

# Homotopy Ends and Thomason Model Categories

Charles Weibel<sup>1</sup>

Institute for Advanced Study  
Princeton, NJ 08540  
weibel@math.ias.edu

In the last year of his life, Robert W. Thomason reworked the notion of a model category, used to adapt homotopy theory to algebra, and used homotopy ends to affirmatively solve a problem raised by Grothendieck: find a notion of model structure which is inherited by functor categories. The axioms for such a Thomason model category were published later in [WT]. In this paper we explain and prove Thomason's results, based on his private notebooks [T78]–[T86].

Each of Thomason's 180-page notebooks are spiral-bound with an index on the inside cover. We have chosen to follow their internal citation method: our citation  $[Tx, y]$  refers to page  $y$  of Thomason's notebook  $x$ .

We first present Thomason's ideas about homotopy ends and its generalizations (sections 1 and 2). These are first formulated for complete simplicial closed model categories in the sense of Quillen [QH], because of their usefulness in this setting, as demonstrated by Dwyer and Kan [DK].

Thomason's axioms and examples come next, followed by a proof that the homotopy category  $\mathrm{ho}\mathcal{C}$  of a Thomason model category  $\mathcal{C}$  exists (sections 3-5). In the last two sections (6-7) we prove the main theorem: the functor category  $\mathcal{C}^{\mathbf{K}}$  inherits a Thomason model structure, at least when  $\mathcal{C}$  is right enriched over simplicial sets, and fibrations are preserved by products and inverse limits. The result was previously known when  $\mathcal{C}$  was the category of simplicial sets (Quillen, Joyal and Jardine) and also when  $\mathbf{K}$  is any partially ordered set (Grothendieck).

---

<sup>1</sup>Partially supported by NSF grants

## §1 Homotopy Ends

Recall from [Mac] that the *end* of a functor  $F: \mathbf{K}^{\text{op}} \times \mathbf{K} \rightarrow \mathcal{C}$  is an object  $e = \int_{\mathbf{K}} F(K, K)$  together with maps  $e \rightarrow F(K, K)$  which are compatible in the sense that for each  $K_1 \rightarrow K_2$  the evident square commutes:

$$\begin{array}{ccc} e & \longrightarrow & F(K_1, K_1) \\ \downarrow & & \downarrow \\ F(K_2, K_2) & \longrightarrow & F(K_1, K_2). \end{array}$$

Moreover,  $e$  is universal with respect to this property. If  $\mathbf{K}$  is a small category and  $\mathcal{C}$  is complete, then ends always exist, and commute with limits [Mac, IX.5].

The “Tot” construction for cosimplicial objects is an example of an end in  $\mathcal{C}$ . Its construction requires that for every finite simplicial set  $K$  there is a *mapping object functor*  $\text{Map}(K, -): \mathcal{C} \rightarrow \mathcal{C}$  which is natural in  $K$ . Given a cosimplicial object  $X^\cdot$  in  $\mathcal{C}$ , we have a functor  $\Delta^{\text{op}} \times \Delta \rightarrow \mathcal{C}$  defined by  $(p, q) \mapsto \text{Map}(\Delta[p], X^q)$ .

**Definition 1.1.** If  $X^\cdot$  is a cosimplicial object in a complete category  $\mathcal{C}$ , and  $\mathcal{C}$  is equipped with a mapping object functor, we define the *total object* of  $X^\cdot$  to be the end

$$\text{Tot}(X^\cdot) = \int_{\Delta} \text{Map}(\Delta[p], X^p).$$

In case  $\mathcal{C}$  is a simplicial closed model category in the sense of Quillen [QH], it is well known that this definition of Tot agrees with the Bousfield-Kan definition in [BK].

In a model category, it is not generally true that an end will preserve weak equivalences. To remedy this, we introduce the notion of a “derived end.”

Given  $F: \mathbf{K}^{\text{op}} \times \mathbf{K} \rightarrow \mathcal{C}$ , with  $\mathcal{C}$  complete and  $\mathbf{K}$  small, the *cosimplicial replacement* of  $F$  is the cosimplicial object  $\prod^* F$  of  $\mathcal{C}$  defined by

$$p \longmapsto \prod^p F = \prod_{K_0 \rightarrow \cdots \rightarrow K_p} F(K_0, K_p),$$

where the indexing set runs over all  $p$ -tuples  $K_0 \rightarrow K_1 \rightarrow \cdots \rightarrow K_p$  of composable maps in  $\mathbf{K}$ , i.e., all functors  $\mathbf{p} \rightarrow \mathbf{K}$ . This is equivalent to the Bousfield-Kan

simplicial replacement of  $F$  [BK, XI.5], except for a change in orientation due to a different orientation of the nerve of  $\mathbf{K}$  on p. 291 [BK].

**Lemma 1.2.** *The end  $\int_{\mathbf{K}} F(K, K)$  is the equalizer of  $\prod^0 F \rightrightarrows \prod^1 F$ .*

*Proof:* ([T78, 115]) Let  $\pi^0 F$  denote the equalizer. The projections  $\pi^0 F \rightarrow \prod^0 F \rightarrow F(K, K)$  have the property that for every map  $K_0 \rightarrow K_1$  in  $\mathbf{K}$  the diagram

$$\begin{array}{ccc} \pi^0 F & \longrightarrow & F(K_0, K_0) \\ \downarrow & & \downarrow \\ F(K_1, K_1) & \longrightarrow & F(K_0, K_1) \end{array}$$

commutes. This gives a map from  $\pi^0 F$  to the end  $\int_{\mathbf{K}} F(K, K)$ . Since the end also equalizes  $\prod^0 F \rightrightarrows \prod^1 F$ , this must be an isomorphism.  $\square$

**Definition 1.3.** If  $\mathcal{C}$  is a complete category equipped with a mapping object functor, the *homotopy end*  $\text{ho} \int F$  of  $F: \mathbf{K}^{\text{op}} \times \mathbf{K} \rightarrow \mathcal{C}$  is the total object of  $\prod^* F$ :

$$\text{ho} \int F = \text{Tot} \left( \prod^* F \right) = \int_{\Delta} \text{Map} \left( \Delta[p], \prod^p F \right) .$$

**Compatibility 1.4.** Suppose that  $F$  factors as the projection  $\mathbf{K}^{\text{op}} \times \mathbf{K} \rightarrow \mathbf{K}^{\text{op}}$  followed by a functor  $\tilde{F}: \mathbf{K}^{\text{op}} \rightarrow \mathcal{C}$ . Then the homotopy end of  $F$  is just the Bousfield-Kan homotopy limit of  $\tilde{F}$  as described in 4.5 and 5.2 of [BK, XI]:

$$\text{ho} \int F = \text{holim } \tilde{F} .$$

Since we have used  $\mathbf{K}^{\text{op}}$ , there is no orientation problem.

**Functoriality 1.5.** A natural transformation  $\eta: F \Rightarrow G$  of functors  $\mathbf{K}^{\text{op}} \times \mathbf{K} \rightarrow \mathcal{C}$  induces a map  $\prod^* F \rightarrow \prod^* G$ , and so a natural map  $\text{ho} \int F \rightarrow \text{ho} \int G$ . The homotopy end functor  $\text{Cat}(\mathbf{K}^{\text{op}} \times \mathbf{K}, \mathcal{C}) \rightarrow \mathcal{C}$  commutes with all limits (because  $\text{Tot}$  and  $F \mapsto \prod^* F$  preserve limits). Given a functor  $\Phi: \mathbf{L} \rightarrow \mathbf{K}$  and  $F: \mathbf{K}^{\text{op}} \times \mathbf{K} \rightarrow \mathcal{C}$ , the diagram

$$\begin{array}{ccc} \text{Cat}(\mathbf{K}^{\text{op}} \times \mathbf{K}, \mathcal{C}) & \xrightarrow{\Phi} & \text{Cat}(\mathbf{L}^{\text{op}} \times \mathbf{L}, \mathcal{C}) \\ \searrow \text{ho} \int & & \swarrow \text{ho} \int \\ & \mathcal{C} & \end{array}$$

commutes up to a natural 2-cell  $\varphi$  which is induced by the “projection”  $\prod_{\mathbf{K}}^* F \rightarrow \prod_{\mathbf{L}}^* F$ : the component indexed by  $L: p \rightarrow \mathbf{L}$  comes from the component indexed by  $\Phi L$ .

We now introduce homotopy theory into the discussion. Suppose to fix ideas that  $\mathcal{C}$  is a simplicial closed model category in the sense of Quillen [QH, II.2.2], and that  $\mathcal{C}$  is complete. We first observe that  $\mathcal{C}$  is cotensored over all simplicial sets, not just finite ones, as in Quillen’s axiom (SM0).

**Lemma 1.6** [T78, 58]. *There is a mapping object functor  $\text{Map}(K, -): \mathcal{C} \rightarrow \mathcal{C}$  for every simplicial set  $K$ , which is natural in  $K$ , and an isomorphism*

$$\text{Hom}_{\mathcal{C}}(X, \text{Map}(K, Y)) \cong \text{Hom}_{\Delta^{\text{op}}\mathbf{Sets}}(K, \text{hom}(X, Y)),$$

which is natural in  $K, X$  and  $Y$ .

Quillen’s axiom (SM7) holds in this setting. If  $i: K \rightarrow L$  is a cofibration in  $\Delta^{\text{op}}\mathbf{Sets}$  (i.e., an injection) and  $f: X \rightarrow Y$  is a fibration in  $\mathcal{C}$ , then

$$\text{Map}(L, X) \rightarrow \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)$$

is a fibration in  $\mathcal{C}$ , and is a weak equivalence if either  $i$  or  $f$  is. Moreover, the functor  $\text{Map}(\_, \_): \Delta^{\text{op}}\mathbf{Sets} \times \mathcal{C} \rightarrow \mathcal{C}$  preserves limits in  $\mathcal{C}$  and converts colimits of simplicial sets to limits in  $\mathcal{C}$ .

*Proof:* Write  $K = \sqcup K_{\alpha}$ , where  $K_{\alpha}$  runs over all finite subspaces of  $K$ , and set  $\text{Map}(K, X) = \lim \text{Map}(K_{\alpha}, X)$ . Checking the properties is routine.  $\square$

Let  $f: X' \rightarrow Y'$  be a map of cosimplicial objects in  $\mathcal{C}$ . We say that  $f$  is a *fibration* if each  $f^p$  is a fibration in  $\mathcal{C}$ , and each  $X^{p+1} \rightarrow Y^{p+1} \times_{M^p Y} M^p X$  is a fibration in  $\mathcal{C}$ . Here  $M^p X \subset \prod_{n+1} X^n$  is the matching object of [BK, X.4.6]. Also recall from [BK, X.3.2] that  $\text{Tot}(X') = \text{hom}(\Delta', X')$ .

**Lemma 1.7** [T78, 92]. *If  $f: X' \rightarrow Y'$  is a fibration of cosimplicial objects in  $\mathcal{C}$ , then  $\text{Tot}(X') \rightarrow \text{Tot}(Y')$  is a fibration in  $\mathcal{C}$ . If in addition each  $f^p$  is a weak equivalence in  $\mathcal{C}$  then it is also a weak equivalence.*

*Proof:* By [BK, X.5], axiom (SM7) for cosimplicial objects holds. The lemma is the case  $A = \emptyset$  and  $B = \Delta'$ :  $\text{Tot}(Y') = \text{hom}(\Delta', Y') \times_* *$ , and  $\text{hom}(\emptyset, Y') = *$ .  $\square$

**Lemma 1.8** [T78, 110]. *If  $\eta: F \Rightarrow G$  is such that  $F(K, K') \rightarrow G(K, K')$  is a fibration in  $\mathcal{C}$  for all  $K, K'$  in  $\mathbf{K}$ , then  $\text{ho}f F \rightarrow \text{ho}f G$  is a fibration in  $\mathcal{C}$ . If in addition each  $F(K, K') \xrightarrow{\sim} G(K, K')$  is a weak equivalence, so is  $\text{ho}f F \rightarrow \text{ho}f G$ .*

*In particular, if each  $F(K, K')$  is fibrant in  $\mathcal{C}$  then  $\text{ho}f F$  is fibrant in  $\mathcal{C}$ .*

*Proof:* If each  $\eta_{KK'}$  is a fibration (resp. trivial fibration) then so is each component of  $\prod^* F \rightarrow \prod^* G$ . Indeed  $\prod^* F \rightarrow \prod^* G$  is a fibration in  $\Delta\mathcal{C}$  [BK, XI.5.3] [T78, 106]. Hence this follows from lemma 1.7.  $\square$

**Corollary 1.8.1.** *The homotopy end preserves (pointwise) homotopy fiber sequences.*

**Lemma 1.9** [T82, 83] [T83, 178]. *Suppose that  $\mathbf{K}$  has an initial object  $K_0$ . Then for any functor  $F: \mathbf{K} \rightarrow \mathcal{C}$ , the natural map is a weak equivalence:*

$$\text{holim}_{\mathbf{K}} F \xrightarrow{\sim} \text{Map}(\Delta[0], F(K_0)).$$

*Proof:* This is a special case of [BK, XI.4.1 (iii)].  $\square$

We conclude this section with a generalization of the above results to a complete “right model category enriched over finite simplicial sets;” see definitions 1.11 and 1.12 below. The reader may easily verify that a complete simplicial closed model category  $\mathcal{C}$  has such a structure, using lemma 1.6 and the adjunction between  $\text{Map}(-, X)$  and  $\underline{\text{Hom}}_{\mathcal{C}}(-, X)$ . We will use this generality in §7 below.

The following result is straightforward, once one checks (see [T78, 68–92]) that (SM7) holds, by the transfinite proof in [BK, X.5].

