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The cyclic homology and K-theory of curves

By *S. Geller*[1]) at New Brunswick, *L. Reid* at Springfield
and *C. Weibel*[2]) at New Brunswick

Introduction

Until recently, very little has been known about the higher algebraic K-theory of anything except for finite fields. The goal of this paper is to compute the K-theory of singular curves in characteristic zero, assuming the K-theory of smooth curves and fields as given. We do this by first computing the Hochschild and cyclic homology of the singular curves (using a new procedure which works more generally), and then invoking several recent results relating K-theory to the cyclic homology of $\mathbb{Q}$-algebras, $\mathbb{Q}$ being the field of rational numbers.

If the curve is seminormal over an algebraically closed field, we give a complete computation of the K-theory of the curve (see Section 6). We also get partial results for K_2 and K_3 for general curves. Otherwise the calculations depend on the validity of the

KABI Conjecture 0. 1. *Let B be a finite integral extension of a $\mathbb{Q}$-algebra A, and let I be an ideal of both A and B, such as the conductor ideal. Then the map $K_n(A, B, I) \to HC_{n-1}(A, B, I)$ is an isomorphism for $n \geqq 1$, where the left-hand term is double relative K-theory and the right-hand term is the double relative cyclic homology taken over $\mathbb{Q}$.*

This conjecture is known to be true when $B = A/J$ ([OW]), when I is nilpotent ([G]), and for $n = 1$ ([GW]). One can also verify the conjecture when A is the coordinate ring of a seminormal curve over an algebraically closed field by using the case $B = A/J$ and analytic isomorphisms for both K-theory and cyclic homology (see [W ai] and the appendix of this paper).

In the absence of the KABI Conjecture, our results can be interpreted as calculations of the cyclic homology of curves in characteristic zero. Using analytic isomorphisms, it is enough to consider the singularities one at a time, and only up to analytic type. For a seminormal curve over an algebraically closed field, it is enough to compute

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the cyclic homology for the coordinate lines in affine space, i.e., for the ring $k[x_0, \ldots, x_b]/(x_i x_j = 0, i \neq j)$. This focusses our attention on the cyclic homology of graded rings, the topic of the first half of this paper.

In Section 1 we present a spectral sequence converging to the Hochschild homology of a graded k-algebra A with $A_0 = k$, and demonstrate in Section 2 how to determine the E_2 terms. This allows us to determine $\tilde{H}\tilde{C}_*(A) = HC_*(A)/HC_*(A_0)$ and hence $HC_*(A)$, since, when a graded k-algebra A contains $\mathbb{Q}$, Goodwillie proved in [G der] that the SBI sequence breaks up into short exact sequences

$$0 \to \tilde{H}\tilde{C}_{n-1}(A) \to \tilde{H}\tilde{H}_n(A; A) \to \tilde{H}\tilde{C}_n(A) \to 0.$$

In Sections 3 and 4, we apply this procedure to compute the Hochschild and cyclic homology of the k-algebras $k[x_0, \ldots, x_b]/(x_i x_j = 0, i \neq j)$ and $k[x, y]/(y^a - x^b)$, $2 \leqq a \leqq b$, when k is a field of char. 0. In Section 5, we indicate how to perform base change to compute the cyclic homology of these rings, considered as $\mathbb{Q}$-algebras.

Although we do not carry out the calculations here, we should note that these calculations enable us to calculate the cyclic homology of *any* seminormal curve over an algebraically closed field k with $\mathrm{char}(k) = 0$. This is because cyclic homology satisfies "analytic isomorphisms", a result we prove in the Appendix. The procedure is entirely analogous to the procedure outlined for K-theory in Section 6.

In the second half of the paper, we turn our attention to calculating the K-theory of curves. In Section 6, we reduce to the calculation of the K-theory of such rings as were considered in our cyclic homology calculations and, assuming the validity of the KABI Conjecture, indicate how to compute the K-theory of these. In Section 7, we compute the K-theory of the coordinate lines in affine space (where the KABI Conjecture is known). In Section 8 we briefly consider the K-theory of seminormal curves over a field which is not algebraically closed. Our results there depend on the validity of the KABI Conjecture. Finally, in Section 9 we consider the K-theory of the cusps $k[t^a, t^b]$, computing it completely under the assumption of the KABI Conjecture, but also computing part of K_2 and K_3 without that assumption.

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$$\Omega X \to X^{S^1} \to X,$$

and they helped us see this analogy. We are grateful to C. Kassel for his help with cyclic homology, and for many discussions about our project.

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§ 1. The Leray spectral sequence for Hochschild homology

We shall begin by setting up a spectral sequence from which we may compute the Hochschild homology $HH_*(R; R)$ of a graded k-algebra R. We shall fix a ring k and write $\otimes$ for $\otimes_k$. In later sections we will further restrict k and write $HH_*^k(R; R)$ when we wish to emphasize over what ring the tensor products are being taken.

We first recall the definition of Hochschild homology from [CE], IX. 6 or [Mac], X. 4. If M is an R-bimodule, then there is a chain complex $\tilde{B}_\bullet = \tilde{B}_\bullet(R; M)$ with $\tilde{B}_0 = M$ and $\tilde{B}_l = M \otimes R^{\otimes l}$ for $l \geqq 1$, where $R^{\otimes l} = R \otimes \cdots \otimes R$ has l factors. The boundary map in $\tilde{B}_\bullet$ is the map $b: \tilde{B}_l \to \tilde{B}_{l-1}$ given by

$$b(m \otimes r_1 \otimes \cdots \otimes r_l) = mr_1 \otimes r_2 \otimes \cdots \otimes r_l + \sum_{0<i<l} (-1)^i m \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_l$$
$$+ (-1)^l r_l m \otimes r_1 \otimes \cdots \otimes r_{l-1}.$$

By definition, $HH_l(R; M) = H_l(\tilde{B}_\bullet(R; M))$. This construction is called the "standard bar construction".

However, it is easier to compute with the "reduced bar construction" (see [CE], p. 176 or [Mac], X. 2). Write $\bar{R}$ for R/k and let $B_\bullet = M \otimes \bar{R}^{\otimes l}$. Since $B_\bullet$ is a quotient of $\tilde{B}_\bullet$ by an acyclic subcomplex, $HH_l(R; M) = H_l(B_\bullet(R; M))$ as well. It is traditional to use the notation $m[r_1| \cdots |r_l]$ for the element $m \otimes r_1 \otimes \cdots \otimes r_l$ of B_l. This notation also applies to $l = 0$ where we write $m[\ ]$ for the element m of $B_0 = M$.

Now suppose that M is a graded bimodule over a graded ring R, and consider the differential module $(B, b) = (B(R; M), b)$. The decomposition $B = \bigoplus B_l$ makes B a differential chain module and allows us to decompose its homology as

$$H(B, b) = \bigoplus HH_l(R; M).$$

We wish to refine this decomposition as follows.

Definitions 1. 1. Let $m \in M$ and $r_i \in R$ be homogeneous elements. Associated to the element $m[r_1| \cdots |r_l]$ of $B(R; M)$ are its

a) *length*: l,

b) *weight*: $\deg(m) + \sum \deg(r_i)$,

c) *topological degree*: $\deg(m) + \sum \deg(r_i) - l = \text{weight} - \text{length}$,

d) *base degree*: $\deg(m)$,

e) *fiber degree*: $\sum \deg(r_i) - l$.

Let ${}_wB^t$ be the submodule of $B(R; M)$ generated by the terms of weight w and topological degree t, and set $B^t = \bigoplus_w {}_wB^t$ and ${}_wB = \bigoplus_t {}_wB^t$. Define

$${}_wHH_l(R; M) = H_l({}_wB_\bullet, b); \quad {}_wHH^t(R; M) = H^t({}_wB^\bullet, b),$$

and $HH^t(R; M) = H^t(B^\cdot, b) = \bigoplus_w {}_wHH^t(R; M)$. Since weight = length + topological degree, it is clear that

$$ {}_wHH^t = {}_wHH_l \text{ if } w = l + t, $$

(1. 1. 1) and

$$ HH(R; M) = H(B, b) = \bigoplus_t HH^t(R; M) = \bigoplus_{w,t} {}_wHH^t(R; M). $$

Remark 1. 2. The complex B^t is described in [Mac], X. 10. It arises from thinking of R as a differential graded cochain algebra (whence $B^\cdot$) whose differentials are all zero.

With these preliminaries, we are ready to construct the Leray spectral sequence. Define L^pB to be the submodule of B generated by the terms $m[r_1|\cdots|r_l]$ with base degree $\geqq p$. Since $R = R_0 \oplus R_1 \oplus \cdots$, we have $b(L^pB) \subseteqq L^pB$, and thus we get a filtration of the differential graded module $B^\cdot$:

$$ B^\cdot = L^0B^\cdot \supseteqq L^1B^\cdot \supseteqq L^2B^\cdot \supseteqq \cdots. $$

We shall call this filtration the "Leray filtration" of B. As in [CE], XV. 1 or [Mac], XI. 3, this filtration gives a (weakly convergent) spectral sequence with

$$ E_0^{pq} = L^pB^{p+q}/L^{p+1}B^{p+q} = \{x \in B^{p+q}\colon x \text{ has base degree } p \text{ and fiber degree } q\}. $$

Proposition 1. 3 (Leray Spectral Sequence). *Suppose that $R = k \oplus R_1 \oplus \cdots$ is a graded k-algebra and that $M = M_0 \oplus M_1 \oplus \cdots$ is a graded R-bimodule. Then there is a first quadrant spectral sequence*

$$ E_1^{pq} = M_p \otimes HH^q(R; k) \Rightarrow H^{p+q}(R; M). $$

Proof. Because $R_0 = k$, the fiber degree q of each term $m[r_1|\cdots|r_l]$ is positive for $l \geqq 1$ and 0 for $l = 0$. This not only puts the spectral sequence in the first quadrant, but also assures that $L^pB^\cdot/L^{p+1}B^\cdot \cong M_p \otimes B^\cdot(R; k)$ as a cochain complex. By op. cit., this gives $E_1^{pq} = H^{p+q}(M_p \otimes B^\cdot(R; k)) = M_p \otimes H^q(B^\cdot(R; k))$ as claimed.

Remark 1. 4. Although the spectral sequence converges to the Hochschild cohomology HH^*, we can often read the Hochschild homology HH_* from the rows of the Leray spectral sequence. To illustrate this remark, suppose that every element of $HH^q(R; k)$ has length $l = l(q)$. We shall see in 1. 9 below that the qth row of E_∞ is a graded R-module, every element of which has length l. Therefore $HH_l(R; M)$ is filtered by those rows of E_∞ which have length l. Thus, if we can compute $HH^q(R; k)$ and E_∞^{pq}, we can compute $HH_l(R; M)$. We shall delay the computation of $HH^q(R; k)$ to Section 2 and devote the rest of this section to computing E_∞^{pq}.

Observation 1. 5. *For R as in Proposition 1. 3, if $rm = mr$ for every $r \in R$ and $m \in M$, then the map $R \to k$ induces a cochain map $B^\cdot(R; M) \to B^\cdot(k; M)$ which is naturally split. Since the spectral sequence for $HH^*(k; M)$ is concentrated in the row $q = 0$, it degenerates with $E_1^{pq} = E_\infty^{pq}$, yielding $HH^*(k; M) = M$. Hence the row $q = 0$ of the spectral sequence for $HH^*(R; M)$ splits off and lives to ∞. In addition, $B(R; M)$ has $B_0 = \bigoplus_p B_0 = \bigoplus_p B^p \cong \bigoplus M_p = M$ as a submodule. The elements in B_0 are the elements in $B_\cdot$ of length zero, and $b: B_1 \to B_0$ is the zero map. Hence, as in Remark 1. 4, the row $q = 0$ of E_∞ is naturally equal to $HH_0(R; M) \cong M$.*

Before analyzing the differentials in the spectral sequence, we give an example illustrating what we have said so far.

Example 1. 6. Let $R=M=k[x]$ with $x \in R_2$ (so that $d_1=0$ and $E_1^{pq}=E_2^{pq}$). It is well-known that $HH_l(k[x];k)$ is k if $l=0$ or 1 and is zero otherwise. The generators are $[\ ] \in B_0$ and $[x] \in B_1$ and have topological degrees 0 and 1 $(=2-1)$ respectively. Hence $HH^n(k[x];k)=HH_n(k[x];k)$. Since $E_2^{pq}=0$ for $q \geqq 2$, and the row $q=0$ lives to ∞, the spectral sequence collapses. Thus $E_2^{pq}=E_\infty^{pq}$ is k if p is even and q is 0 or 1, and 0 otherwise (see Figure 1. 1). In this case the diagonals $p+q=$ constant give HH^* and the rows $q=$ constant give HH_*. Therefore $HH^n(k[x];k[x])=k$ for all n with generators $x^n[\ ]$ for $H^{2n}(k[x];k[x])$ and $x^n[x]$ for $H^{2n+1}(k[x];k[x])$. Since the lengths of the generators are 0 and 1 respectively, $HH_n(k[x];k[x])=k[x]$ for $n=0, 1$, and zero otherwise.

$$\begin{array}{cccccccl}
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
k([x]) & 0 & k(x[x]) & 0 & k(x^2[x]) & 0 & \cdots & \Rightarrow HH_1(R;R) \\
k([\ ]) & 0 & k(x[\ ]) & 0 & k(x^2[\ ]) & 0 & \cdots & \Rightarrow HH_0(R;R)
\end{array}$$

Figure 1. 1. $E_2=E_\infty$ for $R=k[x]$. The notation $k(a)$ means the module is k on generator a.

In more complicated situations we need to compute the differentials. The following observation is essential.

Observation 1. 7. *Let* $Z_r^{pq}=\{z \in L^pB^{p+q} : b(z) \in L^{p+r}B\}$. *Then (see* [Mac]) E_r^{pq} *is a quotient of* Z_r^{pq}, *namely*

$$E_r^{pq}=(Z_r^{pq}+L^{p+1}B^{p+q})/(bZ_{r-1}^{p-r+1,q+r}+L^{p+1}B^{p+q}).$$

Furthermore, if $z \in Z_r^{pq}$, *then* $d_r(z) \equiv b(z) \bmod L^{p+r+1}B$.

For example, an element of E_1^{pq} of length l is represented by a term

$$s=\sum_j m[r_{1j}|\cdots|r_{lj}]$$

with $\deg m=p$ and with $\sum_{i,j}(-1)^i[r_{1j}|\cdots|r_{ij}r_{i+1,j}|\cdots|r_{lj}]=0$. Since $b(s) \in L^{p+1}B$, d_1 is determined by the formula

$$(1.7.1) \qquad d_1(s)=b(s)=\sum_j mr_{1j}[r_{2j}|\cdots|r_{lj}]+(-1)^l r_{lj}m[r_{1j}|\cdots|r_{l-1,j}].$$

This is to be read $\bmod L^{p+2}B$. In particular, if $R_1=0$ then $d_1(s)=0$, so we deduce

Corollary 1. 7. 2. *If in addition* $R_1=0$, *then* $E_1^{pq}=E_2^{pq}$ *and* d_2 *is given by formula* (1. 7. 1).

