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THE HOMOTOPY EXACT SEQUENCE IN ALGEBRAIC K-THEORY

Charles A. Weibel

School of Mathematics
The Institute for Advanced Study
Princeton, New Jersey 08540

This paper investigates the homotopization of the exact sequence

$$(*)\quad K_1(A) \longrightarrow K_1(B) \longrightarrow K_1(C) \longrightarrow K_0(A) \longrightarrow K_0(B) \longrightarrow K_0(C)$$

of Algebraic K-Theory associated to a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of rings, and obtains conditions for its exactness. When the quotient ring C is regular, exactness can fail only at $[K_1]C$, and an example of this failure is provided. The results improve upon those given by Sharma and Strooker in [5].

First, we introduce some notation and terminology, following [3], [6], [7]. We say that a category $\underline{\underline{R}}$ of associative rings is "admissible" if, whenever a ring R is in $\underline{\underline{R}}$, so are the ring $R[t]$ and the morphisms "$t=0$", "$t=1$" : $R[t] \longrightarrow R$, "$t=t_1t_2$" : $R[t] \longrightarrow R[t_1, t_2]$ and $\iota : R \longrightarrow R[t]$. Throughout, A, B, C, R denote rings in such an $\underline{\underline{R}}$.

Let $F : \underline{\underline{R}} \longrightarrow \underline{\underline{A}}$ be a functor, where the category $\underline{\underline{A}}$ has (relevant) coequalizers. The homotopization $[F]$ of F is the functorial coequalizer of the natural transformations "$t=0$", "$t=1$"; $F(R[t]) \longrightarrow F(R)$, and we call F a homotopy functor if $F=[F]$. An alternative characterization [7] is that F is a homotopy functor exactly when $F(R) = F(R[t])$ for all R.

The following convenient notation will be used. If x is an element of $F(R[t])$, we denote its image in $F(R)$ under "$t=i$" $(i=0, 1)$ by $x(i)$. If x is in $F(R)$ we set $\iota(x) = x[t]$ in $F(R[t])$.

Here are some examples from Algebraic K-Theory. When $\underline{\underline{A}}$ is the category of sets, we recover Gersten's definition [2]: $[F] = F/\sim$, where $\sim$ is the equivalence relation on $F(R)$ generated by the requirement that $x(0) \sim x(1)$ for each $x \in F(R[t])$. Thus if $P(R)$ is the set of (isomorphism classes of) finitely generated projective R-modules, the objects of $[P]R$ are equivalence classes of projective modules. When $\underline{\underline{A}} = \underline{\underline{\text{Groups}}}$, there is a nicer description. Recall that $NF(R)$ is the kernel of "$t=0$" : $F(R[t]) \longrightarrow F(R)$.

Lemma. The homotopization of $F : \underline{\underline{R}} \longrightarrow \underline{\underline{\text{Groups}}}$ is the cokernel of the map "$t=1$" : $NF \longrightarrow F$.

Proof: Given $u \in F(R[t])$, $x = (u(0)[t])^{-1}u$ has $x(0) = 1$, $x(1) = u(0)^{-1}u(1)$, so we have only to show that the image of $NF(R)$ is normal in $F(R)$. For $x \in NF(R)$, $y \in F(R)$, observe that $z = y[t]^{-1} x\, y[t]$ is in $NF(R)$ and $z(1) = y^{-1}x(1)y$.

This lemma applies notably to the functors K_0 and K_1. The functor $[K_0]$ may also be computed by taking stable classes of $[P]$. If $U(R)$ is the units of R, then $[U]R$ is obtained by modding out by unipotents. In particular, $[Gl]R = [K_1]R = K_1^{kv}(R)$ is just $Gl(R)/Un(R)$, where $Un(R)$ is the subgroup generated by the unipotent matrices. All the Karoubi-Villamayor functors K_n^{kv} are homotopy functors for $n \geq 1$ [4]. There is a natural map from $[K_n^Q]$ to K_n^{kv} which is the boundary map of a spectral sequence; I do not know if it is an isomorphism or not.