**Proposition 1.10.** *If  $\mathcal{C}$  is a right model category enriched over finite simplicial sets, and  $\mathcal{C}$  is complete, then lemmas 1.6, 1.7, 1.8 and 1.9 hold in  $\mathcal{C}$ .*

**Definition 1.11** [T85, 145]. A *right model category* is a category  $\mathcal{C}$  with a terminal object  $*$  and two subcategories  $\text{we}(\mathcal{C})$  and  $\text{fib}(\mathcal{C})$  so that: every isomorphism of  $\mathcal{C}$  is in both  $\text{we}(\mathcal{C})$  and  $\text{fib}(\mathcal{C})$ ;  $\text{we}(\mathcal{C})$  is saturated and closed under retractions; and

(RM1) The pullback  $B \times_A C \rightarrow B$  of a fibration  $C \rightarrow A$  along any map  $B \rightarrow A$  exists and is a fibration. If either  $B \rightarrow A$  or  $C \rightarrow A$  is in  $\text{we}(\mathcal{C})$ , so is its pullback.

(RM5) Every map  $A \rightarrow B$  factors as  $A \xrightarrow{\sim} A' \rightarrow B$ , the composite of a weak equivalence and a fibration.

Note that the subcategory  $\mathcal{C}_{\text{fib}}$  of *fibrant* objects  $A$  (those for which  $A \rightarrow *$  is a fibration) is a *category of fibrant objects* in the sense of Brown [Br], with the added property that weak equivalences are closed under retractions.

Let  $\mathcal{S}_f$  denote the category of finite simplicial sets.

**Definition 1.12** [T85, 146]. We say that a complete right model category  $\mathcal{C}$  is *right enriched* over  $\mathcal{S}_f$  if there is a “mapping object” functor

$$\text{Map}( , ) : \mathcal{S}_f^{\text{op}} \times \text{fib}(\mathcal{C}) \rightarrow \text{fib}(\mathcal{C})$$

satisfying the following axioms.

(RE1) For each  $K$  in  $\mathcal{S}_f$ :  $\text{Map}(K, *) \cong *$ ; the functor  $\text{Map}(K, -)$  preserves both fibrations and base change along fibrations in  $\mathcal{C}$ .

(RE2) For each  $C$  in  $\mathcal{C}$ :  $\text{Map}(\emptyset, C) \cong *$ ; the functor  $\text{Map}(-, C)$  sends cofibrations in  $\mathcal{S}_f$  to fibrations in  $\mathcal{C}$ , and pushouts along cofibrations in  $\mathcal{S}_f$  to pullbacks along fibrations in  $\mathcal{C}$ .

(RE3) Given a cofibration  $K \xrightarrow{i} L$  in  $\mathcal{S}_f$  and a fibration  $B \xrightarrow{p} C$  in  $\mathcal{C}$ ,

$$\text{Map}(L, B) \rightarrow \text{Map}(K, B) \times_{\text{Map}(K, C)} \text{Map}(L, C)$$

is a fibration. It is a weak equivalence of either  $i$  or  $p$  is.

(RE4) There is a weak equivalence  $\omega : C \xrightarrow{\sim} \text{Map}(\Delta[0], C)$ , natural in  $C$ , so that  $\partial_0 \omega = \partial_1 \omega : C \rightarrow \text{Map}(\Delta[1], C)$ .

(RE5) For each family  $\{C_i\}_{i \in I}$  of fibrant objects in  $\mathcal{C}$ , and each  $K$  in  $\mathcal{S}_f$ , the product  $\prod C_i$  is fibrant and the canonical morphism

$$\text{Map}\left(K, \prod C_i\right) \rightarrow \prod \text{Map}(K, C_i)$$

is an isomorphism in  $\mathcal{C}$ .

**Remark.** Axioms (RE4) and (RE5) are not needed to prove proposition 1.10. They are called (RWN) and (NEP), respectively, in [T85, 148–9].

Here is an elementary result we shall need for theorem 7.2.

**Lemma 1.13** [T85, 168]. *Suppose given a functor  $F: \mathbf{K} \rightarrow \text{fib}(\mathcal{C})$ . Then there is a functor  $(\text{holim } F): \mathbf{K} \rightarrow \mathcal{C}$  defined by  $(\text{holim } F)(K) = \text{holim}_{(K/\mathbf{K})} F$ , and there is a natural (pointwise) weak equivalence*

$$F \xrightarrow{\sim} \text{holim } F.$$

*Proof:* For each  $K$  in  $\mathbf{K}$ ,  $K/\mathbf{K}$  has an initial object  $(K=K)$  and a functor  $K/\mathbf{K} \rightarrow \mathbf{K}$  sending  $K \rightarrow K'$  to  $K'$ . By assumption,  $FK \rightarrow FK'$  is a fibration for each  $K \rightarrow K'$  in  $K/\mathbf{K}$ . Hence (RE4) and lemma 1.9 yield natural weak equivalences for all  $K$ :

$$FK \xrightarrow[\text{RE4}]{\sim} \text{Map}(\Delta[0], FK) \xrightarrow[1.9]{\sim} \text{holim}_{(K/\mathbf{K})} F = (\text{holim } F)(K)F. \quad \square$$

**Remark 1.14.** It is easy to see that

$$\text{holim}_{\mathbf{K}} F = \varprojlim_{K/\mathbf{K}} \text{holim}_{K/\mathbf{K}} F = \varprojlim_K (\text{holim } F)(K).$$

**Definition 1.15.** We say that  $\mathcal{C}$  is a *left model category* if  $\mathcal{C}^{\text{op}}$  is a right model category. That is,  $\mathcal{C}$  has an initial object  $\emptyset$  and two subcategories  $\text{we}(\mathcal{C})$  and  $\text{cof}(\mathcal{C})$ , both containing all isomorphisms, so that  $\text{we}(\mathcal{C})$  is saturated and closed under retractions, and

(LM1) The pushout  $P$  of a cofibration  $c: A \rightarrow C$  with any map  $f: A \rightarrow B$  exists, and  $B \rightarrow P$  is a cofibration. Moreover, if either  $f$  or  $c$  is a weak equivalence, so is its pushout.

(LM5) Every map  $A \rightarrow B$  factors  $A \rightarrow B' \xrightarrow{\sim} B$ .

We say that  $\mathcal{C}$  is *left enriched* over  $\mathcal{S}_f$  if  $\mathcal{C}^{\text{op}}$  is right enriched by a functor

$$\otimes: \mathcal{S}_f \times \text{cof}(\mathcal{C}) \rightarrow \text{cof}(\mathcal{C}).$$

That is,  $\otimes$  should satisfy axioms (LE1)–(LE5) dual to (RE1)–(RE5).

## §2 Homotopy limits of natural systems

Let  $\mathbf{K}$  be a category. The *subdivision category*  $\text{Sd } \mathbf{K}$  is defined to be the category whose objects are morphisms in  $\mathbf{K}$ , and whose morphisms from  $f: A \rightarrow B$  to  $f': A' \rightarrow B'$  are factorizations of  $f'$  through  $f$ :

$$\begin{array}{ccc} B & \longrightarrow & B' \\ f \uparrow & & \uparrow f' \\ A & \longleftarrow & A' \end{array}$$

This construction is due to Quillen, and has many names in the literature. Baues calls  $\text{Sd } \mathbf{K}$  the *category of factorizations* in  $\mathbf{K}$  in [Baues, p. 232]. Dwyer and Kan call it the *Twisted arrow category*  $a\mathbf{K}$  in [DK]. (It is *not* the “subdivision category” of [DK], or of [Mac]).

Following Baues, a *natural system*  $F$  on  $\mathbf{K}$  with values in  $\mathcal{C}$  is a functor  $F: \text{Sd } \mathbf{K} \rightarrow \mathcal{C}$ . If  $\mathcal{C}$  is complete and  $\mathbf{K}$  is small we can form the products in  $\mathcal{C}$ :

$$\prod^p F = \prod_{K_0 \rightarrow \cdots \rightarrow K_p} F(K_0 \rightarrow K_p) .$$

Since the indexing is over all functors  $\mathbf{p} \rightarrow \mathbf{K}$ , the naturality of  $F$  implies that  $\prod^* F$  is a cosimplicial object of  $\mathcal{C}$ . Indeed, for  $i: \mathbf{p} \rightarrow \mathbf{q}$  in  $\Delta$  there is a natural map  $\prod^p F \rightarrow \prod^q F$ , induced by the morphism in  $\text{Sd } \mathbf{K}$  from  $K_{i_0} \rightarrow K_{i_p}$  to  $K_0 \rightarrow K_q$ . Part of the cosimplicial identities depend upon the following observation. Suppose that  $K_0 \rightarrow K_1$  and  $K_p \rightarrow K_{p+1}$  are the identity. Then for every  $K_1 \rightarrow K_p$ :

$$F(K_0 \rightarrow K_p) = F(K_1 \rightarrow K_p) = F(K_1 \rightarrow K_{p+1}) .$$

We remark that  $\prod^* F$  is the analogue of the chain complex used in [PW, 1.2] to define topological Hochschild homology; cf. [JP, 3.1] and [BW]. The Baues-Wirsching cohomology [Baues, IV.5.1] of  $\mathbf{K}$  with coefficients in a natural system  $F: \text{Sd } \mathbf{K} \rightarrow \mathbf{Ab}$  is the cohomology of the cochain complex associated to  $\prod^* F$  by the Dold-Kan correspondence. ([Baues] uses the Bousfield-Kan orientation.)

**Definition 2.1** [T78, 114]. Let  $F: \text{Sd } \mathbf{K} \rightarrow \mathcal{C}$  be a natural system on a small category  $\mathbf{K}$ , with values in a complete category  $\mathcal{C}$  equipped with a mapping object functor. Set

$$\text{holim}^{BW}(F) = \text{Tot} \left( \prod^* F \right) ,$$

where  $\text{Tot}$  is defined in 1.1 above. The name honors Baues and Wirsching, who introduced natural systems in [BW] as a way to study homotopy categories.

**Compatibility 2.2.** There is a natural functor  $\text{Sd } \mathbf{K} \rightarrow \mathbf{K}^{\text{op}} \times \mathbf{K}$  sending  $A \rightarrow B$  to  $(A, B)$ . If  $F$  factors as  $\text{Sd } \mathbf{K} \rightarrow \mathbf{K}^{\text{op}} \times \mathbf{K} \xrightarrow{F'} \mathcal{C}$  then  $\prod^* F$  is the same cosimplicial object as  $\prod^* F'$  in the previous section, so  $\text{holim}^{BW}(F)$  is the homotopy end of  $F'$ :

$$\text{ho} \int F' = \text{holim}^{BW}(F) .$$

As we have seen in 1.4, the homotopy limit is also a special case of this construction. If  $\tilde{F}: \mathbf{K}^{\text{op}} \rightarrow \mathcal{C}$  and  $F$  is the composite of  $\tilde{F}$  with the projection  $\text{Sd}(\mathbf{K}) \rightarrow \mathbf{K}^{\text{op}}$  then

$$\text{holim}(\tilde{F}) = \text{holim}^{BW}(F) .$$

**Functoriality 2.3.** A natural transformation  $F \Rightarrow G$  of natural systems induces a map  $\prod^* F \rightarrow \prod^* G$ , and so a natural map  $\text{holim}^{BW}(F) \rightarrow \text{holim}^{BW}(G)$ . The functor  $\text{holim}^{BW}: \text{Cat}(\text{Sd } \mathbf{K}, \mathcal{C}) \rightarrow \mathcal{C}$  commutes with all limits. Given a functor  $\Phi: \mathbf{L} \rightarrow \mathbf{K}$  and  $F: \text{Sd } \mathbf{K} \rightarrow \mathcal{C}$ , the diagram

$$\begin{array}{ccc} \text{Cat}(\text{Sd } \mathbf{K}, \mathcal{C}) & \xrightarrow{\Phi} & \text{Cat}(\text{Sd } \mathbf{L}, \mathcal{C}) \\ & \searrow \text{holim}^{BW} & \swarrow \text{holim}^{BW} \\ & \mathcal{C} & \end{array}$$

commutes up to a natural 2-cell  $\varphi$  induced by the projection  $\prod_{\mathbf{K}}^* F \rightarrow \prod_{\mathbf{L}}^* F$ .

**Lemma 2.4** [T78, 114] [T81, 164]. *Let  $\mathcal{C}$  be a right model category enriched over  $\mathcal{S}_f$ , or a complete closed model category. If  $F \Rightarrow G$  is such that  $F(k) \rightarrow G(k)$  is a fibration (resp. trivial fibration) in  $\mathcal{C}$  for every  $k$  in  $\text{Sd } \mathbf{K}$ , then  $\text{holim}^{BW}(F) \rightarrow \text{holim}^{BW}(G)$  is a fibration (resp. trivial fibration) in  $\mathcal{C}$ .*

*In particular, if each  $F(k)$  is fibrant then so is  $\text{holim}^{BW}(F)$ . Moreover,  $\text{holim}^{BW}$  preserves homotopy fiber sequences.*

In fact, the proof of lemma 1.8 (or 1.10) goes through in this setting.

