

## On the Cohen–Macaulay and Buchsbaum Property for Unions of Planes in Affine Space

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The coordinate ring  $A$  of a union of planes in affine space is studied and it is asked when  $A$  is a Cohen–Macaulay or Buchsbaum ring. These properties are related to the position of the planes via the notion of seminormality. It is shown that  $A$  is Cohen–Macaulay iff  $A$  is connected in codimension 2 and seminormal in an appropriate sense. Consideration of the Cohen–Macaulification then yields a simple criterion for  $A$  to be Buchsbaum. Methods with several examples are illustrated. © 1985 Academic Press, Inc.

### INTRODUCTION

Let  $V$  be the reducible variety in  $\mathbb{A}^{n+1}(k)$ ,  $k$  a field, formed by the union of planes (2-dimensional linear subspaces) through the origin. Our interest is in the singular point at the origin. More specifically, we wish to discuss when the local ring,  $A$ , of this singular point is a Cohen–Macaulay (C–M) or a Buchsbaum ring.

We may consider these planes as describing a collection,  $\mathcal{L}$ , of lines in the projective space  $\mathbb{P}^n(k)$ . Our discussion will try to relate the Cohen–Macaulay and Buchsbaum properties of the local ring  $A$  described above to the position of the lines in  $\mathbb{P}^n(k)$ .

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Our major result is that when  $\mathcal{L}$  is connected,  $A$  is Cohen–Macaulay if and only if it is seminormal in an overring  $\Pi(A/J_i)$ . This is (8.4). If  $\mathcal{L}$  is a connected union of lines which have linearly independent directions at each vertex of  $\mathcal{L}$ , then  $A$  is Cohen–Macaulay if and only if  $A$  is seminormal. This is (5.9).

If  $\mathcal{L}$  is connected, but  $A$  is not Cohen–Macaulay, we give a simple algorithm for determining whether  $A$  is Buchsbaum. This is (8.8); see also (9.3) and (10.1). We illustrate the method by analyzing some simple “ $m \times n$ ” configurations in  $\mathbb{P}^3$ .

If  $\mathcal{L}$  is not connected,  $A$  cannot be Cohen–Macaulay by a result of Hartshorne. Let  $A_i$  denote the local ring of the connected component  $\mathcal{L}_i$  of  $\mathcal{L}$ . We show that  $A$  is Buchsbaum if and only if both (i) all the  $A_i$  are Buchsbaum, and (ii)  $A$  is seminormal in  $\Pi A_i$ . This is (6.6); in (6.1) we show that (ii) is equivalent to: (ii') the embedding dimension of  $A$  is the sum of the embedding dimensions of the  $A_i$ . We also give a formula for the Buchsbaum type of  $A$  in (7.1).

Our method is to compare the seminormalization of  $A$  (constructed as a pullback ring) to the Buchsbaumification and Cohen–Macaulification of  $A$ . A basic observation is (3.1): any 2-dimensional seminormal ring (of finite type over a field) is Buchsbaum. In order to describe our method, we have organized the paper in the following way:

In Section 1 we give an elementary description of the Cohen–Macaulification of the rings we wish to discuss. In Section 2 we introduce the notion of the Buchsbaumification and discuss its relation to the material in Section 1.

In Section 3 we recall the notion of seminormality. In Section 4 we introduce the notation and the basic facts we shall use to discuss the coordinate rings of lines in  $\mathbb{P}^n$ . Here we introduce the graph associated to a union of lines in  $\mathbb{P}^n$ . We make the first connections in these sections between seminormality, the Buchsbaum property, and the C–M property for unions of lines in  $\mathbb{P}^n$ .

In Section 5 we analyse the case in which the lines in  $\mathbb{P}^n$  are linearly independent at each intersection point of  $\mathcal{L}$ . When  $\mathcal{L}$  is connected, we show that the seminormalization and the Cohen–Macaulification of  $A$  agree. Using work on seminormality, we then give some examples of C–M configurations and some examples of non-C–M configurations.

We discuss the case in which  $\mathcal{L}$  is disconnected in Sections 5–7. In this case,  $A$  cannot be C–M. In Section 6 we give the promised criterion for  $A$  to be Buchsbaum in terms of the connected components of  $\mathcal{L}$ . In Section 7 we give a formula for the Buchsbaum type of  $A$  in terms of the Buchsbaum types of the connected components of  $\mathcal{L}$ .

In Section 8, we return to the case in which  $\mathcal{L}$  is connected, dropping the assumption of linear independence at the vertices. We identify the

Cohen–Macaulification by using a modification of the approach in Section 5. The trick is to preserve the structure at each vertex of  $\mathcal{L}$ .

Knowing the Cohen–Macaulification of  $A$  gives a simple criterion for determining whether  $A$  is Buchsbaum. We apply this criterion in Section 9 to show that an “ $n \times 1$ ” configuration of lines in  $\mathbb{P}^3$  is Buchsbaum if and only if  $n \leq 3$  (and C–M iff  $n \leq 2$ ). In Section 10, we apply this criterion to show that an “ $m \times n$ ” configuration of lines on a quadric surface in  $\mathbb{P}^3$  is Buchsbaum if and only if  $|m - n| \leq 2$  (and C–M iff  $|m - n| \leq 1$ ).

In Section 11 we collect some remarks relating this work to other work in this general area and raise some questions about the material we have discussed.

## 1. COHEN–MACAULIFICATION

In this section  $(A, \mathfrak{m})$  will be a 2-dimensional reduced local ring whose integral closure is a finite  $A$ -module, and  $Q(A)$  will denote the total ring of fractions of  $A$ . We will show that there is a unique smallest Cohen–Macaulay ring containing  $A$  which is contained in  $Q(A)$ . This ring, which is  $C = \bigcup_{n=1}^{\infty} \mathfrak{m}^{-n}$ , is called the Cohen–Macaulification of  $A$ .

The notion of Cohen–Macaulification seems to have first arisen in Zariski’s and Nagata’s work on Hilbert’s 14th Problem (see [15, 24]). (Eisenbud informed us, though, that first glimmerings of the idea are present in Macaulay’s work.) A discussion may be found in EGA IV (5.10) [9] and in [6]. There is also a substantial recent literature on this notion (see, e.g., [5, 12, 1, 2]).

When  $(A, \mathfrak{m})$  is 2-dimensional, however, the rather substantial theory we have referred to may be greatly simplified. Consequently, in this section, we shall give elementary proofs of the facts we shall need.

**LEMMA 1.1.** *A 2-dimensional integrally closed reduced noetherian ring is C–M.*

*Proof.* Such a ring is a finite product of 2-dimensional integrally closed domains, and these are C–M. (See [14, (25.13)]).

**PROPOSITION 1.2.** *Let  $A$  be a 2-dimensional reduced noetherian ring and let  $\mathfrak{m}$  be a height 2 maximal ideal of  $A$ . The ring*

$$C = \bigcup_{n=1}^{\infty} \mathfrak{m}^{-n}$$

*is contained in every C–M ring  $R$  which contains  $A$  and is integral over  $A$ .*

*Proof.* It is easy to see that  $C$  is a ring. If  $R$  is an integral extension of  $A$  then  $m^n R$  has height 2 in  $R$  for every  $n$ . When  $R$  is C-M the grade of  $m^n R$  is 2, and so  $(m^n R)^{-1} = R$ . On the other hand,  $m^{-n} \subset (m^n R)^{-1}$ , since for  $f \in m^{-n} \subset Q(A) \subseteq Q(R)$ , the relation  $f m^n A \subseteq A$  implies that  $f m^n R \subseteq R$ . Thus  $R$  contains  $m^{-n}$  and hence  $C$ .

*Remark.* If  $(A, m)$  is a local domain then  $C$  is the ring

$$A^{(1)} = \bigcap_{ht \mathfrak{p} = 1} A_{\mathfrak{p}} = \Gamma(\text{Spec}(A) - \{m\}, \bar{A}).$$

It is easy to see that  $C \subseteq A^{(1)}$ . Conversely, suppose  $f \in A^{(1)}$ . Then  $I = \{x \in A \mid fx \in A\}$  is an ideal of  $A$ , and is not contained in any height 1 prime of  $A$ . Consequently,  $I$  is primary for  $m$ , i.e., some  $m^n \subset I$ . This implies that  $f \in m^{-n} \subset C$  and shows that  $A^{(1)} \subseteq C$ . (See also [2, (2.11)] and [9 (5.10) 17(ii)]).

**COROLLARY 1.3.** *Let  $(A, m)$  be a 2-dimensional reduced local noetherian ring. Then  $C = \bigcup m^{-n}$  is a semi-local noetherian ring and is contained in the integral closure  $\bar{A}$  of  $A$ . If  $\bar{A}$  is a finite  $A$ -module then  $C = m^{-n}$  for  $n \geq 0$ .*

*Proof.* By (1.1),  $\bar{A}$  is C-M; by (1.2) it contains  $C$ . As  $\bar{A}$  is semilocal, [14, (33.12)] so is  $C$ . The ring  $C$  is noetherian by [6, (1.4)]. Finally, if  $\bar{A}$  is a finite  $A$ -module, the chain  $m^{-1} \subseteq m^{-2} \subseteq \dots$  of submodules must stabilize for  $n \geq 0$ ; i.e.,  $C = m^{-n}$  for  $n \geq 0$ .

*Remark.* The minimal integer  $i$  for which  $m^{-i} = C$  is equal to the minimal integer  $j$  for which  $m^j$  is contained in the conductor from  $C$  to  $A$ . This is the case since  $C = m^{-i}$  holds if and only if  $C m^i \subseteq A$ . In general, however, we cannot say which ideal of  $A$  between  $m^i$  and  $m^{i-1}$  is the conductor without explicit computations.

**LEMMA 1.4.** *Let  $R$  be a reduced noetherian ring in which no minimal prime is maximal. If  $I$  is an ideal of  $R$  with  $I^{-1} = R$  then grade  $I \geq 2$ .*

*Proof.* First suppose  $I \subset Z(R)$ . Then  $I \subset \mathfrak{p}$ , where  $\mathfrak{p}$  is an associated prime of  $(0)$  in  $R$ . Since  $\mathfrak{p} = \text{ann}(x)$  for some  $x \in R$  we have  $x \mathfrak{p} = 0$  and so  $xI = 0$ . Let  $m$  be a maximal ideal of  $R$  which contains  $\mathfrak{p}$  and let  $y \in m$ ,  $y \notin Z(R)$ . Consider the local ring  $R_m$  and note that  $x \neq 0$  in  $R_m$  since  $\mathfrak{p} = \text{ann}(x) \subset m$ . Also  $\bigcap_{n=1}^{\infty} (y^n R_m) = 0$ . Thus there is an integer  $n$  such that  $x \notin (y^n R_m)$ . So  $x \notin y^n R$ . Consequently,  $x/y^n \notin R$  and  $(x/y^n)I = 0$ . Thus if grade  $(I) = 0$  we have  $I^{-1} \not\supseteq R$ .

If grade  $I = 1$ , let  $x \in I$  not be a zero-divisor. Then  $I$  is contained in an associated prime  $\mathfrak{p}$  of  $(x)$  in  $R$ . Since  $\mathfrak{p} = (x) : y$  for some  $y \in R$  with

$y \notin (x)$ , we have  $y\mathfrak{A} \subseteq (x)$  and so  $yI \subseteq (x)$ . Thus  $(y/x)I \subseteq R$ . Since  $y/x \notin R$  we have  $I^{-1} \not\subseteq R$ .

**THEOREM 1.5.** [9, (5.10.17i)]. *Let  $A$  be a 2-dimensional reduced local ring whose integral closure is a finite  $A$ -module. Then  $C = \bigcup m^{-n}$  is a 2-dimensional semilocal Cohen–Macaulay ring, finite over  $A$ . Moreover,  $C$  is contained in every other  $C$ - $M$  ring which contains  $A$  and is integral over  $A$ .*

*Because of this universal property, we call  $C$  the Cohen–Macaulification of  $A$ .*

*Proof.* From (1.2) and (1.3), we see that it will suffice to show that  $C$  is  $C$ - $M$ . As  $C$  is finite over  $A$  it will suffice to show that the ideal  $I = mC$  has grade 2. In view of (1.4), it suffices to show that  $I^{-1} = C$ . If  $f \in I^{-1}$  then  $f m C \subseteq C = m^{-n}$  for  $n \geq 0$ . This implies that  $f \in m^{-(n+1)} = C$ . Thus  $I^{-1} = C$  and  $C$  is  $C$ - $M$ .

**Remark 1.6.** An example of a 2-dimensional local domain  $A$  for which  $C$  is not a finite  $A$ -module is given in [6, (3.3)]. By either [6, (1.1)] or [9, (5.11.11)]  $C$  is finite over  $A$  if and only if the  $m$ -adic completion  $\hat{A}$  has no embedded prime ideals.

