

K_2 -REGULARITY AND NORMALITY

CHRISTIAN HAESEMAYER AND CHARLES A. WEIBEL

ABSTRACT. We take a fresh look at the relationship between K -regularity and regularity of schemes, proving two results in this direction. First, we show that K_2 -regular affine algebras over fields of characteristic zero are normal. Second, we improve on Vorst's K -regularity bound in the case of local complete intersections; this is related to recent work on higher du Bois singularities.

Recall that a ring R is K_n -regular if and only if $NK_s(R) = 0$ for all $s \leq n$, where $NK_n(R)$ is $K_n(R[t])/K_n(R)$. In his 1979 paper [23], Ton Vorst conjectured that an affine k -algebra A of dimension d is regular if, and only if, it is K_{d+1} -regular. This conjecture was proved in characteristic zero by the authors and Cortiñas in [1], and by Kerz, Strunk, and Tamme over perfect fields of characteristic p in [17], drawing on previous work of Geisser and Hesselholt in [11].

In this paper, we make two improvements on existing results. In the first (Theorem 1.9), we generalize Vorst's original result in a different direction by proving that K_2 -regularity implies normality in arbitrary dimensions:

thm A

Theorem 0.1. *Let A be a noetherian commutative ring containing \mathbb{Q} with finite normalization. If A is K_2 -regular then A is normal.*

Remark 0.2. The converse of Theorem 0.1 is false in dimensions larger than 1. Indeed, [26] provides numerous examples of normal surface singularities that are not even K_{-1} -regular, let alone K_2 -regular.

In the second (Theorem 3.3), we begin investigating whether Vorst's bound is the best possible. Combining the K -theoretic calculations from [2] and algebraic-geometric results from [19], we show that much sharper bounds are possible, at least in the case of local complete intersections:

Date: January 3, 2025.

Key words and phrases. Algebraic K -theory, cyclic homology, du Bois complex, algebraic surfaces.

Haesemeyer was supported by ARC DP-210103397.

Weibel was supported by NSF grant 2001417.

thm B

Theorem 0.3. *Let $X \in \text{Sch}/k$ be an affine local complete intersection scheme over a field of characteristic zero, and $p \geq 1$. If X is K_{p+1} -regular, then X is regular in codimension $2p$. In particular, if X has dimension d and is K_q -regular for some $q > d/2$, then X is regular.*

in particular

Example 0.4. Consider the case of a surface X . Theorem 0.1 says that if X is K_2 -regular, then it is normal. Theorem 0.3 says that if X is moreover a local complete intersection, then it is, in fact, non-singular. In contrast, the main result of [1] requires K_3 -regularity for the latter conclusion.

1. K_2 -REGULARITY IMPLIES NORMALITY.**sec:normality**

In this section we prove that a K_2 -regular algebra of finite type over a field of characteristic zero is necessarily normal. This generalizes the arguments that Vorst used in [23] to prove that K_2 -regular curves over fields must be regular. By [23, 1.9], a commutative ring A is K_n -regular iff $A_{\mathfrak{m}}$ is K_n -regular for every maximal ideal \mathfrak{m} of A . This reduces the question to the case in which A is local.

Throughout this section, A will be a reduced noetherian local ring, with nonzero maximal ideal \mathfrak{m} , and B is a finite birational extension of A . An important case is when the normalization of A is finite over A . This is the case for example when A is a localization of some affine algebra over a field. We will write \mathfrak{c} for the conductor of $A \rightarrow B$, that is, the annihilator of the finitely generated A -module B/A .

It is well-known that there is a Milnor square of rings

square:conductor

$$(1.1) \quad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A/\mathfrak{c} & \longrightarrow & B/\mathfrak{c}. \end{array}$$

From this we obtain a diagram of relative K -theory exact sequences, for $n \in \mathbb{Z}$,

ladder

$$(1.2) \quad \begin{array}{ccccccc} NK_{n+1}(A/\mathfrak{c}) & \longrightarrow & NK_n(A, \mathfrak{c}) & \longrightarrow & NK_n(A) & \longrightarrow & NK_n(A/\mathfrak{c}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ NK_{n+1}(B/\mathfrak{c}) & \longrightarrow & NK_n(B, \mathfrak{c}) & \longrightarrow & NK_n(B) & \longrightarrow & NK_n(B/\mathfrak{c}). \end{array}$$

If R is a commutative ring, we shall write $NU(R)$ for the group of units $R[t]^\times/R^\times$; it is in 1–1 correspondence with the set $\text{Nil}(R)[t]$. It is also the top of the γ -filtration of $NK_1(R)$.