Now consider the comma category  $k/\text{Sd } \mathbf{K}$  for a fixed morphism  $k$  in  $\mathbf{K}$ , and write  $F_k$  for the restriction of  $F: \text{Sd } \mathbf{K} \rightarrow \mathcal{C}$  to  $k/\text{Sd } \mathbf{K}$ . Since  $k$  is an initial object of the comma category  $k/\text{Sd } \mathbf{K}$  we have a canonical weak equivalence (as  $F(k)$  is fibrant; cf. [BK, XI.2.5]):

$$F(k) \xrightarrow{\sim} \text{holim}_{k/\text{Sd } \mathbf{K}} F_k .$$

**Remark** [T78, 120]. If  $F: \text{Sd } \mathbf{K} \rightarrow \mathcal{C}$  takes fibrant values, then  $\text{holim}^{BW}(F)$  is weak equivalent to  $\text{holim}_{\text{Sd } \mathbf{K}}(F)$ . This result is suggested by [BW, 4.4] and [DK, 3.3]. In fact, if  $F'(K \rightarrow K')$  is defined to be  $\text{holim}_{K'/\mathbf{K}}$  of  $(K' \rightarrow L) \mapsto G(K \rightarrow L)$ , then  $\text{holim}^{BW}(F') = \text{holim}(F)$ .

**Theorem 2.5** [T78, 117]. *If  $F: \text{Sd } \mathbf{K} \rightarrow \mathcal{C}$  is pointwise fibrant, then*

$$\text{holim}^{BW}(F) \cong \ker \left\{ \prod_{K_0} \text{holim}_{K/\text{Sd } \mathbf{K}} F \rightrightarrows \prod_k \text{holim}_{k/\text{Sd } \mathbf{K}} F_k \right\} .$$

*Since  $\text{holim}^{BW}$  preserves homotopy equivalences of such fibrant valued  $F$ , this shows that  $\text{holim}^{BW}(F)$  is the right derived functor of*

$$F \longmapsto \ker \left\{ \prod_{K_0} F(K_0 = K_0) \rightrightarrows \prod_{K_0 \rightarrow K_1} F(K_0 \rightarrow K_1) \right\}$$

*for  $F: \text{Sd } \mathbf{K} \rightarrow \mathcal{C}$ .*

*Proof:* The right side is

$$\begin{aligned} & \ker \prod_K \left\{ \text{Tot} \left( \prod_{K/\text{Sd } \mathbf{K}}^* F \right) \rightrightarrows \prod_k \text{Tot} \left( \prod_{k/\text{Sd } \mathbf{K}}^* F_k \right) \right\} \\ &= \text{Tot} \left\{ \ker \prod_K \left( \prod_{K/\text{Sd } \mathbf{K}}^* F \right) \rightrightarrows \prod_k \left( \prod_{k/\text{Sd } \mathbf{K}}^* F_k \right) \right\}. \end{aligned}$$

This is Tot of a cosimplicial object in  $\mathcal{C}$  which in degree  $p$  is an equalizer of  $\prod_K \prod_\lambda F(\lambda_p) \rightrightarrows \prod_k \prod_\mu F(\mu_p)$ , where  $\lambda$  and  $\mu$  run over diagrams in  $K/\text{Sd } \mathbf{K}$  and  $k/\text{Sd } \mathbf{K}$ , respectively. This equalizer is easily seen to be  $\prod^p F$ , so the cosimplicial object is just the simplicial replacement  $\prod^* F$ . We are done, since  $\text{holim}^{BW}(F)$  is defined to be  $\text{Tot}(\prod^* F)$ .  $\square$

Thomason defines an even more exotic generalization of a homotopy limit, which we christen  $\text{holim}^T$ . To define it, recall that the objects of the comma category  $\Delta/\mathbf{K}$  are pairs  $(p, K: \mathbf{p} \rightarrow \mathbf{K})$ , i.e., diagrams  $K_0 \rightarrow K_1 \rightarrow \cdots \rightarrow K_p$  in  $\mathbf{K}$ , while a morphism  $(p, K) \rightarrow (q, L)$  consists of a simplicial map  $\sigma: \mathbf{p} \rightarrow \mathbf{q}$  such that  $K = L\sigma$ . It will be convenient to abbreviate  $(p, K)$  as  $K$  and write  $\#K$  for the first entry  $p$  of  $(p, K)$ .

Given a functor  $F: \Delta/\mathbf{K} \rightarrow \mathcal{C}$ , let  $\prod^* F$  be the cosimplicial replacement defined by

$$\prod^p F = \prod_{\#K=p} F(K).$$

If  $\sigma: \mathbf{q} \rightarrow \mathbf{p}$  is a morphism in  $\Delta$ , we define  $\sigma_*: \prod^q F \rightarrow \prod^p F$  by requiring that the component corresponding to  $K: \mathbf{p} \rightarrow \mathbf{K}$  is the projection onto the component  $F(K\sigma)$  followed by the map  $F(K\sigma) \rightarrow F(K)$  associated to the morphism  $K\sigma \rightarrow K$  induced by  $\sigma$ .

$$\mathbf{q} \xrightarrow{\sigma} \mathbf{p} \xrightarrow{K} \mathbf{K}$$

**Definition 2.6** [T85, 36]. Let  $F: \Delta/\mathbf{K} \rightarrow \mathcal{C}$  be a functor, where  $\mathcal{C}$  is a complete category equipped with a mapping object functor. Then

$$\mathrm{holim}^T(F) = \mathrm{Tot} \left( \prod^* F \right) = \int_{p \in \Delta} \mathrm{Map}(\Delta[p], \prod_{\#K=p} F(K)) .$$

**Lemma 2.7** [T83, 158]. *There is a functor  $(K_1, K_2) \mapsto \mathrm{Map}(\Delta[\#K_1], FK_2)$  from  $(\Delta/\mathbf{K})^{\mathrm{op}} \times (\Delta/\mathbf{K})$  to  $\mathcal{C}$ , and  $\mathrm{holim}^T(F) = \mathrm{holim}_{\Delta/\mathbf{K}}^T(F)$  is its end:*

$$\mathrm{holim}^T(F) \cong \int_{K \in \Delta/\mathbf{K}} \mathrm{Map}(\Delta[\#K], FK) .$$

*Proof:* It suffices to check that the two ends have the same universal mapping property with respect to objects of  $\mathcal{C}$ . The details are routine, and duly verified in [T83, 159].  $\square$

There are forgetful functors

$$\begin{aligned} (\Delta/\mathbf{K}) &\longrightarrow \mathrm{Sd} \mathbf{K} \longrightarrow \mathbf{K}^{\mathrm{op}} \times \mathbf{K} \longrightarrow \mathbf{K} \\ (K_0 \rightarrow \cdots \rightarrow K_p) &\longmapsto (K_0 \rightarrow K_p) \longmapsto (K_0, K_p) \longmapsto K_0 . \end{aligned}$$

**Proposition 2.8** [T85, 152]. *For each  $F: \Delta/\mathbf{K} \rightarrow \mathcal{C}$ :*

- (a) *If  $F$  factors through  $F': \mathrm{Sd} \mathbf{K} \rightarrow \mathcal{C}$  then  $\mathrm{holim}^T(F) = \mathrm{holim}^{BW}(F')$ ;*
- (b) *If  $F$  factors through  $F'': \mathbf{K}^{\mathrm{op}} \times \mathbf{K} \rightarrow \mathcal{C}$  then  $\mathrm{holim}^T(F) = \mathrm{hof} F''$ ;*
- (c) *If  $F$  factors through  $F''': \mathbf{K}^{\mathrm{op}} \rightarrow \mathcal{C}$  then  $\mathrm{holim}^T(F) = \mathrm{holim}(F''')$ .*

Indeed, (a) is routine since  $\prod^* F$  is the same cosimplicial object as  $\prod^* F'$ . Parts (b) and (c) follow from Compatibility 2.2.

**Remark 2.8.1.**  $\Delta/(\mathbf{K}^{\mathrm{op}})$  is not  $(\Delta/\mathbf{K})^{\mathrm{op}}$ , and  $\mathrm{Sd}(\mathbf{K}^{\mathrm{op}})$  is not  $(\mathrm{Sd} \mathbf{K})^{\mathrm{op}}$ .

Here is the generalization of lemma 1.8; again the proof of 1.8 goes through.

**Theorem 2.9** [T85, 155]. *Let  $\mathcal{C}$  be a complete right model category enriched over  $\mathcal{S}_f$ . Suppose in addition that towers and products preserve fibrations and trivial fibrations in  $\mathcal{C}$ . Then for every small category  $\mathbf{K}$  and every pointwise fibrant functor*

$$F: \Delta/\mathbf{K} \rightarrow \mathcal{C} ,$$

*the object  $\text{holim}^T(F)$  is fibrant in  $\mathcal{C}$ . If  $\eta: F \rightarrow G$  is a pointwise fibration between pointwise fibration functors then  $\text{holim}^T(\eta): \text{holim}^T(F) \rightarrow \text{holim}^T(G)$  is a fibration. If in addition  $\eta$  is a weak equivalence, so is  $\text{holim}^T(\eta)$ .*

**Variant 2.10** [T85, 6]. Given  $F: (\Delta/\mathbf{K})^{\text{op}} \rightarrow \mathcal{C}_{\text{cof}}$ , we have a simplicial object in  $\mathcal{C}_{\text{cof}}$  whose  $p$ th term is  $\coprod_{\#K=p} F(K)$ . If  $\mathcal{C}$  is cocomplete and left enriched (1.15), one can define  $\text{hocolim}^T(F)$  as the coend  $\int^{\Delta} \Delta[\#K] \otimes \coprod_* F$ . The dual assertions for 2.7–2.8 are explored in [T85]. The dual of theorem 2.9 is given in [T81, 10, 159] and on pp. 46–48 of [T86].

### §3 Thomason's Axioms for homotopy theory

Let  $\mathcal{C}$  be a category. If  $W$  is a distinguished family of morphisms, closed under composition and containing all identity maps, it is convenient to think of  $W$  as the morphisms of subcategory  $\mathcal{W}$  having the same objects as  $\mathcal{C}$ . We say that  $\mathcal{W}$  is *saturated* if given any pair  $(f, g)$  of composable maps in  $\mathcal{C}$ , whenever two of  $f$ ,  $g$  and  $fg$  are in  $\mathcal{W}$  so is the third.

Let  $\mathcal{C}$  be a category equipped with three distinguished subcategories  $\text{cof}(\mathcal{C})$ ,  $\text{fib}(\mathcal{C})$  and  $\text{we}(\mathcal{C})$  having the same objects as  $\mathcal{C}$ . Morphisms in these subcategories will be called *cofibrations* (written  $\rightarrow$ ), *fibrations* (written  $\twoheadrightarrow$ ) and *weak equivalences* (written  $\xrightarrow{\sim}$ ), respectively.

A *trivial cofibration*, or *equicofibration*, (written  $\xrightarrow{\sim}$ ) is a map which is both a cofibration and a weak equivalence; a *trivial fibration*, or *equifibration*, (written  $\xrightarrow{\sim}$ ) is a map which is both a fibration and a weak equivalence.

We suppose that  $\mathcal{C}$  has an initial object  $\emptyset$  and a terminal object  $*$ . Consider the following self-dual axioms [T78, 156].

- (TM0) Every isomorphism of  $\mathcal{C}$  is in  $\text{cof}(\mathcal{C})$ ,  $\text{fib}(\mathcal{C})$  and  $\text{we}(\mathcal{C})$ .
- (TM1) The pushout  $P$  of a cofibration  $c: A \rightarrow C$  with any map  $f: A \rightarrow B$  exists, and  $B \rightarrow P$  is a cofibration. Moreover, if either  $f$  or  $c$  is a weak equivalence, so is its pushout. The dual assertion (RM1) for the pullback  $B \times_A C$  of a fibration  $C \twoheadrightarrow A$  and any map  $B \rightarrow A$  must also hold.
- (TM2) (Quillen's Axiom CM2) The subcategory  $\text{we}(\mathcal{C})$  is saturated.
- (TM3) (Quillen's Axiom CM3w)<sup>†</sup> Any retract of a weak equivalence is a weak equivalence.
- (TM4) (Quillen's Lifting Axiom CM4) Suppose given a commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow i & & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

---

<sup>†</sup>Axiom (TM3) is not needed for cofibrations or fibrations, only for weak equivalences. See [T79, 26] and [T83, 152].

in which  $i$  is a cofibration and  $p$  is a fibration. Then a map  $B \rightarrow X$  exists, factoring both  $A \rightarrow X$  and  $B \rightarrow Y$ , provided that either  $i$  or  $p$  is a weak equivalence. Any such map  $B \rightarrow X$  will be called a *fill-in*, or *factorization*.

(CM5) Any map  $A \xrightarrow{f} B$  factors as  $A \twoheadrightarrow B' \xrightarrow{\sim} B$ , and also as  $A \xrightarrow{\sim} A' \twoheadrightarrow B$ .

(TM5) The following factorizations of  $A \xrightarrow{\sim} B$  can be made functorial in  $f$ :

$$(c) \quad A \twoheadrightarrow B' \xrightarrow{\sim} B$$

$$(f) \quad A \xrightarrow{\sim} A' \twoheadrightarrow B$$

The functoriality in (TM5) may be expressed as follows. Let  $\mathcal{C}^1$  denote the category of arrows in  $\mathcal{C}$ , i.e., functors  $\mathbf{1} \xrightarrow{f} \mathcal{C}$ , and  $\mathcal{C}^2$  the category of composable pairs of arrows in  $\mathcal{C}$ , i.e., functors  $\mathbf{2} \rightarrow \mathcal{C}$ . Then there are functors

$$T, M: \mathcal{C}^1 \rightarrow \mathcal{C}^2$$

sending the arrow  $f: A \rightarrow B$  in  $\mathcal{C}^1$  to the composable pairs (with composition  $f$ )

$$A \twoheadrightarrow T(f) \xrightarrow{\sim} B \quad \text{and} \quad A \xrightarrow{\sim} M(f) \twoheadrightarrow B$$

respectively. See [T79, 72-73] and [T78, 161].

We call an object  $C$  *cofibrant* if the canonical map  $\emptyset \rightarrow C$  is in  $\text{cof}(\mathcal{C})$ ; we call  $C$  *fibrant* if the canonical map  $C \rightarrow *$  is in  $\text{fib}(\mathcal{C})$ .

