K-THEORY OF LINE BUNDLES AND SMOOTH VARIETIES

C. HAESEMEYER AND C. WEIBEL

Abstract. We give a K-theoretic criterion for a quasi-projective variety to be smooth. If $\mathbb{L}$ is a line bundle corresponding to an ample invertible sheaf on X, it suffices that $K_q(X) \cong K_q(\mathbb{L})$ for all $q \le \dim(X)+1$.

Let X be a quasi-projective variety over a field k of characteristic 0. The main result of this paper gives a K-theoretic criterion for X to be smooth. For affine X, such a criterion was given in [2]: it suffices that X be K_{d+1}-regular for $d = \dim(X)$, i.e., that $K_{d+1}(X) \cong K_{d+1}(X \times \mathbb{A}^m)$ for all m. If X is affine, we also showed that K_{d+1}-regularity of X is equivalent to the condition that $K_i(X) \cong K_i(X \times \mathbb{A}^1)$ for all $i \le d+1$.

We also showed that K_{d+1}-regularity is insufficient for quasi-projective X; see [2, Thm. 0.2]. In this paper we prove:

Theorem 0.1. *Let X be quasi-projective over a field k of characteristic 0, of dimension d, and let $\mathbb{L} = \mathrm{Spec}(\mathrm{Sym}\,\mathcal{L})$ be the line bundle corresponding to an ample invertible sheaf $\mathcal{L}$ on X.*

If $K_i(\mathbb{L}) \cong K_i(X)$ for all $i \le n$ then X is regular in codimension $< n$.

If $K_i(\mathbb{L}) \cong K_i(X)$ for all $i \le d+1$, then X is regular.

For example, if $K_i(\mathbb{L}) \cong K_i(X)$ for all $i \le d$, then X has at most isolated singularities.

In the affine case, of course, every line bundle is ample, and when $\mathbb{L} = \mathbb{A}^1_R$ we recover our previous result, proven in [2, 0.1]:

Corollary 0.2. *If R is essentially of finite type over a field of characteristic 0, and $K_i(R) \cong K_i(R[t])$ for all $i \le n$ then R is regular in codimension $< n$.*

The affine assumption in this corollary is critical. In [2], we gave an example of a curve Y which is K_n-regular for all n, but which is not regular; no affine open U is even reduced. However, $K_0(X) \ne K_0(\mathbb{L})$ for the line bundle associated to an ample $\mathcal{L}$; see Example 4.1 below. In Theorem 4.3 we give a surface X which is K_n-regular for all n, but which is not regular and such that $K_0(X) \ne K_0(\mathbb{L})$ for the line bundle associated to an ample $\mathcal{L}$; it is a cusp bundle over an elliptic curve.

As in our previous papers [1, 2, 3], our technique is to compare K-theory to cyclic homology using *cdh*-descent and cyclic homology. The parts of *cdh* descent we need are developed in Section 1, and applied to give a formula for the cyclic

---

Date: July 6, 2017.

Haesemeyer was supported by ARC DP-170102328.

Weibel was supported by NSF grant DMS-146502.

homology of line bundles in Section 2. The main theorem is proven in Section 3, and the two examples are given in Section 4.

Notation. If E is a presheaf of spectra, we write $\pi_n E$ for the presheaf of abelian groups $X \mapsto \pi_n E(X)$; we say that a spectrum E is *n-connected* if $\pi_q E = 0$ for all $q \leq n$. For example, $K_n(X)$ is the homotopy group $\pi_n K(X)$ of the spectrum $K(X)$.

Similarly, if E is a cochain complex of presheaves, we may regard it as a presheaf of spectra via Dold-Kan [16, ch. 10]. Thus $\pi_i E(X)$ is another notation for $H^{-i}E(X)$. We will use the cochain shift convention $E[i]^n = E^{i+n}$, so that the spectrum corresponding to $E[1]$ is the suspension of the spectrum of E, and $\pi_n E[1] = \pi_{n-1}E$. Thus if E is n-connected then $E[1]$ is $(n+1)$-connected.

1. ZARISKI AND *cdh* DESCENT

In this paper, we fix a field of characteristic 0, and work with the category Sch of schemes X of finite type over the field. We will be interested in the Zariski and *cdh* topologies on Sch.

If τ is a Grothendieck topology on Sch, there is an "injective τ-local" model structure on the category $\mathrm{Psh}(\mathbf{Ch}(\mathbf{Ab}))$ of presheaves of cochain complexes of abelian groups on Sch. In this model structure, a map $A \to B$ is a cofibration if $A(X) \to B(X)$ is an injection for all X, and it is a weak equivalence if $H^n A \to H^n B$ induces an isomorphism on the associated τ-sheaves. The fibrant replacement of A in this model structure is written as $A \to \mathbb{H}_\tau(-, A)$. We say that A *satisfies τ-descent* if the canonical map $A(X) \to \mathbb{H}_\tau(X, A)$ is a quasi-isomorphism for all X. There is a parallel notion of τ-descent for presheaves of spectra.

If A is a sheaf then $A \to \mathbb{H}_\tau(-, A)$ is an injective resolution; it follows that $\mathbb{H}^n_\tau(X, A) = H^n\mathbb{H}_\tau(X, A)$ for all n. For a complex A, the hypercohomology group $\mathbb{H}^n_\tau(X, A)$ equals $H^n\mathbb{H}_\tau(X, A)$. See [1, 3.3] for these facts.

The inclusion of complexes of sheaves (for a topology τ) into complexes of presheaves induces an injective τ-local model structure on complexes of sheaves, and the inclusion is a Quillen equivalence; see [8, 5.9].

For the Zariski, Nisnevich and *cdh* topologies, there is a parallel "injective τ-local" model structure on the category $\mathrm{Psh}(\mathbf{Ch}(\mathcal{O}_\tau))$ of presheaves of complexes of $\mathcal{O}_\tau$-modules, and the functor forgetting the module structure is a Quillen adjunction. In particular, if A is a presheaf of complexes of $\mathcal{O}_\tau$-modules, the forgetful functor sends its fibrant $\mathcal{O}_\tau$-module replacement to a presheaf that is objectwise weak equivalent to $\mathbb{H}_\tau(-, A)$.