**COROLLARY 1.7.** *Let  $A$  be a 2-dimensional reduced (non-local) ring whose integral closure is a finite  $A$ -module. Suppose that the localization of  $A$  at any maximal ideal, except possibly  $M$ , is  $C$ - $M$ . Then  $C = \bigcup M^{-n}$  is a  $C$ - $M$  ring, finite over  $A$ , and  $C = M^{-n}$  for  $n \geq 0$ . Moreover, every  $C$ - $M$  ring containing  $A$  and finite over  $A$  also contains  $C$ .*

*Proof.* The proofs of (1.3) and (1.5) go through mutatis mutandis:

If  $P$  is a maximal ideal of  $C$  not containing  $MC$  then  $C_P = A_{P \cap A}$ , since every element  $f$  of  $M^{-n}$  has the form  $xf/x$  for some  $x \in M^n \setminus P$ . Also, if  $S = A - M$ , then  $S^{-1}(M^{-n}) = (S^{-1}M)^{-n}$  as submodules of  $Q(A)$ , since  $S^{-1}A$  is a flat  $A$ -module and

$$M^{-n} = \bigcap \left\{ \ker \left( Q(A) \xrightarrow{x} \frac{Q(A)}{A} \right) \mid x \in M^n \right\}.$$

Thus  $S^{-1}C = \bigcup (S^{-1}M)^{-n}$  is the Cohen–Macaulification of  $A_M$  and so  $C$  is  $C$ - $M$ , since it is  $C$ - $M$  at every maximal ideal.

Now let  $R \supset A$  be a  $C$ - $M$  ring which is finite over  $A$ . Now  $A \subseteq C \subseteq Q(A) \subseteq Q(R)$  and  $A \subset R \subseteq Q(R)$  and the inclusion  $C \rightarrow Q(R)$  induces a homomorphism of  $A$ -modules,  $C \rightarrow Q(R)/R$ . We shall have  $C \subseteq R$  if we can show this map  $C \rightarrow Q(R)/R$  is the zero map. To do that it will suffice to show  $S^{-1}C \subseteq S^{-1}R$  for  $S = A \setminus \mathfrak{A}$ , where  $\mathfrak{A}$  is maximal in  $A$ .

If  $\mathfrak{A} \neq M$  then  $S^{-1}C = S^{-1}A \subseteq S^{-1}R$  and so this case is O.K. If  $\mathfrak{A} = M$  then  $S^{-1}C$  is the Cohen–Macaulification of  $S^{-1}A$  and  $S^{-1}R$  is finite over  $S^{-1}A$ , hence by (1.2),  $S^{-1}R \supseteq S^{-1}C$ .

*Remark 1.8.* In example (3.4) we shall see that the Cohen–Macaulification,  $C$ , of a local ring  $A$  need not be local, even if  $A$  is connected in codimension 2. On the other hand we shall see in Section 5 that when  $A$  is the homogeneous coordinate ring of a union of lines and is connected in codimension 2 then the Cohen–Macaulifications of the local rings of  $A$  are all local.

*Remark 1.9.* If  $S$  is a reduced 2-dimensional graded ring with unique homogeneous maximal ideal  $M$  and if there are elements  $X_0, \dots, X_n$  of degree 1 in  $S$  such that  $\sqrt{(X_0, \dots, X_n)} = M$ , then  $S$  is C–M at every prime  $p \neq M$ . Consequently, if the integral closure of  $S$  is a finite  $S$ -module, (1.7) applies to such an  $S$ .

## 2. BUCHSBAUMIFICATION

In this section we establish the basic criterion for determining whether or not a 2-dimensional reduced local ring  $(A, \mathfrak{m})$  is a Buchsbaum ring (Theorem 2.3). Assuming that the integral closure is a finite  $A$ -module, we show that there is a unique smallest Buchsbaum ring  $B$  containing  $A$  and finite over  $A$ . We will refer to  $B$  as the *Buchsbaumification* of  $A$ . This ring is  $B = A + \mathfrak{m}C$ , where  $C$  is the Cohen–Macaulification of  $A$ .

We first recall the definition of a Buchsbaum ring. A ring is *Buchsbaum* if its localizations are Buchsbaum. A  $d$ -dimensional noetherian local ring  $(A, \mathfrak{m})$  is Buchsbaum if the difference,  $l_A(A/\mathfrak{q}) - e(\mathfrak{q}, A)$ , of the length of  $A/\mathfrak{q}$  and the multiplicity of  $\mathfrak{q}$  is a constant. That is, the difference is independent of the choice of the  $\mathfrak{m}$ -primary ideal  $\mathfrak{q}$  generated by a system of parameters. (See, e.g., [20]). The constant  $l(A/\mathfrak{q}) - e(\mathfrak{q}, A)$  will be referred to as the *Buchsbaum type* of  $A$ . It is well known that C–M rings are Buchsbaum of type 0. Buchsbaum rings have recently been studied extensively. More information is available in [19] or [8] than the facts presented here.

The following characterization of certain Buchsbaum rings will be one of our major tools.

**THEOREM 2.1.** (See [19, Corollary 2.4]) *Let  $(A, \mathfrak{m})$  be a  $d$ -dimensional local ring whose maximal ideal has grade  $r$ . Suppose  $H_{\mathfrak{m}}^i(A) = 0$  for  $i \neq r, d$ . Then  $A$  is a Buchsbaum ring if and only if  ${}_{\mathfrak{m}}H_{\mathfrak{m}}^r(A) = 0$ .*

Here  $H_{\mathfrak{m}}^i(A) = \overset{\text{colim}}{\rightarrow} \text{Ext}_A^i(A/\mathfrak{m}^n, A)$  is the local cohomology of the ring  $A$  (see, e.g., [10]). Clearly, the major difficulty one usually encounters in attempting to apply this cohomological criterion is finding a “good” description of  $H_{\mathfrak{m}}^r(A)$ . Fortunately, this is possible in our situation.

**PROPOSITION 2.2.** *Let  $(A, \mathfrak{m})$  be a 2-dimensional reduced local ring with total ring of fractions  $Q(A)$ . Then,*

$$H_{\mathfrak{m}}^0(A) = 0 \quad \text{and} \quad H_{\mathfrak{m}}^1(A) = \bigcup_{n=1}^{\infty} \left( \frac{\mathfrak{m}^{-n}}{A} \right).$$

*Remark.* Since  $A$  is reduced,  $\mathfrak{m}$  has grade 1 or 2;  $\text{grade}(\mathfrak{m}) = 2$  if and only if  $A$  is C-M. When  $A$  is C-M,  $H_{\mathfrak{m}}^0(A) = H_{\mathfrak{m}}^1(A) = 0$  and  $\mathfrak{m}^{-n} = A$ , so (2.2) is clear if  $\text{grade } \mathfrak{m} = 2$ . The only case of interest is when  $\text{grade } \mathfrak{m} = 1$ .

*Proof.* Since  $\text{grade } (\mathfrak{m}) \neq 0$  we have  $\text{Hom}(A/\mathfrak{m}^n, A) = 0$  for all  $n$  and so  $H_{\mathfrak{m}}^0(A) = 0$ . From the exact sequence

$$0 \rightarrow \mathfrak{m}^n \rightarrow A \rightarrow A/\mathfrak{m}^n \rightarrow 0$$

we obtain the exact sequence

$$0 \rightarrow A \rightarrow \text{Hom}(\mathfrak{m}^n, A) \rightarrow \text{Ext}^1(A/\mathfrak{m}^n, A) \rightarrow 0. \tag{*}$$

There is a natural inclusion of  $I^{-1}$  in  $\text{Hom}(I, A)$  which is an isomorphism for  $I = \mathfrak{m}^n$ . (This is well known if  $A$  is a domain.) To see this in the reduced case, choose  $f \in \text{Hom}(\mathfrak{m}^n, A)$  and  $x \in \mathfrak{m}^n \setminus Z(A)$ . The formula  $xf(y) = f(x)y$  yields  $f = f(x)/x \in \mathfrak{m}^{-n}$ . The sequence (\*) above becomes  $0 \rightarrow A \rightarrow \mathfrak{m}^{-n} \rightarrow \text{Ext}^1(A/\mathfrak{m}^n, A) \rightarrow 0$ , and so  $\text{Ext}^1(A/\mathfrak{m}^n, A) \simeq \mathfrak{m}^{-n}/A$ . The colimit over  $n$  yields the desired formula for  $H_{\mathfrak{m}}^1(A)$ .

**THEOREM 2.3.** *Let  $(A, \mathfrak{m})$  be a 2-dimensional reduced local ring and let  $C = \bigcup \mathfrak{m}^{-n}$ . The following are equivalent:*

- (a)  $A$  is a Buchsbaum ring
- (b)  $\mathfrak{m}C \subseteq A$
- (c)  $\mathfrak{m}C = \mathfrak{m}$
- (d)  $\mathfrak{m}^{-1} = \mathfrak{m}^{-2}$
- (e)  $\mathfrak{m}^{-1} = \mathfrak{m}^{-n}, n \geq 2$
- (f)  $\mathfrak{m}^{-1} = C$
- (g)  $\mathfrak{m}^{-1}$  is a C-M ring.

*In this case,  $\mathfrak{m}^{-1}$  is the Cohen-Macaulification of  $A$ .*

*Proof.* If  $\text{grade } (\mathfrak{m}) = 2$  then  $A$  is C-M and all the statements are clear. So we may assume  $\text{grade } (\mathfrak{m}) = 1$ . From (2.1) and (2.2),  $A$  is a Buchsbaum ring if and only if  $\mathfrak{m}(C/A) = 0$ . This yields the equality of (a) and (b). Since  $ht(\mathfrak{m}) \neq 1$ , we have  $\mathfrak{m}^{-1}\mathfrak{m} = \mathfrak{m}$  and this easily gives that (b)-(f) are equivalent. The rest follows from (1.5).

**COROLLARY 2.4.** *Let  $(A, \mathfrak{m})$  be a 2-dimensional reduced local ring with finite integral closure. Let  $B = A + \mathfrak{m}C$ ,  $C = \bigcup \mathfrak{m}^{-n}$ .*

*Then  $(B, \mathfrak{m}C)$  is a local Buchsbaum ring finite over  $A$ . Every Buchsbaum ring containing  $A$  and finite over  $A$  contains  $B$ .*

*Proof.* Since  $C$  is the smallest extension of  $A$  which is C–M, it is also the smallest extension of  $B$  which is C–M, i.e.,  $C$  is the Cohen–Macaulification of  $B$ . Since  $\mathfrak{m}C \subseteq B$ , (2.3) gives that  $B$  is a Buchsbaum ring. Since  $B \subset C$  and  $C$  is finite over  $A$ ,  $B$  is finite over  $A$ .

Let  $R$  be any Buchsbaum ring which contains  $A$  and is finite over  $A$ .  $R$  is semilocal and we let  $J$  denote the Jacobson radical of  $R$ . By (2.3) (easily modified for semilocal rings) the Cohen–Macaulification of  $R$  is  $J^{-1}$ . By (1.5) we have  $C \subset J^{-1}$ . Thus,  $B = A + \mathfrak{m}C \subseteq A + \mathfrak{m}J^{-1} \subseteq R$  as was to be shown.

It remains only to show that  $B$  is local with maximal ideal  $\mathfrak{m}C$ . We first observe that  $\mathfrak{m}C$  is an ideal of  $C$  and  $\mathfrak{m}C \subseteq B$  so  $\mathfrak{m}C$  is an ideal of  $B$ . Also  $B/\mathfrak{m}C \simeq A/\mathfrak{m}$ , so  $\mathfrak{m}C$  is maximal in  $B$ . Since  $B$  is finite over  $A$  we need only show that  $\mathfrak{m}C$  is the only prime ideal of  $B$  which contains  $\mathfrak{m}B$ . This will follow from the following.

*Claim.*  $\mathfrak{m}C = \sqrt[\mathfrak{m}]{\mathfrak{m}B}$ .

*Proof.* We need only show  $\mathfrak{m}C \subset \sqrt[\mathfrak{m}]{\mathfrak{m}B}$ . We know that  $C = \mathfrak{m}^{-n}$ ,  $\forall n \geq 0$ , so let  $t \in \mathfrak{m}C$ . Then  $t = \sum x_i y_i$ ,  $x_i \in \mathfrak{m}$ ,  $y_i \in \mathfrak{m}^{-n}$ . Then  $t^n = \sum (x_{i_1} \cdots x_{i_n})(y_{i_1} \cdots y_{i_n})$ , which is in  $\mathfrak{m}^n(\mathfrak{m}^{-n})^n \subseteq \mathfrak{m}^n \mathfrak{m}^{-n} \subseteq \mathfrak{m}A$ . So, we obtain the stronger result, namely  $(\mathfrak{m}C)^n \subseteq \mathfrak{m}A \subseteq \mathfrak{m}B$ .

This completes the proof of the claim and hence of the corollary.

**COROLLARY 2.5.** *Let  $A$  be a 2-dimensional reduced (non-local) ring such that  $A_M$  is not Cohen–Macaulay for at most one maximal ideal  $M$  of  $A$ . Assume also that  $\bar{A}$  is a finite  $A$ -module.*

*Then  $B = A + MC$  is the smallest Buchsbaum ring containing  $A$  and finite over  $A$ , where  $C = \bigcup M^{-n}$ . Furthermore  $MC$  is a maximal ideal of  $B$  and is the only prime ideal of  $B$  lying over  $M$ .*

*Proof.* By (1.7)  $C$  is the smallest C–M ring finite over  $A$  which contains  $A$ . One easily shows (as in (1.7)) that, if  $A \subseteq R$  and  $R$  is finite over  $A$  and a Buchsbaum ring, then  $(MR)^{-1}$  is C–M and contains  $C$ . Hence  $B = A + MC \subseteq A + M(MR)^{-1} \subseteq R$ , as was to be shown.