To use the observation to compute d_r, note that from E_{r-1} we are given elements $u \in Z_{r-1}^{pq}$ with $d_{r-1}(u)=0$. In general $u \notin Z_r^{pq}$, but for some $x \in L^{p+1}B^{p+q}$, the element $z=u+x$ will be in Z_r^{pq}. Thus $d_r(u)=d_r(z) \equiv b(z)$.

All of Observation 1. 7 is true without assuming R is commutative. If R is commutative and $M=R$, we can make further simplifications by using the multiplicative structure of the spectral sequence of 1. 3. This structure arises from the fact ([Mac], X. 12. 1 and p. 313) that, when R is commutative, $B(R;R)$ is a differential algebra under the shuffle product

$$\# : B_p(R;R)\otimes B_q(R;R)\to B_{p+q}(R;R),$$

$$a[r_1|\cdots|r_p]\# b[r_{p+1}|\cdots|r_{p+q}]=\sum_{\substack{(p,q)\\ \text{shuffles}}}(-1)^{\operatorname{sgn}(\sigma)}\,ab[r_{\sigma 1}|\cdots|r_{\sigma(p+q)}].$$

Note that $\#$ adds base and fiber degrees as well as topological degree, weight and length. In particular $B^{\cdot}$ is a DG-cochain algebra, and the Leray filtration is by DG-ideals. We cite the following result which may be easily proven using [Mac], XI. 3, but for which we could find no literature reference.

Proposition 1. 8 (Folklore). *If A is a* DG-*cochain algebra, and $\{F^p\}$ is a decreasing filtration of A by* DG-*ideals such that $F^p\# F^q\subseteq F^{p+q}$, then the resulting spectral sequence is multiplicative, i.e. each E_r is a bigraded differential algebra, $E_{r+1}=H(E_r)$ as an algebra, and E_∞ is the associated graded algebra of $H(A)$.*

Corollary 1. 9. *If $R=k\oplus R_1\oplus\cdots$ is commutative, then the spectral sequence $E_1^{pq}=R_p\otimes HH^q(R;k)\Rightarrow HH^{p+q}(R;R)$ is multiplicative. In particular, since the row $q=0$ is R, each row $q=$ constant is a graded R-module. Furthermore,*

$$d_r(u\# v)=d_r(u)\# v+(-1)^p u\# d_r(v),$$

where $p=$ base $\deg u$, *and $d_r(mu)=md_r(u)$ for $m\in R_p$.*

In the rest of this paper, except Section 2, we shall assume that

$$R=k\oplus 0\oplus R_2\oplus\cdots$$

is commutative and $M=R$. Thus the spectral sequence has $E_2^{pq}=R_p\otimes HH^q(R;k)$. Note that the assumption that $R_1=0$ is not really restrictive since any graded ring can be regraded by multiplying the original degrees by 2. The resulting graded ring has $R_{2i+1}=0$ and is isomorphic, as a graded ring, to the original ring.

We conclude the section with two examples to demonstrate the above technique of computing E_∞. We will assume we know $HH^q(R;k)$ which will be computed in Section 2 for these examples.

Example 1. 10 (Folklore). Let $R=k[x]/(x^{n+1})$, where k contains the rational numbers $\mathbb{Q}$, and R is graded by declaring $\deg(x)=2$. Set $1=[\ ]$, $u=[x]$, and $\bar t=[x^n|x]$; these are cycles in $B(R;k)$ but not in $B(R;R)$.

We claim that 1, u and $\bar t$ are nonzero in $HH^*(R;k)=HH_*(R;k)$. This is clear for 1 since $B_1\to B_0$ is zero, and clear for $u\in {}_2B_1$ since ${}_2B_2(R;k)=0$. Finally, if $\bar t$ were the boundary of w, note that w would be a k-linear combination of terms $[x^i|x^j|x^k]$ with $i+j+k=n+1$ and $i,j,k\geqq 1$. Since each x^ix^j and x^jx^k is nonzero in R, the sum of the coefficients of the terms $[x^a|x^b]$ in the boundary of w would be zero, contradicting the assumption that $b(w)=\bar t=[x^n|x]$, and verifying the claim.

We shall now assume the fact (derived in 2. 3 below) that as a graded-commutative k-algebra $HH^*(R;k)=k[\bar{t},u]/(u^2)$. Note that the classes of $1, u$ and $\bar{t}$ belong to HH^0, HH^1 and HH^{2n}, respectively. The spectral sequence is shown in Figure 1. 2; from the given generators and Remark 1. 4 it is easy to see that the terms on the lth nonzero row of E_∞ will have length l and give $HH_l(R;R)$.

$\vdots$	etc.			$\cdots$	etc.	
$4n+1$	$k(t^2u)$	0	$k(xt^2u)$	0 . . .	$k(x^nt^2u)$	$\Rightarrow HH_5$
$4n$	$k(t^2)$	0	$k(xt^2)$	0 . . .	$k(x^nt^2)$	$\Rightarrow HH_4$
$\vdots$	0	0	0	0	0	
$2n+1$	$k(tu)$	0	$k(xtu)$	0 . . .	$k(x^ntu)$	$\Rightarrow HH_3$
$2n$	$k(t)$	0	$k(xt)$	0 . . .	$k(x^nt)$	$\Rightarrow HH_2$
$\vdots$	0	0	0	0 . . .	0	
1	$k(u)$	0	$k(xu)$	0 . . .	$k(x^nu)$	$\Rightarrow HH_1$
0	k	0	$k(x)$	0 . . .	$k(x^n)$	$\Rightarrow HH_0$

d_{2n}

Figure 1. 2. $E_2=E_{2n}$ for $R=k[x]/(x^{n+1})$. The diagonals converge to HH^* and the rows to HH_*.

We now compute d_r using our procedure. Since $b(u)=0$, $d_r(u)=0$ for all r. If $r<2n$, then $d_r(\bar{t})=0$ since the target group is zero, whence $E_2=E_{2n}$ by 1. 9. We now compute $d_{2n}(\bar{t})$ using 1. 7 and then complete the computation of d_{2n} using 1. 9. As suggested in 1. 7 we need to add an element t' of L^1B^{2n} so that $t=\bar{t}+t'$ is in $Z_{2n}^{0,2n}$. This element is

$$t=[x^n|x]+x[x^{n-1}|x]+x^2[x^{n-2}|x]+\cdots+x^{n-1}[x|x].$$

Then $b(t)\in L^{2n}B$, whence $d_{2n}(\bar{t})=b(t)=(n+1)\,x^n[x]$. By multiplicativity (1. 9),

$$d_{2n}(x^it^j)=j(n+1)x^{i+n}t^{j-1}u=0$$

unless $i=0$ (as $x^{n+1}=0$). Similarly, $d_{2n}(x^it^ju)=0$ since $u^2=0$. Thus the only nonzero differentials in E_{2n} are $d(t^i)=i(n+1)x^nt^{i-1}u$. If $r>2n$, $d=0$ because the target group is 0. Thus $E_{2n+1}=E_\infty$ and we see that

$$HH_m(R;R)=\begin{cases} R & \text{on generator } 1=[\ ] \quad \text{if} \quad m=0,\\ R/x^n & \text{on generator } t^iu \quad \text{if} \quad m=2i+1,\\ R/x^n & \text{on generator } xt^i \quad \text{if} \quad m=2i,\ i\neq 0.\end{cases}$$

The last example is a special case of the rings considered in Section 3, namely the coordinate ring of two lines.

Example 1. 11. Let $R = \mathbb{Q}[x, y]/(xy)$ with $\deg(x) = \deg(y) = 2$. In the next section we shall show that as a graded-commutative ring $HH^*(R; \mathbb{Q}) = \mathbb{Q}[u, v, t]$, where the topological degrees of u, v and t are 1, 1 and 2 respectively. Note that $u^2 = v^2 = 0$ and $vu = -uv$. We claim that we can take $u = [x]$, $v = [y]$ and $t = [x|y]$. Since there are no nonzero elements of weight 2 and length 2 (resp. weight 4 and length 3), u, v and t are not boundaries, but are cycles as claimed. Since $b(u) = b(v) = 0$, $d_r(u) = d_r(v) = 0$ for all r. By (1. 7. 1), $d_2(t) = b(t) = x[y] + y[x] = xv + yu$. It is easy to compute the rest of d_2 by multiplicativity, namely: $d_2(t^j) = jt^{j-1}(xv + yu)$; if i or k is nonzero then

$$d_2(x^i t^j u^k) = jx^{i+1} t^{j-1} vu^k \quad \text{and} \quad d_2(y^i t^j v^k) = jy^{i+1} t^{j-1} uv^k;$$

d_2 is zero on the rest of the monomials. Thus, $E_3^{pq} = 0$ if $p \geqq 3$ and $q \geqq 2$. The rest of E_3 is displayed in Figure 1. 3.

By considering the generators, it is easy to see that elements in E_2^{pq} have length q and weight $p + 2q$. Since the weight of elements in $E_r^{p+r, q-r+1}$ is $p + 2q + (2 - r)$, and d_r preserves weight, it follows that $d_r = 0$ for $r \neq 2$. Thus $E_3 = E_\infty$. Since we can read $HH_q(R; R)$ from the qth row, we see that

$$HH_q(R; R) = \mathbb{Q} \quad \text{if } q \neq 0, 1.$$

This example will be discussed in more detail in Section 3.

	etc.					
$\mathbb{Q}(t^2uv)$	0	0	0	etc.		
0	0	$\mathbb{Q}(t^2w)$	0	0	0	0
$\mathbb{Q}(tuv)$	0	0	0	0	0	0
0	0	$\mathbb{Q}(tw)$	0	0	0	0
$\mathbb{Q}(uv)$	0	0	0	0	0	0
$\mathbb{Q}^2(u, v)$	0	$\mathbb{Q}^3(xu, yv, w)$	0	$\mathbb{Q}^2(x^2u, y^2v)$	0	$\mathbb{Q}^2 \cdots$
$\mathbb{Q}(1)$	0	$\mathbb{Q}^2(x, y)$	0	$\mathbb{Q}^2(x^2, y^2)$	0	$\mathbb{Q}^2 \cdots$

Figure 1. 3. $E_3 = E_\infty$ for $R = \mathbb{Q}[x, y]/(xy)$. The generators are shown in parentheses, where $w = xv - yu$. The qth row gives $HH_q(R; R)$; the diagonals give $HH^n(R; R)$.

§ 2. Computing $HH^*(R; k)$

In order to complete our procedure for computing $HH_*(R; R)$, we need to know how to compute $HH^*(R; k)$ when k is a field of characteristic zero. At this point we must introduce topological methods.

Let X be a simply connected topological space such that $R = H^*(X; k)$. It is well-known ([SEM]) that the Eilenberg-Moore spectral sequence for the fibration $\Omega X \to X^I \to X \times X$ collapses to yield the formula

$$H^*(\Omega X; k) \cong \mathrm{Tor}_{**}^{R \otimes R^{op}}(R, k) = HH^*(R; k).$$

(The Tor groups are bigraded by weight and length, so the "total degree" is our topological degree; cf. [Mac], p. 299 or [SEM], p. 64.)

Example 2. 1. Let $R = k[x]$ with $\deg(x) = 2$. Since

$$R = H^*(\mathbb{CP}^\infty; k) \text{ and } \Omega\mathbb{CP}^\infty \cong S^1,$$

we have $HH^*(R; k) = H^*(S^1; k)$, which is k if $* = 0, 1$ and 0 otherwise. The reader may wish to compare this with Example 1. 6.

For most spaces X, $H^*(\Omega X; k)$ is not so easy to determine. We will present two methods here, each of which has advantages over the other for various X. In practice, when we can, we compute $H^*(\Omega X; k)$ by both methods, using one as a check on the other.

Method 1. Rational homotopy and the theory of minimal models.

By the Milnor-Moore theorem ([MM]), $H^*(\Omega X; k)$ is the universal enveloping algebra of the Lie algebra $\pi_*(\Omega X) \otimes k = \pi_{*+1}(X) \otimes k$. Thus the algebra $H^*(\Omega X; k)$ has $\dim_k(\pi_n(\Omega X) \otimes k)$ generators in topological degree n. Since $\pi_*(X) \otimes k$ is finite dimensional when R is a complete intersection, this is a good method to use for complete intersections (see Examples 2. 4 and 2. 5 below). It is more difficult when R is not a complete intersection, in which case Method 2 is often easier.

Thus we have reduced our problem to computing the rational homotopy of X since $\pi_*(X) \otimes k = (\pi_*(X) \otimes \mathbb{Q}) \otimes_{\mathbb{Q}} k$. For this we shall use Sullivan's theory of minimal models ([Sull], [DGMS]).

Definition 2. 2. A minimal model is a DG-cochain algebra $(\mathbb{Q}[x_1, \dots], \partial)$ such that each ∂x_i lies in the subring $\mathbb{Q}[x_1, \dots, x_{i-1}]$.

In this context, the variables $x_1, x_2, \dots$ are homogeneous, and the graded-commutative ring $k[x_1, x_2, \dots]$ is defined to be the polynomial algebra on the x_i in even degrees, tensored with the exterior algebra on the x_i in odd degrees. It should be clear from context whether we are in an ordinary polynomial ring (e.g. when defining a ring R) or the above case.

From [DGMS] and [Sull], we know that every simply connected space X has a unique minimal model such that

i) $H^*(X; \mathbb{Q}) = H^*(\mathbb{Q}[x_1, \dots]; \mathbb{Q})$,

ii) $\dim(\pi_j(X) \otimes \mathbb{Q}) =$ the number of x_i of degree j,

iii) $\mathrm{rank}(\pi_j(X) \to H_j(X; \mathbb{Q})) =$ the number of x_i of degree j with $\partial x_i = 0$,

iv) $[x_i, x_j] = \sum c_{ij}^k x_k$, where c_{ij}^k is the coefficient of $x_i x_j$ in ∂x_k.

Note that property ii) provides the information we require. The following examples illustrate the use of Method 1.

Example 2. 3. The space $X = \mathbb{CP}^n$ has $R = H^*(X; k) = k[x]/(x^{n+1})$, the ring of Example 1. 10. If we let y have degree $2n+1$ (recall that $\deg(x) = 2$), then the minimal model of X is $\mathbb{Q}[x, y]$ with $\partial x = 0$ and $\partial y = x^{n+1}$ $(n \neq 0)$. Hence $\pi_i(X) \otimes k$ is k if $i = 2$ or $2n+1$ and zero otherwise. Let $u \in \pi_1(\Omega X)$ and $t \in \pi_{2n}(\Omega X)$ generate $\pi_*(\Omega X) \otimes k$. This is an abelian Lie algebra, so

$$HH^*(R; k) = H^*(\Omega \mathbb{CP}^n; k) = k[u, t], \quad n \geqq 1.$$

Here is the calculation of $H^*(R; k)$ for the ring in Example 1. 11.