**Definition 3.1.** A *basic model category* is a category  $\mathcal{C}$ , equipped with distinguished subcategories  $\text{cof}(\mathcal{C})$ ,  $\text{fib}(\mathcal{C})$  and  $\text{we}(\mathcal{C})$ , an initial object  $\emptyset$  and a terminal object  $*$ , satisfying axioms (TM0)-(TM4) and (CM5). If in addition it satisfies axiom (TM5), we call  $\mathcal{C}$  a *Thomason model category*.

Comparing definitions, we see that any basic model category is both a left model category and a right model category.

The following consequence of the axioms is of fundamental importance, so we include it here.

**Glueing Lemma 3.2.** *Suppose that  $\mathcal{C}$  satisfies axiom (TM0), (TM1) and (TM2). Given a commutative diagram*

$$\begin{array}{ccccc} B & \longleftarrow & A & \longrightarrow & C \\ \sim \downarrow & & \downarrow \sim & & \downarrow \sim \\ B' & \longleftarrow & A' & \longrightarrow & C' \end{array}$$

*in which the vertical maps are weak equivalences, the pushout  $B \sqcup_A C \rightarrow B' \sqcup_{A'} C'$  is a weak equivalence.*

*Proof:* By (TM1) the map  $C \rightarrow A' \sqcup_A C$  exists and is a weak equivalence. Its composition with  $g: A' \sqcup_A C \rightarrow C'$  is  $C \xrightarrow{\sim} C'$ , so  $g$  is a weak equivalence by (TM2). But then we may use (TM0) and (TM1) twice to get

$$B \sqcup_A C \xrightarrow{\sim} B' \sqcup_A C \cong B' \sqcup_{A'} A' \sqcup_A C \xrightarrow{\sim} B' \sqcup_{A'} C' .$$

The composition is a weak equivalence, as desired.  $\square$

**Remark 3.2.1.** (TM0) is only used for  $\text{we}(\mathcal{C})$  and (TM1) is only used for cofibrations.

## §4 Examples

As usual, the paradigm of a Thomason model category is the category of topological spaces and continuous maps, where the fibrations are Serre fibrations and weak equivalences are homotopy equivalences; cofibrations are defined by the lifting property (TM4).

**Definition 4.1.** More generally, any proper model category in the sense of Quillen [QH] [QR] [BF] [GJ] is a basic model category; the adjective *proper* describes the hard part of axiom (TM1). The easy part of (TM1) is axiom (M4) in Quillen's original definition of a model category [QH]; see [GJ, II.1.3].

I do not know of any cocomplete proper model category in which the factorizations required by axiom (CM5) are not functorial. We will see in 4.8 below that if  $\mathcal{C}$  admits a *small object argument* then  $\mathcal{C}$  satisfies (TM5) and is a Thomason model category. Compare [GJ, I.9.2].

Here is another difference between Quillen's and Thomason's axioms. Thomason's Factorization Axiom (TM5) allows us to factor  $f$  as  $hi$ , where  $i$  is a cofibration and  $h$  is a weak equivalence; in Quillen's Factorization Axiom (CM5)  $h$  must also be a fibration. Therefore if  $f$  is a map having the left lifting property with respect to all fibrations which are isomorphisms, Quillen's axioms (CM3) and (CM5) imply that  $f$  must be a cofibration. This need not be the case in a Thomason model category, as the following example shows.

**Example 4.2.** Here is an example of a Thomason model category which is not a Quillen model category. Let  $\mathbf{Ch} = \mathbf{Ch}^b(R)$  denote the category of bounded chain complexes of modules over a ring  $R$ , with weak equivalences being quasi-isomorphisms (maps inducing isomorphisms on homology).

Let  $\text{cof}(\mathbf{Ch})$  denote the category of injections  $A \hookrightarrow B$  whose cokernel  $B/A$  is a chain complex of projective modules, and let  $\text{fib}(\mathbf{Ch})$  denote the category of surjections whose kernel is a chain complex of injective modules. If  $R$  is a noetherian regular ring of finite global dimension, then  $\mathbf{Ch}^b(R)$  is a Thomason model category

but not a Quillen model category (unless  $R$  is semisimple).

Of course,  $\mathbf{Ch}^b(R)$  has the same homotopy theory as the Quillen model structure of [QH, I.1.2], in which every surjection is a fibration. See also [QH, I.4.12].

**Example 4.3.** Baues' *cofibration category* is another context in which one can do homotopy theory; this theory is developed in [Baues]. By definition, a cofibration category is a category  $\mathcal{C}$  equipped with two classes of morphisms (cof and we) such that Baues' axioms (C1)–(C4) hold. Thus Baues' cofibration categories lack all fibrations present in a Thomason model category.

Baues' axioms (C1), (C2), and (C3) for  $(\mathcal{C}, \text{cof}, \text{we})$  are axioms (TM0), (TM1), (TM2) and (CM5c) for cofibrations and weak equivalences. Baues calls an object  $X$  a *fibrant model* if each trivial cofibration  $X \xrightarrow{\sim} Y$  admits a retraction, and Baues' final axiom is:

(C4) For each object  $A$  in  $\mathcal{C}$  there is a trivial cofibration  $A \xrightarrow{\sim} X$  where  $X$  is a fibrant model.

If  $\mathcal{C}$  is Thomason model category, then  $(\mathcal{C}, \text{cof}, \text{we})$  is a left model category in the sense of 1.13 above. However,  $(\mathcal{C}, \text{cof}, \text{we})$  may not satisfy Baues' axiom (C4), so  $\mathcal{C}$  may not be a cofibration category in the sense of [Baues].

For purposes of comparison, recall from [Baues, I.2.6] that if  $(\mathcal{C}, \text{cof}, \text{fib}, \text{we})$  is a model category in the sense of Quillen [QH], then the subcategory  $\mathcal{C}_{\text{cof}}$  of all cofibrant objects in  $\mathcal{C}$  is a cofibration category.

**Example 4.4.** Recall from [TT, 1.2.4] that a *biWaldhausen category* is a category  $\mathcal{C}$  with a zero object  $0$ , together with subcategories  $\text{cof}(\mathcal{C})$ ,  $\text{fib}(\mathcal{C})$  and  $\text{we}(\mathcal{C})$  so that:

- (i)  $(\mathcal{C}, \text{cof}, \text{we})$  and  $(\mathcal{C}^{\text{op}}, \text{fib}, \text{we})$  are Waldhausen categories [TT, 1.2.3];
- (ii) the canonical map  $A \sqcup B \rightarrow A \times B$  is an isomorphism; and
- (iii) the cofibration sequences in  $\mathcal{C}$  and  $\mathcal{C}^{\text{op}}$  are dual to each other.

Note that every object of  $\mathcal{C}$  is both cofibrant and fibrant by axiom 1.2.1.2 of [TT]. A biWaldhausen category satisfies (TM0) and (TM1) by axioms 1.2.1.1, 1.2.1.3,

1.2.3.1 and 1.2.3.2 of [TT]. We say that  $\mathcal{C}$  is *saturated* if axiom (TM2) holds, i.e., if  $\text{we}(\mathcal{C})$  is a saturated subcategory.

If  $\mathcal{C}$  has mapping cylinders satisfying the cylinder axiom, and mapping path spaces satisfying the path space axiom (see [TT, 1.3.1 and 1.3.2]) then  $\mathcal{C}$  satisfies (TM4) and (TM5). Thus only the retraction axiom (TM3) is needed to make  $\mathcal{C}$  a Thomason model category. Here is a partial converse.

**Lemma 4.5.** *If  $\mathcal{C}$  satisfies (TM0) – (TM3),  $A \sqcup B \simeq A \times B$  for all  $A$  and  $B$ , and every object is both fibrant and cofibrant, then  $\mathcal{C}$  is a saturated biWaldhausen category.*

*Proof:* This is straightforward; since objects are fibrant and cofibrant, axiom (i) follows from (TM0), (TM1) and the gluing lemma 3.2.  $\square$

We caution the reader that the functoriality axiom (TM5) need not give a Waldhausen cylinder functor  $T$ , because the map  $A \sqcup B \rightarrow T(f)$  need not be in  $\text{cof}(\mathcal{C})$ .

We now give a set-theoretic condition under which a proper closed Quillen model category is a Thomason model category. It is based on II.3.4 of [QH].

**Definition 4.6.** Suppose that  $\mathcal{C}$  is a closed model category. We say that a set of trivial cofibrations  $\{A_\alpha \xrightarrow{\sim} B_\alpha\}$  is a set of *test cofibrations* if for each map  $E \xrightarrow{p} F$ ,  $p$  is a fibration if and only if every diagram of the following form admits a factorization  $B_\alpha \rightarrow E$ .

$$\begin{array}{ccc} A_\alpha & \longrightarrow & E \\ \downarrow & & \downarrow p \\ B_\alpha & \longrightarrow & F \end{array}$$

For example, the *anodyne extensions* form a set of test cofibrations for the category  $\mathcal{C}$  of simplicial sets; see [GJ, I.4.3]. The horns  $\{\Lambda_k^n \xrightarrow{\sim} \Delta^n\}$  form a countable set of test fibrations; see [GJ, p. 11].

Let  $\kappa$  be an infinite cardinal number. A poset is called  $\kappa$ -*filtering* if any subset of cardinality  $< \kappa$  has an upper bound.

**Definition 4.7.** An object  $A$  of a category  $\mathcal{C}$  is called  $\kappa$ -small if for every  $\kappa$ -filtering directed poset  $\{C_i\}$  for which  $\varinjlim C_i$  exists we have

$$\varinjlim \mathrm{Hom}_{\mathcal{C}}(A, C_i) \cong \mathrm{Hom}_{\mathcal{C}}(A, \varinjlim C_i) .$$

**Theorem 4.8** (Small Object Argument). *Let  $\mathcal{C}$  be a closed model category with arbitrary coproducts. Suppose  $\mathcal{C}$  has a set  $\{A_\alpha \xrightarrow{\sim} B_\alpha\}$  of test cofibrations in which each  $A_\alpha$  is  $\kappa$ -small for some fixed  $\kappa$ . Then  $\mathcal{C}$  satisfies (TM5). If in addition  $\mathcal{C}$  is a proper closed model category, then  $\mathcal{C}$  is a Thomason model category.*

*Proof:* We construct  $F_\mu: \mathcal{C}^1 \rightarrow \mathcal{C}^2$  by induction on  $\mu \leq \kappa$ . Fix  $f: X \rightarrow Y$ . The functor  $F_0$  sends  $f$  to  $X \xrightarrow{\mathrm{id}} X \rightarrow Y$ . We define  $T_{\mu+1}$  by

$$T_{\mu+1}(f) = \left( \coprod_S B_\alpha \right) \coprod_{(\coprod_S A_\alpha)} T_\mu(f)$$

where  $S$  runs over all test diagrams

$$(*) \quad \begin{array}{ccc} A_\alpha & \longrightarrow & T_\mu(f) \\ \downarrow & & \downarrow \\ B_\alpha & \longrightarrow & Y . \end{array}$$

The map  $T_\mu \rightarrow T_{\mu+1}$  is a trivial cofibration as in Lemma 3 of [QH, II.3.3] and the maps  $B_\alpha \rightarrow Y$  and  $T_\mu \rightarrow Y$  induce a fibration  $T_{\mu+1} \rightarrow Y$ . We define  $F_{\mu+1}$  by

$$X \xrightarrow{\sim} T_\mu \xrightarrow{\sim} T_{\mu+1} \twoheadrightarrow Y .$$

For a limit ordinal  $\mu$ , we observe that  $X \rightarrow \varinjlim T_\nu = T_\mu$  is a trivial cofibration. The map  $T_\mu \rightarrow Y$  is a fibration because for each test diagram (\*) there is a  $\nu < \mu$  so that  $A \rightarrow T_\mu$  factors through a map  $A \rightarrow T_\nu$ ,  $\nu < \mu$  ( $A_\alpha$  is  $\mu$ -small as  $\mu < \kappa$ ), and the factorization  $B \rightarrow T_{\nu+1} \rightarrow T_\mu$  of (\*) exists by the construction of  $T_{\nu+1}$ .  $\square$

## §5 The Homotopy Category

Suppose that  $\mathcal{C}$  is a basic model category. The basic idea needed to construct the homotopy category  $\text{Ho}\mathcal{C}$  is quite simple, and goes back to Quillen [QH]. Given  $A$  cofibrant and  $B$  fibrant, we show in 5.4 that there is a good equivalence relation  $\simeq$  on  $\text{Hom}_{\mathcal{C}}(A, B)$  and  $[A, B]$  is the set of equivalence classes. Given general  $A$  and  $B$ , axiom (CM5) gives factorizations  $\emptyset \rightarrow A' \xrightarrow{\sim} A$  and  $B \xrightarrow{\sim} B' \rightarrow *$ ; we set  $[A, B] = [A', B']$ . As usual, there are lots of technical details to check.

By a *cylinder object* for an object  $A$  of  $\mathcal{C}$ , we mean an object  $C$  of  $\mathcal{C}$ , together with a factorization  $A \sqcup A \rightarrow C \xrightarrow{\sim} A$  of the fold map  $\nabla: A \sqcup A \rightarrow A$  (we assume  $A \sqcup A$  exists). Dually, a *path object* for  $B$  is an object  $M$  together with a factorization  $B \xrightarrow{\sim} M \rightarrow B \times B$  of the diagonal  $\Delta$  (we assume  $B \times B$  exists).