Example 1.1. The Hochschild complex HH/k satisfies Zariski descent by [17, 0.4]. By definition, the cochain complex $HH(X/k)$ is concentrated in negative cohomological degrees and has the (quasi-coherent) Zariski sheaf $\mathcal{O}_X^{\otimes_k n+1}$ in cohomological degree $-n$. When k is understood, we drop the '$/k$' from the notation. We sometimes regard HH as a sheaf of spectra, using Dold-Kan, and use the notation $HH_q(X) = \pi_q HH(X)$ for $\mathbb{H}^{-q}_{\mathrm{zar}}(X, HH)$. Recall from [17, 4.6] that if X is noetherian then $HH_q(X) = 0$ for $q < -\dim(X)$.

If E is a complex of Zariski sheaves of $\mathcal{O}$-modules on Sch/X, we may assume that $\mathbb{H}_{\mathrm{zar}}(-, E)$ is a complex of Zariski sheaves of $\mathcal{O}$-modules, and similarly for

$\mathbb{H}_{\mathrm{cdh}}(-,E)$. (See [8, 8.6].) Thus it makes sense to form the sheaf tensor product $\mathbb{H}_\tau(-,E)\otimes_{\mathrm{zar}}\mathcal{L}$ with a Zariski sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules.

If E is a Zariski sheaf of $\mathcal{O}_X$-modules on X, then there is a Zariski sheaf E' of $\mathcal{O}$-modules on Sch/X, unique up to unique isomorphism, such that for every $f : Y \to X$ in Sch/X the restriction of E' to the small Zariski site of Y is naturally isomorphic to the sheaf f^*E. In this paper we will always work with this sheaf on the big site; so for example "an invertible sheaf $\mathcal{L}$ on X" will indicate the sheaf on the big site associated in this way to an invertible sheaf on X.

Lemma 1.2. *If $\mathcal{L}$ is an invertible sheaf on X, $\otimes_{\mathrm{zar}}\mathcal{L}$ is an auto-equivalence of the category $\mathrm{Sh}(\mathbf{Ch}(\mathcal{O}_{\mathrm{zar}}))/X$ of sheaves of complexes of $\mathcal{O}_{\mathrm{zar}}$-modules on Sch/X which preserves cofibrations, fibrations and weak equivalences.*

Proof. The functor $\otimes_{\mathrm{zar}}\mathcal{L}^{-1}$ is a quasi-inverse to $\otimes_{\mathrm{zar}}\mathcal{L}$. Since $\mathcal{L}$ is flat, $\otimes_{\mathrm{zar}}\mathcal{L}$ preserves injections. Since $\mathcal{L}$ is locally trivial on X (and hence on any X-scheme), and $A\otimes_{\mathrm{zar}}\mathcal{O}_X \cong A$, $\otimes_{\mathrm{zar}}\mathcal{L}$ preserves weak equivalences. Now suppose that $C \to D$ is a Zariski-local fibration; we want to see that $C\otimes_{\mathrm{zar}}\mathcal{L} \to D\otimes_{\mathrm{zar}}\mathcal{L}$ is a Zariski-local fibration. By invertibility, it suffices to observe that if $A \to B$ is a trivial cofibration of $\mathcal{O}_{\mathrm{zar}}$ modules, then so is $A\otimes_{\mathrm{zar}}\mathcal{L}^{-1} \to B\otimes_{\mathrm{zar}}\mathcal{L}^{-1}$, a fact we have just verified. ☐

Corollary 1.3. *If $\mathcal{L}$ is an invertible sheaf on X, and A is a complex of Zariski sheaves of $\mathcal{O}$-modules, then there is a quasi-isomorphism on Sch/X:*

$$\mathbb{H}_{\mathrm{zar}}(-,A)\otimes_{\mathrm{zar}}\mathcal{L} \xrightarrow{\simeq} \mathbb{H}_{\mathrm{zar}}(-,A\otimes_{\mathrm{zar}}\mathcal{L}).$$

Proof. This follows immediately from Lemma 1.2. ☐

We write (a^*,a_*) for the usual adjunction between Zariski and *cdh* sheaves associated to the change-of-topology morphism $a : (Sch/k)_{\mathrm{cdh}} \to (Sch/k)_{\mathrm{zar}}$. Thus if $\mathcal{F}$ is a sheaf of $\mathcal{O}_{\mathrm{cdh}}$-modules on $(\mathrm{Sch}/X)_{\mathrm{cdh}}$, $a_*\mathcal{F}$ is the underlying sheaf of $\mathcal{O}_{\mathrm{zar}}$-modules, and for any Zariski sheaf E of $\mathcal{O}_X$-modules on X, we may form the Zariski sheaf $a_*\mathcal{F}\otimes_{\mathcal{O}_X}E$ on Sch/X.

Recall from [EGA, $0_I(5.4.1)$] that a Zariski sheaf E of $\mathcal{O}_X$-modules is *locally free* if each point of X has an open neighborhood U such that $E|_U$ is a free $\mathcal{O}_U$-module, possibly of infinite rank.

Lemma 1.4. *If E is a locally free sheaf on X, and $\mathcal{F}$ is a cdh sheaf of $\mathcal{O}_{\mathrm{cdh}}$-modules, then $a_*\mathcal{F}\otimes_{\mathcal{O}_X}E$ is a cdh sheaf on (Sch/X).*

Proof. Since the question is local on X, we may replace X by an open subscheme to assume that E is free. Because the *cdh*-topology on Sch/X is quasi-compact, and therefore arbitrary direct sums of sheaves are sheaves, we are reduced to the trivial case $E = \mathcal{O}_X$, when $a_*\mathcal{F}\otimes_{\mathcal{O}_X}E = a_*\mathcal{F}$. ☐

Definition 1.5. If $\mathcal{F}$ is a *cdh* sheaf of $\mathcal{O}_{\mathrm{cdh}}$-modules, we will write $\mathcal{F}\otimes_{\mathrm{zar}}E$ for the *cdh* sheaf $a_*\mathcal{F}\otimes_{\mathcal{O}_X}E$.

Note that $\mathbb{H}^*_{\mathrm{zar}}(X,\mathcal{F}\otimes_{\mathrm{zar}}E) \neq \mathbb{H}^*_{\mathrm{zar}}(X,\mathcal{F})\otimes E(X)$. For example, $E(X) = 0$ does not imply that $(\mathcal{F}\otimes_{\mathrm{zar}}E)(X) = 0$.