1-regular_preserved

Lemma 1.3. *Suppose the conductor \mathfrak{c} is \mathfrak{m} -primary. If A is K_1 -regular, then so is B .*

Proof. By hypothesis, A/\mathfrak{c} and B/\mathfrak{c} are Artinian; therefore $NK_n(A/\mathfrak{c}) = NK_n(B/\mathfrak{c}) = 0$ for $n \leq 0$. By excision, the maps $NK_n(A, \mathfrak{c}) \rightarrow NK_n(B, \mathfrak{c})$ are isomorphisms for $n \leq 0$ and onto for $n = 1$. It now follows from (1.2) and the 5-Lemma that $NK_n(A) \rightarrow NK_n(B)$ is an isomorphism for $n < 0$ and onto for $n = 0$; therefore, B is K_0 -regular.

Since $NK_1(A) = 0$, the map $NK_2(A/\mathfrak{c}) \rightarrow NK_1(A, \mathfrak{c})$ is onto. Because $NK_1(A, \mathfrak{c}) \rightarrow NK_1(B, \mathfrak{c})$ is onto by excision, it follows that $NK_2(B/\mathfrak{c}) \rightarrow NK_1(B, \mathfrak{c})$ is onto as well. Using exactness of the bottom row of (1.2), we conclude that $NK_1(B) \rightarrow NK_1(B/\mathfrak{c})$ is injective.

Since B/\mathfrak{c} is Artinian, $NK_1(B/\mathfrak{c}) = NU(B/\mathfrak{c})$. Since $NK_1(B)$ is a subgroup of $NK_1(B/\mathfrak{c})$, it follows that $NK_1(B) = NU(B)$ as well. But B is birational over A , and in particular is reduced, so $NU(B) = 0$. This shows that $NK_n(B) = 0$ for all $n \leq 1$, that is, B is K_1 -regular. \square

ar_radical_conductor

Corollary 1.4. *Suppose the conductor \mathfrak{c} is \mathfrak{m} -primary. If A is K_1 -regular then $\mathfrak{c} = \mathfrak{m}$ and B/\mathfrak{c} is a product of field extensions of A/\mathfrak{m} .*

Proof. By Lemma 1.3, $NK_1(A) = NK_1(B) = NK_0(A) = NK_0(B) = 0$. Because $NK_0(A, \mathfrak{c}) \rightarrow NK_0(B, \mathfrak{c})$ is an isomorphism by excision, (1.2) implies that $NK_1(A/\mathfrak{c}) \rightarrow NK_1(B/\mathfrak{c})$ is also an isomorphism. In particular, we have an isomorphism $NU(A/\mathfrak{c}) \xrightarrow{\cong} NU(B/\mathfrak{c})$ and hence an isomorphism $\text{Nil}(A/\mathfrak{c}) \xrightarrow{\cong} \text{Nil}(B/\mathfrak{c})$ of nil radicals.

Let $x \in B$ be an element such that the reduction \bar{x} of x modulo \mathfrak{c} is in $\text{Nil}(A/\mathfrak{c}) \cong \text{Nil}(B/\mathfrak{c})$. Then $\bar{x}(B/\mathfrak{c}) \subseteq A/\mathfrak{c}$, and therefore $xB \subseteq A + \mathfrak{c} = A$. Thus, $x \in \mathfrak{c}$ by definition of the conductor, so that $\bar{x} = 0$. It follows that $\text{Nil}(A/\mathfrak{c}) = \text{Nil}(B/\mathfrak{c}) = 0$. Because A/\mathfrak{c} and B/\mathfrak{c} are Artinian rings, this implies that $\mathfrak{c} = \mathfrak{m}$, and that $B/\mathfrak{m} = \prod_i L_i$ is a finite product of field extensions of $K = A/\mathfrak{m}$. \square

We will write K for A/\mathfrak{m} and L for $B/\mathfrak{m} = \prod_i L_i$.