The remaining statements follow immediately from (2.4).

### 3. SEMINORMAL RINGS

In this section we show that “nice” 2-dimensional seminormal rings are always Buchsbaum, but not necessarily C–M. This is a companion result to

the well-known (1.1). We then prove some simple facts about seminormal rings which we shall need later.

We first recall the following characterization of seminormality given by Swan in [21, Theorems 2.5, 3.4]: A ring  $A$  is *seminormal* in an overring  $B$  if for  $b \in B \setminus A$ , one of  $b^2, b^3$  is not in  $A$ ; if  $A$  is noetherian then  $A$  is seminormal if and only if  $A$  is reduced and seminormal in  $Q(A)$ .

**THEOREM 3.1.** *Let  $A$  be a 2-dimensional reduced noetherian seminormal ring. If  $A$  has finite normalization, then  $A$  is a Buchsbaum ring.*

*Proof.* Since  $A_{\mathfrak{p}}$  is C-M for all  $\mathfrak{p}$  of height one, it is enough to show that  $A_{\mathfrak{m}}$  is Buchsbaum for  $\mathfrak{m}$  a maximal ideal of  $A$  with height  $\mathfrak{m} = 2$ . As  $A_{\mathfrak{m}}$  is also seminormal, we may assume that  $A$  is local. Choose any  $x \in \mathfrak{m}, y \in \mathfrak{m}^{-n}$  and consider  $b = xy$  in  $Q(A)$ . By (1.3) we have that  $y^i \in \mathfrak{m}^{-N}$ , where  $N$  is fixed and  $i \geq 0$ . But then for  $i \gg N$  we have  $b^i \in \mathfrak{m}^N \mathfrak{m}^{-N} \subset A$ . As  $A$  is seminormal this gives that  $b \in A$  and so  $\mathfrak{m} \mathfrak{m}^{-n} \subset A$ . Since this is true for any  $n$  we get that  $A$  is Buchsbaum by (2.3).

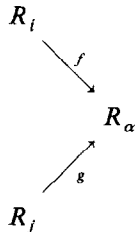
*Remark.* We do not know if the hypothesis on finite normalization is necessary in this theorem.

One frequently constructs seminormal rings by taking pull-backs of directed diagrams of rings. We would now like to be more precise about this notion. First note that the morphisms in a directed diagram of rings impose a partial ordering on the rings involved, which allow us to refer to the "initial" rings in the diagram.

**PROPOSITION 3.2.** *Let  $\{R_{\alpha}\}$  be a partially ordered diagram of reduced rings and suppose  $R_1, \dots, R_s$  are the initial rings in the diagram.*

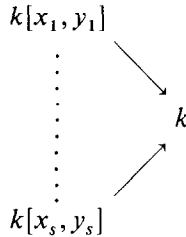
*The pull-back,  $A$ , of this diagram is seminormal in  $\prod_{i=1}^s R_i$ , and if  $R_1, \dots, R_s$  are seminormal then  $A$  is seminormal.*

*Proof.* Let  $b = (b_1, \dots, b_s)$  be an element of  $B = \prod_{i=1}^s R_i$ . To say that  $b \notin A$  is to say that for some subdiagram



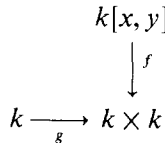
$f(b_i) \neq g(b_j)$ . If  $b^2, b^3 \in A$  then  $f(b_i)^2 = g(b_j)^2$  and  $f(b_i)^3 = g(b_j)^3$  in  $R_{\alpha}$ . Since  $R_{\alpha}$  is reduced this gives  $f(b_i) = g(b_j)$ . This contradiction shows that  $A$  is seminormal in  $B$ . Finally, the last sentence follows from [21, (3.4)].

EXAMPLE 3.3. Let  $A$  be the pull-back of the diagram



It is not hard to see that  $A$  is the coordinate ring of the  $s$  obvious coordinate planes in the  $2s$ -dimensional affine space  $\text{Spec}(k[x_1, y_1, \dots, x_s, y_s])$ . By (3.2)  $A$  is seminormal, so by (3.1)  $A$  is Buchsbaum. It is well known (see (5.5) of this paper) that  $A$  is not C-M, since  $\text{Spec}(A)$  is not connected in codimension 2.

EXAMPLE 3.4. Let  $A$  be the pull-back of the diagram



where  $f(x) = (0, 0)$ ,  $f(y) = (0, 1)$  and  $g(1) = (1, 1)$ . As above  $A$  is a seminormal Buchsbaum ring. Also,  $A$  is a domain and so is connected in codimension 2. In fact,  $A = k[x, xy, y^2 - y, y^3 - y^2]$ . However,  $A$  is not C-M. To see this, consider the maximal ideal  $m = (x, xy, y^2 - y, y^3 - y^2)$  of  $A$ . The element  $y$  of  $k[x, y]$  is not in  $A$  yet  $y m \subset A$ . Thus,  $A$  is not C-M; in fact,  $k[x, y]$  is the Cohen-Macaulification of  $A$ . Note also that  $m k[x, y] = (x, y) \cap (x, y - 1)$ . This example shows that the Cohen-Macaulification of a local ring need not be local.

*Remark 3.5.* Pull-backs of noetherian rings are not, in general, noetherian. However, this will be the case for the pullbacks considered in this paper. (See [4], Corollary 1.5.)

#### 4. PLANES THROUGH THE ORIGIN

In this section we set up some machinery for later sections. We first show that we can work in certain graded rings, rather than local rings. Then, we introduce the graph associated to a union of planes through the origin. Using this graph we construct the seminormalization  $S$  of  $A$ , where  $A$  is the coor-

dinate ring of the original union of planes. This will allow us, in Section 5, to compare the seminormalization and Cohen–Macaulification of  $A$ .

We begin by establishing some notation: Let  $R = k[x_0, \dots, x_n]$  so that  $\mathbb{A}^{n+1}(k) = \text{Spec}(R)$  and  $\mathbb{P}^n(k) = \text{Proj}(R)$ . We consider distinct planes  $V_1, \dots, V_s$  through the origin in  $\mathbb{A}^{n+1}(k)$  and let  $L_1, \dots, L_s$  denote the corresponding lines in  $\mathbb{P}^n(k)$ . By the phrase “ $V_i$  is a plane” we mean that the ideal  $\mathfrak{p}_i$  defining  $V_i$  is a homogeneous prime ideal of  $R$  generated by  $(n - 1)$  linearly independent linear forms. Thus,  $\mathfrak{p}_i$  has height  $= n - 1$ , and  $A = R/\bigcap \mathfrak{p}_i$  is the homogeneous coordinate ring both of  $\bigcup V_i$  and of  $\bigcup L_i$ .

*Note.* If we let  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$  be as above, then either  $\mathfrak{p}_i + \mathfrak{p}_j$  ( $i \neq j$ ) is a prime ideal of height  $n$  in  $R$  generated by  $n$  linearly independent linear forms or else  $\mathfrak{p}_i + \mathfrak{p}_j$  is the homogeneous maximal ideal  $M = (x_0, \dots, x_n)R$ .

**PROPOSITION 4.1.** *Let  $L_1, \dots, L_s$  be lines in  $\mathbb{P}^n(k)$  which all contain a common point  $P$ . If  $A$  is the homogeneous coordinate ring of  $\bigcup_{i=1}^s L_i$ , then  $A$  is a C–M ring.*

*Proof.* With no loss in generality we can assume the point is  $P = [1:0 \cdots 0]$ . If  $\mathfrak{p}_i \leftrightarrow L_i$ , there is no loss of generality in assuming  $x_1 \notin \bigcup_{i=1}^s \mathfrak{p}_i$  and that  $\mathfrak{p}_i \subset (x_0, x_2, \dots, x_n)$ . Then, the coordinate ring of the lines is

$$\frac{k[x_0, x_2, \dots, x_n]}{\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s} [x_1],$$

which is a polynomial ring over a C–M ring and hence is C–M.

**LEMMA 4.2.** *If  $\mathfrak{p} \neq MA$  is a prime ideal of  $A$ , then  $A_{\mathfrak{p}}$  is Cohen–Macaulay.*

*Proof.* If  $\mathfrak{p}$  has height  $= 1$  then  $A_{\mathfrak{p}}$  is C–M, since it is 1-dimensional and  $A$  is reduced.

If  $\mathfrak{p}$  contains a unique  $\mathfrak{p}_i A$  then  $A_{\mathfrak{p}}$  is regular, since it is a localization of the regular ring  $R/\mathfrak{p}_i$ . Finally we consider a maximal ideal  $\mathfrak{p} \neq MA$  which contains at least two of the  $\mathfrak{p}_i A$ . By making a linear change of variables in  $R$  we may assume  $\mathfrak{p}$  contains  $(x_1, \dots, x_n)$  but not  $x_0$ . We may choose  $f \in k[x_0]$  so that  $\mathfrak{p} = (f, x_1, \dots, x_n)A$ . Since  $\mathfrak{p} \neq MA$ ,  $f(0) \neq 0$  and so  $f$  is not a zero-divisor in  $A$ . Thus  $A/fA \simeq (k[x_0]/f)[x_1, \dots, x_n]/\bigcap \bar{\mathfrak{p}}_i$  is a 1-dimensional reduced ring. Hence  $\mathfrak{p}A_{\mathfrak{p}}$  contains a regular sequence of length 2, and so  $A_{\mathfrak{p}}$  is C–M.

**PROPOSITION 4.3.** *Let  $A$  be the coordinate ring of  $s$  planes through the origin, as above, and let  $M$  be the homogeneous maximal ideal of  $A$ . Then,*

- (i) *Each  $A$ -module  $M^{-n}$  is graded*
- (ii)  *$A$  is C–M if and only if  $A = M^{-1}$*

(iii)  $A$  is a Buchsbaum ring if and only if  $M^{-1} = M^{-2}$  if and only if  $M^{-1}$  is  $C$ - $M$ .

*Proof.* By combining (4.2) with (1.7) and (2.5) we obtain (ii) and (iii). Now the integral closure  $\bar{A}$  of  $A$  is graded, since  $\bar{A}$  is the product of  $R/\mathfrak{p}_i$  (see [23, p. 157]). By (1.2),  $C$  is a subring of  $\bar{A}$  and so  $M^{-n}$  is the graded  $A$ -submodule of  $\bar{A}$  consisting of those elements multiplied into  $A$  by the graded ideal  $M^n$ .

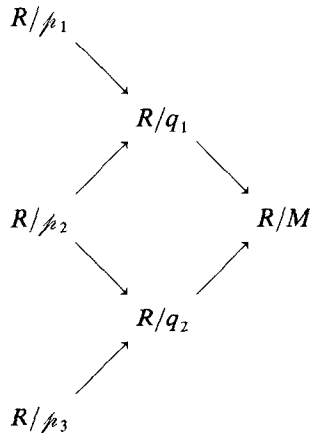
We now associate to  $A$  a directed diagram of quotient rings of  $A$ . Let  $q_1, \dots, q_t$  denote the height  $n$  primes of  $R$  of the form  $\mathfrak{p}_i + \mathfrak{p}_j$ . The set  $\{\mathfrak{p}_i\} \cup \{q_i\} \cup \{M\}$  of prime ideals of  $R$  is partially ordered by inclusion, and the corresponding diagram of rings  $\{R/\mathfrak{p}_i\} \cup \{R/q_i\} \cup \{R/M\}$  is partially ordered in the same way. We refer to this diagram of rings as the *graph associated to  $A$*  (or to the primes  $\mathfrak{p}_i$ , or to the planes  $V_i$ , or to the lines  $L_i$ ).

The vertices in the initial layer of the graph correspond to the planes  $V_i$  (or equivalently, to the lines  $L_i$ ). The vertices in the middle layer correspond to the lines  $V_i \cap V_j$  in  $\mathbb{A}^{n+1}(k)$  (or, equivalently, to the points  $L_i \cap L_j$  of  $\mathbb{P}^n(k)$ ). The final layer has only one vertex,  $R/M$ , which corresponds to the singular point at the origin of  $\mathbb{A}^{n+1}(k)$ .

We shall say that the graph above is *connected in codimension 2* if the graph remains connected after removal of the terminal vertex  $R/M$  (and also of all edges to it). The following are equivalent:

- (i) the graph of  $A$  is connected in codimension 2
- (ii)  $A$  is connected in codimension 2
- (iii)  $\cup L_i$  is connected in  $\mathbb{P}^n$ .

EXAMPLE 4.4. Consider  $\mathfrak{p}_1 = (x_0, x_1)$ ,  $\mathfrak{p}_2 = (x_1, x_2)$ ,  $\mathfrak{p}_3 = (x_2, x_3)$  in  $R = k[x_0, x_1, x_2, x_3]$ . For  $q_1 = (x_0, x_1, x_2)$  and  $q_2 = (x_1, x_2, x_3)$ , we have



In this example,  $A$  is the coordinate ring of three lines  $L_1, L_2, L_3$ , in  $\mathbb{P}^3$ , where  $L_1$  and  $L_3$  are skew and  $L_2$  meets  $L_1$  and  $L_3$ . This graph is connected in codimension 2. We shall see in Example (5.1) that the ring  $A = R/\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3$  is C-M.