We now return to the sequence (*), denoting its termwise homotopization by [*]. This sequence is well-defined, and the composition of any two maps is zero. We know [3] that the sequence is exact at $[K_1]B$ whenever $B \longrightarrow C$ is a Gl-fibration, or more generally when $NK_1B \longrightarrow NK_1C$ is onto. In fact, in the case of a fibration we get a long exact sequence using the K_n^{kv} which ends in $[K_1]C$. For exactness at the $[K_0]$ terms we have the following result.

Theorem 1. Let $R_i \in \underline{\underline{R}}$, $F_i : \underline{\underline{R}} \longrightarrow \underline{\underline{\text{Groups}}}$ be such that the diagram

$$\begin{array}{ccccccc} F_1(R_1[t]) & \longrightarrow & F_2(R_2[t]) & \longrightarrow & F_3(R_3[t]) & \longrightarrow & F_4(R_4[t]) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F_1(R_1) & \longrightarrow & F_2(R_2) & \longrightarrow & F_3(R_3) & \longrightarrow & F_4(R_4) \end{array}$$

commutes with exact rows, where the vertical arrows are evaluation at $t = i$ $(i = 0, 1)$. Then when $F_4(R_4) = F_4(R_4[t])$ the following sequence is also exact:

$$[F_1]R_1 \longrightarrow [F_2]R_2 \longrightarrow [F_3]R_3 \longrightarrow [F_4]R_4 .$$

Corollary. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of rings. Then

(a) If $K_0C = K_0C[t]$ then [*] is exact at $[K_0]A$ and $[K_0]B$.

(b) If $K_0B = K_0B[t]$ then [*] is exact at $[K_1]C$ and $[K_0]A$.

Proof: The hypothesis of the theorem and the lemma above provide exact columns for the diagram:

$$\begin{array}{ccccccc} NF_1(R_1) & \longrightarrow & NF_2(R_2) & \longrightarrow & NF_3(R_3) & \longrightarrow & NF_4(R_4) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F_1(R_1) & \longrightarrow & F_2(R_2) & \longrightarrow & F_3(R_3) & \longrightarrow & F_4(R_4) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [F_1]R_1 & \longrightarrow & [F_2]R_2 & \longrightarrow & [F_3]R_3 & \longrightarrow & [F_4]R_4 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & & 1 & & 1 & & 1 \end{array}$$

Since $NF_4(R_4) = 0$ and the top two rows are exact, we deduce exactness of the bottom row by a simple diagram chase.

Remark. The requirement that $K_0C = K_0C[t]$ is stronger than the requirement $K_0C = [K_0]C$. Sharma and Strooker [5] give an example with $C = 4\,\mathbb{Z}$, so that $K_0C = [K_0]C = 0$, where exactness fails at $[K_0]A$.

Note that we can establish exactness at $[F_3]R_3$ under the weaker condition that $NF_3(R_3) \longrightarrow NF_4(R_4)$ be onto. This result generalizes the above remark about exactness at $[K_1]B$.

Many important applications of (*) in Algebraic K-Theory have C a regular ring. This is also true of the Mayer-Vietoris counterpart. In these cases we have exactness of the homotopy sequence [*] as well, except possibly at $[K_1]C$. Other applications are such that C is an integrally closed domain. In that case the 'homotopy Picard sequence'

$$1 \longrightarrow [U]A \longrightarrow [U]B \longrightarrow [U]C \longrightarrow [\mathrm{Pic}]A \longrightarrow [\mathrm{Pic}]B \longrightarrow [\mathrm{Pic}]C$$

is exact by theorem 1, since $U(C) = U(C[t])$ and $\mathrm{Pic}(C) = \mathrm{Pic}(C[t])$ both hold (again except possibly at the term $[U]C$).

Here is an example that shows failure of exactness at $[K_1]C$, even when C is the complex numbers. Let k be an integral domain of characteristic $\neq 2$. Set $B = k[x, y]$ with $y^2 = (1+x^2)^3$, $A = (1+x^2, y)B$, and $C = B/A = k[i]$, where

$i^2 + 1 = 0$. When $\mathrm{char}(k) = 0$, the sequence [*] is not exact at $[K_1]C$.