KABI

Lemma 1.5. *$NK_1(A, B, \mathfrak{m})$ is zero iff L/K is separable.*

Proof. Now $NK_1(A, B, \mathfrak{m})$ is a direct summand of $K_1(A[t], B[t], \mathfrak{m}[t])$, and the latter group is computed in [12, Theorem 0.2] as

$$K_1(A[t], B[t], \mathfrak{m}[t]) \cong \mathfrak{m}/\mathfrak{m}^2 \otimes_L \Omega_{L[t]/K[t]} \cong \mathfrak{m}/\mathfrak{m}^2 \otimes_L \Omega_{L/K} \otimes_L L[t]$$

It is standard that L/K is separable if and only if $\Omega_{L/K} = 0$, so $K_1(A[t], B[t], \mathfrak{m}[t]) = 0$ if and only if L/K is separable. \square

L/Kseparable

Corollary 1.6. *If A is K_1 -regular and L/K is not separable, then $NK_2(B) \neq 0$.*

Proof. We refer to diagram (1.2). Since A is K_1 -regular, $NK_1(A, \mathfrak{m}) = 0$. Because L/K is a product of fields, $NK_2(B) \cong NK_2(B, \mathfrak{m})$. From the exact sequence $NK_2(B, \mathfrak{m}) \rightarrow NK_1(A, B, \mathfrak{m}) \rightarrow NK_1(A, \mathfrak{m})$, we see that $NK_2(B)$ maps onto $NK_1(A, B, \mathfrak{m})$, which is nonzero by Lemma 1.5. \square

`2-regular_preserved`

Corollary 1.7. *Suppose that the conductor \mathfrak{c} is \mathfrak{m} -primary and A is K_2 -regular. Then B is K_2 -regular iff B/\mathfrak{c} is a product of finite separable field extensions of $A/\mathfrak{m} = A/\mathfrak{c}$.*

Proof. Suppose first that B is K_2 -regular. By Corollary 1.4, $\mathfrak{c} = \mathfrak{m}$ and B/\mathfrak{c} is a product of finite field extensions of A/\mathfrak{m} . Corollary 1.6 implies that the extensions are separable.

proof edited

Now suppose that B/\mathfrak{c} is a product of finite separable field extensions of A/\mathfrak{c} ; by Lemma 1.3, B is K_1 -regular. It remains to prove that $NK_2(B) = 0$. By Corollary 1.4, A/\mathfrak{c} and B/\mathfrak{c} are regular rings, so $NK_n(A/\mathfrak{c}) = NK_n(B/\mathfrak{c}) = 0$ for all $n \in \mathbb{Z}$. Using (1.2), it follows that $NK_n(A) \cong NK_n(A, \mathfrak{c})$ and $NK_n(B) \cong NK_n(B, \mathfrak{c})$ for all integers n . Lemma 1.3 implies that $NK_1(A, \mathfrak{c}) = NK_1(B, \mathfrak{c}) = 0$. It follows that the following sequence is exact:

$$NK_2(A, \mathfrak{c}) \rightarrow NK_2(B, \mathfrak{c}) \rightarrow NK_1(A, B, \mathfrak{c}) \rightarrow 0.$$

Now Lemma 1.5 implies the assertion. \square

We now treat a special case of the main result of this section.

`prop:isolated-normal`

Proposition 1.8. *Suppose that (A, \mathfrak{m}) is a noetherian local ring, B/A is a finite birational extension, and that the conductor \mathfrak{c} is \mathfrak{m} -primary. If A is K_2 -regular, and B/\mathfrak{m} is separable over A/\mathfrak{m} , then $A = B$.*

Proof. Since A is K_2 -regular, it is K_1 -regular. By Corollary 1.4, $\mathfrak{c} = \mathfrak{m}$, and $L = B/\mathfrak{m}$ is a product of (separable, by hypothesis) field extensions of $K = A/\mathfrak{m}$. It suffices to show that $K = L$. Let A' be the strict henselization of A with respect to a separable closure K' of K , $B' = B \otimes_A A'$, and $L' = L \otimes_K K' = B'/\mathfrak{m}B'$. We will show that $K' = L'$ by contradiction.

Since B' is a finite A' -algebra, it is a product $B' = \prod_{i=1}^s B_i$ of local rings each with residue field K' . Write $\pi_i : B' \rightarrow B_i$ for the projections.