Example 2. 4. The cohomology of $X = \mathbb{CP}^\infty \vee \mathbb{CP}^\infty$ is $R = k[x, y]/(xy)$. The minimal model for X is $\mathbb{Q}[x, y, z]$ with $\deg(x) = \deg(y) = 2$, $\deg(z) = 3$, $\partial x = \partial y = 0$ and $\partial z = xy$. Hence $\pi_*(\Omega X) \otimes k$ is 3-dimensional on $u, v \in \pi_1(\Omega X)$ and $t \in \pi_2(\Omega X)$. This yields:

$$HH^*(R; k) = H^*(\Omega X; k) = k[u, v, t]$$

and

$$HH^q(R; k) = \begin{cases} k & \text{on} \quad 1 = [\ ] & \text{if} \quad q = 0, \\ k \oplus k & \text{on} \quad t^i u, t^i v & \text{if} \quad q = 2i+1, \\ k \oplus k & \text{on} \quad t^{i+1}, t^i uv & \text{if} \quad q = 2i+2. \end{cases}$$

The following more general example is sufficient for this paper. It can easily be generalized to any (finite) number of variables, where F is a homogeneous polynomial involving all of the variables.

Example 2. 5. Let $R = \mathbb{Q}[x, y]/F$, where $\deg(x) = a$, $\deg(y) = b \geqq a$ and F is a homogeneous polynomial of degree n. Let us assume a and b are even. Let X denote the fiber of the map from $Y = K(\mathbb{Q}, a) \times K(\mathbb{Q}, b)$ to $K(\mathbb{Q}, n)$ classified by the element F of $[Y, K(\mathbb{Q}, n)] = H^n(Y; \mathbb{Q})$. (This group is the degree n part of the ring

$$H^*(Y; \mathbb{Q}) \cong \mathbb{Q}[x, y].)$$

The minimal model for X is $\mathbb{Q}[x, y, z]$ with $\deg x = a$, $\deg y = b$, $\deg z = n-1$, $\partial x = \partial y = 0$, and $\partial z = F$. Thus $\pi_*(\Omega X) \otimes \mathbb{Q}$ is 3-dimensional on generators $u \in \pi_{a-1}$, $v \in \pi_{b-1}$ and $t \in \pi_{n-2}$. Therefore $HH^*(R; \mathbb{Q}) = H^*(\Omega X; \mathbb{Q}) = \mathbb{Q}[u, v, t]$ where u, v and t are homogeneous of degrees $a-1$, $b-1$ and $n-2$ respectively.

Method 2. Using another spectral sequence.

When R is not a complete intersection $\pi_*(X) \otimes k$ may be nonzero for infinitely many $*$ (e.g. $R = k[x_1, \ldots, x_n]/(x_i x_j = 0$ for $i \neq j)$ when $n \geqq 3$ — see Section 3). This can make rational homotopy calculations cumbersome. Another way to compute $HH^*(R; k)$ is to use the Serre spectral sequence for the fibration $\Omega X \to \mathscr{P}X \to X$, where $\mathscr{P}X$ is the

space of based paths in X. Since X is simply connected, there is a Serre spectral sequence which has $E_2^{pq} = H^p(X; k) \otimes H^q(\Omega X; k) \Rightarrow H^{p+q}(\mathscr{P}X; k)$. Since $\mathscr{P}X$ is contractible, we have $E_\infty^{pq} = 0$ except for $p = q = 0$. The $q = 0$ row of E_2 is $R = H^*(X; k)$. Since the spectral sequence is multiplicative, the qth row is determined by

$$\dim_{\mathbb{Q}} H^q(\Omega X; k)$$

which can be found using the fact that $E_\infty^{pq} = 0$. Basically one works backwards, computing the d_r and $H^q(\Omega X; k)$ from E_∞^{pq}. This method can be tedious but is viable for any R. See Section 3 for an example of its use.

Procedure 2. 6. The following steps summarize the technique presented in Sections 1 and 2 for computing $HH_*(R; R)$ where $R = k \oplus R_2 \oplus R_3 \oplus \cdots$.

1. Determine $HH^*(R; k)$. If $R = H^*(X; k)$ this may be done using rational homotopy theory or the Serre spectral sequence for $\Omega X \to \mathscr{P}X \to X$.

2. Pick generators in $B(R; R)$ for $HH^*(R; k) = HH_*(R; k)$, remembering that the boundary $\bar{b}$ of $B(R; k)$ is not the b of $B(R; R)$.

3. Using the multiplicative structure and changing the generators by elements higher in the filtration if necessary, compute d_r and E_r.

4. Read off $HH_*(R; R)$ from the rows of E_∞ — i.e., rows whose generators have length n determine $HH_n(R; R)$.

§ 3. The homology of the coordinate axes

Let k be a field of characteristic zero, and let R be the k-algebra of functions on the coordinate axes in $(b+1)$-space

$$R = k[x_0, \ldots, x_b]/(x_i x_j = 0 \text{ for } i \neq j).$$

The goals of this section are to compute the Hochschild and cyclic homology of the k-algebra R and to write down explicit representatives for a basis of these homology groups.

We shall use Procedure 2. 6, grading R by setting the x_i in degree 2. We combine steps 1 and 2 in the following theorem.

Theorem 3. 1. *For $q \geqq 1$, $HH^q(R; k) \cong HH_q(R; k)$ is a vector space of dimension $(b+1)b^{q-1}$. A basis is given by the cycles $[x_{i_1}|x_{i_2}|\cdots|x_{i_q}]$, where*

$$i_j \neq i_{j+1} \quad \textit{for} \quad j = 1, 2, \ldots, q-1.$$

Proof. The symbols $[x_{i_1}|\cdots|x_{i_q}]$ are elements of the reduced bar construction $B(R; k)$. They are cycles (under $\bar{b}$) and so represent elements of $HH(R; k)$. Since they have length q, weight $2q$ and topological degree q, they belong to ${}_{2q}HH_q(R; k)$ (see 1.1 above). By definition, ${}_{2q}HH_q(R; k)$ is the homology of

$${}_{2q}B_{q+1}(R; k) \xrightarrow{\bar{b}} {}_{2q}B_q(R; k) \xrightarrow{\bar{b}} {}_{2q}B_{q-1}(R; k).$$

But ${}_{2q}B^{\bullet}_{q+1}(R;k)=0$ because terms of length $q+1$ in the reduced bar construction have weight at least $2(q+1)$. Therefore the $(b+1)b^{q-1}$ cycles described in the statement of the theorem are linearly independent in ${}_{2q}HH_q(R;k)$. To complete the proof of the theorem, it is enough to show that $\dim_k HH^q(R;k)=(b+1)b^{q-1}$.

From topology it is known that $R=H^*(X;k)$ where X is the wedge of $(b+1)$ copies of $\mathbb{CP}^\infty$. We saw in §2 that $H^*(\Omega X;k)\cong H^*(R;k)$. Therefore, Theorem 3. 1 follows from

Proposition 3. 2. *If* $q\geqq 1$, *then* $\dim H^q(\Omega X;k)=(b+1)b^{q-1}$.

We shall prove this using each of the methods in Section 2.

First Proof (Method 1). Since X is simply connected, we can apply rational homotopy theory ([Q]) to see that $H^*(\Omega X;k)$ is the Hopf algebra coproduct of $(b+1)$ copies of

$$H^*(\Omega\mathbb{CP}^\infty;k)\cong H^*(S^1;k)\cong k[\varepsilon]/(\varepsilon^2).$$

Thus $H^*(\Omega X;k)$ is the graded Hopf algebra on generators $\varepsilon_0,\dots,\varepsilon_b$ in degree 1, with $\varepsilon_i^2=0$, $i=0,\dots,b$. A k-module basis for $H^q(\Omega X;k)$ is given by all words of length q in the ε_j with no adjacent ε_i identical; there are $(b+1)b^{q-1}$ such words.

Second Proof (Method 2). Recall that the Serre spectral sequence for

$$\Omega X\to\mathscr{P}X\to X$$

has $E_2^{pq}=R_p\otimes H^q(\Omega X;k)$ and $E_\infty^{pq}=0$ for $(p,q)\neq(0,0)$. We will show by induction on $q\geqq 0$ that

(a) $E_3^{pq}=E_\infty^{pq}$, and

(b) $H^{q+1}(\Omega X;k)$ has a basis of symbols

$$[i_0|\cdots|i_q] \text{ with } i_j\neq i_{j+1} \text{ for } j=0,1,\dots,q-1$$

such that (setting $[\ ]=1$) $d_2([i_0|\cdots|i_q])=x_{i_0}[i_1|\cdots|i_q]$.

The case $q=0$: Since $E_\infty^{20}=E_\infty^{10}=0$, the map d_2 from $E_2^{01}=H^1(\Omega X;k)$ to $R_2=E_2^{02}$ must be an isomorphism. Let $[i]\in H^1(\Omega X)$ be such that $d_2([i])=x_i$, so that (b) is satisfied with $[\ ]=1$. Since the spectral sequence is multiplicative, $d_2(x_i^m[i])=x_i^{m+1}$, whence $d_2: d_2^{2m,1}\to E_2^{2m+2,0}$ is onto. Therefore (a) holds for $q=0$.

Induction step: Now assume (a) and (b) are known for $(q-1)$. It is straightforward that the dimension of the kernel of $d_2^{2p,q}: E_2^{2p,q}\to E_2^{2p+2,q-1}$ is $(b+1)b^q$ for $p\geqq 1$. Since the symbols $x_i^p[i_1|\cdots|i_q]$ with $i\neq i_1$ are linearly independent, they must form a basis for $\ker(d_2^{2p,q})$. Since by induction $E_3^{p,q-1}=0$ for $(p,q-1)\neq(0,0)$, d_2 must be an isomorphism between $E_2^{0,q+1}=H^{q+1}(\Omega X)$ and $\ker(d_2^{2,q})$, proving that (b) holds for q. Since $d_2(x_i^p[i|i_1|\cdots|i_q])=x_i^{p+1}[i_1|\cdots|i_q]$, $E_2^{2p,q+1}$ maps onto $\ker(d_2^{2p+2,q})$, whence condition (a) holds for q as well.

We now know E_2^{pq} of the Leray spectral sequence in terms of generators and need to do step 3, namely compute d_r and E_r. For this ring, our task is somewhat simplified.

Proposition 3. 3. *For $r \geqq 3$, $d_r = 0$. Thus $E_3 = E_\infty$.*

Proof. Since every element of E_2^{pq} has weight $p+2q$, so do the elements of E_r^{pq} for all r. Since d_r preserves weight, for $d_r : E_r^{pq} \to E_r^{p+r,q-r+1}$ to be nonzero we must have $p+2q = p+r+2(q-r+1) = p+2q+2-r$. Thus $d_r = 0$ for $r \geqq 3$ and $E_3 = E_\infty$ as claimed.

Thus we need only compute d_2. To keep track of what happens to the basis elements, it is useful to divide them into different types.

Definition 3. 4. Let $i_0[i_1|\cdots|i_q]$ (resp. $[i_1|\cdots|i_q]$) denote the element

$$x_{i_0}[i_{i_1}|\cdots|x_{i_q}] \quad (\text{resp. } [x_{i_1}|\cdots|x_{i_q}])$$

of the reduced bar construction $B(R;R)$, where $i_1 \neq i_2, \ldots, i_{q-1} \neq i_q$. We say that $i_0[i_1|\cdots|i_q]$ has

Type I if $i_q = i_0 = i_1$;

Type II if $i_q \neq i_0 = i_1$;

Type III if $i_q = i_0 \neq i_1$;

Type IV if $i_q \neq i_0 \neq i_1$ (i_q may or may not equal i_1).

We say that $[i_1|\cdots|i_q]$ has

Type A if $i_1 \neq i_q$;

Type B if $i_1 = i_q$.

Let $A(q)$ denote the number of type A symbols of length q, and $B(q)$ the number of type B symbols of length q.

Lemma 3. 5. *$A(1) = 0$, $B(1) = b+1$ and, for $q \geqq 1$,*

$$A(q) = b^q + (-1)^q b \text{ and } B(q) = A(q-1).$$

Moreover, the number of symbols of each type is given in the following chart:

type	I	II	III	IV	A	B
#	$A(q-1)$	$A(q)$	$A(q)$	$A(q+1)$	$A(q)$	$A(q-1)$

Proof. $B(q) = A(q-1)$ since deleting the last element of a type B symbol yields a type A symbol and $B(1) = b+1$ on $[x_i]$, $i = 0, \ldots, b$. Since

$$A(q) + B(q) = \dim HH^q(R;k) = (b+1)b^{q-1}$$

for $q \geqq 1$, the formula for $A(q)$ follows by induction on q. The rest of the chart is an easy exercise.

By Corollary 1. 7. 2 (and Formula 1. 7. 1),

$$d_2(i_0[i_1|\cdots|i_q]) = i_0 i_1[i_2|\cdots|i_q] + (-1)^q i_q i_0[i_1|\cdots|i_{q-1}]. \tag{3. 6}$$

From this we can quickly compute the following parts of d_2 and most of E_3.

Proposition 3. 7. (a) $d_2^{p1} = 0$ *for all* p.

(b) *For* $p \geqq 2$ *and* $q \geqq 2$, *the map* $d_2^{pq}: R_p \otimes HH^q(R; k) \to R_{p+2} \otimes HH^{q-1}(R; k)$ *has rank* $(b+1)b^{q-1}$.

(c) *For* $p \geqq 4$ *and* $q \geqq 2$, $E_3^{pq} = E_\infty^{pq} = 0$.

(d) *For* $p \geqq 4$, p *even, the symbols* $x_i^{p/2}[x_i]$, $i = 0, \ldots, b$, *form a basis of the* $(b+1)$-*dimensional vector space* $E_3^{p1} = E_\infty^{p1}$.

Proof. (a) Substitute $q = 1$ into Formula 3. 6, or use 1. 5.

(b) For $p \geqq 2$, p even and $q \geqq 2$, we have $\dim(E_2^{pq}) = (b+1)^2 b^{q-1}$ on a basis of all symbols $i_0[i_1|\cdots|i_q]$. We analyze the effect of d_2 by type of element and obtain the following matrix for d_2:

type \ type	I	II	III	IV
I	0	0	0	0
II	$(-1)^q I$	0	0	0
III	I	0	0	0
IV	0	I	$(-1)^q I$	0

Using 3. 5, we see that the rank of d_2 is $A(q-1) + A(q) = (b+1)b^{q-1}$.

(c) When $p \geqq 4$, p even, we have $E_3^{pq} = 0$ because

$$\operatorname{rank}(d_2^{p-2,q+1}) + \operatorname{rank}(d_2^{pq}) = (b+1)^2 b^{q-1} = \dim(E_2^{pq}).$$

(d) Since $d_2^{p1} = 0$ and $\operatorname{rank}(d_2^{p-2,2}) = (b+1)b$, $\dim(E_3^{p1}) = b+1$. Mapping R to $k[x]$ by sending x_i to x and using Example 1. 6, we see that the $x_i^{p/2}[x_i]$ are linearly independent in E_∞^{p1}, whence the claim.

Thus we need only compute $E_3^{p,q}$ for $p = 0$ and 2 and $q \geqq 1$. These can be determined from Proposition 3. 7 and an analysis of d_2^{0q}, $q \geqq 2$, namely we need to analyze Formula 3. 6 with $i_0 = 1$.