**THEOREM 4.5.** *The pull-back ring,  $S$ , of the graph associated to  $A$  is the seminormalization of  $A$ . Moreover,  $S$  is graded and noetherian.*

*Proof.*  $S$  is seminormal by (3.2) and the seminormalization of  $A$  must contain  $S$  by [16]. The ring  $S$  is noetherian by Remark (3.5), and is graded since it is the inverse limit of a directed system of graded  $k$ -algebras and graded homomorphisms.

*Remark 4.6.* The ring  $S$  is a natural extension of the ring  $A$  of “polynomial” functions on  $V_i$  in the sense that  $S$  is the ring of “Piecewise polynomial” functions on  $V_i$ . In particular,  $\text{Spec}(S)$  is topologically homeomorphic to  $\text{Spec}(A)$  and  $\text{Proj}(S)$  is topologically homeomorphic to  $\text{Proj } A$ . It might be useful to note that as a graded ring  $S$  is not, in general, generated by forms of degree 1. (See, e.g., Proposition (10.1).) However, the unique homogeneous maximal ideal of  $S$  is the radical of an ideal generated by forms of degree 1. (See Remark (1.9).)

*Remark 4.7.* The graph associated to  $A$  is the same as the graph associated to  $S$ . Specifically, the minimal primes of  $S$  are  $I_i = \ker(S \rightarrow R/\mathfrak{p}_i)$ ,  $i = 1, \dots, s$ . Also,  $S/I_i \simeq R/\mathfrak{p}_i$  and  $S/I_i + I_j \simeq R/\mathfrak{p}_i + \mathfrak{p}_j$ . This is easy to see from the fact that the surjections  $A \rightarrow A/\mathfrak{p}_i A$  and  $A \rightarrow A/(\mathfrak{p}_i + \mathfrak{p}_j) A$  factor through the inclusions  $A \hookrightarrow S \hookrightarrow \prod R/\mathfrak{p}_i$ .

### 5. SEMINORMAL CONFIGURATIONS

In this section we consider unions of lines in  $\mathbb{P}^n$  which are linearly independent at each intersection point. This condition ensures that the homogeneous coordinate ring of these lines is locally seminormal, except possibly at the homogeneous maximal ideal. We then show that the seminormalization,  $S$ , of  $A$

(a) is the Cohen-Macaulification of  $A$  when the union of lines in  $\mathbb{P}^n$  is connected.

(b) contains the Buchsbaumification of  $A$ , is Buchsbaum, and is strictly contained in the Cohen-Macaulification of  $A$  when the union of lines in  $\mathbb{P}^n$  is not connected. (We shall identify the Buchsbaumification more precisely in §6).

In particular, a seminormal coordinate ring for a union of lines in  $\mathbb{P}^n$  is always Buchsbaum, and is C-M precisely when the lines are connected.

Before proving the statements above we would like to illustrate how one can use these results. These interesting examples are drawn from [4].

EXAMPLE 5.1. Consider the quadric surface  $x_0x_2 - x_1x_3$  in  $\mathbb{P}^3$ . Choose  $m$  lines from one ruling and  $n$  lines from the other ruling on this surface. If  $m, n \neq 0$  then this configuration of lines is always connected. Dayton and Roberts showed in [3, Example 7] that the coordinate ring of such a configuration is seminormal precisely when  $|m - n| \leq 1$ . From (5.6) we will obtain that, in this case, the coordinate ring is C-M. For example, the ring  $A$  of Example (4.4) is C-M since  $m = 2$  and  $n = 1$ .

For any  $m, n$ , the coordinate ring is always seminormal in codimension 1 by (5.3). It will follow from (5.9) and the Dayton-Roberts result that, if  $|n - m| \geq 2$ , the coordinate ring is never C-M. (We shall show in Section 10 below that the coordinate ring is Buchsbaum if  $|n - m| = 2$ , and never Buchsbaum if  $|n - m| \geq 3$ ).

EXAMPLE 5.2. In [3, Example 16] an example of a connected union of 10 lines in  $\mathbb{P}^3$  was given, whose coordinate ring is seminormal. From (5.6) we obtain that this coordinate ring is C-M. The configuration Dayton and Roberts describe is a "double 5" which is *not* on any cubic surface. (See Fig. 1.) This configuration is most intriguing since, if one removes any line from it, the resulting configuration (though still connected) no longer has a C-M coordinate ring.

On the other hand, the coordinate ring of a "double 5" configuration which *does* lie on a cubic surface is not C-M. This again follows from (5.6) and [3, Example 14].

These two "double 5's" point out that the graph is not always sufficient to determine whether or not the coordinate ring is C-M. The least degree of a non-singular surface which contains the configuration seems to play a subtle rôle (Fig. 1).

We now proceed to prove the statements we made at the beginning of this section.

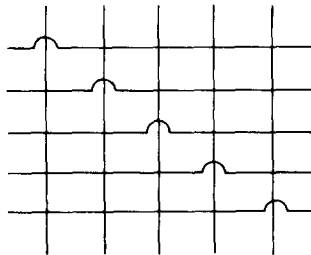


FIG. 1. The double 5.

**PROPOSITION 5.3.** *Let  $\mathcal{L}$  be a union of lines in  $\mathbb{P}^n$  for which the lines through each vertex have linearly independent directions. Let  $A$  be the homogeneous coordinate ring of these lines and let  $\mathfrak{p}$  be prime in  $A$ .*

*If  $\mathfrak{p}$  is not the homogeneous maximal ideal of  $A$  then  $A_{\mathfrak{p}}$  is seminormal.*

*Proof.* If necessary, perform a change of variables so that  $x_0 \notin \mathfrak{p}$ . Then  $A_{\mathfrak{p}}$  is a localization of  $A[x_0^{-1}] = A_0[x_0, x_0^{-1}]$ , where  $A_0$  is a quotient of  $k[x_1/x_0, \dots, x_n/x_0]$ .  $A_0$  is the coordinate ring of an affine open subset of the projective variety  $\mathcal{L}$ . By [18], the tangent directions through each vertex are linearly independent if and only if  $A_0$  is seminormal. This being the case,  $A_0[x_0]$  and its localization  $A_{\mathfrak{p}}$  are also seminormal.

**COROLLARY 5.4.** *If  $A$  is as in (5.3) then*

$$A \subseteq B \subseteq S \subseteq C.$$

*Here  $B$  is the Buchsbaumification of  $A$ ,  $S$  is the seminormalization of  $A$  and  $C$  is the Cohen–Macaulayfication of  $A$ .*

*Proof.* Let  $I$  denote the conductor from  $S$  to  $A$ . All of the minimal associated primes of  $I$  fail to be seminormal. To see this let  $\mathfrak{p}$  be such a prime and note that  $A_{\mathfrak{p}} \neq S_{A-\mathfrak{p}}$  and that  $S_{A-\mathfrak{p}}$  is the seminormalization of  $A_{\mathfrak{p}}$  [22, (1.1) and (2.2)]. Hence  $I$  is primary for  $M$ , the homogeneous maximal ideal of  $A$ . Thus  $M^n \subseteq I$ , so that  $M^n S \subseteq IS \subset A$  and so  $S \subseteq M^{-n} \subseteq C$  for  $n \gg 0$ . By (3.1) we know that  $S$  is Buchsbaum, so we must have  $B \subset S$ .

Here is the reason for the connectedness hypotheses that has appeared from time to time:

**THEOREM 5.5.** [11]. *Let  $G$  be the homogeneous coordinate ring of a union of irreducible curves in  $\mathbb{P}^n$ . If  $G$  is  $C$ – $M$  then the union of curves is connected.*

*This applies, in particular, to  $A$  and, via Remark 4.6, to  $S$ . If the graph associated to  $A$  is not connected in codimension 2 then neither  $A$  nor  $S$  is  $C$ – $M$  (although  $S$  is Buchsbaum by (3.1)).*

Example 5.2 above, and other examples that we shall see later in Sections 9 and 10, show that the converse of this theorem is not true:  $A$  can fail to be  $C$ – $M$  (or even Buchsbaum) and yet have a graph which is connected in codimension 2.

However, we do have a partial converse to Hartshorne’s theorem. For this theorem we make no assumption about linear independence of lines through a vertex.

**THEOREM 5.6.** *Let  $\mathcal{L}$  be any connected union of lines in  $\mathbb{P}^n$ , and let  $S$  be the pull-back of the associated graph. Then  $S$  is Cohen–Macaulay.*

*In particular, if  $A$  is seminormal, then  $A$  is Cohen–Macaulay.*

**EXAMPLE 5.7.** The ring  $A = k[x, y, z]/xy(x - y)$  shows that  $A$  can be  $C$ - $M$  without being seminormal. In this case  $S$  is the coordinate ring of three coordinate axes in  $\mathbb{P}^3$  while  $A$  is the coordinate ring of three coincident lines in  $\mathbb{P}^2$ . Both  $A$  and  $S$  are  $C$ - $M$ . The reason that  $A \neq S$  is that  $A_{\mathfrak{p}}$  fails to be seminormal for the height one prime  $\mathfrak{p} = (x, y)$ . We shall return to this discussion in Section 8.

*Remark 5.8.* If the graph associated to  $A$  is connected in codimension 2 then  $S$  is also the pull-back of the “deleted” graph, i.e., of the graph obtained by removing the top vertex  $R/M$ . This is an easy exercise in diagram chasing. Thus, to check that an element  $(b_1, \dots, b_s)$  of  $\Pi(R/\mathfrak{p}_i)$  lies in  $S$ , it is enough to check the consistency of  $(b_1, \dots, b_s)$  at the “ $R/q$ -level.”

*Proof of 5.6.* In view of Remarks (4.6) and (1.9) it will be enough to show that the unique homogeneous maximal ideal of  $S$  contains a regular sequence of length two. The most straightforward proof seems to be a direct construction of a regular sequence  $(f, g)$  in the homogeneous maximal ideal of  $S$ . Choose a linear form  $f \in R$  not in any  $q_j$  and let  $f = (f_1, \dots, f_s)$  denote the image of  $f$  in  $S$ . For each  $i$  choose  $g_i \in R/\mathfrak{p}_i$  to be an element of  $\bigcap_j \{q_j/\mathfrak{p}_i\}$  not in any associated prime of  $f(R/\mathfrak{p}_i)$ . This is possible because  $R/\mathfrak{p}_i$  is a UFD and the  $q_j/\mathfrak{p}_i$  are height one primes not containing  $f$ . Because every  $g_i$  vanishes in every  $q_j$ , the element  $g = (g_1, \dots, g_s)$  of  $\Pi(R/\mathfrak{p}_i)$  actually lives in  $S$ . By construction,  $(f, g)$  is a regular sequence in  $\Pi(R/\mathfrak{p}_i)$ .

Thus, if  $gh \in fS$  we have  $h = rf$  for some  $r = (r_1, \dots, r_s) \in \Pi(R/\mathfrak{p}_i)$ . In order to show that  $(f, g)$  form a regular sequence in  $S$  it is enough to show that if  $h \in S$  then  $r \in S$ . By (5.8) it is enough to check that  $r_i$  and  $r_j$  agree modulo  $q = \mathfrak{p}_i + \mathfrak{p}_j$  whenever  $q \neq M$ . But, since  $h = (h_1, \dots, h_s)$  is in  $S$ , we have that  $h_i$  and  $h_j$  agree modulo  $q$ . If we compute in the domain  $R/q$ , we obtain that

$$\bar{r}_i \bar{f} = \bar{h}_i = \bar{h}_j = \bar{r}_j \bar{f}.$$

As  $\bar{f} \neq 0$  in  $R/q$  we obtain  $\bar{r}_i = \bar{r}_j$ , which finishes the proof.

**COROLLARY 5.9.** Let  $\mathcal{L}$  be a union of lines in  $\mathbb{P}^n$  which are linearly independent at each vertex and let  $A$  be the coordinate ring of this configuration. Then

(a) if  $\mathcal{L}$  is connected, the seminormalization and Cohen–Macaulification of  $A$  coincide. In particular,  $A$  is seminormal if and only if  $A$  is  $C$ - $M$ ;

(b) if  $\mathcal{L}$  is not connected then the seminormalization  $S$  of  $A$  is a Buchsbaum ring which is strictly contained in the Cohen–Macaulification of  $A$ . Neither  $A$  nor  $S$  is  $C$ - $M$ .

*Proof.* This is immediate from (5.4), (5.5), (5.6) and (4.7).

6. THE DISCONNECTED CASE

The purpose of this section is to reduce the analysis of a disconnected union of lines in  $\mathbb{P}^n$  to the analysis of its connected pieces. Our main result is this:

**THEOREM 6.1.** *Let  $A$  be the homogeneous coordinate ring of a disconnected union  $\mathcal{L}$  of lines in  $\mathbb{P}^n$ . Let  $A_1, \dots, A_c$  be the homogeneous coordinate rings of the connected components  $\mathcal{L}_1, \dots, \mathcal{L}_c$  of  $\mathcal{L}$ . Then  $A$  is a Buchsbaum ring if and only if the following two conditions are satisfied:*

- (a) *Each  $A_i$  is a Buchsbaum ring*
- (b) *The embedding dimension of  $A$  is the sum of the embedding dimensions of the  $A_i$ . (We shall give a formula for the Buchsbaum type of  $A$  in Section 7.)*

**Remark 6.2.** Before we prove the theorem we wish to remark on the geometric meaning of condition (b). A union of lines in  $\mathbb{P}^n$  has embedding dimension  $r + 1$  if there is a linear subspace  $\mathbb{P}^r$  of  $\mathbb{P}^n$  which contains the lines, and no linear subspace of  $\mathbb{P}^r$  also contains the lines. The translation of this geometric condition to the algebraic notion of the embedding dimension can be easily seen by making an appropriate linear change of variables in  $k[x_0, \dots, x_n]$ .