**Definition 5.1.** For  $f, g \in \text{Hom}_{\mathcal{C}}(A, B)$ , define  $f \simeq_L g$  to hold if the map  $f \sqcup g: A \sqcup A \rightarrow B$  factors as  $A \sqcup A \xrightarrow{\vartheta} C \xrightarrow{H} B$ , where there is a weak equivalence  $q: C \xrightarrow{\sim} A$  such that  $q\partial$  is the fold map, i.e.,  $C$  is a cylinder object for  $A$ .

**Lemma 5.2** [T78, 166]. *If  $A$  is cofibrant then  $\simeq_L$  is an equivalence relation on the set  $\text{Hom}(A, B)$ . We define  $\pi(A, B)$  to be  $\text{Hom}(A, B)_{\simeq_L}$ .*

If  $A$  is cofibrant and  $B \rightarrow B'$  arbitrary, it is easy to see that  $\text{Hom}(A, B) \rightarrow \text{Hom}(A, B')$  induces a quotient map  $\pi(A, B) \rightarrow \pi(A, B')$ .

*Proof:* (Quillen [QH, I.1.8]) Axiom (CM5) implies that the fold map  $\nabla$  factors as  $A \sqcup A \rightarrow C \xrightarrow{\sim} A$ ; the composition with  $f: A \rightarrow B$  is clearly  $f \sqcup f$ , so  $f \simeq_L f$ . To see that the relation is symmetric, precompose the factorization of  $f \sqcup g$  with the permutation  $\tau$ ; the composition

$$A \sqcup A \xrightarrow{\tau} A \sqcup A \xrightarrow{\vartheta} C \xrightarrow{H} B$$

will then be  $g \sqcup f$ , and  $q\partial\tau = \nabla\tau = \nabla$ . Finally, to see that  $\simeq_L$  is transitive, suppose given factorizations of  $f \sqcup g$  and  $g \sqcup h$ :

$$A \sqcup A \rightarrow C_1 \rightarrow B \quad \text{and} \quad A \sqcup A \rightarrow C_2 \rightarrow B .$$

Since  $A$  is cofibrant,  $A = \emptyset \sqcup A \twoheadrightarrow A \sqcup A \twoheadrightarrow C_1$  and  $A = A \sqcup \emptyset \twoheadrightarrow A \sqcup A \twoheadrightarrow C_2$  are cofibrations. Hence we may apply the Glueing Lemma to

$$\begin{array}{ccccc} C_1 & \longleftarrow & A & \longrightarrow & C_2 \\ q_1 \downarrow & & \downarrow = & & \downarrow q_2 \\ A & \xlongequal{\quad} & A & \xlongequal{\quad} & A. \end{array}$$

This shows that the pushout  $a: C_1 \sqcup_A C_2 \rightarrow A$  is a weak equivalence. Moreover, the composites  $A \twoheadrightarrow C_i \rightarrow B$  are both  $g$ , so they induce a map  $h_1 \sqcup_A h_2: C_1 \sqcup_A C_2 \rightarrow B$ . If we let  $\partial$  be the map

$$A \sqcup A = (A \sqcup \emptyset) \sqcup (\emptyset \sqcup A) \twoheadrightarrow C_1 \sqcup C_2 \rightarrow C_1 \sqcup_A C_2$$

then clearly  $q\partial = \nabla$  and  $(h_1 \sqcup_A h_2) \circ \partial = f \sqcup h$ . Finally,  $\partial$  is a cofibration because it is the composition of two cofibrations:  $\partial_0 \sqcup A: A \sqcup A \twoheadrightarrow C_1 \sqcup A$ , the pushout of  $A \twoheadrightarrow C_1$  by  $A = A \sqcup \emptyset \twoheadrightarrow A \sqcup A$ , and the map  $C_1 \sqcup A \twoheadrightarrow C_1 \sqcup_A C_2$ , which is the pushout of  $A \sqcup A \twoheadrightarrow C_2$  by  $\partial_0 \sqcup A: A \sqcup A \twoheadrightarrow C_1 \sqcup A$ .  $\square$

$$\begin{array}{ccccc} A & \longrightarrow & A \sqcup A & \twoheadrightarrow & C_2 \\ \downarrow & & \downarrow \partial_0 \sqcup A & & \downarrow \\ C_1 & \longrightarrow & C_1 \sqcup A & \twoheadrightarrow & C_1 \sqcup_A C_2 \end{array}$$

If  $A$  is cofibrant, the following argument shows that we can fix the choice of cylinder object in the definition of  $\simeq_L$ . This fact allows us to simplify some set-theoretic considerations later on.

**Lemma 5.3.** *If  $A$  is cofibrant,  $B$  is fibrant and  $C$  is a cylinder object for  $A$ , then whenever two maps  $f, g: A \rightarrow B$  satisfy  $f \simeq_L g$  there is a map  $C \rightarrow B$  so that  $A \sqcup A \twoheadrightarrow C \rightarrow B$  is  $f \sqcup g$ .*

*Proof:* [T83, 77] We are given a factorization  $A \sqcup A \twoheadrightarrow C' \xrightarrow{H'} B$  of  $f \sqcup g$ , where  $C'$  is a different cylinder object for  $A$ . Let  $C''$  denote the pushout of  $C$  and  $C'$  along  $A \sqcup A$ , and factor the pushout  $C'' \rightarrow A$  as  $C'' \twoheadrightarrow C''' \xrightarrow{\sim} A$ . Axiom (TM2) shows that the composite  $C' \twoheadrightarrow C'' \twoheadrightarrow C'''$  is a trivial cofibration. Since  $B$  is fibrant,

axiom (TM4) applied to the right side of

$$\begin{array}{ccccc}
 A \sqcup A & \twoheadrightarrow & C' & \xrightarrow{H'} & B \\
 \downarrow & & \downarrow \sim & \nearrow \text{---} & \downarrow \\
 C & \twoheadrightarrow & C''' & \longrightarrow & *
 \end{array}$$

shows that  $H'$  factors through  $C'''$ . But then  $f \sqcup g$  factors as

$$A \sqcup A \twoheadrightarrow C \longrightarrow C''' \longrightarrow B . \quad \square$$

Dually, if  $B$  is fibrant, we get an equivalence relation  $\simeq_R$  on  $\text{Hom}_{\mathcal{C}}(A, B)$  by declaring  $f \simeq_R g$  if there is a diagram:

$$\begin{array}{ccccc}
 A & \longrightarrow & M & \xleftarrow{\sim} & B \\
 \searrow f \times g & & \downarrow & & \swarrow \Delta \\
 & & B \times B & & 
 \end{array}$$

**Lemma 5.4.** *If  $A$  is cofibrant and  $B$  fibrant, then for all  $f, g \in \text{Hom}_{\mathcal{C}}(A, B)$  we have:  $f \simeq_L g$  if and only if  $f \simeq_R g$ .*

*Proof:* Suppose that  $f \simeq_L g$  via  $A \sqcup A \twoheadrightarrow C \xrightarrow{h} B$ . By (CM5) we can factor  $\Delta$  as  $B \xrightarrow{\sim} M \twoheadrightarrow B \times B$ . By (TM4), there is a fill-in  $\varphi: C \rightarrow M$  for the diagram:

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{\sim} & M \\
 \partial_0 \downarrow & & \downarrow & & \downarrow \\
 C & \xrightarrow{(fg, h)} & B \times B & & 
 \end{array}$$

and  $\varphi \partial_1: A \rightarrow M$  is the map needed to show that  $f \simeq_R g$ . By duality,  $f \simeq_R g$  implies  $f \simeq_L g$ .  $\square$

**Proposition 5.4** [T78, 170]. *Let  $\mathcal{C}$  be a basic model category. If  $A$  and  $A'$  are cofibrant and  $B$  is fibrant, then for every weak equivalence  $w: A' \rightarrow A$  there is a bijection*

$$w^*: \pi(A, B) \rightarrow \pi(A', B)$$

**Remark.** This is [QH, I.1.10] if  $w$  is a trivial cofibration.

*Proof:* The map  $w^*$  is well defined since by definition  $f \simeq_R g$  implies  $fw \simeq_R gw$ . To see that  $w^*$  is injective, suppose given  $f, g: A \rightarrow B$  so that  $fw \simeq_L gw$ . That is,

$fw \sqcup gw$  factors as  $A' \sqcup A' \twoheadrightarrow C \rightarrow B$ . Form the pushout  $P$  of  $C$  and  $A \sqcup A$  along  $A' \sqcup A'$ ; the map  $f \sqcup g$  and  $C \rightarrow B$  agree on  $A' \sqcup A'$  and induce a map  $G: P \rightarrow B$ .

$$\begin{array}{ccccc}
 A' \sqcup A' & \xrightarrow{\vartheta'} & C & \xrightarrow[q']{\sim} & A' \\
 w \sqcup w \downarrow \sim & & \downarrow & & \downarrow w \\
 A \sqcup A & \xrightarrow{\vartheta} & P & \xrightarrow{q} & A \\
 & \searrow f \sqcup g & \downarrow G & & \\
 & & B & & 
 \end{array}$$

Similarly  $\nabla$  and  $wq'$  induce a map  $q: P \rightarrow A$  so that  $q\vartheta = \nabla$ . Since  $A$  and  $A'$  are cofibrant, the glueing lemma shows that  $w \sqcup w: A' \sqcup A' \rightarrow A \sqcup A$  is a weak equivalence. By axiom (TM1), its pushout  $C \xrightarrow{\sim} P$  is a weak equivalence. Its composition with  $q$  is a weak equivalence, so  $q$  is a weak equivalence by (TM2). Thus  $P$  gives  $f \simeq_L g$ .

To see that  $w^*$  is onto we have to lift any  $A' \xrightarrow{f} B$  to  $A$ . If  $w$  were a trivial cofibration, this would follow from (TM4), as in [QH, I.1.10]. In the general case, factor  $w \sqcup 1$  as  $A' \sqcup A \twoheadrightarrow A'' \xrightarrow{\sim} A$ . Since  $A$  and  $A'$  are cofibrant, so are the evident maps  $A \twoheadrightarrow A''$  and  $A' \twoheadrightarrow A''$ . They are weak equivalences, since their composition with  $A'' \xrightarrow{\sim} A$  is. By the trivial cofibration case, we have the desired isomorphism.  $\square$

$$\begin{array}{ccccc}
 \pi(A', B) & \xleftarrow{\cong} & \pi(A'', B) & \xrightarrow{\cong} & \pi(A, B) \\
 & \swarrow w^* & \uparrow & \searrow = & \\
 & & \pi(A, B) & & 
 \end{array}$$

**Lemma 5.5.** *Let  $\mathcal{C}$  be a basic model category, and  $A$  any object. Suppose given weak equivalences  $w': A' \xrightarrow{\sim} A$ ,  $w'': A'' \xrightarrow{\sim} A$  with  $A'$  and  $A''$  cofibrant, and a fibrant object  $B$ . Then for any two maps  $a_1, a_2: A' \rightarrow A''$  so that  $w' = w''a_1 = w''a_2$  we have*

$$a_1^* = a_2^*: \pi(A'', B) \rightarrow \pi(A', B).$$

*Proof:* Given  $f: A'' \rightarrow B$ , factor  $(f, w'')$  as  $A'' \xrightarrow{\sim} P \twoheadrightarrow B \times A$ . As  $B \twoheadrightarrow *$  is a fibration and  $w'$  is a weak equivalence, its pullback  $B \times A' \rightarrow B \times A$  exists and is a weak equivalence by (TM1). In turn, the pullback of this map along  $P \twoheadrightarrow B \times A$  is a weak equivalence  $Q \xrightarrow{\sim} P$ , and  $Q \twoheadrightarrow B \times A'$  is a fibration. We have a diagram in which the bottom square is a pullback, so the fold map on  $A'$  factors through the dotted arrow.

$$\begin{array}{ccccc}
 A \sqcup A' & \xrightarrow{a_1 \sqcup a_2} & A'' & & \\
 \downarrow \text{dotted} & & \downarrow \sim & \searrow f & \\
 Q & \xrightarrow{\sim} & P & \twoheadrightarrow & B \\
 \downarrow & & \downarrow & & \\
 A' & \xrightarrow{\sim} & A & & 
 \end{array}$$

Factor the dotted arrow as  $A' \sqcup A' \twoheadrightarrow C \xrightarrow{\sim} Q$ . Now the map  $A' = A' \sqcup \emptyset \rightarrow A' \sqcup A' \rightarrow C$  is a weak equivalence, since its composite with  $C \xrightarrow{\sim} P$  is  $A' \xrightarrow{\sim} A'' \xrightarrow{\sim} P$ . Its composite with  $C \rightarrow A'$  is the identity on  $A'$ , so  $C \xrightarrow{\sim} A'$  is also a weak equivalence.

We have shown that the map  $f a_1 \sqcup f a_2$  factors as

$$\begin{array}{ccccccc}
 A \sqcup A' & \twoheadrightarrow & C & \xrightarrow{\sim} & Q & \xrightarrow{\sim} & P \twoheadrightarrow B \\
 \searrow \nabla & & \downarrow \sim & & & & \\
 & & A' & & & & 
 \end{array}$$

which shows that  $f a_1 \simeq_L f a_2$ , as required.  $\square$

In order to have an absolutely functorial set  $[A, B]$ , it is useful to parametrize the cofibrant resolutions of  $A$ , and the fibrant resolutions of  $B$ .

**Definition 5.6.** Given  $A$  in  $\mathcal{C}$ , let  $I_A$  denote the category whose objects are weak equivalences  $A' \xrightarrow{\sim} A$  with  $A'$  cofibrant. There is a (unique) morphism from  $A' \xrightarrow{\sim} A$  to  $A'' \xrightarrow{\sim} A$  in  $I_A$  if there exists a (backwards!) arrow  $A'' \rightarrow A'$  so that  $A'' \rightarrow A' \rightarrow A$  equals  $A'' \rightarrow A$ .