If $[i_1|\cdots|i_q]$ is of type A, then both terms on the right-hand side of 3. 6 have type IV and differ by a cyclic permutation. This leads us to consider the orbits of $[i_1|\cdots|i_q]$ and $i_1[i_2|\cdots|i_q]$ under the action of the cyclic group. If the orbits have m elements, the matrix of d_2 restricted to these orbits is the $m \times m$ matrix

$$M(m,q)=\begin{pmatrix} 1 & 0 & 0 & (-1)^q \\ (-1)^q & 1 & \ddots & 0 \\ 0 & \ddots & 1 & 0 \\ 0 & 0 & (-1)^q & 1 \end{pmatrix}$$

(only three diagonals have nonzero entries).

Exercise 3. 8. If m is odd and q is even, $M(m,q)$ has kernel 0. If m and q are both odd, or are both even, the kernel of $M(m,q)$ is 1-dimensional on the vector

$$(1,(-1)^{q-1},(-1)^{2(q-1)},\ldots,(-1)^{(m-1)(q-1)}),$$

and the vector $(1,0,\ldots,0)$ is a nonzero element of the 1-dimensional cokernel. (The case m even and q odd cannot occur because $m|q$.)

If $[i_1|\cdots|i_q]$ is of type B, the right-hand side of 3. 6 has one element of type 2 and one of type 3. Thus the matrix for d_2^{0q}, $q\geqq 2$, is

	type A	type B
I	0	0
II	0	$(-1)^q I$
III	0	I
IV	T	0

where T is the block sum over the $\mathbb{Z}_q$-orbits of the matrices $M(m,q)$, where m is the size of the orbit.

Lemma/Definition 3. 9. *Let* $c(b,q)=\dim\operatorname{Ker}(T)$. *Then* $c(b,q)$ *is the number of* $\mathbb{Z}_q$-*orbits of the type* A *symbols in which the orbit size is congruent to* q modulo 2.

Proof. Exercise 3. 8 above.

Remark 3. 9. 1. We may also describe $c(b,q)$ as the number of ways of using the alphabet $\{0,\ldots,b\}$ to write words of q letters around in a circle so that (i) no two adjacent letters are the same and (ii) if rotation by m positions fixes the word then $m\equiv q \pmod 2$. (For example, the word 123123 is disallowed when $q=6$.)

Having analyzed d_2^{0q}, $q\geqq 2$, we now complete the computation of E_3. Since the kernel of d_2^{0q} has dimension $c(b,q)$, $\dim E_3^{0q}=c(b,q)$ for $q\geqq 2$, and

$$\operatorname{rank}(d_2^{0q})=(b+1)b^{q-1}-c(b,q).$$

Therefore,

$$\begin{aligned}\dim(E_3^{2q})&=\dim(E_2^{0q})-\operatorname{rank}(d_2^{2q})-\operatorname{rank}(d_2^{0,q+1})\\ &=(b+1)^2b^{q-1}-(b+1)b^{q-1}-((b+1)b^q-c(b,q+1))\\ &=c(b,q+1).\end{aligned}$$

Finally, since $d_2^{p1}=0$ by 3. 7(a), $\dim(E_3^{01})=b+1$ and

$$\dim(E_3^{21})=(b+1)^2-((b+1)b-c(b,2))=(b+1)+c(b,2).$$

This finishes the calculation of E_3 (see Figure 3. 1).

$\cdots$		$\cdots$				
$c(b,4)$	0	$c(b,5)$	0	0	0	0
$c(b,3)$	0	$c(b,4)$	0	0	0	0
$c(b,2)$	0	$c(b,3)$	0	0	0	0
$b+1$	0	$b+1+c(b,2)$	0	$b+1$	0	$b+1\cdots$
1	0	$b+1$	0	$b+1$	0	$b+1\cdots$

Figure 3. 1. The dimensions of $E_3^{pq}=E_\infty^{pq}$.

Remark 3. 10. The $c(b,q)$ basis elements of E_∞^{0q} correspond to the $c(b,q)$ one-dimensional kernels of the matrices $M(m,q)$; the $c(b,q+1)$ basis elements of E_∞^{2q} correspond to the $c(b,q+1)$ one-dimensional cokernels of the matrices $M(m,q+1)$.

We can now determine $HH_*(R;R)$. Since every element of $HH^q(R;k)$ has length q, the same is true for $E_2^{pq}=R_p\otimes HH^q(R;k)$, and hence for E_∞^{pq}. Thus $HH_q(R;R)$ has a filtration with associated graded groups E_∞^{pq}. Since E_∞^{pq} has weight $p+2q$, $HH_q(R;R)$ is actually the sum of the E_∞^{pq}. For $q=0$, we have $HH_0(R;R)=\bigoplus R_p=R$. For $q=1$, the graded R-module $HH_1(R;R)$ is naturally isomorphic to the graded R-module $\Omega_{R/k}$, with $[x_i]$ corresponding to dx_i. Letting ck denote the direct sum of c copies of k, we have

Theorem 3. 11.

$$HH_q(R;R)\cong\begin{cases}R & \text{if } q=0,\\ \Omega_{R/k} & \text{if } q=1,\\ c(b,q)k\oplus c(b,q+1)k & \text{for } q\geqq 2.\end{cases}$$

For $q\geqq 2$ a basis for $HH_q(R;R)$ is represented by cycles

$$[i_1|\cdots|i_q]+(-1)^{q-1}[i_q|i_1|\cdots|i_{q-1}]+\cdots+(-1)^{m(q-1)}[i_2|\cdots|i_q|i_1]$$

of type A (*one for each $\mathbb{Z}_q$-orbit of size $m\equiv q\pmod 2$*), *and by cycles $i_0[i_1|\cdots|i_q]$ of type* IV (*one for each of the $c(b,q)$ $\mathbb{Z}_{q+1}$-orbits of size $m\equiv(q+1)\pmod 2$*).

Corollary 3. 12. $HC_0(R)=R$.

$$HC_q(R)\cong HC_q(k)\oplus c(b,q+1)k \quad \text{for} \quad q\geqq 1.$$

A basis for $\tilde{H}\tilde{C}_q(R)\cong c(b,q+1)k$ is given by the cyclic q-cycles

$$x_{i_0}\otimes\cdots\otimes x_{i_q},\quad i_0\neq i_1,\ldots,i_{q-1}\neq i_q,\quad i_q\neq i_0,$$

one for each of the $c(b,q+1)$ orbits of $\mathbb{Z}_{q+1}$ of size $m\equiv(q+1)\pmod 2$.

Proof. Since R is graded and $\operatorname{char}(k)=0$, the SBI sequence relating

$$\tilde{H}\tilde{C}_q(R)=HC_q(R)/HC_q(k)$$

to $HH_q(R;R)$ breaks up for $q\geqq 1$ into short exact sequences ([G der])

$$0\longrightarrow \tilde{H}\tilde{C}_{q-1}(R)\xrightarrow{B} HH_q(R;R)\xrightarrow{I}\tilde{H}\tilde{C}_q(R)\longrightarrow 0.$$

In our notation, we have $I(r_0[r_1|\cdots|r_q])=r_0\otimes\cdots\otimes r_q$, and

$$\begin{aligned}B(r_1\otimes\cdots\otimes r_q)=&[r_1|\cdots|r_q]+(-1)^{q-1}[r_q|r_1|\cdots|r_{q-1}]\\&+\cdots+(-1)^{(q-1)(q-1)}[r_2|\cdots|r_q|r_1].\end{aligned}$$

For $q\geqq 2$, the image under B of the cyclic $(q-1)$ cycles described in the statement of this corollary forms a basis of the first summand of $HH_q(R;R)$ (Theorem 3. 11), while the cyclic q-cycles in $\tilde{H}\tilde{C}_q(R)$ are the images under I of the basis elements of the second summand of $HH_q(R;R)$. For $q=1$, the map B from $\tilde{H}\tilde{C}_0(R)=R/k$ to $HH_1(R;R)$ is graded, injecting R_p into $E_\infty^{p-2,1}$. By counting dimensions (see Figure 3. 1), we see that $\dim\tilde{H}\tilde{C}_1(R)=c(b,2)$. Since the $c(b,2)$ cyclic cycles $x_i\otimes x_j$ $(i<j)$ are detected by B, they must form a basis for $\tilde{H}\tilde{C}_1(R)$.

We conclude this section with a formula for $c(b,q)$. Fix q, and let $f(m)$ denote the number of type A q-tuples $[i_1|\cdots|i_q]$ which lie in a $\mathbb{Z}_q$-orbit of m elements. Since

$$\sum_{m|q} f(m)=A(q)=b^q+(-1)^q b,$$

the Möbius inversion formula yields $f(m)=\sum\mu\left(\frac{m}{d}\right)A(d)$, the sum being over all d dividing m. Since there are $f(m)/m$ orbits of size m, we have the formula

$$c(b,q)=\sum_{\substack{m|q\\ m\equiv q\,(\mathrm{mod}\,2)}}\frac{1}{m}\sum_{d|m}\mu\left(\frac{m}{d}\right)\{b^d+(-1)^d b\}. \tag{3. 13}$$

For example, $c(b,1)=0$,

$$\begin{aligned}c(b,2)&=(b^2+b)/2,\\c(b,3)&=(b^3-b)/3,\\c(b,4)&=(b^4-b^2)/4+(b^2+b)/2,\\c(b,5)&=(b^5-b)/5\end{aligned}$$

and

$$c(b,6)=(b^6-b^3-b^2+b)/6+(b^2+b)/2.$$

In general, $c(b,q)$ grows like b^q/q as $q\to\infty$.

§ 4. The homology of $k[x, y]/(y^a - x^b)$

Another family of interesting curves consists of the planar cusps $k[t^a, t^b]$ where a and b are relatively prime. (When $(a, b) = d$ is not 1, $k[t^a, t^b] \cong k[s^{a/d}, s^{b/d}]$ for $s = t^d$.) More generally, one can consider the planar curves $R = k[x, y]/(y^a - x^b)$ for $2 \leqq a < b$. The goal of this section is to compute the Hochschild and cyclic homology of these k-algebras when k is a field of characteristic zero.

Theorem 4. 1. *Let k be a field of characteristic zero, $2 \leqq a < b$, and*

$$R = k[x, y]/(y^a - x^b),$$

considered as a k-algebra. Then

$$HH_0(R) = R,$$
$$HH_1(R) \cong \Omega_{R/k} \cong (R \oplus R/y^{a-1}R \text{ as a } k[x]\text{-module}),$$
$$HH_n(R) \cong R/(x^{b-1}, y^{a-1})R \quad \text{for} \quad n > 1.$$

The cyclic homology of R is

$$HC_0(R) = R,$$
$$HC_n(R) = HC_n(k) = k \quad \text{if} \quad n \geqq 2 \text{ is even},$$
$$HC_n(R) \cong R/(x^{b-1}, y^{a-1})R \quad \text{if} \quad n \text{ is odd}.$$

Remark. As a k-vector space, $R/(x^{b-1}, y^{a-1})R$ has dimension $(a-1)(b-1)$. In the case a, b relatively prime, Dan Burghelea and Michele Vigué-Poirrer [BV] have obtained the same results via different methods.

We will use our Procedure 2. 6, grading R by letting x and y have degrees $2a$, $2b$ respectively. Since R can be defined over $\mathbb{Q}$, we know by 2. 5 that $HH^*(R; k)$ is the graded-commutative k-algebra $k[u, v, w]$ with u, v and w in topological degrees $2a-1$, $2b-1$ and $2(ab-1)$ respectively. Note that $u^2 = v^2 = 0$ and $uv + vu = 0$, because u and v are in odd degrees, and that w is central in $HH^*(R; k)$.

In the spectral sequence, the qth row of E_2 is $R \otimes HH^q(R; k)$. We leave the elementary verification of the following result to the reader.

Lemma 4. 2. *The rows of the E_2 term in the Leray spectral sequence are periodic of order $2(ab-1)$. The qth row is zero, except for the following 4 cases, when the qth row is R:*

$$q \equiv 0 \mod 2(ab-1), \text{ generated by } w^i,$$
$$q \equiv 2a-1 \mod 2(ab-1), \text{ generated by } uw^i,$$
$$q \equiv 2b-1 \mod 2(ab-1), \text{ generated by } vw^i,$$
$$q \equiv 2a+2b-2 \mod 2(ab-1), \text{ generated by } uvw^i.$$

In order to compute the differentials in E_2, we need concrete representatives in $B(R; R)$ for u, v and w.

Lemma 4. 3. *In $B(R; R)$, $u=[x]$ and $v=[y]$ with $b(u)=b(v)=0$. Moreover,*

$$w=\sum_{i=0}^{a-2} y^i[y^{a-i-1}|y]-\sum_{j=0}^{b-2} x^j[x^{b-j-1}|x]$$

and is congruent $\bmod L^1 B$ *to* $w_1=[y^{a-1}|y]-[x^{b-1}|x]$. *Finally,*

$$b(w)=a y^{a-1} v-b x^{b-1} u \quad in \quad B(R; R).$$

Proof. Clearly, u, v and w_1 are cycles in $B(R; k)$. To see that they are not boundaries in $B(R; k)$, note that the images of u and w_1 in $B(R/yR; k)$ (resp., of v and w_1 in $B(R/xR; k)$) are not boundaries by Example 1.10. By Lemma 4. 2 these must be the generators of $HH^*(R; k)$. The calculation of $b(u)$, $b(v)$ and $b(w)$ in $B(R; R)$ is an easy exercise left to the reader.

Corollary 4. 4. *$E_2=E_r$ for $r=2b(a-1)$, and d_r is determined by multiplicativity, $d_r(u)=d_r(v)=0$, and*

$$d_r(w)=a y^{a-1} v.$$

Proof. This follows from 1. 7 and the observation that, for $r=2b(a-1)$, $b(w) \in L^r B$ and $b(w) \equiv a y^{a-1} v \bmod L^{r+1} B$.

Fix $r=2b(a-1)$. Using multiplicativity and the fact that y is a nonzerodivisor in R, we see that d_r injects the rows generated by w^i and uw^i into the rows generated by vw^i and uvw^i, while d_r is zero on these latter rows.

Thus E_{r+1} has only two rows still isomorphic to R, namely the rows $q=0$ and $q=2a-1$ on generators 1 and u. If $q \geqq 2a$ and $q \equiv 2b-1$ or $q \equiv 2a+2b-2 \bmod 2(ab-1)$, then the qth row of E_{r+1} is $R/y^{a-1}R=k[x, y]/(x^b, y^{a-1})$ on generators vw^i and uvw^{i-1}, respectively. The rest of the rows are zero.

Because E_{r+1} has so many zero rows, we must have $E_{r+1}=E_s$ for $s=2ab-2a$. Clearly $d_s(1)=d_s(u)=0$. By multiplicativity, we see that d_s is completely determined by $d_s(vw^i)$ and $d_s(uvw^i)$. At this point, we write $\#$ for the shuffle product in $B(R; R)$ for emphasis, and observe that by 4. 3 we have

$$b(v \# w^{\# i})=-i v \# b(w) \# w^{\# i-1}=i b x^{b-1} u \# v \# w^{\# i-1},$$
$$b(u \# v \# w^{\# i})=i u \# v \# b(w) \# w^{\# i-1}=0.$$

Again by 1. 7, it follows that d_s is zero except on the qth row for $q=2i(ab-1)+2b-1$, when $d_s: R/y^{a-1}R \to R/y^{a-1}R$ is multiplication by ibx^{b-1}. The cokernel is $R/(x^{b-1}, y^{a-1})R$ and the kernel is $x(R/y^{a-1}R)$ which, as an R-module, is isomorphic to $R/(x^{b-1}, y^{a-1})R$ and, as a k-module, has dimension $(a-1)(b-1)$.