**EXAMPLE 6.3.** Since the embedding dimension of  $A_i$  (or of  $\mathcal{L}_i$ ) is at least two, a union of  $c > (n + 1)/2$  skew lines in  $\mathbb{P}^n$  can never have a Buchsbaum coordinate ring. For example, the coordinate ring  $R/I$ ,  $I = (x_0, x_1) \cap (x_2, x_3) \cap (x_0 + x_2, x_1 + x_3)$ , of three skew lines in  $\mathbb{P}^3$  is not Buchsbaum. One can compute that the Cohen–Macaulification of this ring is  $M^{-2}$ . It follows from (6.1) that the set of 2 skew lines is the *only* disconnected configuration of lines in  $\mathbb{P}^3$  with a Buchsbaum coordinate ring.

In order to prove Theorem 6.1 we begin by establishing some notation. First we assume that the configuration of lines has  $c$  connected components. Let  $J_j$  denote the intersection of the primes  $\mathfrak{p}_i$  corresponding to the lines of the  $j$ th component. The homogeneous coordinate ring of the  $j$ th component is  $R/J_j = A_j$ . Now  $\bigcap J_j = \bigcap \mathfrak{p}_i$  and  $A \subseteq \prod_{i=1}^c R/J_i \subseteq \prod_{i=1}^c R/\mathfrak{p}_i$ .

**LEMMA 6.4.** *The Cohen–Macaulification  $C$  of  $A$  is the product of the Cohen–Macaulifications  $C_i$  of the  $A_i$ .*

*Proof.* By (1.7)  $C \subseteq \prod C_i$  and we need only show that  $\prod A_i \subseteq C$  to have that  $C = \prod C_i$ .

To obtain this inclusion it will be enough to prove that the conductor from  $\prod A_i$  to  $A$  is primary for the maximal ideal  $M$  of  $A$ , since then  $M^n(\prod A_i) \subset A$  for some  $n$ , and so  $\prod A_i \subseteq M^{-n} \subseteq C$ .

Now it is well known (see, e.g., [3]) that this conductor is  $\bigcap_j (J_j + \bigcap_{j \neq i} J_i)$ . It suffices to show that  $J_j + \bigcap_{j \neq i} J_i$  is  $M$ -primary for each  $j$ . But this last fact is clear from the definition of the  $J$ 's: the ideals  $J_j$  and  $\bigcap_{i \neq j} J_i$  correspond to disjoint projective varieties  $\mathcal{L}_j$  and  $\bigcap_{i \neq j} \mathcal{L}_i$ , so the sum of these ideals must be  $M$ -primary.

**PROPOSITION 6.5.** *Let  $B_1, \dots, B_c$  be Buchsbaum local rings of dimension 2 and assume that their residue class fields  $k_1, \dots, k_c$  each contain the field  $k$ . The pull-back,  $B$ , of the following diagram*

$$\begin{array}{ccc} B & \hookrightarrow & \prod B_i \\ \downarrow & & \downarrow \\ k & \hookrightarrow & \prod k_i \end{array}$$

is a Buchsbaum local ring.

*Proof.* Let  $C$  be the Cohen–Macaulification of  $B$ . The conductor from  $\prod B_i$  to  $B$  is  $\prod \mathfrak{m}_{B_i} = \mathfrak{m}_B$ . Hence  $\prod B_i \subset \mathfrak{m}_B^{-1} \subset C$ . As in (6.4),  $C$  is the product  $\prod C_i$  of the Cohen–Macaulifications of the  $B_i$ . Since the height of  $\mathfrak{m}_i$  is not one,  $\mathfrak{m}_{B_i} C_i = \mathfrak{m}_{B_i}$  for each  $i$ . Thus,  $\mathfrak{m}_B C = (\prod \mathfrak{m}_{B_i})(\prod C_i) = \prod (\mathfrak{m}_{B_i} C_i) = \prod \mathfrak{m}_{B_i} = \mathfrak{m}_B$ . By (2.3c),  $B$  is a Buchsbaum ring.

**COROLLARY 6.6.** *Let  $A$  be the homogeneous coordinate ring of a disconnected union  $\mathcal{L}$  of lines in  $\mathbb{P}^n$ . Let  $A_1, \dots, A_c$  denote the coordinate rings of the components of  $\mathcal{L}$ . Then the following are equivalent:*

- (a)  $A$  is Buchsbaum
- (b) Each  $A_i$  is Buchsbaum, and the conductor from  $\prod A_i$  to  $A$  is  $MA$
- (c) Each  $A_i$  is Buchsbaum, and the following diagram is a pull-back:

$$\begin{array}{ccc} A & \hookrightarrow & \prod A_i \\ \downarrow & & \downarrow \\ k & \hookrightarrow & \prod k \end{array}$$

*Proof.* The equivalence of (b) and (c) is an easy exercise. By the obvious graded version of (6.5) we have that (c) implies (a).

Now assume (a) holds, i.e., that  $A$  is Buchsbaum. By (2.5),  $MC \subset A$ , where  $C$  is the Cohen–Macaulification of  $A$ . By (6.4),  $C = \prod C_i$ , and so  $\prod A_i \subset C$ . Hence  $M(\prod A_i) \subset A$ , i.e., the conductor of  $\prod A_i$  to  $A$  is  $MA$ . From (2.3c) we see that  $MA_i \subset MA$  for each  $i$ , and so  $MA = \prod (MA_i)$ . We then have  $(MA_i) C_i = (MC_i) A_i \subset (MC) A_i \subset AA_i = A_i$  for each  $i$ . By (2.5) the  $A_i$  are each Buchsbaum rings. This shows that (a) implies (b).

*Proof of (6.1).* In view of (6.6) we need only show that the following are equivalent:

- (i) The conductor from  $\prod A_i$  to  $A$  is  $MA = \prod MA_i$
- (ii)  $\text{emb dim}(A) = \sum \text{emb dim}(A_i)$ .

If  $MA = \prod MA_i$  then  $M^2A = \prod M^2A_i$  and so  $M/M^2 = \prod (MA_i)/(MA_i)^2$ . Thus (i) implies (ii).

Now  $A \rightarrow \prod A_i$  is a graded homomorphism, so  $M_1$ , the degree 1 part of  $MA$ , is a vector subspace of  $\prod (MA_i)_1$ , the degree 1 part of  $\prod (MA_i)$ . Since  $MA$  and  $MA_i$  are generated by their degree 1 parts, we always have  $\text{emb dim } A = \dim_k M_1 \leq \sum \dim_k (MA_i)_1$ . Equality holds if and only if  $M_1 = \prod (MA_i)_1$ , i.e., if and only if  $MA = \prod MA_i$ .

**THEOREM 6.7.** *Let  $A$  be the homogeneous coordinate ring of a disconnected union  $\mathcal{L}$  of lines in  $\mathbb{P}^n$ , and let  $A_1, \dots, A_c$  denote the coordinate rings of the components of  $\mathcal{L}$ . Let  $B_i$  denote the Buchsbaumification of  $A_i$  for  $i = 1, \dots, c$ . Then the Buchsbaumification  $B$  of  $A$  is the pull-back in the following diagram:*

$$\begin{array}{ccc}
 B & \hookrightarrow & \prod B_i \\
 \downarrow & & \downarrow \\
 k & \hookrightarrow & \prod k
 \end{array}$$

*Proof.* By (6.5) the ring  $B$  is a Buchsbaum ring with maximal ideal  $\prod M_{B_i}$ . By standard pull-back arguments,  $B = A + M_B$ . By (4.2) and (2.5),  $M_{B_i} = M_{A_i} C_i = M_A C_i$  where  $C_i$  is the Cohen–Macaulification of  $B_i$ . By (6.3)  $C = \prod C_i$ . These facts combine to yield

$$\prod M_{B_i} = \prod (M_A C_i) = M_A \left( \prod C_i \right) = M_A C.$$

Thus  $B = A + M_A C$ , which implies (by (2.5)) that  $B$  is the Buchsbaumification of  $A$ .

### 7. THE BUCHSBAUM TYPE OF A DISCONNECTED UNION OF LINES

In the last section we saw precisely when the disconnected union of lines in  $\mathbb{P}^n$  has a Buchsbaum coordinate ring. In this section we compute the Buchsbaum type of this ring.

**THEOREM 7.1.** *Let  $A$  be the homogeneous coordinate ring of a disconnected union,  $\mathcal{L}$  of lines in  $\mathbb{P}^n$ , and let  $A_1, \dots, A_c$  be the coordinate rings of the connected components of  $\mathcal{L}$ .*

If  $A$  is a Buchsbaum ring then each  $A_i$  is a Buchsbaum ring and the Buchsbaum type of  $A$ , denoted  $Bbm(A)$ , is given by:

$$Bbm(A) = (c - 1) + \sum_{i=1}^c Bbm(A_i).$$

EXAMPLES 7.2. (a) From (6.2) we see that the union of  $c$  skew lines in  $\mathbb{P}^{2c-1}$  is Buchsbaum if and only if the embedding dimension of this configuration is  $2c$ . By (7.1) the coordinate ring of such a configuration has Buchsbaum type  $c - 1$ .

(b) In  $\mathbb{P}^4$  let  $\mathcal{L} = L_1 \cup L_2 \cup L_3$ , where  $L_1$  and  $L_2$  meet and  $L_3$  is skew to  $L_1$  and  $L_2$ .  $\mathcal{L}$  has two connected components whose embedding dimensions are 3 and 2, respectively. Each component has a C–M coordinate ring. Applying (7.1) we obtain that the coordinate ring of this configuration is

(a) a Buchsbaum ring of type 1 if  $L_3$  does not meet the plane spanned by  $L_1$  and  $L_2$

(b) is not Buchsbaum otherwise.

*Proof of (7.1).* The fact that if  $A$  is a Buchsbaum ring then each  $A_i$  is a Buchsbaum ring is contained in (6.1). In order to calculate the Buchsbaum type of  $A$  we must first select a system of parameters for the homogeneous maximal ideal of  $A$ . To this end, we first select a system of parameters  $x_i, y_i$  of degree one in  $A_i$ . Then  $x = (x_1, \dots, x_c), y = (y_1, \dots, y_c)$  lie in  $A$  by (6.6b) and clearly form a system of parameters. Write  $q = (x, y)A$  and  $q_i = (x_i, y_i)A_i$ .

LEMMA 7.3.  $length(A/q) = (c - 1) + \sum length(A_i/q_i)$ .

*Proof.* We decompose the ideals  $q$  and  $q_i$  into  $k$ -vector spaces as follows:

$$q_i = (x_i, y_i)A_i \oplus M_i(x_i, y_i)$$

$$q = (x, y)A \oplus \prod M_i(x_i, y_i).$$

This yields

$$length(A_i/q_i) = 1 + (emb\ dim(A_i) - 2) + \dim(M_i^2/M_i q_i)$$

$$length(A/q) = 1 + (emb\ dim(A) - 2) + \sum \dim(M_i^2/M_i q_i).$$

Thus

$$\begin{aligned} length(A/q) - \sum length(A_i/q_i) &= (emb\ dim(A) - 1) - \sum \left( \sum emb\ dim(A_i) - 1 \right). \end{aligned}$$

By (6.1) we find this expression equals  $(c - 1)$ , completing the proof of the lemma.

LEMMA 7.4.  $\text{length}(q^n/q^{n+1}) = (c - 1) + \sum \text{length}(q_i^n/q_i^{n+1})$ .

*Proof.* Let  $[I]_r$  denote the  $r$ th graded piece of a homogeneous ideal  $I$ . Then

$$q_i^n = (x_i^n k \oplus x_i^{n-1} y_i k \oplus \cdots \oplus y_i^n k) \oplus [M_i q_i^n]_{n+1} \oplus M_i^2 q_i^n$$

and

$$q^n = (x^n k \oplus x^{n-1} y k \oplus \cdots \oplus y^n k) \oplus [M q^n]_{n+1} \oplus M^2 q^n.$$

We thus obtain

$$\begin{aligned} \text{length}(q^n/q^{n+1}) &= \dim_k[q^n]_n + \dim_k[M q^n]_{n+1} \\ &\quad - \dim_k[q^{n+1}]_{n+1} + \dim_k(M^2 q^n/M^2 q^{n+1}) \end{aligned}$$

and

$$\begin{aligned} \text{length}(q_i^n/q_i^{n+1}) &= \dim_k[q_i^n]_n + \dim_k[M_i q_i^n]_{n+1} \\ &\quad - \dim_k[q_i^{n+1}]_{n+1} + \dim_k(M_i^2 q_i^n/M_i^2 q_i^{n+1}). \end{aligned}$$

So

$$\begin{aligned} &\text{length}(q^n/q^{n+1}) - \sum \text{length}(q_i^n/q_i^{n+1}) \\ &= \left( \dim_k[q^n]_n - \sum \dim_k[q_i^n]_n \right) \\ &\quad - \left( \dim_k[q^{n+1}]_{n+1} - \sum \dim_k[q_i^{n+1}]_{n+1} \right) \\ &= (n + 1)(1 - c) - (n + 2)(1 - c) = c - 1. \end{aligned}$$

This completes the proof of Lemma (7.4).