Dually, given  $B$  in  $\mathcal{C}$ , let  $J_B$  denote the category of weak equivalences  $B \xrightarrow{\sim} B'$  with  $B'$  fibrant, whose opposite category is  $I_{(A^{\text{op}})}$ .

**Lemma 5.7.** (a)  $I_A$  is a cofiltering category with at most one morphism between any two objects.

(b) Given  $A$  and  $B$ , there is an induced functor  $\pi(, ) : I_A^{\text{op}} \times J_B \rightarrow \mathbf{Sets}$ . Moreover, each morphism in  $I_A^{\text{op}} \times J_B$  induces a bijection on  $\pi(, )$ .

*Proof:* (a) is obvious, once we note that for each  $A'$  and  $A''$  we can factor  $A \sqcup A'' \twoheadrightarrow A''' \xrightarrow{\sim} A$ . Part (b) follows from 5.4 and 5.5.  $\square$

**Definition 5.8.** Given  $A$  and  $B$  in  $\mathcal{C}$ , set  $[A, B] = \lim_{I_A \times J_B} \pi(A', B')$ .

**Lemma 5.9.** If  $A$  and  $B$  are cofibrant, and  $C'$  is fibrant, then a weak equivalence  $w : B \xrightarrow{\sim} B'$  determines a well-defined “composition” pairing

$$\pi(A, B') \times \pi(B, C') \rightarrow \pi(A, C').$$

*Proof:* Suppose given  $f : A \rightarrow B'$  and  $g : B \rightarrow C'$ . Since  $A$  and  $B$  are cofibrant, we can form  $f \sqcup w : A \sqcup B \rightarrow B'$ . Factoring it as  $A \sqcup B \twoheadrightarrow B'' \xrightarrow{\sim} B'$ , axiom (TM2) shows that the composite  $B \twoheadrightarrow A \sqcup B \twoheadrightarrow B''$  is a trivial cofibration. Since  $C'$  is fibrant,  $g$  lifts to  $g'' : B'' \rightarrow C'$  by (TM4). Since  $\pi(B'', C') \cong \pi(B, C')$  by 5.4,  $g''$  is well-defined up to  $\simeq_L$ . We let  $g \circ f$  be the composite of  $A \twoheadrightarrow A \sqcup B \twoheadrightarrow B''$  and  $g''$ .

$$\begin{array}{ccccc}
 & & B & \xrightarrow{g} & C' \\
 & & \downarrow \sim & \nearrow \text{---} & \downarrow \\
 A \sqcup B & \twoheadrightarrow & B'' & \longrightarrow & * \\
 \uparrow & & \downarrow \sim & & \\
 A & \xrightarrow{f} & B' & & 
 \end{array}$$

Replacing  $A$  by a cylinder object in this argument shows that if  $f \simeq_L f'$  then  $g \circ f \simeq_L g \circ f'$ . Dually, if  $g \simeq_R g'$  then  $g \circ f \simeq_R g' \circ f$ .  $\square$

**Porism 5.10.** Suppose given  $f' : A \rightarrow B'$  and  $g : B \rightarrow C'$ , with  $w : B \xrightarrow{\sim} B'$ .

(a) If  $f = w f_0$  for some  $f_0 : A \rightarrow B$ , then  $g \circ f$  is the class of  $g f_0$  in  $\pi(A, C')$ .

(b) If  $g = g' w$  for some  $g' : B' \rightarrow C'$ , then  $g \circ f$  is the class of  $g' f$  in  $\pi(A, C')$ .

In particular,  $g \circ w = g$  and  $w \circ f = f$ . Hence the class  $1_B$  of  $w$  in  $\pi(B, B)$  is a 2-sided unit for the composition pairing.

**Lemma 5.11.** The composition pairing is associative.

*Proof:* Suppose given  $f: A \rightarrow B'$ ,  $g: B \rightarrow C'$  and  $h: C \rightarrow D'$  with  $A, B$  cofibrant and  $B', C', D'$  fibrant, and weak equivalences  $w: B \xrightarrow{\sim} B'$  and  $v: C \xrightarrow{\sim} C'$ . The proof of 5.9 shows that we can replace  $B$  by  $B''$  and  $g$  by  $g''$  to assume that  $f = wf_0$  for some  $f_0: A \rightarrow B$ . Similarly, we may replace  $C$  by  $C''$  and  $h$  by  $h''$  to assume that  $g = vg_0$  for some  $g_0: B \rightarrow C$ . But then the formula

$$(h \circ g) \circ f = (hg_0) \circ f = (hg_0)f_0 = h(g_0f_0) = h \circ (gf_0) = h \circ (g \circ f)$$

holds in  $\pi(A, D')$ , as required.  $\square$

**Definition 5.12.** Suppose that  $\mathcal{C}$  is a basic model category. Then  $\text{Ho}\mathcal{C}$  is the category with the same objects as  $\mathcal{C}$ , but the Hom-set of morphisms  $A \rightarrow B$  in  $\text{Ho}\mathcal{C}$  is  $[A, B]$ . Composition is defined in 5.9, and is associative with identity by 5.10 and 5.11.

Also, there is a canonical functor  $\mathcal{C} \rightarrow \text{Ho}\mathcal{C}$ , defined by 5.10.

**Theorem 5.13.** If  $\mathcal{C}$  is a basic model category then  $\text{Ho}\mathcal{C}$  is the localization  $\mathcal{C}[w^{-1}]$  of  $\mathcal{C}$  at the family  $w = \text{we}(\mathcal{C})$  of weak equivalences.

*Proof:* We first show that every weak equivalence  $f: A \xrightarrow{\sim} B$  in  $\mathcal{C}$  becomes an isomorphism in  $\text{Ho}\mathcal{C}$ . Choosing weak equivalences  $A' \xrightarrow{\sim} A$  and  $B \xrightarrow{\sim} B'$  with  $A'$  cofibrant and  $B'$  fibrant, we see that  $\pi(A', B')$  is canonically isomorphic to  $[A, A]$ ,  $[A, B]$ ,  $[B, B]$  and  $[B, A]$ . The class of the composite map  $f': A' \rightarrow B'$  represents  $1_A$ ,  $[f]$ ,  $1_B$  and a morphism  $g \in [B, A]$ , respectively. The composition pairing

$$\pi(A', B') \times \pi(A', B') \rightarrow \pi(A', B')$$

satisfies  $f' \circ f' = f'$  by 5.10. By definition 5.8, this shows that  $[f] \circ g = 1_B$  and  $g \circ [f] = 1_A$ , i.e.,  $[f]$  is an isomorphism in  $\text{Ho}\mathcal{C}$ .

Conversely, suppose that  $\phi: \mathcal{C} \rightarrow \mathcal{C}'$  is a functor sending weak equivalences to isomorphisms. If  $C$  is a cylinder object for  $A$  then the two composites  $i_0, i_1: A \rightarrow A \sqcup A \rightarrow C$  are both left inverses to  $q: C \xrightarrow{\sim} A$ . Hence  $\phi(i_0) = \phi(i_1)$ . Therefore if  $f \simeq_L g$  via a map  $H: C \rightarrow B$  (notation as in 5.1), then  $\phi(f) = \phi(g)$  because

$$\phi(f) = \phi(H)\phi(i_0) = \phi(H)\phi(i_1) = \phi(g).$$

It follows that  $\phi$  factors uniquely through  $\mathcal{C} \rightarrow \text{Ho}\mathcal{C}$ .  $\square$

## §6 Functor Categories

If  $\mathbf{K}$  is a small category and  $\mathcal{C}$  is a model category, we would like to make the functor category  $\mathcal{C}^{\mathbf{K}}$  into a model category. There are at least two model structures one could consider: a local structure and a global structure. We shall focus on the global structure.

**Definition 6.1.** Let  $\mathbf{K}$  be a small category and  $\mathcal{C}$  a Thomason model category. We say that a natural transformation  $\eta: F \rightarrow G$  of functors  $\mathbf{K} \rightarrow \mathcal{C}$  is a *(global) cofibration* (written  $F \twoheadrightarrow G$ ) if for each  $K$  in  $\mathbf{K}$  the map  $\eta_K: F(K) \rightarrow G(K)$  is a cofibration in  $\mathcal{C}$ ; *weak equivalence* (written  $F \xrightarrow{\sim} G$ ) if for each  $K$  in  $\mathbf{K}$  the map  $\eta_K: F(K) \rightarrow G(K)$  is a weak equivalence in  $\mathcal{C}$ ; *(global) fibration* (written  $F \twoheadrightarrow G$ ) for for each  $K$  in  $\mathbf{K}$  the map  $\eta_K$  is a fibration in  $\mathcal{C}$ , and if  $\eta$  has the right lifting property with respect to transformations  $A \xrightarrow{\sim} B$  which are simultaneously global cofibrations and weak equivalences. That is, every diagram of the following form admits a factorization  $B \rightarrow F$  making the resulting diagram commute.

$$\begin{array}{ccc} A & \longrightarrow & F \\ \sim \downarrow & & \eta \downarrow \\ B & \longrightarrow & G \end{array}$$

We shall say that  $F \rightarrow G$  is a *pointwise fibration* (resp. has pointwise  $P$ ) if each  $F(i) \rightarrow G(i)$  is a fibration (resp. has property  $P$ ) in  $\mathcal{C}$ .

**Proposition 6.2** [T85, 32]. *If  $\mathcal{C}$  is a basic model category and  $\mathbf{K}$  is a small category, then  $\mathcal{C}^{\mathbf{K}}$  with the global structure above satisfies axioms (TM0)–(TM3). If  $\mathcal{C}$  satisfies the factorization axiom (TM5c) then  $\mathcal{C}^{\mathbf{K}}$  satisfies (TM4) and (TM5c).*

*Proof:* The constant functors  $F(K) = \emptyset$  and  $F(K) = *$  are the initial and terminal objects of  $\mathcal{C}^{\mathbf{K}}$ . Axioms (TM0), (TM2) and (TM3) for  $\mathcal{C}^{\mathbf{K}}$  are immediate from the pointwise definitions. Because limits and colimits in  $\mathcal{C}^{\mathbf{K}}$  are defined pointwise, there is only one nontrivial point to check for (TM1): if  $F \twoheadrightarrow G$  is a fibration in  $\mathcal{C}^{\mathbf{K}}$ , does the pullback  $\eta: F \times_G H \rightarrow H$  have the lifting property with respect to a trivial

cofibration  $A \xrightarrow{\sim} B$ ? To see that it does, consider the diagram

$$\begin{array}{ccccc} A & \longrightarrow & F \times_G H & \longrightarrow & F \\ \sim \downarrow & & \downarrow & & \downarrow \eta \\ B & \longrightarrow & H & \longrightarrow & G. \end{array}$$

The lifting property for the global fibration  $\eta$  yields a fill-in  $B \rightarrow F$  which must factor through a map to the pullback  $B \rightarrow F \times_G H$ . So (TM1) holds for  $\mathcal{C}^{\mathbf{K}}$ .

If  $\mathcal{C}$  satisfies (TM5c), any  $\eta: A \rightarrow B$  in  $\mathcal{C}^{\mathbf{K}}$  determines factorizations which are functorial in  $K$ :

$$A(K) \twoheadrightarrow T(\eta_K) \xrightarrow{\sim} B(K).$$

This gives the functorial factorization  $A \twoheadrightarrow T \xrightarrow{\sim} B$  in  $\mathcal{C}^{\mathbf{K}}$ .

To check axiom (TM4), consider a square in  $\mathcal{C}^I$  of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y. \end{array}$$

If  $A \xrightarrow{\sim} B$  is a weak equivalence, the definition of global fibration yields a fill-in  $B \rightarrow X$ . Suppose then that  $X \xrightarrow{\sim} Y$  is a weak equivalence. By (TM1) the pushout  $X \twoheadrightarrow C$  of  $A \twoheadrightarrow B$  exists, and there is a canonical map  $C \rightarrow Y$ . By (LM5) we can factor it as  $C \twoheadrightarrow D \xrightarrow{\sim} Y$ . By saturation (TM2), the cofibration  $X \twoheadrightarrow D$  is also a weak equivalence since  $D \xrightarrow{\sim} Y$  and  $X \xrightarrow{\sim} Y$  are. But then the definition of global fibration yields a fill-in  $D \rightarrow X$  of the right square in the diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & \xlongequal{\quad} & X \\ \downarrow & & \sim \downarrow & & \downarrow \sim \\ B & \longrightarrow & D & \xrightarrow{\sim} & Y, \end{array}$$

and  $B \rightarrow D \rightarrow X$  provides the desired factorization. This completes the verification of (TM4).  $\square$

**Remark 6.3.** If  $\mathcal{C}$  is a Thomason model category, all that is missing is the factorization part (f) of (CM5). We do have a functorial factorization  $F \xrightarrow{\sim} M \rightarrow G$  from

(RM5) on  $\mathcal{C}$ , and  $M \rightarrow G$  is a pointwise fibration, i.e., each  $M(K) \rightarrow G(K)$  is a fibration in  $\mathcal{C}$ . However, there appears to be no reason for  $M \rightarrow G$  to satisfy the right lifting property required of a global fibration.

Recall that an object  $C$  of  $\mathcal{C}$  is called *fibrant* if  $C \rightarrow *$  is a fibration. We write  $\mathcal{C}_{\text{fib}}$  for the full subcategory of fibrant objects in  $\mathcal{C}$  and  $\mathcal{C}_{\text{fib}}^{\mathbf{K}}$  for the category of all pointwise fibrant functors, i.e., all functors  $\mathbf{K} \rightarrow \mathcal{C}_{\text{fib}}$ . Note that  $(\mathcal{C}^{\mathbf{K}})_{\text{fib}}$  is strictly contained in  $\mathcal{C}_{\text{fib}}^{\mathbf{K}} = (\mathcal{C}_{\text{fib}})^{\mathbf{K}}$ . The following argument allows us to reduce the verification of the fibration half (TM5f) of axiom (TM5) to pointwise fibrations in  $\mathcal{C}_{\text{fib}}^{\mathbf{K}}$ .