Proposition 4. 5. *The qth row of E_∞ in the Leray spectral sequence is:*

R	*if* $q=0$ *or* $q=2a-1$	(*on* 1 *and* u),
$R/y^{a-1}R$	*if* $q=2b-1$	(*on* v),
$R/(x^{b-1}, y^{a-1})$	*if* $q=2i(ab-1)+2b-1,\ i\geqq 1$	(*on* xvw^i),
$R/(x^{b-1}, y^{a-1})$	*if* $q=2i(ab-1)+2a+2b-2,\ i\geqq 0$	(*on* uvw^i),
0	*otherwise.*	

Proof. From the preceding discussion, we know that these are the rows of E_{s+1}, $s=2ab-2a$, so it is enough to see that $E_{s+1}=E_\infty$. For this, recall that d_t must lower length by 1 since b does. But v has length 1, xvw^i has length $2i+1$ and uvw^i has length $2i+2$, so by inspection, $d_t=0$ for $t>s$.

We are now ready to determine $HH_*(R;R)$. Since every row of E_∞ has constant length, $HH_l(R;R)$ has a filtration by rows of E_∞. The only row with length 0 is the row $q=0$, giving $HH_0(R;R)=R$. Two rows have length 1, so we have a filtration

$$0 \longrightarrow R \xrightarrow{v} HH_1(R;R) \longrightarrow R/y^{a-1}R \longrightarrow 0.$$

In fact, $HH_1(R;R)$ is naturally isomorphic to

$$\Omega_{R/k}=(R\,dx\otimes R\,dy)/(a\,y^{a-1}\,dy-b\,x^{b-1}\,dx)R$$

with $u=[x]$ and $v=[y]$ corresponding to dx and dy, respectively. For each $l\geqq 2$ there is a unique row with length l, so we must have $HH_l(R;R)\cong R/(x^{b-1}, y^{a-1})R$ for $l\geqq 2$. This proves the first half of Theorem 4. 1.

To finish the proof of Theorem 4. 1, we need to calculate $HC_*(R)$. Since R is an augmented k-algebra, $HC_*(R)=HC_*(k)\oplus\tilde{H}\tilde{C}_*(R)$, and we only need to calculate $\tilde{H}\tilde{C}_*(R)$. Since R is graded and $\operatorname{char}(k)=0$, the SBI sequence breaks up into short exact sequences of k-modules by Goodwillie's result [G der]:

$$0 \longrightarrow \tilde{H}\tilde{C}_{q-1}(R) \xrightarrow{B} HH_q(R;R) \xrightarrow{I} \tilde{H}\tilde{C}_q(R) \longrightarrow 0, \quad q\geqq 1.$$

Since $HH_q(R;R)$ is a finite-dimensional vector space for $q\geqq 2$, it is enough to show that

$$HC_1(R)\cong HH_2(R;R).$$

But R and $\Omega_{R/k}$ are graded vector spaces finite in each degree, and $R\to\Omega_{R/k}$ preserves the grading (weight). From Goodwillie's result, we know the kernel of this map is $HC_0(k)=k$. A straightforward calculation of the Hilbert functions of R and $\Omega_{R/k}$, which we omit, shows that the graded vector space $HC_1(R)=\operatorname{coker}(R\to\Omega_{R/k})$ has dimension $(a-1)(b-1)$. Since this equals $\dim HH_2(R;R)$, it follows that

$$HC_1(R)\cong HH_2(R;R),$$

completing the proof of Theorem 4. 1.

Remark 4. 6. Leslie Roberts showed us another method of proving that

$$HC_1(R) = \Omega_{R/k}/dR$$

is isomorphic to $HH_2(R; R) = \Omega^2_{R/k}$. One observes that, since R is graded, the higher de Rham cohomology of R vanishes, i.e., $H^*_{DR}(R) \cong H^*_{DR}(k)$. Therefore, the augmented de Rham complex

$$0 \longrightarrow k \longrightarrow R \longrightarrow \Omega_{R/k} \longrightarrow \Omega^2_{R/k} \longrightarrow 0$$

is exact, whence the result.

Remark 4. 7. The element $axvw^i$ of $B(R; R)$ is not a cycle, but it is congruent mod $L^{2a+1}B$ to the cycle $(axv - byu)w^i$. Thus $HH_{2i+1}(R; R)$ is generated as R-module by the cycle $(axv - byu)w^i$ for $2i+1 \geqq 3$, and the same is true for the cyclic homology of R since, for $2i+1 \geqq 3$, we have

$$I : HH_{2i+1}(R; R) \xrightarrow{\cong} HC_{2i+1}(R).$$

§ 5. Base change formula for cyclic homology

If A is a k-algebra and l an extension of k, we can consider $A \otimes l$ as an l-algebra or a k-algebra when computing cyclic homology. If l is flat over k, it is easy to see that $HC^l_*(A \otimes l) \cong l \otimes HC^k_*(A)$, where the superscript indicates whether we are considering k-algebras or l-algebras. In this section we will compute $HC^k_*(A \otimes l)$ from $HC^k_*(A)$, a result which will be used in computing K-theory from cyclic homology.

We begin by recalling the well-known formula for Hochschild homology,

$$HH^k_n(A \otimes l; A \otimes l) \cong \coprod_{p+q=n} HH^k_p(A; A) \otimes HH^k_q(l; l),$$

assuming A and l are flat over k ([CE]). The isomorphism is given by the shuffle product $\#$. For example, since $H^k_q(k[t]; k[t]) = 0$ for $q \geqq 2$, we have

$$HH^k_n(A[t]; A[t]) \cong HH^k_n(A; A) \otimes k[t] \oplus HH^k_{n-1}(A; A) \otimes \Omega_{k[t]/k}. \tag{5. 1}$$

Of course, $\Omega_{k[t]/k} \cong k[t]$ on generator dt (cf. Example 1. 6).

The basic formula we shall use for cyclic homology was given by Kassel in [Kass], 3. 2. For simplicity let k be a field, and suppose that

$$HC_*(A) \cong (HC_*(k) \otimes U_*) \oplus V_*$$

as a graded module, where $S : HC_n(A) \to HC_{n-2}(A)$ sends $HC_p(k) \otimes U_q$ $(p+q=n)$ to $HC_{p-2}(k) \otimes U_q$ and V_n to 0. Then for any k-algebra l,

$$HC^k_*(A \otimes l) \cong (HC^k_*(l) \otimes U_*) \oplus (HH^k_*(l; l) \otimes V_*) \tag{5. 2}$$

as a graded k-module, and S maps $HH_*(l; l) \otimes V_*$ to zero ([Kass], 3. 2).

Corollary 5. 3 (Kassel [Kass], 3. 3, 3. 6). *For any k-algebra l,*

a) $HC_n(l[x]) \cong HC_n(l) \oplus (HH_n(l; l) \otimes (xk[x]))$,

b) $HC_n(l[x]/x^2) \cong HC_n(l) \oplus HH_n(l; l) \oplus HH_{n-2}(l; l) \oplus \cdots$.

One special case of (5. 2) is of particular importance to us.

Theorem 5. 4. *Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded algebra over a field k of characteristic 0, and let $\tilde{H}\tilde{C}_n(A)$ denote the kernel of $HC_n(A) \to HC_n(A_0)$. Then for every k-algebra l we have $HC_n(A \otimes l) \cong HC_n(A_0 \otimes l) \oplus \tilde{H}\tilde{C}_n(A \otimes l)$, where*

$$\tilde{H}\tilde{C}_n(A \otimes l) \cong \coprod_{p+q=n} \tilde{H}\tilde{C}_p(A) \otimes HH_q(l; l).$$

Proof. Kassel's formula (5. 2) applies equally well to $\tilde{H}\tilde{C}_*$ (cf. [Kass], 3. 1). Since $S = 0$ on $\tilde{H}\tilde{C}_*(A)$ by [G der], $U_* = 0$ and we have $\tilde{H}\tilde{C}_*(A) = V_*$ in (5. 2).

Remark 5. 4. 1. We will see below that this is not a natural decomposition. However, it will be clear from the discussion below that there is a natural descending filtration on the k-module $\tilde{H}\tilde{C}_n(A \otimes l)$ whose p^{th} associated graded module is

$$\tilde{H}\tilde{C}_p(A) \otimes HH_q(l; l).$$

In particular, the summand $\tilde{H}\tilde{C}_0(A) \otimes HH_n(l; l)$ of $HC_n(A \otimes l)$ is natural in A.

We shall give another proof of 5. 4, since we need a more concrete description of the decomposition of $HC_*(A \otimes l)$. For this, we shall adopt the notation of 5. 4 and recall that $I: \tilde{H}\tilde{H}_p(A; A) \to \tilde{H}\tilde{C}_p(A)$ is onto for all p, where $\tilde{H}\tilde{H}_p(A; A)$ denotes the kernel of $HH_p(A; A) \to HH_p(A_0; A_0)$. Choose a k-module splitting

$$s: \tilde{H}\tilde{C}_p(A) \to \tilde{H}\tilde{H}_p(A; A)$$

of I, and define

$$x * y = I(s(x) \# y), \qquad x \in \tilde{H}\tilde{C}_p(A),\ y \in HH_q(l; l).$$

This is an element of $\tilde{H}\tilde{C}_{p+q}(A \otimes l)$ since $s(x) \# y$ is in $HH_{p+q}(A \otimes l; A \otimes l)$. To see the dependence on the choice of splitting, we shall need the following result from p. 576 of [LQ].

Lemma 5. 5. *If A, l are k-algebras and $x \in HH_p(A; A)$, $y \in HH_q(l; l)$, then the map $BI: HH_{p+q}(A \otimes l; A \otimes l) \to HH_{p+q+1}(A \otimes l; A \otimes l)$ satisfies the Leibniz formula*

$$BI(x \# y) = BI(x) \# y + (-1)^p x \# BI(y).$$

Corollary 5. 5. 1. *If $s, s': \tilde{H}\tilde{C}_p(A) \to \tilde{H}\tilde{H}_p(A; A)$ are two splittings of I in the setting of Theorem 5. 4, then the image of $I(s(x) \# y) - I(s'(x) \# y)$ under the injection $B: \tilde{H}\tilde{C}_{p+q}(A \otimes l) \to \tilde{H}\tilde{H}_{p+q+1}(A \otimes l; A \otimes l)$ is $(-1)^p (s(x) - s'(x)) \# BI(y)$.*

Proof. Choose $z \in \tilde{H}\tilde{H}_{p-1}(A; A)$ so that $BI(z) = s(x) - s'(x)$ in $\tilde{H}\tilde{H}_p(A; A)$. Since $I(s(x) \# y) - I(s'(x) \# y) = I(BI(z) \# y)$, the result follows from $IB = 0$ and

$$BI(BI(z) \# y) = BIBI(z) \# y + (-1)^p BI(z) \# BI(y).$$

Corollary 5. 5. 2. *If BI maps $y \in HH_q(l; l)$ to zero in $HH_{q+1}(l; l)$ (e.g., y is a "closed" q-form in $\Omega^q_{l/k}$), then $x * y \in \tilde{H}\tilde{C}_{p+q}(A \otimes l)$ is independent of the choice of the splitting of I, and the image of $x * y$ in $\tilde{H}\tilde{H}_{p+q+1}(A \otimes l; A \otimes l)$ is the shuffle product*

$$B(x) \# y.$$

Proof. Since $BI(y) = 0$, we have $I(s(x) \# y) = I(s'(x) \# y)$ by 5. 5. 1. By Lemma 5. 5 we have

$$BI(s(x) \# y) = BI(s(x)) \# y + 0 = B(x) \# y.$$

Remark 5. 5. 3. If $y = B(y')$ for some $y' \in HC_{q-1}(l)$, then our $x * y$ is the $*$-product $x * y'$ of [LQ], (3. 2).

Theorem 5. 6. *If A is a graded algebra over a field k of characteristic zero, and l is a k-algebra, then the map*

$$*: \coprod_{p+q=n} \tilde{H}\tilde{C}_p(A) \otimes HH_q(l; l) \to \tilde{H}\tilde{C}_n(A \otimes l)$$

is an isomorphism for every choice of a k-module splitting s of I.

Proof. For notational convenience, let us write H_q for $HH_q(l; l)$. We will show that there is a commutative diagram with exact rows:

(5. 6. 1)

$$\begin{tikzcd}
0 \arrow[r] & \coprod_{p+q=n} \tilde{H}\tilde{C}_{p-1}(A) \otimes H_q \arrow[r, "\varphi"] \arrow[d, "*"] & \coprod_{p+q=n} \tilde{H}\tilde{H}_p(A; A) \otimes H_q \arrow[r, "\psi"] \arrow[d, "\#", "\cong"'] & \coprod_{p+q=n} \tilde{H}\tilde{C}_p(A) \otimes H_q \arrow[r] \arrow[d, "*"] & 0 \\
0 \arrow[r] & \tilde{H}\tilde{C}_{n-1}(A \otimes l) \arrow[r, "B"] & \tilde{H}\tilde{H}_n(A \otimes l, A \otimes l) \arrow[r, "I"] & \tilde{H}\tilde{C}_n(A \otimes l) \arrow[r] & 0.
\end{tikzcd}$$

Since the shuffle product $\#$ is an isomorphism for all n, the five-lemma tells us that in (5. 6. 1) the left-hand map $*$ is injective and the right-hand map $*$ is surjective for all n. Comparing the cases for n and $n+1$, we see that $*$ is an isomorphism.

To construct (5. 6. 1) we need to define the maps φ and ψ. For $x \in \tilde{H}\tilde{C}_{p-1}(A)$ and $y \in H_q$ we define

$$\varphi(x \otimes y) = B(x) \otimes y + (-1)^{p-1} s(x) \otimes BI(y).$$

This defines φ; since B is an injection it is easy to see (by induction on q, say) that φ is an injection. To define ψ, recall that every element of $\tilde{H}\tilde{H}_p(A;A)$ can be written uniquely as $B(x)+s(z)$ for $x \in \tilde{H}\tilde{C}_{p-1}(A)$ and $z \in \tilde{H}\tilde{C}_p(A)$. We define ψ by

$$\psi(B(x)\otimes y)=(-1)^p x\otimes BI(y),$$
$$\psi(s(z)\otimes y)=z\otimes y.$$

From the latter formula it is clear that ψ is onto and split by $s\otimes 1$. Hence the kernel of ψ is generated by elements of the form $w-(s\otimes 1)\,\psi(w)$, $w\in \coprod \tilde{H}\tilde{H}_p(A;A)\otimes H_q$. Since $s(z)\otimes y=(s\otimes 1)\,\psi(s(z)\otimes y)$, the kernel of ψ is generated by the elements

$$\begin{aligned} B(x)\otimes y-(s\otimes 1)\,\psi(B(x)\otimes y) &= B(x)\otimes y-(-1)^p s(x)\otimes BI(y)\\ &= \varphi(x\otimes y). \end{aligned}$$

This proves that the top row of (5. 6. 1) is exact.