For  $n \gg 0$ ,  $\text{length}(q^n/q^{n+1})$  is a linear function of  $n$ , having leading coefficient  $e(q, A)$ . Similarly,  $\text{length}(q_i^n/q_i^{n+1})$  is a linear polynomial function of  $n$  with leading coefficient  $e(q_i, A_i)$ . From (7.4) we obtain  $e(q, A) = \sum e(q_i, A_i)$ . From (7.3) we deduce that

$$\begin{aligned} \text{Bbm}(A) &= \text{length}(A/q) - e(q, A) \\ &= \left[ (c - 1) + \sum \text{length}(A_i/q_i) \right] - \sum e(q_i, A_i) \\ &= (c - 1) + \sum \text{Bbm}(A_i). \end{aligned}$$

### 8. COHEN-MACAULIFICATION OF UNIONS OF LINES

In Section 5 we identified the Cohen-Macaulification of a union  $\mathcal{L}$  of lines in  $\mathbb{P}^n$  under the restrictions that (a)  $\mathcal{L}$  was connected, and (b) at each

vertex of  $\mathcal{L}$  the lines of  $\mathcal{L}$  through the vertex were in linearly independent directions. In this section we identify the Cohen–Macaulification without these restrictions. The trick is to accept the situation at the height one primes (which are always C–M even if not necessarily seminormal). We implement the trick by adding an extra layer to the graph associated to  $\mathcal{L}$  (introduced in Section 4) and by deleting the terminal vertex to decompose the graph into its connected pieces.

CONSTRUCTION (8.1). Let  $\mathcal{L} = L_1 \cup \dots \cup L_s$  be a configuration of lines in  $\mathbb{P}^n$  with corresponding primes  $\mathfrak{p}_1, \dots, \mathfrak{p}_s$  in  $R = k[x_0, \dots, x_n]$ . We adopt the notation of Section 4 so that in particular  $q_1, \dots, q_t$  are the primes of  $R$  corresponding to the “vertices” of  $\mathcal{L}$ . In order to avoid ambiguous usage of the word “vertex,” we shall refer to those points of  $\mathbb{P}^n$  of the form  $L_i \cap L_j$  as the *cluster points* of the configuration. Thus each cluster point  $Q_j$  corresponds both to a prime  $q_j$  of  $R$  and to a radical ideal

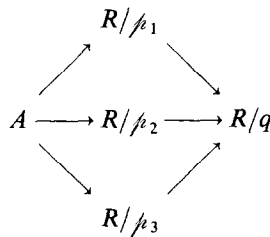
$$J_j = \bigcap \{ \mathfrak{p}_i \mid \mathfrak{p}_i \subset q_j \}$$

of height  $n - 1$  of  $R$ . The union of the subset of lines in  $\mathcal{L}$  through the cluster point  $Q_j$  has homogeneous coordinate ring  $R/J_j$ , a C–M ring. Note that there is a 1–1 correspondence between the ideals  $q_j$  and the ideals  $J_j$ .

The partially ordered set  $\{J_j\} \cup \{\mathfrak{p}_i\} \cup \{q_j\}$  of ideals of  $R$  gives rise to a directed diagram  $\Gamma$  of rings  $\{R/J_j\} \cup \{R/\mathfrak{p}_i\} \cup \{R/q_j\}$  and we let  $C$  denote the pull-back of this diagram. Our aim in this section is to show that  $C$  is the Cohen–Macaulification of  $A$ , even if  $\mathcal{L}$  is not connected.

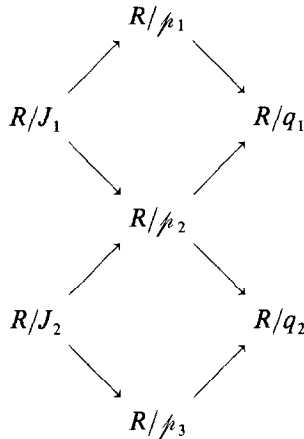
Remark 8.2. If  $\mathcal{L}$  has connected components  $\mathcal{L}_1, \dots, \mathcal{L}_c$ , then  $\Gamma$  decomposes as a disjoint union of diagrams  $\Gamma_1, \dots, \Gamma_c$ . As  $C$  is the product of the pull-backs of the  $\Gamma_i$ , and each  $\Gamma_i$  is the diagram constructed by (8.1) for the component  $\mathcal{L}_i$ , it is enough to consider the case in which  $\mathcal{L}$  is connected. This is because the Cohen–Macaulification of  $A$  is the product of the Cohen–Macaulifications of the  $A_i$  by (6.3). In the connected case, we could augment  $\Gamma$  by adding a terminal vertex  $R/M$  without affecting the pull-back ring, as noted in Remark (5.8).

EXAMPLE 8.3. (a) For the ring  $A$  of Example (5.7) the directed diagram is



The pull-back ring is clearly  $A$ .

(b) For the configuration of Example (4.4) the diagram is



The rings  $R/J_1, R/J_2$  are the pull-backs of the relevant portions of the diagram, so the pull-back ring  $C$  could also be obtained by deleting the  $R/J$ 's from the diagram, and then taking the pull-back. The pull-back of the deleted diagram is  $A$  by (5.1) and (5.9), so once again  $A = C$ .

LEMMA 8.4. *If  $\mathcal{L}$  is connected,  $C$  is a subring of the seminormalization  $S$  of  $A$ .*

*Proof.* By deleting the  $R/J$ 's, we obtain a diagram whose pull-back ring is  $S$  by (4.5) and (5.8). This gives a map  $C \rightarrow S$ . If we think of  $C$  as a subring of  $\prod (R/J_j)$  and  $S$  as a subring of  $\prod (R/\mathfrak{p}_i)$ , then an element  $(c_1, \dots, c_t)$  of  $C$  maps to zero only if for each  $j$  the element  $c_j$  is in the kernel of  $R/J_j \rightarrow \prod (R/\mathfrak{p}_i)$ . But  $R/J_j$  is a subring of  $\prod \{R/\mathfrak{p}_i | J_j \subset \mathfrak{p}_i\}$ , so  $c_j = 0$ . This shows that  $C \subset S$ .

LEMMA 8.5.  *$C$  is a graded ring. If  $P$  is a prime ideal of  $C$  other than the graded maximal ideal, then  $A_{P \cap A} \cong C_P$ .*

*Proof.*  $C$  is graded as it is the pull-back of a system of graded ring homomorphisms. Using (8.4), we think of  $C$  as a subring of  $\prod (R/\mathfrak{p}_i)$  and write its elements as  $c = (c_1, \dots, c_s)$ . The graded maximal ideal  $M_C$  is the set of all elements of  $C$  mapping to 0 in  $R/M$ .

If  $P$  does not contain any prime  $q_j$ , then  $C_P \cong S_P \cong A_{P \cap A}$  since  $A_{P \cap A}$  is seminormal (in fact, regular). If  $P \neq M_C$  contains any  $q_j$ , it contains exactly one  $q_j$ . In this case we consider the diagram  $\Gamma_P$  obtained from  $\Gamma$  by localizing at the multiplicative set  $A - (P \cap A)$ . We claim that  $C_P$  is the pull back of  $\Gamma_P$ . This is because localization, being a directed colimit, commutes with finite limits, i.e., with pull backs [13, p. 211].

Now the composition  $A_{P \cap A} \rightarrow C_P \rightarrow (R/J_j)_{P \cap A}$  is an isomorphism by the definition of  $J_j$ , since  $P \cap A$  meets every prime  $\not\subseteq_i A$  not contained in  $q_j A$ . Thus the map  $A_{P \cap A} \rightarrow C_P$  of local rings is an isomorphism.

**THEOREM 8.6.** *The pull back ring  $C$  constructed in (8.1) is  $C$ - $M$ .*

*Proof.* By (8.5) and (1.5) it is enough to see that  $M_C^{-1} = C$ . By (8.4) and (5.6) we have  $M_C^{-1} \subset \prod_{i=1}^s R/\not\subseteq_i$ . Let  $b = (b_1, \dots, b_s)$  be an element of  $M_C^{-1}$  not in  $C$ . Then we can reindex so that  $J_1 = \not\subseteq_1 \cap \dots \cap \not\subseteq_k$  and  $(b_1, \dots, b_k)$  is an element of  $\prod_{i=1}^k R/\not\subseteq_i$  not in the subring  $R/J_1$ . For  $a = (a_1, \dots, a_s)$  in  $M_C$  we have  $ab \in C$ , so in particular  $(a_1 b_1, \dots, a_k b_k) \in R/J_1$ . This shows that  $(b_1, \dots, b_k)$  lies in  $(MR/J_1)^{-1}$ , and not in  $R/J_1$ . This contradicts the fact that  $R/J_1$  is  $C$ - $M$ , proving that  $C = M_C^{-1}$ , and that  $C$  is  $C$ - $M$ .

**COROLLARY 8.7.** *The pull-back ring  $C$  constructed in (8.1) is the Cohen-Macaulification of  $A$ .*

*Proof.* By (8.5), the conductor from  $C$  to  $A$  is primary for the maximal ideal  $M_A$ . Hence  $C$  is the Cohen-Macaulification of  $A$  by (1.7).

**COROLLARY 8.8.** *If  $\mathcal{L}$  is connected, then  $A$  is Buchsbaum if and only if the subring  $A$  of  $C$  contains  $M_C$ .*

### 9. “ $n \times 1$ ” CONFIGURATIONS IN $\mathbb{P}^3$

In this section we consider “ $n \times 1$ ” configurations  $\mathcal{L}$  of lines in  $\mathbb{P}^3$ . This means  $\mathcal{L} = L_0 \cup L_1 \cup \dots \cup L_n$ , where  $L_i \cap L_j = \emptyset$  for  $1 \leq i \neq j \leq n$  and  $L_0$  meets each of  $L_1, \dots, L_n$ . We will show that

- (9.1) If  $n \leq 2$  the “ $n \times 1$ ” configuration is  $C - M$ .
- (9.4) The “ $3 \times 1$ ” configuration is Buchsbaum but not  $C - M$
- (9.6), (9.9) If  $n \geq 4$ , the “ $n \times 1$ ” configuration is not even Buchsbaum.

(By the statement “the configuration is Buchsbaum,” we of course mean that the homogeneous coordinate ring  $A$  of  $\mathcal{L}$  is a Buchsbaum ring.)

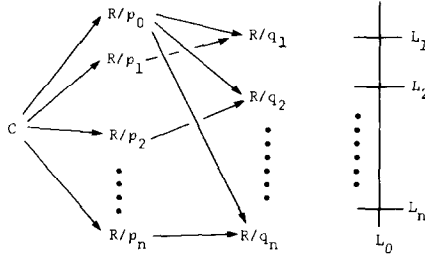
Our interest in these configurations grew out of the examples of [3], where they considered such configurations lying on a quadric surface and studied the seminormality. A completely different approach to “ $n \times 1$ ” configurations lying on a quadric surface is given in [7].

We also analyse a related family of configurations in Example (9.7). These configurations are obtained by deforming the “ $4 \times 1$ ” configurations so that  $L_3$  and  $L_4$  meet. The (5-dimensional) family of all such configurations has two non-empty components, only one of which consists of Buchsbaum

configurations. This example illustrates that the graph of intersections of a configuration is not always sufficient to determine whether or not the coordinate ring of the configuration is Buchsbaum (see also (5.2) above).

We adopt the notation of Section 4. Since only two lines meet at any cluster point, the Cohen–Macaulification  $C$  of  $A$  is the pull back ring of the associated graph by (5.9).

So, for the “ $n \times 1$ ” configuration we have:



*Remark 9.1.* When  $n = 1$ , the configuration has (after a suitable change of variable) coordinate ring  $k[x, y, z]/x(x - y)$ . When  $n = 2$ , the configuration was discussed in (4.4). Both configurations are seminormal and C–M by (5.1), since they lie on (many) quadric surfaces.

**LEMMA 9.2.** *Let  $I$  be the ideal of all elements  $(0, b_1, \dots, b_n)$  of  $\prod_{i=0}^n R/\mathfrak{p}_i$  which are in  $C$ . Then  $C = A + I$ .*

*Proof.* If  $b = (b_0, b_1, \dots, b_n) \in C$ , choose  $a \in A$  mapping onto  $b_0 \in A/\mathfrak{p}_0$ . Then  $b - a = (0, b_1 - b_0, \dots, b_n - b_0)$  is an element of  $I$ .