Lemma 1.6. *If E is locally free on X then $\otimes_{\mathrm{zar}}E$ preserves weak equivalences and cofibrations for complexes of cdh sheaves of $\mathcal{O}_{\mathrm{cdh}}$-modules on Sch/X.*

Proof. As in the proof of Lemma 1.4, we may replace X by an open subscheme and assume that E is a sheaf of free modules. Since $A \otimes_{\mathrm{zar}} E$ is a sum of copies of A, it follows that $A \mapsto A \otimes_{\mathrm{zar}} E$ preserves weak equivalences and cofibrations. ☐

Definition 1.7. Given a cochain complex A of presheaves of abelian groups on Sch, we write $F_A(X)$ for the homotopy fiber (the shifted mapping cone) of the canonical map $A(X) \to \mathbb{H}_{\mathrm{cdh}}(X, A)$, so for each X there is a long exact sequence

$$\cdots \mathbb{H}^{n-1}_{\mathrm{cdh}}(X, A) \to H^n F_A(X) \to H^n A(X) \to \mathbb{H}^n_{\mathrm{cdh}}(X, A) \to H^{n+1} F_A(X) \cdots .$$

If A is a complex of sheaves (in some topology) of $\mathcal{O}$-modules, then $\mathbb{H}_{\mathrm{cdh}}(X, A)$ can be represented by a complex of sheaves of $\mathcal{O}$-modules as well (see [8, 8.1]), and hence so can F_A. We also write $F_K(X)$ for the homotopy fiber of $K(X) \to KH(X)$.

It is well known that HH, HC and K-theory satisfy Zariski descent; it follows that F_{HH}, F_{HC} and F_K also satisfy Zariski descent.

Proposition 1.8. *If $\mathcal{L}$ is an invertible sheaf on X and A is a complex of Zariski sheaves of $\mathcal{O}$-modules on* Sch/X*, then:*

$$\mathbb{H}_{\mathrm{cdh}}(-, A) \otimes_{\mathrm{zar}} \mathcal{L} \xrightarrow{\simeq} \mathbb{H}_{\mathrm{cdh}}(-, A \otimes_{\mathrm{zar}} \mathcal{L}).$$

Consequently, $F_A \otimes_{\mathrm{zar}} \mathcal{L} \xrightarrow{\simeq} F_{A\otimes\mathcal{L}}$.

Proof. Arguing as in the proof of Lemma 1.2, Lemma 1.4 shows that $\otimes_{\mathrm{zar}}\mathcal{L}$ preserves *cdh*-local fibrations (in addition to cofibrations and weak equivalences). The first statement follows immediately from this. Because $\otimes_{\mathrm{zar}}\mathcal{L}$ is exact, the second statement follows from the triangles

$$F_A \to A \to \mathbb{H}_{\mathrm{cdh}}(-, A) \to \quad \text{and} \quad F_{A\otimes\mathcal{L}} \to A \otimes \mathcal{L} \to \mathbb{H}_{\mathrm{cdh}}(-, A \otimes \mathcal{L}) \to . \qquad ☐$$

Lemma 1.9. *let A_i be cochain complexes of presheaves on* Sch/X*. Then for every X-scheme Y, the canonical maps*

$$\bigoplus_i \mathbb{H}_{\mathrm{zar}}(Y, A_i) \to \mathbb{H}_{\mathrm{zar}}(Y, \bigoplus_i A_i)$$

and

$$\bigoplus_i \mathbb{H}_{\mathrm{cdh}}(Y, A_i) \to \mathbb{H}_{\mathrm{cdh}}(Y, \bigoplus_i A_i)$$

are quasi-isomorphisms.

Proof. These sites are quasi-compact, and thus cohomology in them commutes with direct limits. ☐

2. Homology of line bundles

Suppose that R is a (commutative) noetherian algebra over a field k of characteristic 0. In [3, 3.2, 4.1], we showed that $NK(R) = K(R[t])/K(R)$ is isomorphic to $NF_{HC/\mathbb{Q}}(R)[1]$ as well as $F_{HH/\mathbb{Q}}(R)[1] \otimes_R tR[t]$. In this section, we replace $R[t]$ by the symmetric algebra $R[L] = \mathrm{Sym}_R(L)$ of a rank 1 projective R-module, and the ideal $tR[t]$ by $LR[L]$. More generally, if $\mathcal{L}$ is an invertible sheaf on a scheme X, we replace $X \times \mathbb{A}^1$ by the line bundle $\mathbb{L} = \mathrm{Spec}(\mathrm{Sym}_X \mathcal{L})$.

Lemma 2.1. *Let L be a rank 1 projective R-module. Then the symmetric algebra $R[L] = \mathrm{Sym}_R(L)$ satisfies:*

$$\begin{aligned} HH(R[L]) &\simeq HH(R) \otimes_R R[L] \ \oplus\ HH(R)[1] \otimes_R LR[L] \\ HC(R[L]) &\simeq HC(R) \ \oplus\ HH(R) \otimes_R LR[L]. \end{aligned}$$

Sinilarly, if X is a scheme over R and $X[L]$ denotes $X \times_R \mathrm{Spec}(R[L])$, then

$$\begin{aligned} HH(X[L]) &\simeq HH(X) \otimes_R R[L] \ \oplus\ HH(X)[1] \otimes_R LR[L]) \\ HC(X[L]) &\simeq HC(X) \ \oplus\ HH(X) \otimes_R LR[L]. \end{aligned}$$

Note that, as an R-module, $LR[L] = R[L] \otimes_R L$ is just $\oplus_{j=1}^{\infty} L^{\otimes j}$.

Proof. The cochain complex $HH(R[L])$ ends: $\to R[L] \otimes R[L] \xrightarrow{0} R[L] \to 0$. Therefore there are natural maps from $R[L]$ and $R[L] \otimes L[1]$ to $HH(R[L])$. Using the shuffle product, we get a natural map $\mu(R)$ from the direct sum of $HH(R) \otimes_R R[L]$ and $HH(R) \otimes_R (R[L] \otimes L)[1]$ to $HH(R[L])$. For each prime ideal $\wp$ of R, we have $R_\wp[L] \cong R_\wp[t]$ and $\mu(R_\wp)$ is a quasi-isomorphism by the Künneth formula [16, 9.4.1]. It follows that $\mu(R)$ is a quasi-isomorphism. The formula for $HC(R[L])$ follows by induction on the SBI sequence, just as it does for $HC(R[t])$.