Let us check that the diagram (5. 6. 1) commutes. It is easy to see that $B*=\#\varphi$, and by definition

$$*\psi(s(z)\otimes y)=I(s(z)\# y)=(I\#)(s(z)\otimes y).$$

To see that $*\psi(B(x)\otimes y)=(-1)^p x * BI(y)$ equals $(I\#)(B(x)\otimes y)=I(B(x)\# y)$, we apply the injection B and use Lemma 5. 5 twice, noting that $IB=0$:

$$\begin{aligned} B(x*BI(y)) &= BI(s(x)\# BI(y))\\ &= BI(s(x))\# BI(y)+(-1)^{p-1}\, s(x)\# BIBI(y)\\ &= B(x)\# BI(y); \end{aligned}$$

$$\begin{aligned} BI(B(x)\# y) &= BIB(x)\# y+(-1)^p\, B(x)\# BI(y)\\ &= (-1)^p\, B(x)\# BI(y). \end{aligned}$$

Hence (5. 6. 1) commutes, concluding the proof of 5. 6.

Example 5. 7. Let k be a field of characteristic zero, l a field containing k, and $A=k[x_0,\dots,x_b]/(x_i x_j=0,\, i\neq j)$. In 3. 12 we showed that $\tilde{H}\tilde{C}_p(A)$ is a vector space of dimension $c(b,p+1)$ if $p\geqq 1$, and constructed an explicit splitting s sending the cyclic p-cycle $\alpha x_{i_0}\otimes\cdots\otimes x_{i_p}$ to the Hochschild p-cycle $\alpha x_{i_0}[x_{i_1}|\cdots|x_{i_p}]$. The isomorphism $I:\tilde{H}\tilde{H}_0(A)\cong\tilde{H}\tilde{C}_0(A)$ is canonically split. Therefore

$$\begin{aligned} &\tilde{H}\tilde{C}_n(l[x_0,\dots,x_b]/(x_i x_j=0,\, i\neq j))\\ &\qquad\cong \coprod_{c(b,n+1)} l\oplus \coprod_{c(b,n)} \Omega_{l/k}\oplus\cdots\oplus \coprod_{c(b,2)} \Omega^{n-1}_{l/k}\oplus(\tilde{A}\otimes\Omega^n_{l/k}). \end{aligned}$$

Explicitly, the $c(b,n+1)$ copies of l associate $\alpha\in l$ to the cyclic cycles

$$(x_{i_0}\otimes\cdots\otimes x_{i_n})*\alpha=\alpha x_{i_0}\otimes x_{i_1}\otimes x_{i_n}.$$

For $1 \leqq p \leqq n-1$ and $q=n-p$, the $c(b, p+1)$ copies of $\Omega^q_{l/k}$ associate the form $\alpha d\beta_1 \cdots d\beta_q$ to the cyclic cycles

$$(x_{i_0} \otimes \cdots \otimes x_{i_p}) * (\alpha d\beta_1 \cdots d\beta_q) = I(\alpha x_{i_0}[x_{i_1}| \cdots |x_{i_p}] \# (d\beta_1 \cdots d\beta_q))$$
$$= (\alpha x_{i_0} \otimes \cdots \otimes x_{i_p}) * (d\beta_1 \cdots d\beta_q).$$

Finally, this decomposition associates $a \otimes (\alpha d\beta_1 \cdots d\beta_n) \in \tilde{A} \otimes \Omega^n_{l/k}$ to the cyclic cycle $I(a\alpha d\beta_1 \cdots d\beta_n)$ coming from the summand $\Omega^n_{A \otimes l/k}$ of $HH_n(A \otimes l; A \otimes l)$.

Example 5. 8. Let k be a field of characteristic zero, l a field containing k, and $R = k[x, y]/(y^a - x^b)$ for $2 \leqq a < b$. In 4. 1 we showed that $\tilde{H}\tilde{C}_p(R) = 0$ for p even ($p \neq 0$) and that $\tilde{H}\tilde{C}_p(R) \cong R/(x^{b-1}, y^{a-1})R$ for p odd. In 4. 7 we showed that $r \in R/(x^{b-1}, y^{a-1})R$ corresponded to the cyclic cycle

$$I(r(axv - byu)w^{(p-1)/2}).$$

Therefore

$$\tilde{H}\tilde{C}_n(l[x, y]/(y^a - x^b)) \cong \coprod_{i=0}^{[\frac{n-1}{2}]} R/(x^{b-1}, y^{a-1}) \otimes \Omega^{n-2i-1}_{l/k} \oplus (\tilde{R} \otimes \Omega^n_{l/k}).$$

For $r \in R/(x^{b-1}, y^{a-1})R$ and $\theta \in \Omega^{n-2i-1}_{l/k}$, the corresponding cyclic cycle is

$$I(r(axv - byu)w^i \# \theta).$$

§ 6. *K*-theory of curves

Let X be a quasiprojective curve over a field of characteristic zero. In this section we shall show how to reduce the calculation of $K_*(X)$ to calculations of the K-theory of smooth curves and fields and calculations of cyclic homology of the type discussed in the previous sections of this paper, at least for seminormal curves over an algebraically closed field. For general curves X, our reduction partially depends on the validity of the KABI Conjecture 0. 1 that double relative cyclic homology and double relative K-theory coincide.

We first reduce to the case when X is affine. Because X has infinitely many closed points, we can find two disjoint, finite sets S and T of closed points of X such that $X-S$ and $X-T$ are affine, and S supports a Cartier divisor of X. Let A, B and B_S denote the coordinate rings of the affine curves $X-S$, $X-T$ and $X-T-S$, respectively. Then the map $(X-T) \to X$ is an "analytic isomorphism" along S. (The terminology "analytic isomorphism" is from [W ai] and is described below.)

Hence there is a long exact sequence in K-theory

$$\cdots \to K_{n+1}(B_S) \to K_n(X) \to K_n(A) \oplus K_n(B) \to K_n(B_S) \to \cdots.$$

This reduces the calculation of $K_*(X)$ to the calculation of the K-theory of the affine curves $X-S$, $X-T$ and $X-S-T$.

We next reduce to the case in which $X = \operatorname{Spec}(A)$ is reduced. Let N be the nilradical of A, so that $X_{\mathrm{red}} = \operatorname{Spec}(A/N)$. In this case the exact sequence of an ideal is

$$\cdots \to K_{n+1}(X_{\mathrm{red}}) \to K_n(A, N) \to K_n(X) \to K_n(X_{\mathrm{red}}) \to \cdots$$

and $K_n(A, N) \cong HC^{\mathbb{Q}}_{n-1}(A, N)$ by Goodwillie [G]. So, in the calculation of $K_*(X)$, we may assume that X is reduced and affine.

At this point, we break up the K-theory of the reduced ring A into two pieces: the "Karoubi-Villamayor theory" (with groups $KV_*(A)$) and the "nil K-theory" (with groups $K_*^{\mathrm{nil}}(A)$). These theories are related by the long exact sequence

$$\cdots \to KV_{n+1}(A) \to K_n^{\mathrm{nil}}(A) \to K_n(A) \to KV_n(A) \to \cdots.$$

To compute the KV-theory of A, we recall that A is a reduced 1-dimensional ring, essentially finitely generated over a field. Therefore the normalization B of A is a Dedekind ring which is finite over A, and the conductor I from B to A is a height one ideal of both rings. The square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A/I \arrow[d] \\ B \arrow[r] & B/I \end{tikzcd} \tag{6. 1}$$

is cartesian, B is Dedekind, and the rings A/I and B/I are artinian. If $m_1, \ldots$ (resp. $n_1, \ldots$) are the maximal ideals of A (resp. B) containing I, then the reduced rings corresponding to A/I and B/I are the finite products of fields $\prod A/m_i$ and $\prod B/n_j$. We therefore have a long exact sequence

$$\begin{aligned} 0 \to A^* \to B^* \times (A/I)^* \to (B/I)^* \to K_0(A) \to K_0(B) \oplus \textstyle\prod K_0(A/m_i) \\ \to \textstyle\prod K_0(B/n_j) \to K_{-1}(A) \to 0 \end{aligned}$$

from [Bass], IX, which computes $K_0(A) \cong KV_0(A)$ and $K_{-1}(A) = KV_{-1}(A)$. Moreover, $KV_1(A) = A^* \oplus SKV_1(A)$. By [W nilp], Thm. 3. 9, there is a long exact sequence

$$\cdots \textstyle\prod K_{n+1}(B/n_j) \to KV_n(A) \to K_n(B) \oplus \prod K_n(A/m_i) \to \prod K_n(B/n_j) \cdots$$

ending in

$$K_2(B) \oplus \textstyle\prod K_2(A/m_i) \to \prod K_2(B/n_j) \to SKV_1(A) \to SK_1(B) \to 0.$$

Thus, if we know the K-theory of the smooth curve $\operatorname{Spec}(B)$ and of the fields A/m_i and B/n_j, we can recover (at least up to extension), the KV-theory of A. This procedure for calculating $KV_*(A)$ was first given by L. Roberts in [R].

This leaves us with the calculation of the groups $K_*^{\mathrm{nil}}(A)$. We may reduce this to the calculation of the nil K-groups of simpler rings using the observation that $K_*^{\mathrm{nil}}(R)=0$ if R is a regular ring. Recall from [W ai] that an *analytic isomorphism* along nonzerodivisors S is a ring map $A \to B$ sending a central multiplicative set S of nonzerodivisors of A to a central set of nonzerodivisors of B, such that $A/sA \cong B/sB$ for every $s \in S$.

Theorem 6. 2. *If $A \to B$ is an analytic isomorphism along nonzerodivisors S with both $S^{-1}A$ and $S^{-1}B$ regular, then $K_*^{\mathrm{nil}}(A) \cong K_*^{\mathrm{nil}}(B)$.*

Proof. Let D denote the cartesian square of rings

$$\text{(D)} \qquad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ S^{-1}A & \longrightarrow & S^{-1}B. \end{array}$$

There is an infinite loop spectrum $\mathscr{K}^{\mathrm{nil}}(A)$ for every A with $K_n^{\mathrm{nil}}(A)=\pi_n \mathscr{K}^{\mathrm{nil}}(A)$ [W map]. Since $S^{-1}A$ and $S^{-1}B$ are regular, $K_*^{\mathrm{nil}}(S^{-1}A)=K_*^{\mathrm{nil}}(S^{-1}B)=0$. Thus we need only show that $\mathscr{K}^{\mathrm{nil}}(D)$ is a homotopy cartesian square. If $\mathscr{K}(A)$ and $\mathscr{K}V(A)$ denote the spectra whose homotopy groups give $K_*(A)$ and $KV_*(A)$ respectively, then there is a sequence of diagrams

$$\mathscr{K}^{\mathrm{nil}}(D) \to \mathscr{K}(D) \to \mathscr{K}V(D)$$

which is a fibration in every entry by definition of $\mathscr{K}^{\mathrm{nil}}$ ([W map]). But $\mathscr{K}(D)$ and $\mathscr{K}V(D)$ are homotopy cartesian by [W ai], 1. 3, whence $\mathscr{K}^{\mathrm{nil}}(D)$ is also.

Corollary 6. 3. *Let A be a reduced, 1-dimensional ring with only finitely many singular primes $\{m_1, \ldots, m_n\}$. Then $K_*^{\mathrm{nil}}(A) \cong \oplus K_*^{\mathrm{nil}}(A_{m_i})$.*

Proof. Choose a nonzero-divisor $s \in \bigcap m_i$ and let B be the product of all A_m with $s \in m$. Then $A \to B$ is an analytic isomorphism along $\{s^n\}$ with $A[s^{-1}]$ and $B[s^{-1}]$ regular.

Corollary 6. 4. *Let A be a reduced, 1-dimensional local ring essentially finite over a field, with completion $\hat{A}$. Then $K_*^{\mathrm{nil}}(A) \cong K_*^{\mathrm{nil}}(\hat{A})$.*

Proof. $A \to \hat{A}$ is an analytic isomorphism along any nonzero-divisor s in m, and $\hat{A}$ is reduced ([Mat], 34. C). Thus $A[s^{-1}]$ and $\hat{A}[s^{-1}]$ are regular.

The above two corollaries complete our reductions for a reduced affine curve $X=\mathrm{Spec}(A)$ in that they show that $K_*^{\mathrm{nil}}(A)$ is the direct sum over the singular primes m of $K_*^{\mathrm{nil}}(A_m)$, and these groups depend only on the analytic type of the singularity in the sense that $K_*^{\mathrm{nil}}(A_m) \cong K_*^{\mathrm{nil}}(\hat{A}_m)$.

Before we compute nil K-theory from cyclic homology, we pause to illustrate most of the above reductions. Using the computation for $B=l[x, y]/(xy)$, we analyze

Example 6. 5 (Node). Let $A = l[x, y]/(y^2 = x^2 - x^3)$ and

$$X = \operatorname{Proj}(l[X, Y, Z]/(Y^2 Z = X^2 Z - X^3)),$$

where l is any field of characteristic zero. Then $K_n(X) = K_n(A) \oplus K_n(l)$ and

$$K_n(A) = KV_n(A) \oplus V_n,$$

where $KV_n(A) = K_n(l) \oplus K_{n+1}(l)$ and

$$V_n = \begin{cases} 0, & n \leqq 1, \\ l \oplus \Omega_l^2 \oplus \Omega_l^4 \oplus \cdots \oplus \Omega_l^{n-2}, & n \text{ even}, n \geqq 2, \\ \Omega_l \oplus \Omega_l^3 \oplus \cdots \oplus \Omega_l^{n-2}, & n \text{ odd}, n \geqq 3. \end{cases}$$

Proof. Let S be the point $(0:0:1)$ and T the point $(0:1:0)$, whence

$$X - T = \operatorname{Spec}(A), \quad X - S = \operatorname{Spec}(l[t, t^{-1}]), \quad \text{and} \quad X - S - T = \operatorname{Spec}(l[t, t^{-1}, (t-1)^{-1}]).$$

We know that $K_n(X - S)$ injects into $K_n(X - S - T)$ with cokernel $K_{n-1}(l)$. Letting $\pi : \mathbb{P}_l^1 \to X$ be the normalization and $\tilde{S} = \pi^{-1}(S)$, we obtain an exact diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K_n(l) \arrow[r] \arrow[d, equal] & K_n(X) \arrow[r] \arrow[d] & K_n(A) \arrow[r] \arrow[d] & K_{n-1}(l) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & K_n(l) \arrow[r, "\partial"] & K_n(\mathbb{P}_l^1) \arrow[r] & K_n(l[t]) \arrow[r] & K_{n-1}(l) \arrow[r] & \cdots .
\end{tikzcd}$$

Since the lower left arrow ∂ is known to be a split injection ([Q 341]),

$$K_n(X) \cong K_n(A) \oplus K_n(l).$$

The computation of $KV_n(A)$ is from [R]; as a $K_*(l)$-algebra it is generated by $1 \in K_0(A)$ and an element $\tau \in K_{-1}(A)$ with $\tau^2 = 0$ in $K_{-2}(A)$ (cf. [DW], 5. 5). Thus the $K_*(l)$-algebra map $K_*(A) \to KV_*(A)$ is a split surjection, whence

$$K_n(A) = KV_n(A) \oplus K_*^{\mathrm{nil}}(A).$$

Furthermore, $K_n^{\mathrm{nil}}(A) \cong K_n^{\mathrm{nil}}(B)$ since $\hat{A} = \hat{B}$ (Corollary 6. 4). This completes our reductions. We shall see in 7. 1 that $K_n^{\mathrm{nil}}(B) = V_n$.