**COROLLARY 9.3.** *Let  $A$  be the coordinate ring of an “ $n \times 1$ ” configuration. For each  $i = 1, \dots, n$  there is a linear form  $z_i$  of  $R$  for which  $\mathfrak{p}_i + \mathfrak{p}_0 = (\mathfrak{p}_i, z_i)$ . Let  $\zeta_i = (0, \dots, 0, z_i, 0, \dots, 0)$ . Then*

*$C = A[\zeta_1, \dots, \zeta_n]$  and the maximal ideal of  $C$  is  $(M, \zeta_1, \dots, \zeta_n)$ . In particular,  $A$  is Buchsbaum if and only if  $M\zeta_i \subset A$ ,  $1 \leq i \leq n$ .*

*Proof.* It is clear that the  $\zeta_i$  are in  $I$ . If  $(0, b_1, \dots, b_n) \in I$ , then each  $b_i \in R/\mathfrak{p}_i$  is in the kernel of  $R/\mathfrak{p}_i \rightarrow R/q_i$ . By (4.1), this kernel is principal and generated by a linear form  $z_i$ . Some  $a_i \in A$  exists so that  $b_i = a_i z_i$  and hence  $(0, b_1, \dots, b_n) = \sum a_i \zeta_i$ . Thus,  $I = \sum A\zeta_i$  so that  $C = A[\zeta_1, \dots, \zeta_n]$  by (9.2).

Finally, the last statement in the corollary is a restatement of (2.3) b) for the situation at hand.

**PROPOSITION 9.4.** *The “ $3 \times 1$ ” configuration of lines in  $\mathbb{P}^3$  is Buchsbaum but not C–M.*

*Proof.* Choose coordinates  $(x, y, z, w)$  for  $\mathbb{P}^3$  such that  $x=0$  on  $L_0 \cup L_1$ ,  $y=0$  on  $L_0 \cup L_2$ ,  $z=0$  on  $L_1$  and  $w=0$  on  $L_2$ . Then  $A = R/\mathfrak{p}_0 \cap \dots \cap \mathfrak{p}_3$ , where  $\mathfrak{p}_0 = (x, y)$ ,  $\mathfrak{p}_1 = (x, z)$ ,  $\mathfrak{p}_2 = (y, w)$  and  $\mathfrak{p}_3 = (ax + y, cx + dy + w)$  with  $ad \neq 0$ . In  $\prod R/\mathfrak{p}_i$  we have the elements:

$$\begin{aligned} x &= (0, 0, x, x) \\ y &= (0, y, 0, -ax) \\ z &= (z, 0, z, -d^{-1}(w + cx)) \\ w &= (w, w, 0, w) \\ \zeta_1 &= (0, y, 0, 0) \\ \zeta_2 &= (0, 0, x, 0) \\ \zeta_3 &= (0, 0, 0, x). \end{aligned}$$

We have:

$$\begin{aligned} x\zeta_2 &= (0, 0, x^2, 0) = x^2 + a^{-1}xy \\ -ax\zeta_3 &= (0, 0, 0, -ax^2) = xy \\ y\zeta_1 &= (0, y^2, 0, 0) = y^2 + axy \\ y\zeta_3 &= (0, 0, 0, -ax^2) = xy \\ z\zeta_2 &= (0, 0, xz, 0) = xz + a^{-1}yz \\ z\zeta_3 &= (0, 0, 0, -d^{-1}x(w + cx)) = a^{-1}yz \\ w\zeta_1 &= (0, yw, 0, 0) = yw + axw \\ w\zeta_3 &= (0, 0, 0, xw) = xw. \end{aligned}$$

By (9.3) this is sufficient to prove that  $A$  is Buchsbaum,  $A$  is not C-M by (5.1), or more directly since  $\zeta_1 \notin A$ .

*Remark.* A tedious computation shows that the Buchsbaum type is 1.

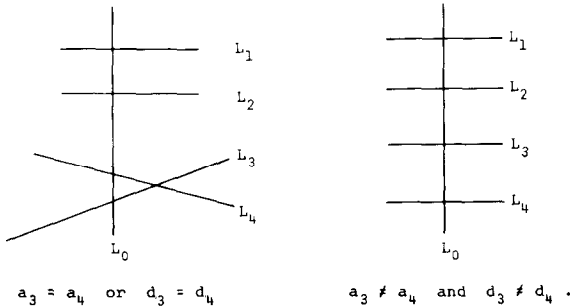
**LEMMA 9.5.** *If an “ $n \times 1$ ” configuration in  $\mathbb{P}^3$  has a Buchsbaum coordinate ring, then for each  $i \geq 1$  there is a quadric surface in  $\mathbb{P}^3$  containing all the lines of the configuration except for  $L_i$ .*

*Proof.* In the notation of (9.3),  $z_i\zeta_i = (0, \dots, 0, z_i^2, 0, \dots, 0)$  must be an element of  $A$  if  $A$  is Buchsbaum. For this to be the case there must be a quadratic form of  $R$  which is in  $\mathfrak{p}_j$  for  $j \neq i$ , but which does not belong to  $\mathfrak{p}_i$ . The quadric surface is given by the vanishing of this form.

**PROPOSITION (9.6).** *If  $n \geq 5$ , the coordinate ring of an " $n \times 1$ " configuration in  $\mathbb{P}^3$  cannot be Buchsbaum.*

*Proof.* Any " $3 \times 1$ " subconfiguration lies on a unique quadratic surface. (This is clear from the proof of (9.4).) This quadric surface must contain all the lines  $L_j$  except  $L_i$  by (9.5) and this is impossible for all  $i$  if  $n \geq 5$ .

**EXAMPLE (9.7).** Consider the prime ideals  $\mathfrak{p}_0 = (x, y)$ ,  $\mathfrak{p}_1 = (x, z)$ ,  $\mathfrak{p}_2 = (y, w)$  and  $\mathfrak{p}_i = (a_i x + y, c_i x - d_i z + w)$  for  $i = 3, 4$ . We assume  $a_3 a_4 d_3 d_4 \neq 0$ . Then each  $L_0 \cup L_1 \cup L_2 \cup L_i$  forms a " $3 \times 1$ " configuration in  $\mathbb{P}^3$  and  $L_0 \cup L_1 \cup L_2 \cup L_3 \cup L_4$  is a " $4 \times 1$ " configuration if and only if  $a_3 \neq a_4$  and  $d_3 \neq d_4$ .



We will show that  $A = R/\mathfrak{p}_0 \cap \dots \cap \mathfrak{p}_4$  is Buchsbaum if and only if  $a_3 = a_4$ .

In  $\prod (R/\mathfrak{p}_i)$ , we have:

$$\begin{aligned} x &= (0, 0, x, x, x) \\ y &= (0, y, 0, -a_3 x, -a_4 x) \\ z &= (z, 0, z, d_3^{-1}(w + c_3 x), d_4^{-1}(w + c_4 x)) \\ w &= (w, w, 0, w, w) \\ \zeta_2 &= (0, 0, x, 0, 0) \\ \mathfrak{m} &= (0, 0, 0, x(c_4 x - d_4 z + w), 0). \end{aligned}$$

It is easy to see that  $\zeta_2, \mathfrak{m}$  are elements of the pullback ring  $C$ , while  $\zeta_2 \notin A$ . Thus,  $A$  is not  $C$ - $M$ . The following lemma shows that  $A$  is not Buchsbaum unless  $a_3 = a_4$ , for otherwise,  $\mathfrak{m}^{-1} \neq C$ .

**LEMMA 9.8.** *With the notation above, assume  $d_3 \neq d_4$ . Then,*

- (i)  $x\zeta_2 \in A \Rightarrow a_3 = a_4$  or  $a_3 d_4 = a_4 d_3$
- (ii)  $z\zeta_2 \in A \Rightarrow (a_3 = a_4$  or  $a_3 c_4 = a_4 c_3)$ , and  $a_3 d_4 \neq a_4 d_3$  in either case.
- (iii)  $\zeta_2 \in \mathfrak{m}^{-1} \Rightarrow a_3 = a_4$ .

*Proof.* In order that  $x\zeta_2$  belong to  $A$  we must have  $u_{ij} \in k$  such that  $x\zeta_2 = u_{11}x^2 + u_{12}xy + \dots + u_{44}w^2$ . As all elements are in  $\prod_{i=0}^4 R/\mathfrak{p}_i$  this amounts to 5 equations in the  $u_{ij}$ . The first three coordinate equations force most  $u_{ij}$  to be equal to zero. We also find that  $u_{11} = 1$  and that the coordinate equations give

$$-x^2 = (-a_i x^2) u_{12} + (xw) u_{14} - a_i d_i^{-1} (xw + c_i x^2) u_{23}.$$

Equating coefficients of  $x^2$  and  $xw$  gives four equations in the three variables  $u_{12}, u_{14}, u_{23}$ . A simple determinant argument shows that if there is a solution we must have either  $a_3 = a_4$  or  $a_3 d_4 = a_4 d_3$ . This proves (i).

The proof of (ii) is similar. In this case an inspection of the first three coordinates shows that  $z\zeta_2 \in A$  iff there are  $u_{ij} \in k$  so that

$$z\zeta_2 - xz = u_{12}(xy) + u_{14}xw + u_{23}yz \quad \text{in } \prod_{i=0}^4 R/\mathfrak{p}_i.$$

We find, again by considering determinants, that if there is a solution then we must have  $a_3 = a_4$ , or  $a_3 c_4 = a_4 c_3$ . Also since  $d_3 \neq d_4$  we must have  $a_4 d_3 \neq d_4 a_3$ .

Part (iii) is immediate from (i) and (ii) since  $y\zeta_2 = w\zeta_2 = 0$  in any case.

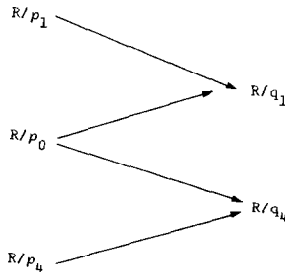
*Note.* With the notation of Example 9.7 it is easy to see that if  $a_3 = a_4$  then  $x\zeta_2 \in A$  and  $z\zeta_2 \in A$ .

**COROLLARY 9.9.** *The “ $4 \times 1$ ” configuration in  $\mathbb{P}^3$  does not have a Buchsbaum coordinate ring.*

*Proof.* One may make a change of variables to conform with the notation of Example 9.7. We must have  $a_3 \neq a_4$  and  $d_3 \neq d_4$ . The result is then immediate from (9.8) and (9.3).

**LEMMA 9.10.** *With the notation of Example (9.7) let  $a_3 = a_4$ . Then  $C = A[\zeta_2, \nu]$ .*

*Proof.* This is similar to the proof of (9.2). By (9.1) the coordinate ring of  $L_0 \cup L_1 \cup L_4$  is the pull-back of the diagram



This coordinate ring is a quotient of  $A$ , so any element of  $C$  is congruent to an element of the form  $b = (0, 0, b_2, b_3, 0)$  modulo  $A$ . To be consistent at  $R/\mathfrak{p}_3 + \mathfrak{p}_4$  the element  $b_3$  of  $R/\mathfrak{p}_3$  must be divisible by the image in  $R/\mathfrak{p}_3$  of  $c_4x - d_4z + w$ . Choose  $r \in R$  with  $r(c_4x - d_4z + w)$  mapping onto  $b_3$ . Then,  $b = b_2\zeta_2 + r\mathfrak{z}$ .

**COROLLARY 9.11.** *If  $a_3 = a_4$  in Example (9.7), then  $A$  is a Buchsbaum ring.*

*Proof.* By (9.8),  $A[\zeta_2] \subset m^{-1}$  and so by (9.10) it is enough to show that  $n \in m^{-1}$ .

We have

$$\begin{aligned} x\mathfrak{z} &= -a^{-1}xy(c_4x - d_4z + w) \\ y\mathfrak{z} &= xy(c_4x - d_4z + w) \\ z\mathfrak{z} &= yz(c_4x - d_4z + w) \\ w\mathfrak{z} &= xw(c_4x - d_4z + w). \end{aligned}$$

### 10. LINES ON A QUADRIC SURFACE IN $\mathbb{P}^3$

In this section we consider “ $m$  by  $n$ ” configurations  $\mathcal{L}$  of lines on a quadric surface in  $P^3$ . Such a surface has two rulings, and  $\mathcal{L}$  consists of  $m$  lines from one ruling and  $n$  from the other. We will show

- if  $|m - n| \leq 1$ , the “ $m$  by  $n$ ” configuration is  $C-M$ ,
- if  $|m - n| = 2$ , the configuration is Buchsbaum but not  $C - M$ ,
- if  $|m - n| \geq 3$ , the configuration is not even Buchsbaum.

In Example (5.1), we showed that (the coordinate ring of) the configuration was  $C-M$  if and only if  $|m - n| \leq 1$ . We may assume that  $n \geq m + 2$  and write  $n = m + 1 + k$ . For convenience, we number the lines of  $\mathcal{L}$  so that  $L_{-k}, \dots, L_{-1}, L_0, L_1, \dots, L_m$  are chosen from one ruling, and  $L_{m+1}, \dots, L_{2m}$  are chosen from the other ruling.