If X is a scheme over R, the same argument applies to $\pi_* HH(\mathcal{O}_X[L])$, the direct image along $X[L] \xrightarrow{\pi} X$ of the cochain complex $HH(\mathcal{O}_X[L])$ on $X[L]$ of quasi-coherent sheaves described in Example 1.1. Because π is affine, we have a quasi-isomorphism

$$\mathbb{H}_{\mathrm{zar}}(X[L], HH(\mathcal{O}_X[L])) \cong \mathbb{H}_{\mathrm{zar}}(X, \pi_* HH(\mathcal{O}_X[L]).$$

Now the assertions about $X[L]$ follow from Corollary 1.3 and Lemma 1.9. □

Corollary 2.2. $F_{HC}(R[L]) \cong F_{HC}(R) \ \oplus\ \bigoplus_{j=1}^{\infty} \left(F_{HH} \otimes_R L^{\otimes j}\right)(R)$.

Proof. This follows from Lemma 2.1, Proposition 1.8, and Lemma 1.9. □

Now suppose that X is a scheme of finite type over a field of characteristic 0, containing k, and write HH, HC, etc for HH/k, HC/k, etc.

Lemma 2.3. *Let $\mathbb{L}$ be a line bundle over X, and write $\mathcal{F}_{HH}$ for the cochain complex of Zariski sheaves on X associated to the complex of presheaves $U \mapsto F_{HH}(\mathbb{L}|_U)$. Then $F_{HH}(\mathbb{L}) \xrightarrow{\simeq} \mathbb{H}_{\mathrm{zar}}(X, \mathcal{F}_{HH})$.*

Proof. As observed after 1.7, the presheaf of complexes F_{HH} satisfies Zariski descent: $F_{HH}(\mathbb{L}) \simeq \mathbb{H}_{\mathrm{zar}}(\mathbb{L}, F_{HH})$. By [11, 1.56], $\mathbb{H}_{\mathrm{zar}}(\mathbb{L}, F_{HH}) \xrightarrow{\simeq} \mathbb{H}_{\mathrm{zar}}(X, \mathcal{F}_{HH})$. □

In what follows, we write $\otimes$ for the tensor product of $\mathcal{O}_X$-modules.

Proposition 2.4. *Let $\mathbb{L}$ be the line bundle* $\mathrm{Spec}(\mathrm{Sym}\,\mathcal{L})$ *on X associated to an invertible sheaf $\mathcal{L}$, and $p : \mathbb{L} \to X$ the projection. Then we have quasi-isomorphisms:*

$$\begin{aligned} HC(\mathbb{L}) &\simeq HC(X) \oplus \mathbb{H}_{\mathrm{zar}}(X, HH \otimes \mathrm{Sym}(\mathcal{L}) \otimes \mathcal{L}); \\ \mathbb{H}_{\mathrm{cdh}}(X, p_* HC) &\simeq \mathbb{H}_{\mathrm{cdh}}(X, HC) \oplus \mathbb{H}_{\mathrm{cdh}}(X, HH \otimes \mathrm{Sym}(\mathcal{L}) \otimes \mathcal{L}); \\ F_{HC}(\mathbb{L}) &\simeq F_{HC}(X) \ \oplus\ \bigoplus_{j=1}^{\infty} \left(F_{HH} \otimes \mathcal{L}^{\otimes j}\right)(X); \\ K(\mathbb{L}, X) &\simeq F_{HC}(\mathbb{L}, X)[1]. \end{aligned}$$

Proof. Using Zariski descent, we may assume that $X = \operatorname{Spec}(R)$ for some R. The first two quasi-isomorphisms are immediate from Lemma 2.1, while the third is immediate from Corollary 2.2. By Theorem 1.6 of [2],

$$K(\mathbb{L})/K(X) \cong F_K(\mathbb{L})/F_K(X) \simeq F_{HC/\mathbb{Q}}(\mathbb{L})[1]/F_{HC/\mathbb{Q}}(X)[1].$$

Now use the formula for $F_{HC}(\mathbb{L})$ to get the final quasi-isomorphism. □

Now suppose that R is a commutative $\mathbb{Q}$-algebra. Then $K_n(R[L], R)$ is a $\mathbb{Q}$-module [13], and the Adams operations give an R-module decomposition

$$K_n(R[L], R) \cong \oplus_{i=0}^{\infty} K_n^{(i)}(R[L], R)$$

with $K_n^{(0)}(R[L], R) = 0$ for all n. The relative terms $F_K(R) \cong F_{HC}(R)[1]$ have a similar decomposition, and $F_K^{(i)}(R[L], R) \simeq F_{HC}^{(i-1)}(R[L], R)[1]$.

As in [3, 5.1], we define the *typical piece* $TK_n(R)$ to be $H^{1-n}(F_{HH}(R))$, and set $TK_n^{(i)}(R) = H^{1-n}(F_{HH}^{(i-1)}(R))$. Since these groups were detemined in [3], we may rephrase the last part of Proposition 2.4 as follows:

Corollary 2.5. *If R is a commutative $\mathbb{Q}$-algebra, $K_n(R[L], R) \cong TK_n(R) \otimes_R LR[L]$ and*

$$K_n^{(i)}(R[L]) \cong K_n^{(i)}(R) \oplus TK_n^{(i)}(R) \otimes_R LR[L].$$

Moreover,

$$TK_n^{(i)}(R) \cong \begin{cases} HH_{n-1}^{(i-1)}(R), & \text{if } i < n, \\ H_{\text{cdh}}^{i-n-1}(R, \Omega^{i-1}), & \text{if } i \geq n+2. \end{cases}$$

(The formulas for $TK_n^{(n)}$ and $TK_n^{(n+1)}$ are more complicated; see *loc. cit.*) The following special case $n = 0$ of 2.5, which is an analogue of [3, (0.5)], shows that we cannot twist out the example in [2, Theorem 0.2]

Corollary 2.6. *Let L be a rank 1 projective R-module, where R is a d-dimensional commutative $\mathbb{Q}$-algebra, with seminormalization R^+, and $R[L]$ the twisted polynomial ring. Then*

$$K_0(R[L], R) \cong \left((R^+/R) \oplus \bigoplus_{p=1}^{d-1} \mathbb{H}_{\text{cdh}}^p(R, \Omega^p) \right) \otimes_R LR[L].$$

In particular, $K_n(R) = K_n(R[t])$ if and only if $K_n(R) = K_n(R[L])$.