We can generalize Example 6. 5 to analyze the K-theory of any seminormal curve X over an algebraically closed field l, $\operatorname{char}(l) = 0$. By [Davis], the local rings A_α of the singularities of X have the same completion as $B_\alpha = l[x_0, \ldots, x_b]/(x_i x_j = 0, i \neq j)$, where the αth singularity has $b_\alpha + 1$ branches. Thus

$$K_*^{\mathrm{nil}}(A) \cong \bigoplus K_*^{\mathrm{nil}}(B_\alpha) = \bigoplus \tilde{K}_*(B_\alpha),$$

and $\tilde{K}_*(B_\alpha)$ will be calculated in Theorem 7. 1.

If X is a seminormal curve over a field l which is not algebraically closed, then the local rings A_α have the same completion as the rings $B_\alpha = l_\alpha \oplus \prod_\beta t\, l_{\alpha\beta}[t]$, where the l_α and $l_{\alpha\beta}$ are finite field extensions of l. The K-theory of the B_α will be discussed in Section 8, and the remarks there apply since $K_*^{\mathrm{nil}}(B_\alpha) \cong \tilde{K}_*(B_\alpha)$.

6. 6. We conclude this section by connecting K^{nil}-theory to cyclic homology. Let A be the local ring of a singular point of a curve over a field of characteristic zero. Associated to the conductor square (6. 1) is an exact diagram of K^{nil}-groups

$$
\begin{tikzcd}
 & \cdots \arrow[d] & & & \\
 & K_n^{\mathrm{nil}}(A, B, I) \arrow[d] & & & \\
\cdots \arrow[r] & K_n^{\mathrm{nil}}(A, I) \arrow[r] \arrow[d] & K_n^{\mathrm{nil}}(A) \arrow[r] \arrow[d] & K_n^{\mathrm{nil}}(A/I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_n^{\mathrm{nil}}(B, I) \arrow[r] \arrow[d] & K_n^{\mathrm{nil}}(B) \arrow[r] & K_n^{\mathrm{nil}}(B/I) \arrow[r] & \cdots \\
 & \cdots & & &
\end{tikzcd}
\tag{6. 6. 1}
$$

There is also an associated diagram of cyclic homology groups over the rational numbers $\mathbb{Q}$:

$$
\begin{tikzcd}
 & \cdots \arrow[d] & & & \\
 & HC_{n-1}(A, B, I) \arrow[d] & & & \\
\cdots \arrow[r] & HC_{n-1}(A, I) \arrow[r] \arrow[d] & HC_{n-1}(A) \arrow[r] \arrow[d] & HC_{n-1}(A/I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & HC_{n-1}(B, I) \arrow[r] \arrow[d] & HC_{n-1}(B) \arrow[r] & HC_{n-1}(B/I) \arrow[r] & \cdots \\
 & \cdots & & &
\end{tikzcd}
\tag{6. 6. 2}
$$

By [W map], there is a natural homomorphism v from (6. 6. 1) to (6. 6. 2). Furthermore, since A/I is an artinian local ring containing a field L of characteristic zero, it has the form $L \oplus m$, where m is a nilpotent ideal. Thus, by Goodwillie [G],

$$v: K_n^{\mathrm{nil}}(A/I) \to HC_{n-1}(A/I)/HC_{n-1}(L)$$

is an isomorphism. Similarly, B/I is a finite product of artinian local rings $L_\alpha \oplus M_\alpha$. Since B is regular, $K_n^{\mathrm{nil}}(B)=0$ and we obtain

$$K_n^{\mathrm{nil}}(B, I) \cong K_{n+1}^{\mathrm{nil}}(B/I) \cong HC_n(B/I)/\prod HC_n(L_\alpha).$$

If we assume the validity of the KABI Conjecture 0.1, then from the knowledge of $HC_*(A)$ we can use (6. 6. 2) to compute $K_n(A, B, I)$. Via (6. 6. 1) we can then compute $K_n^{\mathrm{nil}}(A, I)$ and finally $K_n^{\mathrm{nil}}(A)$. We will do this type of analysis in Section 9 for the cusps $l[t^a, t^b]$, $2 \leqq a < b$, a special case of which is

Example 6. 7. Let $A = \mathbb{Q}[t^2, t^3]$, the coordinate ring of the simple cusp. Then $B = \mathbb{Q}[t]$ and $I = t^2 B$, the augmentation ideal of A. Thus $K_n^{\mathrm{nil}}(A, I) \cong K_n^{\mathrm{nil}}(A)$, a fact which simplifies 6. 6. 1. Furthermore, if $n \geqq 2$, then $K_n^{\mathrm{nil}}(B)$ and $HC_{n-1}(B)/HC_{n-1}(\mathbb{Q})$ are zero, whence by the five-lemma, $K_n^{\mathrm{nil}}(B, I) \cong HC_{n-1}(B, I)$ for $n \geqq 2$. If the KABI Conjecture is valid, i.e., $K_n(A, B, I) \cong HC_{n-1}(A, B, I)$ for all n, then (using the five-lemma and 4. 7), we would obtain an isomorphism for $n \geqq 2$

$$\begin{aligned} v: K_n(A)/K_n(\mathbb{Q}) &\cong K_n^{\mathrm{nil}}(A) \cong HC_{n-1}(A)/HC_{n-1}(\mathbb{Q}) \\ &\cong \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases} \end{aligned}$$

The groups $K_0(A)$ and $K_1(A)$ are known to agree with this formula ([Kr], 12. 1).

§ 7. *K*-theory of coordinate axes

For the rest of this paper, we shall focus on various special curves, studying their K-theory via their cyclic homology. In this section, we consider the coordinate axes in affine $(b+1)$-space over a field l of characteristic zero. The coordinate ring is

$$R_b = l[x_0, \ldots, x_b]/(x_i x_j = 0,\ i \neq j).$$

From Example 5. 7 we know that the cyclic homology of R_b over $k = \mathbb{Q}$ is

$$HC_{n-1}(R_b) = HC_{n-1}(l) \oplus V_n \oplus (\tilde{R}_b \otimes \Omega_l^{n-1}).$$

Here $\tilde{R}_b$ is the augmentation ideal generated by the x_i, Ω_l is the l-module of absolute Kähler differentials of l, and

$$V_n = \begin{cases} \coprod_{c(b,n)} l \oplus \coprod_{c(b,n-1)} \Omega_l \oplus \cdots \oplus \coprod_{c(b,2)} \Omega_l^{n-2} & \text{if } n \geqq 2, \\ 0 & \text{if } n \leqq 1. \end{cases}$$

The number of Ω_q summands in V_n was determined in (3. 13) to be

$$c(b, q) = \sum_{\substack{e|q \\ q-e \text{ even}}} \sum_{d|e} \frac{1}{e} \mu(e/d) [b^d + (-1)^d b].$$

Theorem 7. 1. *Let l be a field of characteristic* 0, *and R_b the coordinate ring of the axes in affine $(b+1)$-space, as above. Then for all n*

$$K_n(R_b) = K_n(l) \oplus V_n.$$

We will explicitly describe the elements of V_n as symbols in $K_n(R_b)$ when we come to Theorem 7. 2 below. To prove Theorem 7. 1, we shall use induction on b. The case $b=0$ is clear because $R_0 = l[x]$. The inductive step uses the cartesian square

$$\begin{tikzcd} R_b \arrow[r] \arrow[d] & R_{b-1} \arrow[d] \\ l[x_b] \arrow[r] & l, \end{tikzcd} \tag{7. 1. 1}$$

where the ring R_{b-1} is R_b/J, $J = x_b R_b$, and $l[x_b]$ is R_b/I, $I = (x_0, \ldots, x_{b-1}) R_b$.

Part of Theorem 7. 1 is that $K_n(R_b) = K_b(l)$ for $n \leqq 1$. This is an easy consequence of the Mayer-Vietoris sequence for (7. 1. 1) which starts at $K_2(R_b)$ (see [Mil], 6. 2).

To prove that $K_n(R_b) \cong K_n(l) \oplus V_n$ for $n \geqq 2$, we follow the method described in § 6. We will show that $KV_n(R_b) = K_n(l)$ and that $K_n^{\text{nil}}(R_b) \cong V_n$. Suppose more generally that A is a graded ring with $A_0 = l$. Then $KV_n(A) \cong K_n(l)$ and $K_n^{\text{nil}}(A) \cong \tilde{K}_n(A)$ for $n \geqq 1$ by [W map], 5. 8. Therefore, by [W map] and [G der], there is a commutative diagram with exact rows for $n \geqq 1$:

$$\begin{tikzcd} 0 \arrow[r] & \tilde{K}_n(A) \arrow[r] \arrow[d, equal] & K_n(A) \arrow[r] \arrow[d] & K_n(l) \arrow[r] \arrow[d, equal] & 0 \\ \cdots \arrow[r] & K_n^{\text{nil}}(A) \arrow[r] \arrow[d, "v"] & K_n(A) \arrow[r] \arrow[d, "D"] & KV_n(A) \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & \tilde{H}\tilde{C}_{n-1}(A) \arrow[r, "B"] & \tilde{H}\tilde{H}_n(A; A) \arrow[r, "I"] & \tilde{H}\tilde{C}_n(A) \arrow[r] & 0. \end{tikzcd}$$

The cyclic and Hochschild homology in this diagram are taken over $k = \mathbb{Q}$.

Thus $v: \tilde{K}_n(A) \to \tilde{H}\tilde{C}_{n-1}(A)$ is a natural lift of the Dennis trace map D. Similarly, there is a relative version $v: K_n(A, I) \to HC_{n-1}(A, I)$ defined for every graded ideal I of A.

In our case, when $A = R_b$ and $I = (x_0, \dots, x_{b-1})$ we have $A/I \cong l[x_b]$. Since $\tilde{H}\tilde{C}_{n-1}(l[x_b]) \cong x_b l[x_b] \otimes_l \Omega_l^{n-1}$ by 5. 3. a), and the kernel of $\tilde{R}_b \to x_b l[x_b]$ is naturally isomorphic to $\tilde{R}_{b-1}$, the computation of $HC_{n-1}(R_b)$ mentioned at the beginning of this section shows that

$$HC_{n-1}(R_b, I) \cong V_n \oplus (\tilde{R}_{b-1} \otimes_l \Omega_l^{n-1}).$$

On the other hand, $K_n(R_b, I) \cong \tilde{K}_n(R_b)$ because $\tilde{K}_n(l[x_b]) = 0$. Therefore, the relative version of v maps $\tilde{K}_n(R_b)$ to $HC_{n-1}(R_b, I)$, and our goal is to show that it is injective with cokernel $\tilde{R}_{b-1} \otimes_l \Omega_l^{n-1}$.

The map $R_b \to R_{b-1}$ in (7. 1. 1) maps I isomorphically onto the kernel of $R_{b-1} \to l$, so $K_n(R_{b-1}, I) \cong \tilde{K}_n(R_{b-1})$ and $HC_n(R_{b-1}, I) = \tilde{H}\tilde{C}_n(R_{b-1})$. Inductively, the map v from $\tilde{K}_n(R_{b-1})$ to $\tilde{H}\tilde{C}_{n-1}(R_{b-1})$ is known to be an injection with cokernel $\tilde{R}_{b-1} \otimes_l \Omega_l^{n-1}$. For notational simplicity, write X_n for the double relative group $K_n(R_b, R_{b-1}, I)$, which is isomorphic to $HC_{n-1}(R_b, R_{b-1}, I)$ by [OW]. From the exact sequences of the ideal I, we obtain the following exact diagram:

(7. 1. 2)

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & & & U_n \arrow[d] & 0 \arrow[d] & \\
 & \tilde{K}_{n+1}(R_{b-1}) \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d, "\cong"] & \tilde{K}_n(R_b) \arrow[r] \arrow[d] & \tilde{K}_n(R_{b-1}) \arrow[r] \arrow[d] & X_{n-1} \arrow[d, "\cong"] \\
HC_n(R_b, I) \arrow[r] \arrow[d] & \tilde{H}\tilde{C}_n(R_{b-1}) \arrow[r] \arrow[d] & X_n \arrow[r] & HC_{n-1}(R_b, I) \arrow[r] \arrow[d] & \tilde{H}\tilde{C}_{n-1}(R_{b-1}) \arrow[r] \arrow[d] & X_{n-1} \\
W_{n+1} \arrow[r] \arrow[d] & \tilde{R}_{b-1} \otimes \Omega_l^n \arrow[d] & & W_n \arrow[r] \arrow[d] & \tilde{R}_{b-1} \otimes \Omega_l^{n-1} \arrow[d] & \\
0 & 0 & & 0 & 0. &
\end{tikzcd}$$

From 5. 4. 1 and the fact that $HC_0(R_b, I) \cong \tilde{H}\tilde{C}_0(R_{b-1}) = \tilde{R}_{b-1}$, we see that $HC_{n-1}(R_b, I)$ maps onto $\tilde{R}_{b-1} \otimes \Omega_l^{n-1}$. A diagram chase now reveals that

$$W_n \cong \tilde{R}_{b-1} \otimes \Omega_l^{n-1}.$$

Another chase shows that $U_n = 0$. This completes the proof of Theorem 7. 1.

We shall now describe the elements of $\tilde{K}_n(R_b) \cong V_n$ in terms of symbols. Recall from [L sym] that if $x_1, \ldots, x_q$ are elements of a ring A such that

$$x_1 x_2 = x_2 x_3 = \cdots = x_{q-1} x_q = x_q x_1 = 0,$$

then there is an element $\langle\langle x_1, \ldots, x_q \rangle\rangle$ of $K_q(A)$ which is invariant under cyclic permutation in the sense that

$$\langle\langle x_1, \ldots, x_q \rangle\rangle = (-1)^{q-1} \langle\langle x_2, \ldots, x_q, x_1 \rangle\rangle.$$

We call these elements *Loday symbols*. If A is an l-algebra, then the map $l \to K_q(A)$ sending $\alpha \in l$ to $\langle\langle \alpha x_1, x_2, \ldots, x_q \rangle\rangle$ is an abelian group homomorphism. Finally, we shall need the fact, derived in [Ogle T], 3. 4 and [Ogle I], p. 243, that in $HH_q(A; A)$

$$D(\langle\langle x_1, \ldots, x_q \rangle\rangle) = B(x_1 \otimes \cdots \otimes x_q) = BI(x_1 [x_2 | \cdots | x_q]).$$

Let us apply these general remarks to the ring R_b. For each sequence $i_1, \ldots, i_q$ such that $i_1 \neq i_2, \ldots, i_{q-1} \neq i_q$ and $i_q \neq i_1$, there is an abelian group homomorphism $l \to K_q(R_b)$ sending $\alpha \in l$ to $\langle\langle \alpha x_{i_1}, x_{i_2}, \ldots, x_{i_q} \rangle\rangle$. Since

$$\langle\langle \alpha x_{i_1}, x_{i_2}, \ldots, x_{i_q} \rangle\rangle = (-1)^{q-1} \langle\langle \alpha x_{i_2}, x_{i_3}, \ldots, x_{i_q}, x_{i_1} \rangle\rangle,$$

this homomorphism does not depend on the cyclic ordering of the i_j, at least up to sign. If q is even and $(i_1, \ldots, i_q)$ has an odd number of distinct q-tuples in its orbit under the action of the cyclic group, this Loday symbol must be zero.