Now assume for a contradiction that $s > 1$. Choose elements $f, g \in \mathfrak{m}$ such that $\pi_1(f) \neq 0$, $\pi_i(f) = 0$ for all $i \neq 1$, $\pi_2(g) \neq 0$, and $\pi_i(g) = 0$ for all $i \neq 2$, and such that f and g are linearly independent modulo \mathfrak{m}^2 . Since the elements $f, gt \in A'[t]$ satisfy $fgt = 0$, the Dennis-Stein symbol $\langle f, gt \rangle \in NK_2(A')$ is defined (see [6] and [27, Theorem III.5.11.1]).

The Dennis trace $K_2(A'[t]) \rightarrow \Omega_{A'[t]}^2$ sends $\langle f, gt \rangle \in NK_2(A')$ to $-df \wedge d(gt)$ in $\Omega_{A'[t]}^2$; see e.g. [13, 2.2]. Because f and g are linearly independent modulo \mathfrak{m}^2 , $df \wedge d(gt) \neq 0$ in $\Omega_{A'[t]}^2$, and it follows that the Dennis trace of $\langle f, gt \rangle \in NK_2(A')$ is non-zero. Therefore $NK_2(A') \neq 0$.

On the other hand, it follows from [22, Theorem 3.2] that $NK_2(A) = 0$ implies $NK_2(A') = 0$. Thus, we have arrived at a contradiction. \square

`n:K_2-regular-normal`

Theorem 1.9. *Let A be a noetherian commutative ring containing a field and with finite normalization. Assume either that $\mathbb{Q} \subseteq A$, or that $\dim(A) = 2$ and A/\mathfrak{m} is perfect for all maximal ideals \mathfrak{m} . If A is K_2 -regular, then A is normal.*

Proof. Assume first that $\mathbb{Q} \subseteq A$. We proceed by noetherian induction. Since A is K_2 -regular, it is K_1 -regular and hence reduced. It follows that the localization of A at any minimal prime is a field, and hence normal. Now let \mathfrak{p} be a prime ideal such that $A_{\mathfrak{q}}$ is normal for all primes \mathfrak{q} properly contained in \mathfrak{p} . Then the local ring $A_{\mathfrak{p}}$ is K_2 -regular by Vorst [23, 1.9], and satisfies the hypotheses of Proposition 1.8. We conclude that $A_{\mathfrak{p}}$ is normal. It follows by induction that all local rings of A are normal, and hence that A is normal.

Now suppose that $\dim(A) = 2$. By [23], A is regular in codimension 1. It follows that the conductor of the normalization is \mathfrak{m} -primary. If, moreover, the residue field is perfect, then we can again apply Proposition 1.8 to finish the proof. \square

2. HIGHER DU BOIS SINGULARITIES, K -THEORY AND REGULARITY.`sec:duBois`

In this section we point out that the proof of Vorst's conjecture for an affine scheme X of finite type over a field of characteristic zero contained in [1] does not actually require the full strength of the K -regularity hypothesis; instead, it suffices to assume only an algebraic-geometric consequence of this hypothesis, namely, that X have *higher du Bois singularities*.

We begin by briefly recalling (in a form convenient for our purposes) the definition of the du Bois complexes, $\underline{\Omega}_X^p$, and the notion of du Bois and higher du Bois singularities. Du Bois complexes were introduced in [7] to study mixed Hodge structures on possibly singular quasiprojective complex varieties, and du Bois singularities were introduced by Steenbrink in [21] as a weakening of the notion of rational singularities better behaved in families.

Throughout this section, k will be a field of characteristic zero, and Sch/k will be the category of separated schemes essentially of finite type over k . Following [3], we write $a : (\text{Sch}/k)_{\text{cdh}} \rightarrow (\text{Sch}/k)_{\text{zar}}$ for the morphism of sites given by comparing the *cdh* and Zariski topologies.

`defn:duBois-complex`

Definition 2.1. Let $X \in \text{Sch}/k$, and $p \geq 0$. The p -th du Bois complex of X is the complex $\underline{\Omega}_{X/k}^p = \mathbb{R}a_* a^* \Omega_{/k}^p|_X$ of Zariski sheaves.

Unless it is necessary to emphasize the base field, we will suppress it from the notation and simply write Ω_X^p and $\underline{\Omega}_X^p$ for differentials and du Bois complexes over k , respectively.

A scheme X is said to have *du Bois singularities* if $\mathcal{O}_X \rightarrow \underline{\Omega}_X^0$ is an isomorphism.