Proof. This follows from the fact that $\mathbb{H}_{\text{cdh}}(X, HH^{(i)}) \cong Ra_* a^* \Omega^i[-i]$, so that when $i > 1$ we have $K_0^{(i)}(R[L], R) \cong \mathbb{H}_{\text{cdh}}^{i-1}(R, \Omega^{i-1}) \otimes_R LR[L]$; see [2, 2.2]. □

Remark 2.7. Corollary 2.5 shows that $K_*(R[L], R)$ is a graded $R[L]$-module. As in [3], this reflects the fact that locally $R[L]$ is a polynomial ring, and $K_*(R[t], R)$ has a continuous module structure over the ring of big Witt vectors $W(R)$, compatible with the operations V_n and F_n; when $\mathbb{Q} \subset R$, such modules are graded $R[t]$-modules. Since $H^0(\operatorname{Spec} R, \widetilde{W}) = W(R)$, patching the structures via Zariski descent proves that $K_*(R[L], R)$ is a graded $R[L]$-module.

When X is no longer affine, this Zariski descent argument shows that

$$K_n(\mathbb{L}, X) = \oplus H^{1-n}(X, F_{HH} \otimes_{\text{zar}} \mathcal{L}^{\otimes i})$$

is a graded module over $S = \oplus H^0(X, \mathcal{L}^{\otimes i})$. This is clear from Proposition 2.4. Previously, using [13], it was only known that the $K_n(\mathbb{L}, X)$ are continuous modules over $H^0(X, \widetilde{W}) = W(k) = \prod_1^{\infty} k$.

3. Proof of Theorem 0.1

In order to use Proposition 2.4, we need to analyze $\mathbb{H}^n_{\mathrm{zar}}\left(X, F_{HH/\mathbb{Q}} \otimes \mathcal{L}^j\right)$. For this, we use the hypercohomology spectral sequence. (See [16, 5.7.10].)

$$E_2^{p,q} = H^p_{\mathrm{zar}}(X, H^q E) \Rightarrow \mathbb{H}^{p+q}_{\mathrm{zar}}(X, E). \tag{3.1}$$

Here E is a cochain complex which need not be bounded below and (by abuse of notation) the E_2 term denotes cohomology with coefficients in the Zariski sheaf associated to $H^q E$; the spectral sequence converges if X is noetherian and finite dimensional. When $E = F_{HH} \otimes \mathcal{L}^j$, we have $H^q E = H^q(F_{HH}) \otimes \mathcal{L}^j$, because $\mathcal{L}^j$ is flat.

Lemma 3.2. *If X is noetherian and finite dimensional, and E is a complex of Zariski sheaves such that $H^p_{\mathrm{zar}}(X, H^q E) = 0$ for $1 \le p \le \dim(X)$ and $p+q = s, s+1$ then $\mathbb{H}^s_{\mathrm{zar}}(X, E) \cong H^0_{\mathrm{zar}}(X, H^s E)$.*

Proof. This is immediate from the hypercohomology spectral sequence (3.1). □

In the remainder of this section, we will write $H^p(X, -)$ for $H^p_{\mathrm{zar}}(X, -)$. By a "quasi-coherent" (or "coherent") sheaf on Sch/k we mean a Zariski sheaf whose restriction to every small Zariski site is quasi-coherent (or coherent). When discussing Hochschild homology (or cyclic homology, or differentials, etc.) relative to $\mathbb{Q}$, we will suppress the base from the notation. For example, if X is a k-scheme then $HH_n(X)$ and Ω^n_X will mean $HH_n(X/\mathbb{Q})$ and $\Omega^n_{X/\mathbb{Q}}$.

Recall that when $\mathbb{Q} \subseteq k$, the Hochschild homology complex relative to k decomposes into a direct sum of weight pieces $HH^{(j)}(-/k)$; this induces decompositions on $\mathbb{H}_{\mathrm{cdh}}(-, HH(/k))$, the fiber $F_{HH(/k)}$, and on their cohomology sheaves and hypercohomology groups as well. As in [2], we use versions of a spectral sequence introduced by Kassel and Sletsjœ in [9] to obtain information about $F_{HH(/k)}$ from information about F_{HH}.

Lemma 3.3. *(Kassel-Sletsjœ) Let $\mathbb{Q} \subseteq k$ and $p \ge 1$ be fixed, and X a scheme over k. Then there are bounded cohomological spectral sequences of quasi-coherent sheaves on Sch/k ($p > s \ge 0$):*

$$E_1^{s,t} = \Omega^s_k \otimes_k H^{2s+t-p} HH^{(p-s)}(-/k) \Rightarrow H^{s+t-p} HH^{(p)}(-/\mathbb{Q})$$

(for $s + t \le 0$) and

$$E_1^{s,t} = \Omega^s_k \otimes_k H^{s+t}(Ra_* \Omega^{(p-s)}_{(-/k),\mathrm{cdh}}) \Rightarrow H^{s+t}(Ra_* \Omega^p_{\mathrm{cdh}})$$

and a morphism of spectral sequences between them. If k has finite transcendence degree, then both spectral sequences are spectral sequences of coherent sheaves.

We remark that the second spectral sequence is just the sheafification of the spectral sequence in [2, 4.2].

Proof. If $X = \mathrm{Spec}(R)$, the homological spectral sequence in [9, 4.3a] is

$${}_pE^1_{-i,i+j} = \Omega^i_k \otimes_k HH^{(p-i)}_{p-i+j}(R/k) \Rightarrow HH^{(p)}_{p+j}(R)$$

($0 \le i < p$, $j \ge 0$); see [2, 4.1].