To see this, let $R = k[x, z]$ with $z^2 = 1 + x^2$, and identify B as a subring of R by equating $y = z^3$. It is easy to see that A is the conductor from R to B and that $R/A = C[\varepsilon]$, where $\varepsilon^2 = 0$. Hence we have the diagram of cartesian squares

$$(**)\qquad \begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & R \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C & \longrightarrow & R/A \end{array} .$$

We first give the criterion to be used.

<u>Theorem</u> 2. Given a diagram (**) of cartesian squares and commutative rings with C reduced. Suppose u is a unit of C not coming from $U(B)$, and that r is a unit of R for which $\overline{ru}$ is unipotent in R/A. Then the following two sequences are <u>not</u> exact:

$$[U]B \longrightarrow [U]C \xrightarrow{\partial} [\mathrm{Pic}]A \quad \text{and} \quad [K_1]B \longrightarrow [K_1]C \longrightarrow [K_0]A .$$

<u>Proof</u>: It is enough to check that $\partial[u] = 0$ in $[\mathrm{Pic}]A$. Consideration of (**) shows that ∂ factors through $[U]R/A$. But in that group we have $[\overline{r}][\overline{u}] = 1$, so in $[\mathrm{Pic}]A$ we have the identity $\partial[u] = -\partial[\overline{r}] = 0$.

We now verify the hypotheses of theorem 2 for the specific choice of A, B, C, R above. We pick $r = z - x$ and $u = i$, noting that $\overline{ru} = 1 + i\varepsilon$ in R/A. As $\mathrm{char} \neq 2$ and C is reduced, all we have to show is that i is not in the image of $U(B)$.

<u>Theorem</u> 3. Every unit of R is of the form $u(x+z)^n$ for some unit u of k and some integer n. The inverse of $(x+z)$ is $(z - x)$.

<u>Proof</u>: R is a free $k[x]$-module on basis $\{1, z\}$ and has an involution, $(f_0 + f_1 z)^* = (f_0 - f_1 z)$. Let $f = f_0 + f_1 z$ be a unit of R; then ff^* is a unit of $k[x]$, hence of k. As $ff^* = f_0^2 + (1 + x^2)f_1^2$, either $f_1 = 0$ (done) or degree $(f_0) = 1 + \mathrm{degree}(f_1)$. Inspection of ff^* shows that (replacing f by f^* if needed) the two leading coefficients of f_0 and f_1 agree. Then $(z - x)f$ has lower degree in x, and the result follows by induction.

<u>Corollary</u>. If $\mathrm{char}(k) = 0$, the image of $U(B)$ in $U(C)$ is just $U(k)$. Hence the kernel of $\partial : U(C) \longrightarrow \mathrm{Pic}(A)$ is $U(k)$. In particular, $\partial(i) \neq 0$.

<u>Proof</u>: As B is the pullback of R and C in (**), it suffices to check that the images of $U(C)$ and $U(R)$ in $U(R/A)$ intersect in $U(k)$. The image of $u(x+z)^n$ in R/A is $ui^n(1 - ni\varepsilon)$. As $\mathrm{char}(k) = 0$, this lies in C only when $n = 0$.

Note that this argument fails in characteristic p. In fact, if k is a finite field, theorem 4 below shows that the sequence [*] is indeed exact.

We can measure the obstruction to exactness, and delimit a number of instances in which the homotopy sequence [*] is truly exact. As before, we give the general argument first.

<u>Proposition</u>. Let $R_i \in \underline{\underline{R}}$, $F_i : \underline{\underline{R}} \longrightarrow \underline{\underline{\text{Groups}}}$ be such that $F_1(R_1) \longrightarrow F_2(R_2) \longrightarrow F_3(R_3)$ is exact. Then $[F_1]R_1 \longrightarrow [F_2]R_2 \longrightarrow [F_3]R_3$ is exact if and only if the images of $F_2(R_2)$ and $NF_3(R_3)$ in $F_3(R_3)$ intersect in the image of $NF_2(R_2)$. We also assume that the diagram below commutes.

$$\begin{array}{ccccc} & & NF_2(R_2) & \longrightarrow & NF_3(R_3) \\ & & \downarrow & & \downarrow \\ F_1(R_1) & \longrightarrow & F_2(R_2) & \longrightarrow & F_3(R_3) \end{array}$$

<u>Proof</u>: Chase the above diagram and use the definition of homotopization.