`rem:duBois`

Remark 2.2. (a) This definition of $\underline{\Omega}_X^p$ agrees with the original one given by du Bois in [7] (using hyper-resolutions); see [3, Lemma 2.1] for a proof of this fact.

(b) By adjunction, there is a natural homomorphism $\Omega_X^p \rightarrow \underline{\Omega}_X^p$. This homomorphism is an isomorphism when X is smooth.

(c) The complex of sheaves of \mathcal{O}_X -modules $\underline{\Omega}_X^p$ has coherent cohomology sheaves (see [7, Propn. 4.4]).

The following generalization of the definition of du Bois singularity has been studied recently, see [14], [19], [20], [8], [9].

`defn:dB-sing`

Definition 2.3. Let X be a k -scheme, and $p \geq 0$. We say X has (only) *p -du Bois singularities* if the natural homomorphisms

$$\Omega_X^s \longrightarrow \underline{\Omega}_X^s$$

are isomorphisms for all $s \leq p$.

Remark 2.4. As pointed out in the Introduction of [20], this notion (called “strict p -du Bois” in loc. cit.) is poorly understood for $p > 0$, unless X is a local complete intersection. We will return to this in section 3.

We will connect du Bois singularities to K -regularity using descent for Hochschild homology. Recall that $HH(/k)$ satisfies p -cdh descent on a k -scheme X if the fiber of the map from $HH(X/k)$ to its cdh -hypercohomology is p -connected.

lem:tech1

Lemma 2.5. [1, Lemma 2.3] *Let R be a k -algebra essentially of finite type, and let $p \geq 0$. Then $HH(/k)$ satisfies p -cdh descent on $X = \text{Spec}(R)$ if and only if the following three conditions hold simultaneously:*

hhkvanish

$$(2.5a) \quad HH_m^{(q)}(R/k) = 0 \quad \text{if } 0 \leq q < m \leq p;$$

agreek

$$(2.5b) \quad \Omega_{R/k}^q \rightarrow H_{\text{cdh}}^0(X, \Omega_{/k}^q) \text{ is bijective if } q \leq p, \text{ onto if } q = p + 1;$$

cdhkvanish

$$(2.5c) \quad H_{\text{cdh}}^s(X, \Omega_{/k}^q) = 0 \quad \text{if } s > 0 \text{ and } 0 \leq q \leq s + p + 1.$$

The authors and Cortiñas proved in [1, Theorem 3.1] that a finite type k -algebra satisfying the equivalent conditions of Lemma 2.5 is regular in codimension p . The following theorem shows that the hypothesis can be weakened:

thm:dB-Vorst

Theorem 2.6. *Let $X \in \text{Sch}/k$. If X has p -du Bois singularities, then X is regular in codimension p .*

Proof. Let $x \in X$ be a point of codimension $d \leq p$. By [1, Lemma 4.10], we can represent the local ring $A = \mathcal{O}_{X,x}$ as the localization of a finite type F -algebra at a maximal ideal, for some field F containing k . The proofs of [1, Lemma 4.4 and Prop. 4.8] go through, noting that the hypotheses of Theorem 2.6 suffice, showing that if X is p -du Bois (over k) then $\text{Spec}(A)$ is p -du Bois (over F). Thus we may assume, without loss of generality, that $X = \text{Spec}(A)$ is the spectrum of a local ring of a d -dimensional k -algebra of finite type at a maximal ideal, and x is its closed point. The hypothesis that X has p -du Bois singularities then is equivalent to the following subset of the conditions of Lemma 2.5 (recall that $d \leq p$):

$$(2.6a) \quad \Omega_{A/k}^q \rightarrow H_{\text{cdh}}^0(X, \Omega_{/k}^q) \text{ is an isomorphism if } q \leq d;$$

$$(2.6b) \quad H_{\text{cdh}}^s(X, \Omega_{/k}^q) = 0 \quad \text{if } s > 0 \text{ and } 0 \leq q \leq d.$$

By inspection, the proof of [1, Theorem 3.1] only needs the weaker hypotheses given here, not all of Lemma 2.3 loc. cit. Indeed, the properties above suffice to conclude, as in the proof of [1, Theorem 3.1],

that

$$H^*(\Omega_{A/k}^{\leq d}, d) \cong \mathbb{H}_{\text{cdh}}^*(X, \Omega_{/k}^{\leq d}) \cong \mathbb{H}_{\text{cdh}}^*(X, \Omega_{/k}^\bullet)$$

and, as explained there, this implies that $\Omega_{A/k}^{d+1} = 0$. This proves the assertion. \square

The following is extracted from [1].