We claim that this is a spectral sequence of R-modules, compatible with localization of R. Indeed, following the construction in [9, Theorem 3.2], the exact couple underlying the spectral sequence is constructed by choosing $\mathbb{Q}$-cofibrant simplicial resolutions $P_\bullet \to k$ and $Q_\bullet \to R$ and then filtering the differential modules $\Omega^p_{Q_\bullet/\mathbb{Q}}$

by certain $Q_\bullet$-submodules, leading to a filtration of $\Omega^p_{Q_\bullet/\mathbb{Q}} \otimes_{Q_\bullet} B$ by B-modules. (Although the filtration steps are defined as certain $P_\bullet$-submodules in [9, Section 3], they are in fact $Q_\bullet$-submodules.) The identification of the associated graded via [9, Lemma 3.1] is easily checked to be a B-module isomorphism. The whole construction commutes with localization because forming differential modules does.

Setting $\ell = i + j$, the spectral sequence is

$$_pE^1_{-i,\ell} = \Omega^i_k \otimes_k HH^{(p-i)}_{p+\ell-2i}(R/k) \Rightarrow HH^{(p)}_{p+\ell-i}(R), \quad \ell \le i.$$

As this spectral sequence is a spectral sequence of R-modules, compatible with localization and natural in R, we may sheafify it for the Zariski topology to obtain a spectral sequence of quasi-coherent sheaves. Reindexing cohomologically, with $s = i$ and $t = -\ell$, we have

$$^pE^{s,t}_1 = \Omega^s_k \otimes_k H^{2s+t-p}(HH^{(p-s)})(-/k) \Rightarrow H^{s+t-p}(HH^{(p)}).$$

This yields the first spectral sequence. If we sheafify it for the *cdh* topology, and use the isomorphism $HH^{(p)}_{\mathrm{cdh}} \cong \Omega^p_{\mathrm{cdh}}[p]$, we get the second spectral sequence. That it is still a spectral sequence of quasi-coherent sheaves follows from [2, lemma 2.8]. The morphism between the spectral sequences is just the change-of-topology map.

Finally, if k has finite transcendence degree, then the E_1-terms of both spectral sequences are coherent (apply [2, lemma 2.8] again for the second one) and hence so are the abutments. □

Corollary 3.4. *There is a bounded spectral sequence of quasi-coherent sheaves*

$$E^{s,t}_1 = \Omega^s_k \otimes_k H^{2s+t-p}(F^{(p-s)}_{HH/k}) \Rightarrow H^{s+t-p}(F^{(p)}_{HH}).$$

If k has finite transcendence degree, this is a spectral sequence of coherent sheaves.

Proof. The morphism of spectral sequences in Lemma 3.3 comes from a morphism $HH^{(p)} \to HH^{(p)}_{\mathrm{cdh}}$ of filtered complexes of quasi-coherent sheaves on Sch/k. By a lemma of Eilenberg–Moore [16, Ex. 5.4.4], there is a filtration on the [shifted] mapping cone $F^{(p)}_{HH}$ of $HH^{(p)} \to HH^{(p)}_{\mathrm{cdh}}$, yielding a spectral sequence converging to $H^*(F_{HH})$. This is the displayed spectral sequence. □

Proposition 3.5. *Assume that k has finite transcendence degree. If $\mathcal{L}$ is an ample line bundle on X, then for every n and $p \ge 0$ there is an $N_0 = N_0(n,p)$ such that for all $N > N_0$ the Zariski sheaf $H^nF^{(p)}_{HH} \otimes \mathcal{L}^{\otimes N}$ is generated by its global sections, and $H^q(X, H^nF^{(p)}_{HH} \otimes \mathcal{L}^{\otimes N}) = 0$ for all $q > 0$.*

Proof. The complex $F^{(0)}_{HH}$ is quasi-isomorphic to the cone of the map from the structure sheaf $\mathcal{O}$ to $Ra_*a^*\mathcal{O}$ and thus has coherent cohomology by [1, Lemma 6.5]. If $p > 0$, then by Corollary 3.4 the cohomology sheaves in question are coherent as well. Now apply Serre's Theorem B. □

Let $\mathcal{L}$ be an ample sheaf on X and $\mathbb{L}$ the line bundle $\mathrm{Spec}(\mathrm{Sym}\,\mathcal{L})$. Recall that for any Y, $F_{HC}(Y)$ is n-connected if and only if $F_{HH}(Y)$ is n-connected; see [2, 1.7]. If $\mathbb{L}$ is a line bundle over X, we define $F_{HH/k}(\mathbb{L}, X)$ to be the cokernel of the canonical split injection $F_{HH/k}(X) \to F_{HH/k}(\mathbb{L})$, and similarly for cyclic homology.

Theorem 3.6. *If $F_{HC}(\mathbb{L}, X)$ is n-connected for some ample line bundle $\mathcal{L}$ on X, then $F_{HH}(\mathbb{L}, X)$ is n-connected and:*

(1) *The Zariski sheaf F_{HH} is n-connected.*

(2) *X is regular in codimension $\le n$.*
(3) *If $F_{HC}(\mathbb{L},X)$ is d-connected for $d=\dim(X)$, then X is regular.*

Proof. There is a finitely generated subfield k_0 of k, a k_0-scheme X_0 and an ample line bundle $\mathcal{L}_0$ such that $X=X_0\otimes_{k_0}k$ and $\mathcal{L}=\mathcal{L}_0\otimes_{k_0}k$. The Künneth formula for Hochschild homology implies that $F_{HH}(\mathbb{L},X)=F_{HH}(\mathbb{L}_0,X_0)\otimes\Omega^*_{k/k_0}$, whence $F_{HH}(\mathbb{L},X)$ is n-connected if and only if $F_{HH}(\mathbb{L}_0,X_0)$ is. Thus we may assume that k has finite transcendence degree.

(1) Recall [2, 2.1] that $F_{HH}(\mathbb{L},X)=\prod F^{(p)}_{HH}(\mathbb{L},X)$. Thus it suffices to fix p and show that $F^{(p)}_{HH}$ is n-connected. Set $\mathcal{G}_N=\mathcal{L}^N\otimes F^{(p)}_{HH}$, and note that $H^q\mathcal{G}_N=\mathcal{L}^N\otimes H^qF^{(p)}_{HH}$. By Proposition 3.5 and Lemma 3.2, $H^s(X,\mathcal{G}_N)\cong H^0(X,H^s\mathcal{G}_N)$ for large N and all $s\ge -n$.