<u>Corollary</u>. If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence of rings, then [*] is exact at $[K_1]C$ just in case the images of K_1C and NK_0A intersect in the image of NK_1C in K_0A. This is the case if any of the following hold:

(a) $[K_1]B \longrightarrow [K_1]C$ is onto

(b) $K_1(B) \longrightarrow K_1(C)$ is onto

(c) $K_0(A) = [K_0]A$

(d) $K_0(B) = K_0(B[t])$

(e) $NK_0(B) \longrightarrow NK_0(C)$ is into.

<u>Proof</u>: We verify the criterion in each case. (b) and (c) state that the intersection is 0. (a) says that $NK_1C \longrightarrow K_1C$ is onto, modulo the kernel of $K_1C \longrightarrow K_0A$. From the exact sequence

$$NK_1C \longrightarrow NK_0A \longrightarrow NK_0B \longrightarrow NK_0C$$

and conditions (d) or (e) we deduce that $NK_1C \longrightarrow NK_0A$ is onto.

Cases (b) and (d) are implicitly stated in [<u>4</u>], but proven using different techniques.

Finally, we determine another instance in which [*] is exact, using the obstruction in a conductor situation with finite characteristic.

<u>Theorem</u> 4. In the conductor situation (**) with all rings commutative, suppose that R is seminormal, C is reduced of prime characteristic $p \neq 0$, and that U(C) is a torsion group. Then the following sequence is exact:

$$1 \longrightarrow [U]A \longrightarrow [U]B \longrightarrow [U]C \longrightarrow [Pic]A .$$

If in addition $SK_1B \longrightarrow SK_1C$ is onto and $K_0C = K_0C[t]$ then [*] is exact. This is true when C is a finite field.

Proof: Since $U(C) = U(C[t])$, exactness at $[U]A$ and $[U]B$ follows from theorem 1. If we verify exactness at $[U]C$ we will be done, as the last statement follows from it and the observation that $K_1 C = U(C) \oplus SK_1(C)$.

By the proposition, it is enough to show that, if $x \in U(C)$ and $y \in NPic(A)$ have the same image in $Pic(A)$, then x comes from $U(B)$. Consideration of (**) shows that the following diagram commutes, has exact rows, and that the lower left square is cartesian:

$$\begin{array}{ccccc} & & NU(R/A) & \longrightarrow & NPic(A) & \longrightarrow & NPic(R) \\ & & \downarrow & & \downarrow & & \\ U(R) & \longrightarrow & U(R/A) & \longrightarrow & Pic(A) & & \\ \uparrow & & \uparrow & & \| & & \\ U(B) & \longrightarrow & U(C) & \longrightarrow & Pic(A) & & \end{array}$$

Since R is seminormal [8], we have $NPic(R) = 0$. The diagram shows that for some unit r of R the element $\overline{xr} = z$ of R/A is unipotent.

Let C_0 be the smallest ring containing 1 and x inside C. The hypothesis on C shows that C_0 is a direct sum of fields of characteristic p. Hence we can choose some large $q = p^n$ so that $x^q = x$ and $z^q = 1$. But then the unit (x, r^{-q}) of B maps onto x, as desired.

Corollary. Let A be a maximal ideal in commutative ring B, with finite residue field C. Suppose that A is the conductor from B to its normalization. Then the long exact sequence for Karoubi-Villamayor K-Theory, ending in [*], is exact.

The technique of theorem 4 generalizes to certain other squarefree finite characteristics. Let n be squarefree, and let z be unipotent in some ring of characteristic n; then $z^{n^k} = 1$ for sufficiently large k. This may be seen by decomposing the ring into components of prime characteristic, and checking it there. Theorem 4 will then follow for those n such that, given a unit x in C, we can find a large $q = n^k$ for which $x^q = x$. If $\phi(n)$ denotes the Euler function, it is a straightforward matter to check that this condition is equivalent to n being relatively prime to $\phi(n)$. Hence

Corollary. In the situation of theorem 4, the conclusions remain valid if the characteristic of C is relatively prime to its Euler number.

This condition is satisfied by all prime numbers as well as $n = 15, 33$, etc. The smallest squarefree numbers for which it fails are (besides all even numbers) $n = 21, 39, 55$.