**Lemma 6.4** [T85, 166]. *Suppose that there is a functorial factorization in  $\mathcal{C}^{\mathbf{K}}$*

$$F' \xrightarrow{\sim} H' \twoheadrightarrow G'$$

*for every pointwise fibration  $F' \rightarrow G'$  of pointwise fibrant functors. Then axiom (TM5f) holds in  $\mathcal{C}^{\mathbf{K}}$ , and  $\mathcal{C}^{\mathbf{K}}$  is a Thomason model category.*

*Proof:* Suppose given  $F \rightarrow G$  in  $\mathcal{C}^{\mathbf{K}}$ . Using functorial factorization in  $\mathcal{C}$ , we have a factorization  $G \xrightarrow{\sim} G' \twoheadrightarrow *$ , where  $G'$  is pointwise fibrant, and then a factorization of  $F \rightarrow G'$

$$\begin{array}{ccc} F & \longrightarrow & G \\ \sim \downarrow & & \sim \downarrow \\ F' & \longrightarrow & G' \end{array}$$

in which  $F' \rightarrow G'$  is pointwise fibrant. By assumption, it factors as  $F' \xrightarrow{\sim} H' \twoheadrightarrow G'$ . By (TM1) the pullback  $H = F' \times_{G'} G$  exists,  $H \twoheadrightarrow G$  is a fibration and  $H \xrightarrow{\sim} H'$  is a weak equivalence. By saturation (TM2),  $F \xrightarrow{\sim} H$  is a weak equivalence, and we have the required functorial factorization  $F \xrightarrow{\sim} H \twoheadrightarrow G$ .  $\square$

$$\begin{array}{ccccc} F & \longrightarrow & H & \twoheadrightarrow & G \\ \sim \downarrow & & \downarrow \sim & & \sim \downarrow \\ F' & \xrightarrow{\sim} & H' & \twoheadrightarrow & G' \end{array}$$

## §7 Enriched Functor Categories

If  $\mathcal{C}$  is a Thomason model category, we saw in the last section that  $\mathcal{C}^{\mathbf{K}} = \text{Cat}(\mathbf{K}, \mathcal{C})$  satisfies all the axioms for a model category except for (TM5f), that every map  $F \rightarrow G$  in  $\mathcal{C}^{\mathbf{K}}$  has a factorization  $F \xrightarrow{\sim} H \twoheadrightarrow G$ . In order to do this we shall use homotopy limits and homotopy ends in  $\mathcal{C}$ ; this requires that  $\mathcal{C}$  be not only complete but enriched over finite simplicial sets. Recall from 1.12 that this means that  $\mathcal{C}$  has a mapping object functor satisfying axioms (RE1) – (RE5). We will also need  $\mathcal{C}$  to satisfy the following fibration conditions.

**Definition 7.1** [T85, 147]. We say that *towers preserve fibrations* in  $\mathcal{C}$  if given a map between towers of fibrations in  $\mathcal{C}$

$$\begin{array}{ccccccccccc} \cdots & \twoheadrightarrow & C_n & \twoheadrightarrow & C_{n-1} & \twoheadrightarrow & \cdots & \twoheadrightarrow & C_0 & \twoheadrightarrow & * \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ \cdots & \twoheadrightarrow & D_n & \twoheadrightarrow & D_{n-1} & \twoheadrightarrow & \cdots & \twoheadrightarrow & D_0 & \twoheadrightarrow & * \end{array}$$

in which  $C_0 \rightarrow D_0$  is a fibration and each  $C_{n+1} \rightarrow C_n \times_{D_n} D_{n+1}$  is a fibration ( $n \geq 0$ ), then  $\varprojlim C_n \rightarrow \varprojlim D_n$  is a fibration. If in addition each  $C_n \xrightarrow{\sim} D_n$  is a weak equivalence, so is  $\varprojlim C_n \xrightarrow{\sim} \varprojlim D_n$ . Note that this implies that  $\varprojlim C_n$  is fibrant (take  $D_n = *$ ).

We say that *products preserve fibrations (and equifibrations)* in  $\mathcal{C}$  if for every set  $I$  and every family  $\{C_i\}_{i \in I}$  of fibrant objects of  $\mathcal{C}$ : (a)  $\prod C_i$  exists in  $\mathcal{C}$ , and (b) if  $B_i \rightarrow C_i$  are fibrations (resp. equifibrations) then  $\prod B_i \rightarrow \prod C_i$  is a fibration (resp. an equifibration) in  $\mathcal{C}$ . Note that (b) applied to  $C_i \rightarrow *$  implies that  $\prod C_i$  is fibrant.

**Theorem 7.2** [T85, 160]. *Let  $\mathcal{C}$  be a complete Thomason model category with a right enrichment over finite simplicial sets (1.12). Suppose that products and towers in  $\mathcal{C}$  preserve fibrations and equifibrations. If  $\mathbf{K}$  is a small category then  $\mathcal{C}^{\mathbf{K}}$  has the structure of a Thomason model category.*

*Moreover,  $\mathcal{C}^{\mathbf{K}}$  is right enriched, and products and towers in  $\mathcal{C}^{\mathbf{K}}$  preserve fibrations and equifibrations.*

The proof of theorem 7.2 will be given after lemma 7.6 below. It requires a simplicial *hom* construction.

**Definition 7.3.** Given  $B$  and  $C$  in  $\mathcal{C}$ , the simplicial set  $\text{hom}_{\mathcal{C}}(B, C)$  is defined to be  $[\text{hom}_{\mathcal{C}}(B, C)]_p = \text{Hom}_{\mathcal{C}}(B, \text{Map}(\Delta[p], C))$ . Thus  $\text{hom}_{\mathcal{C}}$  is a functor from  $\mathcal{C}^{\text{op}} \times \mathcal{C}$  to simplicial sets, and we may take its homotopy end.

**Lemma 7.4** [T85, 121]. *If  $A \twoheadrightarrow B$  is a cofibration in  $\mathcal{C}$ , then*

$$\text{hom}_{\mathcal{C}}(B, C) \twoheadrightarrow \text{hom}_{\mathcal{C}}(A, C)$$

*is a Kan fibration of simplicial sets.*

*Proof:* For  $0 \leq k \leq n$ , let  $V = V(n, k)$  denote the simplicial subcomplex of  $\Delta[n]$  which is the union of all faces except the  $k$ th one. By axiom (RE2) the right vertical map is a fibration in  $\mathcal{C}$  in the diagram:

$$\begin{array}{ccc} A & \longrightarrow & \text{Map}(\Delta[n], C) \\ \downarrow & \dashrightarrow & \downarrow \\ B & \longrightarrow & \text{Map}(V(n, k), C). \end{array}$$

By (TM4), a fill-in exists. Thus the function

$$\text{Hom}_{\mathcal{C}}(B, \text{Map}(\Delta[n], C)) \rightarrow \text{Hom}_{\mathcal{C}}(A, \text{Map}(\Delta[n], C)) \times_{\text{Hom}_{\mathcal{C}}(B, \text{Map}(V, C))} \text{Hom}_{\mathcal{C}}(A, \text{Map}(V, C))$$

is surjective. By (RE2),  $\text{Map}(-, C)$  sends the pushout  $V(m, k)$  of the cofibrations  $\Delta[n-2] \twoheadrightarrow \Delta[n-1]$  to the pullback in  $\mathcal{C}$  of the fibrations  $\text{Map}(\Delta[n-1], C) \twoheadrightarrow \text{Map}(\Delta[n-2], C)$ . Applying  $\text{Hom}_{\mathcal{C}}(B, -)$ , we see that  $\text{Hom}_{\mathcal{C}}(B, \text{Map}(V, C))$  is the set-theoretic pullback of functions  $\text{hom}(B, C)_{n-1} \rightarrow \text{hom}(B, C)_{n-2}$ . That is,

$$\text{Hom}_{\mathcal{C}}(B, \text{Map}(V, C)) \cong \text{Hom}_{\Delta^{\text{op}} \mathbf{Sets}}(V, \text{hom}_{\mathcal{C}}(B, C)).$$

Since  $\text{Hom}_{\mathcal{C}}(B, \text{Map}(\Delta[n], C)) = \text{hom}_{\mathcal{C}}(B, C)_n$  by definition, this shows that  $\text{hom}_{\mathcal{C}}(B, C) \rightarrow \text{hom}_{\mathcal{C}}(A, C)$  is a Kan fibration.  $\square$

If  $C$  is fibrant in  $\mathcal{C}$ , then each  $\text{Map}(\Delta[p], C)$  is fibrant by (RE1).

**Theorem 7.5** [T85, 158]. *For  $B$  in  $\mathcal{C}$  and  $F: \mathbf{K} \rightarrow \mathcal{C}_{\text{fib}}$ , there is a natural bijection:*

$$\text{Hom}_{\mathcal{C}} \left( B, \text{holim}_{\mathbf{K}} F \right) \cong \left[ \text{holim}_{K \in \mathbf{K}} \text{hom}_{\mathcal{C}}(B, FK) \right]_0.$$

*For  $B$  in  $\mathcal{C}^{\mathbf{K}}$ , there is a natural bijection of sets:*

$$\text{Hom}_{\mathcal{C}^{\mathbf{K}}} (B, \text{holim } F) \cong \left[ \text{ho} \int \text{hom}_{\mathcal{C}}(BK', FK) \right]_0.$$

*Proof* ([T85, 123–130]): By 1.4 and 1.14, part a) is a special case of b). By [Mac, p. 219], the left side is the end

$$\int_{\mathbf{K}} \text{Hom}_{\mathcal{C}} \left( BK, \text{holim}_{K/\mathbf{K}} F \right) = \int_{\mathbf{K}} \text{Hom}_{\mathcal{C}} \left( BK, \int_{\Delta} \text{Map} \left( \Delta[p], \prod_I FK_p \right) \right),$$

where the indexing set  $I$  runs over all diagrams  $K \rightarrow K_0 \rightarrow \cdots \rightarrow K_p$  in  $\mathbf{K}$ . By the universal property of ends, this equals the double end

$$\int_{\mathbf{K}} \int_{\Delta} \text{Hom}_{\mathcal{C}} \left( BK, \text{Map} \left( \Delta[p], \prod_I FK_p \right) \right).$$

By the definition of  $\text{hom}_{\mathcal{C}}$ , this equals

$$\iint_{\mathbf{K} \times \Delta} \text{hom}_{\mathcal{C}} \left( BK, \prod_I FK_p \right)_p = \iint_{\mathbf{K} \times \Delta} \text{Hom}_{\Delta^{\text{op}} \mathbf{Sets}} \left( \Delta[p], \text{hom}_{\mathcal{C}} \left( BK, \prod_I FK_p \right) \right).$$

Because  $\text{Hom}_{\Delta^{\text{op}} \mathbf{Sets}}(X, Y)$  is the set of 0-simplices in  $\text{Map}(X, Y)$ , and limits in  $\Delta^{\text{op}} \mathbf{Sets}$  are formed pointwise, this equals

$$\iint_{\mathbf{K} \times \Delta} [\text{Map}(\Delta[p], Y)]_0 = \left[ \iint_{\mathbf{K} \times \Delta} \text{Map}(\Delta[p], Y) \right]_0,$$

where  $Y = Y(K', K)$  denotes  $\text{hom}_{\mathcal{C}}(BK, \prod_I FK_p)$ . Hence it suffices to show that we have an isomorphism of simplicial sets between

$$\iint_{\mathbf{K} \times \Delta} \text{Map}(\Delta[p], Y) \quad \text{and} \quad \text{ho} \int \text{hom}_{\mathcal{C}}(BK', FK).$$

Because  $\text{Map}(\Delta[p], -)$  preserves limits of simplicial sets and ends commute, the left side equals

$$\int_{\Delta} \int_{\mathbf{K}} \text{Map}(\Delta[p], Y) = \int_{\Delta} \text{Map} \left( \Delta[p], \int_{\mathbf{K}} Y \right).$$

Now axiom (RE5) in  $\mathcal{C}$  implies that (for each  $p$  and  $K$ )

$$Y(K', K) = \text{hom}_{\mathcal{C}} \left( BK', \prod_{K \rightarrow \cdots \rightarrow K_p} FK_p \right) \cong \prod_{K \rightarrow \cdots \rightarrow K_p} \text{hom}_{\mathcal{C}}(BK', FK_p).$$

But now  $\text{hof} \text{hom}_{\mathcal{C}}(BK', FK)$  is defined (1.3) to be the simplicial set

$$\int_{\Delta} \text{Map} \left( \Delta[p], \prod_{K_0 \rightarrow \cdots \rightarrow K_p} \text{hom}_{\mathcal{C}}(BK_0, FK_p) \right).$$

Hence it suffices to show that for all  $p$

$$\prod_{K_0 \rightarrow \cdots \rightarrow K_p} \text{hom}_{\mathcal{C}}(BK_0, FK_p) \cong \int_{\mathbf{K}} \prod_{K \rightarrow K_0 \rightarrow \cdots \rightarrow K_p} \text{hom}_{\mathcal{C}}(BK', FK_p).$$