By assumption and Lemma 2.3, the groups

$$\pi_sF^{(p)}_{HH}(\mathbb{L},X)=\mathbb{H}^{-s}_{\text{zar}}(X,F^{(p)}_{HH}(\mathbb{L},X))=\mathbb{H}^{-s}_{\text{zar}}(X,F^{(p)}_{HH}(\mathbb{L})/F^{(p)}_{HH})$$

vanish for $s\le n$. By Lemma 2.4, this implies that for all $N>0$:

$$H^0(X,H^{-s}\mathcal{G}_N)\cong H^{-s}(X,\mathcal{G}_N)=H^{-s}(X,\mathcal{L}^N\otimes F^{(p)}_{HH})=0,s\le d.$$

Since $\mathcal{L}$ is ample, the sheaves $H^s\mathcal{G}_N=\mathcal{L}^N\otimes H^sF^{(p)}_{HH}$ are generated by their global sections $H^0(X,H^s\mathcal{G}_N)$ for large N and $s\ge -n$. This implies that the sheaves $\mathcal{L}^N\otimes H^sF^{(p)}_{HH}$ vanish, and hence that the sheaves $H^sF^{(p)}_{HH}$ vanish for $s\ge -n$. This proves (1).

Given (1), the stalks $F_{HH}(\mathcal{O}_{X,x})$ are n-connected. We proved in [2, 4.8] that this implies that each $F_{HH/k}(\mathcal{O}_{X,x})$ is n-connected. If $\dim(\mathcal{O}_{X,x})\ge n$, we proved in [2, 3.1] that $\mathcal{O}_{X,x}$ is smooth over k, and hence regular. ☐

Variant 3.7. *Let X, $\mathcal{L}$ and $\mathbb{L}$ be as in Proposition 3.6. Suppose that $F_{HC/k}(\mathbb{L},X)$ is n-connected. Then the proof of Theorem 3.6 goes through to show that:*

(1) *The sheaf $\mathcal{F}_{HH/k}$ is n-connected.*
(2) *X is regular in codimension $\le n$.*
(3) *If $F_{HH/k}(\mathbb{L},X)$ is d-connected for $d=\dim(X)$, then X is regular.*

Proof of Theorem 0.1. Suppose that $K_i(\mathbb{L})\cong K_i(X)$ for all $i\le n$. By Proposition 2.4, $F_{HC/\mathbb{Q}}(\mathbb{L},X)$ is $(n-1)$-connected. By Theorem 3.6, $F_{HH/\mathbb{Q}}(\mathbb{L},X)$ is $(n-1)$-connected and X is regular in codimension $<n$ ☐

4. Two examples

We conclude with two quick examples. Let E be an elliptic curve over $\mathbb{Q}$ with basepoint Q, and P a point such that $P-Q$ does not have finite order in $\mathrm{Pic}(E)$.

Example 4.1. Consider the non-reduced scheme $Y=\mathrm{Spec}(\mathcal{O}_E\oplus J)$, where J is the invertible sheaf $\mathcal{O}(P-Q)$. We showed in [2, 0.2] that Y is K_n-regular for all n, because $K_n(Y\times\mathbb{A}^1)\cong K_n(Y)\cong K_n(E)$ for all n.

Let $\mathcal{L}$ be the sheaf $\mathcal{O}(Q)$ and set $\mathbb{L}=\mathrm{Spec}_Y(\mathrm{Sym}\,\mathcal{L})$. Then

$$K_0(\mathbb{L})\cong K_0(Y)\oplus\mathbb{Q}[x,y].$$

For our second example, recall that if R is a regular $\mathbb{Q}$-algebra and J is a rank 1 projective R-module and A is the subring $R[J^2,J^3]$ of $R[J]=\mathrm{Sym}_R(J)$

then $\operatorname{Spec}(A)$ is an affine cusp bundle over $\operatorname{Spec}(R)$. For $n \geq 2$, set

$$V_n(R) = \begin{cases} J^{6(i-1)} \oplus (J^{6(i-2)} \otimes \Omega^2_R) \oplus \cdots \oplus (R \otimes \Omega^{n-2}_R), & n = 2i \geq 2; \\ J^{6(i-1)} \otimes \Omega^1_R \oplus (J^{6(i-2)} \otimes \Omega^3_R) \oplus \cdots \oplus (R \otimes \Omega^{n-2}_R), & n = 2i+1 \geq 3. \end{cases}$$

In particular, $V_2(R) = R$ and $V_3(R) = \Omega^1_R$. Let us write $\widetilde{K}_n(A)$ for $K_n(A)/K_n(R)$.

Proposition 4.2. *If $A = R[J^2, J^3]$ and R is a regular $\mathbb{Q}$-algebra then*

$$\widetilde{K}_n(A) \cong (J^5 \oplus J^6) \otimes V_n(R) \oplus (J \otimes \Omega^n_R).$$

In particular, $\widetilde{K}_0(A) \cong J$, $\widetilde{K}_1(A) \cong J \otimes \Omega^1_R$ and

$$\widetilde{K}_2(A) \cong (J^5 \oplus J^6) \oplus (J \otimes \Omega^2_R).$$

Proof. For $J = R$, this is Theorem 9.2 of [7], which holds for any regular $\mathbb{Q}$-algebra R (not just for any field). In order to pass to $R[J^2, J^3]$, we need more detail. Using the classical Mayer-Vietoris sequence for $A \subset R[J]$, it is easy to see that $K_0(A)/K_0(R) \cong J$ and $K_1(A)/K_1(R) \cong J \otimes \Omega^1_R$.

For $n \geq 2$ the factors in $K_n(A)$ come from $HH_{n-1}(A)$ via the maps $HH_*(A) \to HC_*(A)$ and $\widetilde{K}_n(A) \to \widetilde{HC}_{n-1}(A)$. The summand $J \otimes \Omega^n_R$ of $K_n(A)$ comes from the $J \otimes \Omega^1_R$ in $K_1(A)$ (or $HH_0(A, R[J], J)$) by multiplication by $HH_{n-1}(R) \cong \Omega^{n-1}_R$.

The V_n factors come from the explicit description of the corresponding cyclic homology cycles (coming from cycles in Hochschild homology $HH_{n-1}(A)$) in 4.3, 4.7 and 5.8 of [7]. Locally, J is generated by an element t; we set $x = t^2 \in J^2$, $y = t^3 \in J^3$ so that $y^2 = x^3$. The summands J^5 and J^6 of $K_2(A)$ are locally generated by the cycles $z = 2x[y] + 3y[x]$ and $tz = 2y[y] + 3x^2[x]$ in $HH_1(A)$. Multiplication by Ω^{n-2}_R gives the summands $(J^5 \oplus J^6) \otimes \Omega^{n-2}_R$ in $K_n(A)$.