K-regular-to-duBois

Proposition 2.7. *Let $X \in \text{Sch}/k$ be an affine scheme. If X is K_{p+1} -regular, then X has p -du Bois singularities.*

Proof. By [1, Corollary 1.7], the K -regularity hypothesis is equivalent to the assertion that $HH(/\mathbb{Q})$ satisfies p -cdh-descent on X . Now [1, Propn. 4.8] implies that $HH(/k)$ satisfies p -cdh-descent on X as well. Finally Lemma 2.5 gives an explicit description of this descent property; parts (2.5b) and (2.5c) of this description in particular imply that X has p -du Bois singularities. \square

Combining the above yields a new proof of the main theorem of [1]. Recall that k is a field of characteristic 0.

thm:Vorst-redux

Theorem 2.8. *Let $X \in \text{Sch}/k$ be an affine scheme, and $p \geq 1$. If X is K_{p+1} -regular, then X is regular in codimension p .*

Proof. Proposition 2.7 asserts that X is p -du Bois. It now follows from Theorem 2.6 that X is regular in codimension p . \square

sec:lci

3. THE CASE OF LOCAL COMPLETE INTERSECTIONS.

In this short section, we combine Hodge theoretic results of Musta and Popa [19] with the previous section to prove that the K -regularity bound for singularities given by Vorst's conjecture and proven in [1] can be significantly improved if the singularity is a local complete intersection (*lci*). We also formulate some questions.

The following theorem is proved by Musta and Popa in [19] and is stated in the form used here (with $k = \mathbb{C}$) as [9, Theorem 3.8]. The hypersurface case of this theorem also follows from bounds proved in [18] and [14].

thm:dB-smooth

Theorem 3.1 (Musta–Popa). *Let X/k be locally a complete intersection with p -du Bois singularities, $p \geq 1$. Then X is normal and regular in codimension $2p$.*

Question 3.2. Let A be a k -algebra, essentially of finite type over k . As can be seen using Lemma 2.5 and [1, Propn. 4.8], A is K_2 -regular if, and only if, $\text{Spec}(A)$ is 1-du Bois, the natural homomorphism $\Omega_{A/k}^2 \rightarrow H_{\text{cdh}}^0(X, \Omega_{/k}^2)$ is onto, and certain cohomology groups of the higher du

Bois complexes vanish. We proved in Theorem 1.9 that a K_2 -regular algebra is normal, without the assumption that A is a local complete intersection.

This suggests the following question: Suppose A is a 1-du Bois algebra, not necessarily a local complete intersection. Is A normal? We note that A is regular in codimension 1 by Theorem 2.8, so this question really asks if A necessarily satisfies Serre's condition (S2).

The following result provides the promised sharpening of Vorst's bound in the case of a local complete intersection.

`thm:Vorst-lci`

Theorem 3.3. *Let $X \in \text{Sch}/k$ be an affine scheme, and $p \geq 1$. If X is K_{p+1} -regular and a local complete intersection, then X is normal and regular in codimension $2p$.*

Proof. This follows immediately from Proposition 2.7 and Theorem 3.1. \square

Question 3.4. We wonder if the lci hypothesis is needed to prove the sharper bound. Typically, schemes that are not local complete intersection are not even 1-du Bois, since their sheaves of differentials are usually not torsion free. On the other hand, the arguments developed in [19] employ the Hodge structure on (top) local cohomology and rely crucially on the lci condition.

The first case of interest is those of surfaces, which suggests the question: Suppose X is an affine surface over a field of characteristic zero. If X is K_2 -regular, is it necessarily regular?