We can apply lemma 7.6 below to  $I = \mathbf{K}^{\text{op}}$ ,  $i_0 = K$ ,  $\mathcal{S} = \text{simplicial sets}$  and  $G(K') = \prod_{K_0 \rightarrow \cdots \rightarrow K_p} \text{hom}_{\mathcal{C}}(BK', FK_p)$ , where the indexing set runs over diagrams of length  $p$  in  $\mathbf{K}$ . The result is the required isomorphism:

$$\begin{aligned} G(K_0) &\cong \int_{\mathbf{K}^{\text{op}}} (K, K') \longmapsto \prod_{\substack{K \rightarrow K_0 \\ \text{in } \mathbf{K}}} G(K') \\ &\cong \int_{\mathbf{K}} (K', K) \longmapsto \prod_{K \rightarrow K_0} G(K') \\ &= \int_{\mathbf{K}} (K', K) \longmapsto \prod_{K \rightarrow K_0 \rightarrow \cdots \rightarrow K_p} \text{hom}_{\mathcal{C}}(BK', FK_0). \quad \square \end{aligned}$$

**Lemma 7.6** [T85, 129]. *Let  $I$  be a small category,  $i_0$  an object of  $I$ , and  $G: I \rightarrow \mathcal{S}$  a functor to a complete category  $\mathcal{S}$ . If  $F: I^{\text{op}} \times I \rightarrow \mathcal{S}$  denotes the functor  $F(i, j) = \prod_{i_0 \rightarrow i} G(j)$  then there is a natural isomorphism*

$$G(i_0) \cong \int_I F = \int_I (i, j) \longmapsto \prod_{i_0 \rightarrow i} G(j).$$

*Proof:* The diagonal  $G(i_0) \rightarrow \prod_{i_0 \rightarrow i} G(i_0) \rightarrow \prod_{i_0 \rightarrow i} G(i)$  is dinatural, and induces a map  $G(i_0) \rightarrow \int_I F$ . Conversely, the projection onto the coordinate indexed by the identity of  $i_0$  yields a map  $\int_I F \rightarrow \prod_{i_0 \rightarrow i_0} G(i_0) \rightarrow G(i_0)$ . These maps are inverses to each other.  $\square$

*Proof* (of theorem 7.2): As remarked above, we are reduced by 6.2 and 6.4 to producing a functorial factorization of  $\eta: F \rightarrow G$  under the assumption that each  $FK \rightarrow GK \rightarrow *$  is a fibration. From the diagram using lemma 1.13:

$$\begin{array}{ccc} F & \xrightarrow{\sim} & \text{holim } F \\ \eta \downarrow & & \downarrow \text{holim } \eta \\ G & \xrightarrow{\sim} & \text{holim } G. \end{array}$$

Now  $\text{holim } \eta$  is a pointwise fibration by lemma 1.8. Let  $H$  denote the pull-back of  $G$  along  $\text{holim } \eta$ ; axiom (TM1) in  $\mathcal{C}$  implies that  $H \xrightarrow{\sim} \text{holim } F$  is a weak equivalence and  $H \rightarrow G$  is a pointwise fibration. Saturation (TM2) gives  $F \xrightarrow{\sim} H$ . We claim that in the resulting functorial factorization of  $\eta$ ,

$$F \xrightarrow{\sim} H \rightarrow G,$$

the map  $H \rightarrow G$  is a global fibration, i.e., it satisfies the right lifting property.

Let  $A \xrightarrow{\sim} B$  be a cofibration and weak equivalence. We need to show that a fill-in exists for any diagram

$$\begin{array}{ccccc} A & \longrightarrow & H & \longrightarrow & \text{holim } F \\ \sim \downarrow & & \downarrow & & \downarrow \text{holim } \eta \\ B & \longrightarrow & G & \longrightarrow & \text{holim } G. \end{array}$$

As the right square is a pullback, it suffices to find a fill-in  $B \rightarrow \text{holim } F$ , i.e., to show that  $\text{holim } \eta$  is a global fibration. That is, it suffices to show that the following map is a surjection.

$$\text{Hom}_{\mathcal{C}\kappa}(B, \text{holim } F) \rightarrow \text{Hom}_{\mathcal{C}\kappa}(A, \text{holim } F) \Big]_{\text{Hom}(A, \text{holim } G)} \times \text{Hom}_{\mathcal{C}\kappa}(B, \text{holim } G).$$

By proposition 7.5,  $\text{Hom}(B, \text{holim } F)$  is naturally isomorphic to the degree zero part of the homotopy end  $\text{ho} \int_{\mathbf{K}} \text{hom}_{\mathcal{C}}(BK', FK)$ , because each  $FK$  is fibrant.

As the homotopy end formed in  $\Delta^{\text{op}}\text{-Sets}$  preserves limits, such as pullbacks along a fibration, the map we need to show surjective is identified with the degree zero part of the simplicial map

$$\begin{aligned} & \text{ho} \int \text{hom}_{\mathcal{C}}(BK', FK) \\ & \rightarrow \left[ \text{ho} \int \text{hom}(AK', FK) \right] \times_{\text{ho} \int \text{hom}(AK', GK)} \left[ \text{ho} \int \text{hom}(BK', GK) \right] \\ & \cong \text{ho} \int \left[ \text{hom}(AK', FK) \times_{\text{hom}(AK', GK)} \text{hom}(BK', GK) \right]. \end{aligned}$$

Because each  $AK' \twoheadrightarrow BK'$  is a cofibration, each simplicial map

$$\text{hom}_{\mathcal{C}}(BK', GK) \rightarrow \text{hom}_{\mathcal{C}}(AK', GK)$$

is a Kan fibration by lemma 7.4. Since each  $FK \twoheadrightarrow GK \twoheadrightarrow *$  is a fibration, the map

$$\text{hom}_{\mathcal{C}}(BK', FK) \rightarrow \text{hom}(AK', FK) \times_{\text{hom}(AK', GK)} \text{hom}(BK', GK)$$

is a fibration and a weak equivalence of simplicial sets. By lemma 1.8, applying  $\text{ho} \int$  yields a Kan fibration and weak equivalence of simplicial sets. But any such map is surjective on 0-simplices. This completes the proof that (TM5f) holds, and that  $\mathcal{C}^{\mathbf{K}}$  is a Thomason model category.

To see that products preserve fibrations and equifibrations in  $\mathcal{C}^{\mathbf{K}}$ , suppose given a family  $\{F_i \twoheadrightarrow G_i\}_{i \in I}$  of fibrations (resp. equifibrations). Since  $\mathcal{C}$  is complete, the maps  $\prod F_i K \twoheadrightarrow \prod G_i K$  exist and are fibrations (resp. equifibrations) in  $\mathcal{C}$  for all  $K$ . It suffices to show that  $\prod F_i \rightarrow \prod G_i$  has the right lifting property. But for any  $A \xrightarrow{\sim} B$ , each square

$$\begin{array}{ccc} A & \longrightarrow & \prod F_i \\ \downarrow & & \downarrow \\ B & \longrightarrow & \prod G_i \end{array}$$

will admit a lift since lifts  $B \rightarrow F_i$  exist for each  $i \in I$  and  $\prod F_i$  is a product.

To see that towers preserve fibrations in  $\text{Cat}(\mathbf{K}, \mathcal{C})$ , suppose given a map between towers of fibrations in which  $F_0 \twoheadrightarrow G_0$  and all  $F_{n+1} \twoheadrightarrow G_{n+1} \times_{G_n} F_n$  are fibrations.

$$\begin{array}{ccccccccccc} \cdots & \twoheadrightarrow & F_n & \twoheadrightarrow & F_{n-1} & \twoheadrightarrow & \cdots & \twoheadrightarrow & F_0 & \twoheadrightarrow & * \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ \cdots & \twoheadrightarrow & G_n & \twoheadrightarrow & G_{n-1} & \twoheadrightarrow & \cdots & \twoheadrightarrow & G_0 & \twoheadrightarrow & * \end{array}$$

Since towers preserve fibrations in  $\mathcal{C}$ , each  $\varprojlim F_n K$  and  $\varprojlim G_n K$  is fibrant and  $\varprojlim F_n K \rightarrow \varprojlim G_n K$  is a fibration (and is a weak equivalence when each  $F_n K \rightarrow G_n K$  is). To show it is a fibration it suffices to check the lifting property. Let  $A \xrightarrow{\sim} B$  be an equi-cofibration and suppose given a square

$$\begin{array}{ccccc} A & \longrightarrow & \varprojlim F_n & \longrightarrow & F_n \\ \sim \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & \varprojlim G_n & \longrightarrow & G_n \end{array}$$

We need to find a lift  $B \rightarrow \varprojlim F_n$ .

We proceed by induction to produce a coherent tower of lifts  $B \rightarrow F_n$ ; passing to the inverse limit will yield the desired lift  $B \rightarrow \varprojlim F_n$ . For  $n = 0$ , a lift  $B \rightarrow F_0$  exists because  $F_0 \rightarrow G_0$  is a fibration. Inductively, there is a lift  $B \rightarrow F_n$ . It factors uniquely through the pullback  $G_{n+1} \times_{G_n} F_n$ , and we have a diagram

$$\begin{array}{ccccc} A & \longrightarrow & F_{n+1} & \longrightarrow & F_n \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & G_{n+1} \times_{G_n} F_n & \longrightarrow & G_n \end{array}$$

Since the middle vertical is a fibration, there is a lift  $B \rightarrow F_{n+1}$  compatible with the preceding lift  $B \rightarrow F_n$ . This completes the inductive step, showing that the map  $\varprojlim F_n \rightarrow \varprojlim G_n$  is a fibration, and that towers preserve fibrations in  $\text{Cat}(\mathbf{K}, \mathcal{C})$ .

Finally, we need to show that  $\text{Cat}(\mathbf{K}, \mathcal{C})$  is right enriched over  $\mathcal{S}_f$ . Because  $\text{Cat}(\mathbf{K}, \mathcal{C})_{\text{fib}} \subseteq \text{Cat}(\mathbf{K}, \mathcal{C}_{\text{fib}})$ , there is a functor

$$\text{Map}(, ): \mathcal{S}_f^{\text{op}} \times \text{Cat}(\mathbf{K}, \mathcal{C})_{\text{fib}} \longrightarrow \text{Cat}(\mathbf{K}, \mathcal{C}_{\text{fib}})$$

We need to show that this lands in  $\text{Cat}(\mathbf{K}, \mathcal{C})_{\text{fib}}$  and makes  $\text{Cat}(\mathbf{K}, \mathcal{C})$  right enriched over  $\mathcal{S}_f$ . This is straightforward, and left to the reader. See [T86, 61].

This finishes the proof of Theorem 7.2  $\square$

## Acknowledgements

This work was done at the Institute for Advanced Study in Princeton, but the main sources were located at the Université de Paris 7. The author is deeply grateful to Liliane Barengi, Bruno Kahn and Max Karoubi for their help in giving me access to Thomason's private notebooks [T78]–[T86]. I am also grateful to Luca Barbieri-Viale, Marco Grandis and Ieke Moerdijk for providing me with their notes of seminar talks in which Thomason presented his ideas in Genova and Utrecht. Finally, I am grateful to Rick Jardine and Bill Dwyer for their comments on an earlier draft.

## References

- [Baues] H. Baues, *Algebraic Homotopy*, Cambridge Univ. Press, 1989.
- [BF] A. Bousfield and E. Friedlander, *Homotopy theory of  $\Gamma$ -spaces, spectra and bisimplicial sets*, Lecture Notes in Math. 658, 1978, 80–130.
- [BK] A. Bousfield and D. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. 304, Springer-Verlag, 1972.
- [Br] K. S. Brown *Abstract Homotopy Theory and Generalized Sheaf Cohomology*, Trans. AMS 186 (1973), 419–458.
- [BW] H. Baues and G. Wirsching, *The cohomology of small categories*, J. Pure Appl. Alg. 38 (1985), 187–211.
- [DK] W. Dwyer and D. Kan, *Hochschild-Mitchell cohomology of simplicial categories and the cohomology of simplicial diagrams of simplicial sets*, Nederl. Akad. Wetensch. Indag. Math. 50 (1988), 111–120.
- [GJ] P. Goerss and J. F. Jardine, *Simplicial Homotopy Theory*, Birkhäuser, 1999.
- [JP] M. Jibladze and T. Pirashvili, *Cohomology of algebraic theories*, J. Alg. 137 (1991), 253–296.
- [Mac] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, 1971.
- [Mau] L. Mauri, *Thomason models for homotopy*, preprint, 2000.
- [PW] T. Pirashvili and F. Waldhausen, *MacLane homology and topological Hochschild homology*, J. Pure Appl. Alg. 82 (1992), 81–98.
- [QH] D. Quillen, *Homotopical Algebra*, Lecture Notes in Math. 43, Springer-Verlag, 1967.
- [QR] D. Quillen, *Rational homotopy theory*, Annals Math. 90 (1969), 205–295.
- [T78] R. Thomason, private notebook, Paris, Jan.-Feb., 1995, 180 pp.
- [T79] R. Thomason, private notebook, Paris, March, 1995, 180 pp.
- [T81] R. Thomason, private notebook, Paris, June, 1995, 180 pp.
- [T82] R. Thomason, private notebook, Paris, July 1995, 180 pp.
- [T83] R. Thomason, private notebook, Paris, Aug., 1995, 180 pp.
- [T85] R. Thomason, private notebook, Paris, Sept. 1995, 180 pp.
- [T86] R. Thomason, private notebook, Paris, Oct., 1995, 74 pp.

- [TT] R. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift III, Progress in Math. 88, Birkhäuser, 1990, 247–435.
- [Wa] F. Waldhausen, *Algebraic K-theory of spaces*, Lecture Notes in Math. 1126, Springer-Verlag, 1985, 318–419.
- [WT] C. Weibel, *The Mathematical Enterprises of Robert Thomason*, Bull. AMS 34 (1997), 1–13.