**LEMMA (10.1).** *For  $j < 0$  let  $f_j \in R/\mathfrak{p}_j$  denote a form of degree  $m$  which generates the kernel of  $R/\mathfrak{p}_j \rightarrow \prod_{i=m+1}^{2m} R/(\mathfrak{p}_i + \mathfrak{p}_j)$ , and let  $\zeta_j = (0, \dots, f_j, 0, \dots, 0)$  denote the corresponding element of  $\prod_{i=-k}^{2m} R/\mathfrak{p}_i$ . Then  $C$  is generated as an  $A$ -module by  $\zeta_{-k}, \dots, \zeta_{-1}$ , i.e.,*

$$C = A[\zeta_{-k}, \dots, \zeta_{-1}].$$

*Proof.* Let  $A'$  denote the pull-back of the diagram obtained by deleting  $R/\mathfrak{p}_{-k}, \dots, R/\mathfrak{p}_{-1}$  from the graph associated to  $A$ . By (5.1),  $R$  (and  $A$ ) maps

onto  $A'$ . This shows that if  $(c_{-k}, \dots, c_{2m}) \in C$  then there is an element of  $A$  of the form  $(a_{-k}, \dots, a_{-1}, c_0, \dots, c_{2m})$ . Thus every element of  $C$  is congruent mod  $A$  to an element of the form  $c = (c_{-k}, \dots, c_{-1}, 0, \dots, 0)$ . Each  $c_j$  vanishes in all  $R/(\mathfrak{p}_i + \mathfrak{p}_j)$ , so  $f_j$  divides  $c_j$ , and so  $c = \sum (c_j/f_j)\zeta_j$ , as claimed.

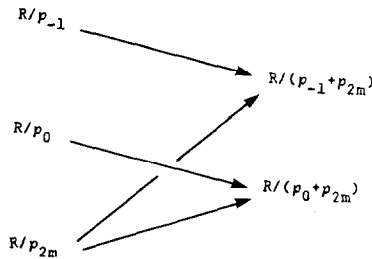
**THEOREM (10.2).** *For all  $m \geq 0$ , the “ $m$  by  $(m + 2)$ ” configuration of lines on a quadric surface in  $\mathbb{P}^3$  has a Buchsbaum coordinate ring.*

*Proof.* We will proceed by induction on  $m$ , the cases  $m = 0, 1$  being (6.3) and (9.4). By (10.1) and (4.3),  $A$  is Buchsbaum iff  $\zeta_{-1}M \subset A$ , where  $\zeta_{-1} = (f_{-1}, 0, \dots, 0)$ . Let  $f'$  be a form of degree  $(m - 1)$  in  $R/\mathfrak{p}_{-1}$  which generates the kernel of  $R/\mathfrak{p}_{-1} \rightarrow \prod_{i=m+1}^{2m-1} R/(\mathfrak{p}_i + \mathfrak{p}_{-1})$ , and  $f''$  the linear form so that  $f_{-1} = f'f''$ .

By removing  $L_0$  and  $L_{2m}$  from  $\mathcal{L}$ , we obtain an “ $(m - 1)$  by  $(m + 1)$ ” configuration  $\mathcal{L}'$  whose coordinate ring  $A'$  is a quotient of  $A$ . The Cohen–Macaulification  $C'$  of  $A'$  is the pullback of the graph obtained by removing  $R/\mathfrak{p}_0$  and  $R/\mathfrak{p}_{2m}$  from the graph associated to  $\mathcal{L}$ . The element  $\zeta_{-1}$  of  $C'$  maps to  $f''\zeta'_{-1}$ . For  $a \in M$  the element  $a\zeta_{-1}$  of  $MC$  maps to  $f'' a\zeta'_{-1}$  in  $M'C'$ . By induction  $M'C' \subset A'$ , so there is an element  $b$  of  $A$  mapping to  $a\zeta'_{-1}$ . As an element of  $\prod_{i=-1}^{2m} R/\mathfrak{p}_i$  we have

$$b = (af', b_0, 0, \dots, 0, b_{2m}).$$

Now consider the configuration  $\mathcal{L}'' = L_{-1} \cup L_0 \cup L_{2m}$ . By (5.1), the coordinate ring  $A''$  of this configuration is the pullback of the diagram



The element  $(f'', 0, 0)$  of  $R/\mathfrak{p}_{-1} \times R/\mathfrak{p}_0 \times R/\mathfrak{p}_{2m}$  is in this pullback. As  $A''$  is a quotient of  $A$ , there is an element  $c = (f'', 0, c_1, \dots, c_{2m-1}, 0)$  of  $A$ . In  $A$  we have  $bc = (af'f'', 0, \dots, 0) = a\zeta_{-1}$ . This shows that  $a\zeta_{-1} \in A$  for every  $a \in M_j$ , i.e., that  $\zeta_{-1}M \subset A$ .

**THEOREM (10.3).** *If  $|m - n| \geq 3$ , the “ $m$  by  $n$ ” configuration of lines on a quadric surface in  $\mathbb{P}^3$  has a coordinate ring which is not Buchsbaum.*

*Proof.* We need to introduce coordinates for  $\mathbb{P}^3$ . We can assume that the quadric surface is given by the equation  $yz - xw = 0$ , and that the lines are given by the primes

$$\begin{aligned} \not\! /_i &= (\mu_i x - z, \mu_i y - w), & -k \leq i \leq m \\ \not\! /_i &= (\lambda_i x - y, \lambda_i z - w), & m + 1 \leq i \leq m. \end{aligned}$$

A coordinate change  $y \rightarrow y + \alpha x, w \rightarrow w + \alpha z$  allows us to assume that all the  $\lambda_i$  are non-zero. With this choice we have in  $\prod_{i=-k}^{2m} R/\not\! /_i = \prod_{i=-k}^m k[x, y] \times \prod_{i=m+1}^{2m} k[x, z]$ :

$$\begin{aligned} x &= (x, x, \dots, x) \\ y &= (y, \dots, y, \lambda_{m+1}x, \dots, \lambda_{2m}x) \\ z &= (\mu_{-k}x, \dots, \mu_mx, z, \dots, z) \\ w &= (\mu_{-k}y, \dots, \mu_my, \lambda_{m+1}z, \dots, \lambda_{2m}z) \\ \zeta_{-k} &= (f, 0, \dots, 0) \end{aligned}$$

where  $f = \prod_{i=m+1}^{2m} (\lambda_i x - y)$ .

We will show that  $x\zeta_{-k} \notin A$  for  $k \geq 2$ . This will show that  $\zeta_{-k} \notin M^{-1}$ , and hence that  $C \neq M^{-1}$ . By (4.3), this will show that  $A$  is not Buchsbaum.

Suppose that  $x\zeta_{-k} \in A$ . Then there are  $\alpha(q)$  in  $k$  so that

$$x\zeta_{-k} = \sum \{ \alpha(q) q \mid q \text{ is a monomial of degree } (m + 1) \text{ in } R \}.$$

This is an equation in  $\prod (R/\not\! /_i)$  of forms of degree  $m + 1$ . In equations  $-k$  through  $m$ , we consider the coefficients of  $x^{m+1}$ . This yields the system of  $(m + k + 1)$  equations in  $(m + 2)$  unknowns:

$$\begin{bmatrix} \prod \lambda_i \\ 0 \\ \dots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \mu_{-k} & \mu_{-k}^2 & \dots & \mu_{-k}^{m+k} \\ 1 & \mu_{-k+1} & \mu_{-k+1}^2 & \dots & \mu_{-k+1}^{m+1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \mu_m & \mu_m^2 & \dots & \mu_m^{m+1} \end{bmatrix} \begin{bmatrix} \alpha(x^{m+1}) \\ \alpha(x^m z) \\ \dots \\ \alpha(z^{m+1}) \end{bmatrix}.$$

The last set of  $(m + 2)$  equations has the unique solution  $\alpha(x^{m+1}) = \dots = \alpha(z^{m+1}) = 0$  when  $k \geq 2$ . Because  $\prod \lambda_i \neq 0$ , this solution does not satisfy the first equation. This contradiction shows that  $x\zeta_{-k} \notin A$ , as desired.

*Remark.* A more geometric approach to the content of (10.2) and (10.3) is given in [7]. The proof there uses the notion of liaison and is entirely different from the above linear algebra proof.

## 11. CONCLUDING REMARKS AND QUESTIONS

Some readers may have noted that many of our techniques about pull-backs of rings are borrowed from the paper [4]. Unfortunately, that paper was written prior to Swan's work on seminormality [21] and the simple algebraic definition he worked with. Swan's work thus allowed a substantial simplification of some of the results of [4] that we used and so we decided to give the new proofs.

The problem of determining when a configuration of lines in  $\mathbb{P}^3$  is C-M or Buchsbaum is still a tempting and seemingly quite difficult question. The next obvious step is to consider configurations on cubic surfaces in  $\mathbb{P}^3$  and to decide how the minimal degree of a surface containing configurations with the same graph affects the Cohen-Macaulay and Buchsbaum properties for the configuration. The same question applies as well, given the examples of Dayton and Roberts, to questions concerning seminormality. The work of Goto [8] may be of great use here.

In higher dimensional spaces the problems seem more difficult. We have not been able to decide, e.g., if the Buchsbaum type of a Buchsbaum configuration of lines in  $\mathbb{P}^n$  is always  $<n/2$ . Such unanswered simple questions are just one indication of the work yet to be done.

There is, of course, the possibility of considering questions analogous to those we have considered for higher dimensional linear subspaces of  $\mathbb{P}^n$  (where  $n$  is large enough, relative to the dimensions of the subspaces, to make the questions non-trivial). Some work has already begun in that direction by Reisner in [17] for "co-ordinate" subspaces. The methods there are quite sophisticated. Interestingly enough, Reisner gives an example (a union of 3-dimensional subspaces of  $\mathbb{A}^6$ ) in which the decision as to whether the coordinate ring is C-M or not depends on the characteristic of the base field. We have discovered no such phenomena for planes in  $\mathbb{A}^4$ . This result of Reisner's suggests interesting things for higher-dimensional C-M phenomena.

## REFERENCES

1. M. P. BRODMANN, A "Macaulayfication" of unmixed domains, *J. Algebra* **44** (1977), 221-234.
2. M. P. BRODMANN, Finiteness of ideal transforms, *J. Algebra* **63**, (1980), 162-185.
3. B. DAYTON AND L. ROBERTS, "Seminormality of Unions of Planes," Lecture Notes in Mathematics No. 854, pp. 93-126, Springer-Verlag, Berlin/Heidelberg/New York, 1981.
4. B. DAYTON AND C. WEIBEL, "A Spectral Sequence for the  $K$ -Theory of Affine Glued Schemes," Lecture Notes in Mathematics No. 854, pp. 24-92, Springer-Verlag, Berlin/Heidelberg/New York, 1981.
5. G. FALTINGS, Über Macaulayfizierung, *Math. Ann.* **238** (1978), 175-192.
6. D. FERRAND AND M. RAYNAUD, Fibres formelles d'un anneau local noetherien, *Ann. Sci. Ecole Norm. Sup.* **3** (1970), 295-311.

7. A. GERAMITA, P. MAROSCIA, AND W. VOGEL, On curves linked to lines in  $\mathbb{P}^3$ , Queen's University Math. preprint No. 1981-31.
8. S. GOTO, On Buchsbaum rings obtained by glueing, preprint.
9. A. GROTHENDIECK, EGA IV, *Pub. Math. Inst. Hautes Études Sci.* **24** (1965).
10. R. HARTSHORNE, "Local Cohomology," Lecture Notes in Mathematics No. 41 Springer-Verlag, New York/Berlin/Heidelberg, 1970.
11. R. HARTSHORNE, Complete intersections and connectedness, *Amer. J. Math.* **84** (1962), 497-508.
12. B. HERZOG, On the Macaulayfication of local rings, *J. Algebra* **67** (1980), 305-317.
13. S. MACLANE, "Categories for the Working Mathematician," Springer-Verlag, New York, 1971.
14. M. NAGATA, "Local Rings," Interscience Tracts No. 13, Interscience, New York, 1962.
15. M. NAGATA, "Lectures on the Fourteenth Problem of Hilbert, Notes by P. Murthy," Tata Inst. of Fund. Res., Lect. Math. No. 31, 1965.
16. F. ORECCHIA, Sulla seminormalità di certe Varietà affini riducibili, *Boll. Un. Mat. Ital.* (2) **BB** (1976), Theorem 1.3.
17. G. A. REISNER, Cohen-Macaulay quotients of polynomial rings, *Adv. in Math.* **21** (1976), 30-49.
18. L. ROBERTS, "A Combinatorial Approach to the  $K$ -Theory of some Reducible Affine Curves," Lecture Notes in Mathematics, No. 551, Springer-Verlag, New York/Berlin/Heidelberg, 1976.
19. P. SCHENZEL, J. STÜCKRAD, AND W. VOGEL, Foundations of Buchsbaum modules and applications, in preparation.
20. J. STÜCKRAD UND W. VOGEL, Eine Verallgemeinerung der Cohen-Macaulay Ringe und Anwendungen auf ein Problem der Multiplizitätstheorie, *J. Math. Kyoto Univ.* **13** (1973), 513-528.
21. R. SWAN, On seminormality, *J. Algebra* **67** (1980), 210-229.
22. C. TRAVERSO, Seminormality and Picard group, *Ann. Scuola Norm. Sup. Pisa* **24** (1970), 585-595.
23. O. ZARISKI AND P. SAMUEL, "Commutative Algebra," Vol. I, Van Nostrand, Princeton, N. J., 1960.
24. O. ZARISKI, Interprétations algébrique-géométriques du 14-ième problème de Hilbert, *Bull. Sci. Math.* **78** (1954), 155-168.