Now consider the summand J^6 in the degree 2 part $A^{\otimes 3}$ of the Hochschild complex for A, locally generated by the element $w = [y|y] - x[x|x] - [x^2|x]$. The product zw^{i-1} is a cycle in $HH_{2i-1}(A)$, and locally generates a summand $J^{5+6(i-1)}$ of $HH_{2i-1}(A)$, corresponding to the factor $J^{5+6(i-1)}$ of the summand $J^5 \otimes V_{2i}(R)$ of $K_{2i}(A)$. As above, multiplication by Ω^*_R gives the rest of the summands. □

Remark 4.2.1. In the spirit of Corollary 2.5, we note that $NK_n(A) \cong TK_n(A) \otimes_R LR[L]$, where

$$TK_n(A) = \widetilde{K}_n(A) \oplus \widetilde{K}_n(A).$$

Theorem 4.3. *Let J be the invertible sheaf $\mathcal{O}(P-Q)$ on the ellipic curve E and let X denote the affine cusp bundle $\operatorname{Spec}_E(\mathcal{O}_E[J^2, J^3])$ over E. (X has a codimension 1 singular locus.) If J does not have finite order in $\operatorname{Pic}(E)$ then X is K_n-regular for all integers n: for all $m \geq 0$ we have*

$$K_n(X) \cong K_n(X \times \mathbb{A}^m) \cong K_n(E)$$

On the other hand, if $\mathbb{L} = \operatorname{Sym}_E(\mathcal{O}(Q))$ then $K_{-1}(\mathbb{L}) \neq K_{-1}(X)$ and $K_0(\mathbb{L}) \neq K_0(X)$.

Proof. Since $\Omega_E \cong \mathcal{O}_E$, $V_n(\mathcal{O}_E)$ is a sum of terms J^i for $i > 0$; the same is true for the pushforward of the sheaf $V_n(\mathcal{O}_E[t_1, ..., t_m])$ to E. Recall that $H^p(E, J^r) = 0$ for all $r \neq 0$. From the Zariski descent spectral sequence

$$E_2^{p,q} = H^p(E, K_{-q}(\mathcal{O}_E[J^2, J^3][t_1, ..., t_m])/K_{-q}(\mathcal{O}_E)) \Rightarrow K_{-p-q}(X \times \mathbb{A}^m)/K_{-p-q}(E)$$

we see that $K_n(X \times \mathbb{A}^m) \cong K_n(E)$ for all n.

On the other hand, Proposition 4.2 yields $\widetilde{K}_{-1}(\mathbb{L}) \cong \oplus_{j\geq 1} H^1(E, J \otimes \mathcal{L}^j)$ and

$$\widetilde{K}_0(\mathbb{L}) \cong \oplus_{j\geq 1} H^0(E, J \otimes \mathcal{L}^j) \oplus \widetilde{K}_{-1}(\mathbb{L}).$$

These groups are nonzero because $\mathcal{L}$ is ample. ☐

Acknowledgements. The authors would like to thank the Tata Institute, the University of Melbourne and the Hausdorff Institute for Mathematics for their hospitality during the preparation of this paper.

References

[1] G. Cortiñas, C. Haesemeyer, M. Schlichting and C. Weibel, Cyclic homology, *cdh*-cohomology and negative K-theory, *Annals of Math.* **167** (2008), 549–563.

[2] G. Cortiñas, C. Haesemeyer and C. Weibel, K-regularity, *cdh*-fibrant Hochschild homology, and a conjecture of Vorst, *J. AMS* **21** (2008), 547–561.

[3] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, Bass' NK groups and *cdh*-fibrant Hochschild homology, *Inventiones Math.* 181 (2010), 421–448.

[4] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, K–theory of cones of smooth varieties, *J. Alg. Geom.* 22 (2012), 13–34.

[5] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, A negative answer to a question of Bass, *Proc. AMS* 139 (2011), 1187–1200.

[6] P. du Bois, Complexe de de Rham filtré d'une variété singulière, *Bull. Soc. Math. France* 109 (1981), 41–81.

[7] S. Geller, L. Reid and C. Weibel, The cyclic homology and K-theory of curves, J. reinie angew. Math. 393 (1989), 39–90.

[8] J. F. Jardine, *Local Homotopy Theory*, Springer Monographs in Math., 2015.

[9] C. Kassel, A. B. Sletsjœ. Base change, transitivity and Künneth formulas for the Quillen decomposition of Hochschild homology. *Math. Scand.*, 70:186–192, 1992.

[10] R. Michler, Hodge-components of cyclic homology for affine quasi-homogeneous hypersurfaces, *Astérisque* 226 (1994), 321–333.

[11] R. W. Thomason. Algebraic K-theory and étale cohomology. *Annales Sc. Éc. Norm. Sup.* (Paris), 18:437–552, 1985.

[12] R. W. Thomason and T. Trobaugh, Higher algebraic K-theory of schemes and of derived categories, pp. 247–435 in *The Grothendieck Festschrift III,* Prog. in Math. 88, Birkhäuser. 1990.

[13] C. Weibel, Module structures on the K–theory of graded rings, *J. Algebra*, 105 (1987), 465–483.

[14] C. Weibel, Homotopy Algebraic K-theory, *Contemporary Math.* 83 (1989), 461–488.

[15] C. Weibel, Cyclic homology for schemes, *Proc. AMS* **124** (1996), 1655–1662.

[16] C. Weibel, *An introduction to homological algebra*, Cambridge Univ. Press, 1994.

[17] C. Weibel and S. Geller, Étale descent for Hochschild and cyclic homology, *Comm. Math. Helv.* 66 (1991), 368–388.

[EGA] A. Grothendieck, Éléments de gémétrie algébrique:I. Le langage des schémas, Publ. Math. IHES (Paris) 4 (1960), 5–228.

School of Mathematics and Statistics, University of Melbourne, VIC 3010, Australia
E-mail address: `christian.haesemeyer@unimelb.edu.au`

Dept. of Mathematics, Rutgers University, New Brunswick, NJ 08901, USA
E-mail address: `weibel@math.rutgers.edu`