j \leq \frac{1}{2}m$.

Proof. $K_3(R_j)/\mathrm{im}\,K_3(R_m)$ is a subgroup of $K_2(R_m, t^j)$, and r maps to some element $\langle at^{m-1}, t\rangle$, $a \in S$. The assumption on r implies that $a \neq 0$. By Theorem 1.3, $\langle at^{m-1}, t\rangle$ is nonzero if $m < 2j$ and zero if $m = 2j$. Finally, if $m = 2j+1$ we argue as in Lemma 3.3:

$$\begin{aligned}
\langle at^{m-1}, t\rangle &= \langle at^j, t^{m-j}\rangle\langle t^j, at^{m-j}\rangle \\
&= 2\langle at^j, t^{m-j}\rangle \\
&= 2(m-j)\langle at^{m-1}, t\rangle \\
&= m\langle (a/m)t^{m-1}, t\rangle = 0.
\end{aligned}$$

We conclude with the following amusing consequence of part (b). Assume either case (A) or (B) with $\mathbf{Q} \subset k$. Form the chain complex $C_0 = 0$ and

$$C_m = K_3(R_m)/\mathrm{image}(K_3(R_{m+2})),$$

$m \geq 1$, with the obvious maps $C_m \to C_{m-1}$. By part (c) we have $C_1 = 0$. Now $K_2(R_4, t^2) \cong \Omega_{R_2}$ and the Tate map sends f dg to $\mathrm{d}(ft^2) \wedge \mathrm{d}g$ in $R_3 \otimes \Omega^2_A$. An analysis entirely similar to the proof of part (a) of Theorem 3.1 shows that $e: S \to \Omega_{R_2}$, $e(s) = s\,\mathrm{d}t + t\,\mathrm{d}s/3$, is an injection and that $e(S) = \mathrm{Ker}(\iota_*)$. Thus $C_2 \cong S$ and the map $C_3 \to C_2$ is induced by $K_2(R_5, t^3) \to K_2(R_4, t^2)$; this map is zero since $\langle rt^3, t\rangle + \langle t^4/4, r\rangle = 0$ in $K_2(R_4, t^2)$. In summary, we have

Corollary 3.9. *The complex $C_* = K_3(R_*)/\mathrm{im}\, K_3(R_{*+2})$ is*

$$\cdots \xrightarrow{t} R_2 \xrightarrow{t} R_2 \xrightarrow{0} S \to 0 \to 0.$$

Hence $H_m(C_*) = 0$ except for $H_3 = H_2 = S$.

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