REFERENCES

- | | |
|-------------------------|---|
| <code>chw-vorst</code> | [1] G. Cortiñas, C. Haesemeyer, and C. Weibel, K -regularity, cdh -fibrant Hochschild homology, and a conjecture of Vorst, <i>J. AMS</i> 21 (2008), 547–561. |
| <code>chww-bass</code> | [2] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, Bass' NK groups and cdh -fibrant Hochschild homology, <i>Invent. Math.</i> 181 (2010), 421–448. |
| <code>chww-bass2</code> | [3] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, A negative answer to a question of Bass, <i>Proc. AMS</i> 139 (2011), 1187–1200. |
| <code>chww-toric</code> | [4] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, The K -theory of toric varieties, <i>Trans. AMS</i> 361 (2009), 3325–3341. |
| <code>chww-cones</code> | [5] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, The K -theory of cones of smooth varieties, <i>J. Alg. Geom.</i> 22 (2013), 13–34. |
| <code>DS73</code> | [6] R. K. Dennis and M. Stein, K_2 of radical ideals and semilocal rings revisited, <i>Lecture Notes in Mathematics</i> 342 (1973), 281–303. |
| <code>duBois</code> | [7] Ph. du Bois, Complexe de de Rham filtré d'une variété singulière, <i>Bull. SMF</i> 109 (1981), 41–81. |
| <code>FL24a</code> | [8] R. Friedman and R. Laza, The higher Du Bois and higher rational properties for isolated singularities, <i>J. Algebraic Geom.</i> 33 (2024), 493–520. |

- [9] R. Friedman and R. Laza, Higher Du Bois and higher rational singularities, *Duke Math. J.* 173 (2024), 1839–1881.
- [10] T. Geisser and L. Hesselholt, On the vanishing of negative K -groups, *Math. Ann.* 348 (2010), 707–736.
- [11] T. Geisser and L. Hesselholt, On a conjecture of Vorst, *Math. Z.* 270 (2012), 445–452.
- [12] S. C. Geller and C. A. Weibel, $K_1(A, B, I)$, *J. reine angew. Math.* 342 (1983), 12–34.
- [13] S. C. Geller and C. A. Weibel, Hodge decompositions of Loday symbols in K -Theory and cyclic homology, *K-theory* 8 (1994), 587–632.
- [14] Seung-Jo Jung, In-Kyun Kim, Morihiko Saito, and Youngho Yoon, Higher Du Bois singularities of hypersurfaces, *Proc. Lond. Math. Soc. (3)* 125 (2022), 543–567.
- [15] C. Kassel, Cyclic homology, comodules, and mixed complexes, *J. Alg.* 107 (1987), 195–216.
- [16] M. Kerz, F. Strunk and G. Tamme, Algebraic K -theory and descent for blow-ups, *Invent. Math.* 211 (2018), 523–577.
- [17] M. Kerz, F. Strunk and G. Tamme, Towards Vorst’s conjecture in positive characteristic, *Compos. Math.* 157 (2021), 1143–1171.
- [18] M. Mustață, S. Olano, and M. Popa, and J. Witaszek, The Du Bois complex of a hypersurface and the minimal exponent, *Duke Math. J.* 172 (2023), 1411–1436.
- [19] Mustață, Mircea and Popa, Mihnea, Hodge filtration on local cohomology, Du Bois complex and local cohomological dimension, *Forum Math. Pi* 10 (2022), Paper No. e22.
- [20] Wanchun Shen, Sridhar Venkatesh, and Anh Duc Vo, On k -Du Bois and k -Rational Singularities, Preprint 2023. arXiv 2306.03977
- [21] J. H. M. Steenbrink, Mixed Hodge structures associated with isolated singularities, in *Singularities, Part 2 (Arcata, Calif., 1981)*, Proc. Sympos. Pure Math. 40 (Amer. Math. Soc., Providence, 1983), 513–536.
- [22] W. van der Kallen, Descent for the K -Theory of Polynomial Rings, *Math. Z.* 191 (1986), 405–415.
- [23] Ton Vorst, Localization of the K -theory of polynomial extensions, *Math. Ann.*, 244:33–54, 1979.
- [24] Ton Vorst, Polynomial extensions and excision for K_1 , *Math. Ann.* 244, 193–204, 1979.
- [25] C. Weibel, *An introduction to homological algebra*, Cambridge U. Press, 1994.
- [26] C. Weibel, The negative K -theory of normal surfaces, *Duke Math. J.* 108 (2001), 1–35.
- [27] C. Weibel, *The K-book*, Grad. Studies in Math. 145, AMS, 2013.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MELBOURNE,
VIC 3010, AUSTRALIA

Email address: christian.haesemeyer@unimelb.edu.au

MATH. DEPT., RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ 08901, USA

Email address: weibel@math.rutgers.edu

URL: <http://math.rutgers.edu/~weibel>