For each of the remaining cyclic orbits of q-tuples $(i_1, \ldots, i_q)$ we have a map $l \to K_q(R_b)$ which is nontrivial, since the composite

$$l \longrightarrow K_q(R_b) \xrightarrow{D} HH_q^{\mathbb{Q}}(R_b; R_b) \longrightarrow HH_q^l(R_b; R_b)$$

sends α to $B(\alpha x_{i_1} \otimes \cdots \otimes x_{i_q})$ and is a split injection by 3. 11. By 3. 9, there are $c(b, q)$ such cyclic orbits, hence $c(b, q)$ such maps. In fact, more is true. By 3. 12 we know that $\tilde{H}\tilde{C}^l_{q-1}(R_b)$ is the direct sum of $c(b, q)$ copies of l and that the $x_{i_1} \otimes \cdots \otimes x_{i_q}$ form an l-basis of $\tilde{H}\tilde{C}^l_{q-1}(R_b)$. Hence we have proven

Lemma 7. 2. *When $q \geqq 2$, the map $v: \tilde{K}_q(R_b) \to \tilde{H}\tilde{C}^l_{q-1}(R_b) \cong \coprod_{c(b,q)} l$ is a split surjection, split by the $c(b, q)$ maps $l \to \tilde{K}_q(R_b)$ sending $\alpha \in l$ to the Loday symbol $\langle\langle \alpha x_{i_1}, x_{i_2}, \ldots, x_{i_q} \rangle\rangle$.*

Remark 7. 2. 1. It is important to observe that the l-module structure on $\tilde{H}\tilde{C}^l_{q-1}(R_b)$ comes from the l-module structure on $HH^l_{q-1}(R_b; R_b)$ via the split surjection $I: HH^l_{q-1}(R_b; R_b) \to \tilde{H}\tilde{C}^l_{q-1}(R_b)$, and *not* from the split injection

$$B: \tilde{H}\tilde{C}^l_{q-1}(R_b) \to HH^l_q(R_b; R_b)$$

which we use to detect elements in K-theory.

Remark 7. 2. 2. When l is an algebraic extension of $\mathbb{Q}$, so that $\Omega_l = 0$, this lemma completely describes $\tilde{K}_q(R_b)$, because by Theorem 7. 1 when $q \geqq 2$

$$\tilde{K}_q(R_b) \cong V_q = \coprod_{c(b,q)} l.$$

To describe the remaining elements of V_n as symbols in $\tilde{K}_n(R_b)$, we shall take the cup product of the above Loday symbols with the Steinberg symbols $\{\beta_1, \dots, \beta_p\}$ in $K_p(l)$.

Theorem 7. 3. *Under the isomorphism of* 7. 1 *we have*

$$\tilde{K}_n(R_b) \cong \coprod_{c(b,n)} l \oplus \coprod_{c(b,n-1)} \Omega_l \oplus \cdots \oplus \coprod_{c(b,2)} \Omega_l^{n-2},$$

the element α *in the* $(i_1, \dots, i_n)$*-copy of* l *corresponds to the Loday symbol*

$$\langle\langle \alpha x_{i_1}, \dots, x_{i_n} \rangle\rangle,$$

and the element $\alpha d\beta_1 \cdots d\beta_p$ *in the* $(i_1, \dots, i_q)$*-copy of* Ω_l^p $(p+q=n)$ *corresponds to the cup product*

$$\langle\langle \alpha(\textstyle\prod \beta_i) x_{i_1}, x_{i_2}, \dots, x_{i_q} \rangle\rangle \cup \{\beta_1, \dots, \beta_p\}.$$

As a first step in the proof, we have

Lemma 7. 4. *If* $\beta_1, \dots, \beta_p \in l^*$, $\alpha \in l$ *and* $i_1 \neq i_2, \dots, i_q \neq i_1$ *then the cup product*

$$\langle\langle \alpha x_{i_1}, \dots, x_{i_q} \rangle\rangle \cup \{\beta_1, \dots, \beta_p\}$$

in $K_{p+q}(R_b)$ *maps under the Dennis trace map to*

$$\begin{aligned} BI(\alpha x_{i_1}[x_{i_2}| \cdots |x_{i_q}] \# ((\textstyle\prod \beta_i)^{-1} d\beta_1 \cdots d\beta_p)) \\ = BI(x_{i_1}[x_{i_2}| \cdots |x_{i_q}] \# \alpha(\textstyle\prod \beta_i)^{-1} (d\beta_1 \cdots d\beta_p)) \end{aligned}$$

in $HH_{p+q}(R_b; R_b)$.

Proof. Recall that the Dennis trace map $D: K_p(l) \to \Omega_l^p$ sends $\{\beta_1, \dots, \beta_p\}$ to the closed p-form

$$w \equiv (\textstyle\prod \beta_i)^{-1} d\beta_1 \cdots d\beta_p.$$

Since the Dennis trace map commutes with products by [Igusa], 5. c. 3,

$$D(\langle\langle \alpha x_{i_1}, \dots, x_{i_q} \rangle\rangle \cup \{\beta_1, \dots, \beta_p\}) = BI(\alpha x_{i_1}[x_{i_2}| \cdots |x_{i_q}]) \# w.$$

In order to complete the computation we apply the Leibniz formula 5. 5 for the map $BI: HH_n(R; R) \to HH_{n+1}(R; R)$ and use the fact that $BI(w) = dw = 0$.

Proof of Theorem 7. 3. For fixed $\beta_1, \dots, \beta_p$, there are $c(b, q)$ maps

$$l \longrightarrow \tilde{K}_q(R_b) \xrightarrow{\cup\{\beta_1, \dots, \beta_p\}} \tilde{K}_{p+q}(R_b) \xrightarrow{\nu} \tilde{H}\tilde{C}_{p+q-1}(R_b).$$

Since B is an injection, 7.4 states that each map sends γ to

$$I(x_{i_1}[x_{i_2}|\cdots|x_{i_q}] \# \gamma(\prod \beta_i)^{-1}(d\beta_1 \cdots d\beta_p)),$$

i.e. to $\gamma(\prod \beta_i)^{-1} d\beta_1 \cdots d\beta_p$ in one of the $c(b, q)$ summands Ω_l^p of $\tilde{H}\tilde{C}_{p+q-1}(R_b)$. Letting γ, the β_i and $(i_1, \ldots, i_q)$ vary, we see that we can find a symbol in $K_{p+q}(R_b)$ hitting every element in the copy of Ω_l^p induced by $(i_1, \ldots, i_q)$. In particular, if $\gamma = \alpha \prod \beta_i$, then we see that the symbol $\langle\langle \alpha(\prod \beta_i) x_{i_1}, \ldots, x_{i_q} \rangle\rangle \bigcup \{\beta_1, \ldots, \beta_q\}$ hits $\alpha d\beta_1 \cdots d\beta_p$.

§ 8. The cyclic homology and K-theory of seminormal curves

Not every seminormal curve has singularities of the analytic type considered in the previous section, i.e., of the analytic type of $R_b = l[x_0, \ldots, x_b]/(x_i x_j = 0, i \neq j)$. By [Davis], however, every other analytic type is of the form

$$R = l \oplus t(\prod l_i)[t],$$

where $\prod l_i$ is a finite product of finite extension fields of l. In this section we consider those R such that $l_i \bar{\mathbb{Q}} = l\bar{\mathbb{Q}}$ as subfields of the algebraic closure $\bar{l}$ of l. For those rings there is a finite Galois extension F of $k = l \cap \bar{\mathbb{Q}}$ so that, if $L = Fl$, then

$$R \otimes_k F \cong L[x_0, \ldots, x_b]/(x_i x_j = 0, i \neq j).$$

We will prove that $\tilde{H}\tilde{C}_*(R) \cong \tilde{H}\tilde{C}_*(R \otimes_k F)^G$, where G is the Galois group of F over k, and deduce the following result:

Theorem 8. 1. *Let $R = l \oplus t(\prod l_i)[t]$, where l and the l_i are fields of characteristic zero such that $l_i \bar{\mathbb{Q}} = l\bar{\mathbb{Q}}$ for all i. Assume that $b = -1 + \sum \dim_l(l_i)$ is finite. Then for $n \geqq 0$,*

$$HC_n^{\mathbb{Q}}(R) \cong HC_n^{\mathbb{Q}}(l) \oplus V_{n+1} \oplus (\tilde{R} \otimes_l \Omega_l^n).$$

Here V_n is as in Theorem 7. 1, i.e., $V_n = 0$ if $n \leqq 1$ and

$$V_n = \coprod_{c(b,n)} l \oplus \coprod_{c(b,n-1)} \Omega_l \oplus \cdots \oplus \coprod_{c(b,2)} \Omega_l^{n-2} \quad \text{if} \quad n \geqq 2.$$

Furthermore, if the KABI *Conjecture* 0. 1 *is valid, then for all n*

$$K_n(R) \cong K_n(l) \oplus V_n.$$

We begin with the following technical lemma, which is well-known but for which we could find no reference:

Lemma 8. 2 (Folklore). *Suppose that k is a separable algebraic field extension of a field K, and let A be a k-algebra. Then $HC_*^K(A) \cong HC_*^k(A)$. If M is an A-bimodule, then*

$$HH_*^K(A; M) \cong HH_*^k(A; M).$$

Proof. Write k^e and A^e for $k \otimes_K k^{\mathrm{op}}$ and $A \otimes_K A^{\mathrm{op}}$, respectively, and note that $A \otimes_k A^{\mathrm{op}} = A^e \otimes_{k^e} k$. If k is a finite separable extension of K, then k is a projective k^e-module; in general, since k is the union of finite separable extensions of K, we still know that k is a flat k^e-module. Therefore, any free $A \otimes_k A^{\mathrm{op}}$ resolution of an A-bimodule M is also a flat A^e resolution of M. Tensoring such a resolution with A over either A^e or $A \otimes_k A^{\mathrm{op}}$ yields the same chain complex, since

$$\begin{aligned} A \otimes_{A^e} (A \otimes_k A^{\mathrm{op}}) &\cong A \otimes_{A^e} (A^e \otimes_{k^e} k) \cong A \\ &\cong A \otimes_{A \otimes_k A^{\mathrm{op}}} (A \otimes_k A^{\mathrm{op}}). \end{aligned}$$

Taking the homology of this chain complex, we find that

$$HH_*^K(A; M) = \mathrm{Tor}_*^{A^e}(A, M) \cong \mathrm{Tor}_*^{A \otimes_k A^{\mathrm{op}}}(A, M) = HH_*^k(A; M).$$

Setting $M = A$, and using induction on $*$ in the SBI sequence, we find that

$$HC_*^K(A) \cong HC_*^k(A),$$

as claimed.

Lemma 8. 3. *Let l be a field of characteristic zero, $k = l \cap \bar{\mathbb{Q}}$, and F a Galois extension of k with Galois group G. Let L denote the field $lF = l \otimes_k F$. Then for every graded l-algebra A*

$$\tilde{H}\tilde{C}_*(A) \cong \tilde{H}\tilde{C}_*(A \otimes_l L)^G.$$

Here the cyclic homology may be taken over $\mathbb{Q}$ or k.

Proof. Using 8. 2, we shall take cyclic homology over k. Since $\Omega_{F/k} = 0$, the base-change formula 5. 4 yields

$$\tilde{H}\tilde{C}_*(A \otimes_k F) = \tilde{H}\tilde{C}_*(A) \otimes_k F.$$

Since G acts trivially on $HC_*(A)$ and $F^G = k$, the result follows.

Let us now turn to the situation that concerns us. Let $l_1, l_2, \ldots$ be a finite number of finite field extensions of l such that $l_\alpha \bar{\mathbb{Q}} \cong l\bar{\mathbb{Q}}$, and set $R = l \oplus t(\prod l_\alpha)[t]$. Let F be a Galois extension of $k = l \cap \bar{\mathbb{Q}}$ such that $L = lF$ contains every l_α, and let G be the Galois group of F over k. Then there are $[l_\alpha : l]$ idempotents $e_{\alpha\beta}$ in $l_\alpha \otimes_l L$ so that $l_\alpha \otimes_l L \cong \prod L e_{\alpha\beta}$. Hence

$$R \otimes_l L = L \oplus t(\prod (l_\alpha \otimes_k F))[t] \cong L[x_0, \ldots, x_b]/(x_i x_j = 0, i \neq j),$$

where the x_i are of the form $e_{\alpha\beta} t$ for some α, β. Note that elements of G permute the $\{e_{\alpha\beta}\}$, hence the $\{x_i\}$, and that $R \cong (R \otimes_l L)^G$.

By 5. 7, we know that there are l-vector spaces M_p of dimension $c(b, p+1)$ such that the cyclic homology over k is

$$\tilde{H}\tilde{C}_n(R \otimes_l L) \cong \coprod_{q=0}^{n-1} (M_{n-q} \otimes_l \Omega_{L/k}^q) \oplus (\tilde{R} \otimes_l \Omega_{L/k}^n).$$

The explicit description of this decomposition given in 5. 7 shows that each summand is fixed under the action of the Galois group G. Since $\Omega^q_{L/k} = \Omega^q_{l/k} \otimes_l L$, we see from 8. 3 that

$$\tilde{H}\tilde{C}_n(R) \cong \coprod_{q=0}^{n-1} (M_{n-q} \otimes_l L)^G \otimes_l \Omega^q_{l/k} \oplus (\tilde{R} \otimes_l \Omega^n_{l/k}).$$

The group G acts on the $M_p \otimes_l L$ diagonally. Since L is a free $l[G]$-module, so is $M_p \otimes_l L$. It follows that

$$\dim_l (M_p \otimes_l L)^G = \dim_l (M_p) = c(b, p+1).$$

This proves the first half of Theorem 8. 1.

We now turn our attention to the K-theory of R, following the outlines of the computation in § 7. The analogue of the square (7. 1. 1) is

$$\begin{array}{ccc} R & \longrightarrow & \prod l_i[t] \\ \downarrow & & \downarrow \\ l & \longrightarrow & \prod l_i. \end{array}$$

Since $\Omega_{l_i/l} = 0$, the obstruction $K_1(R, \prod l_i[t], \tilde{R})$ vanishes by [GW], (0. 2), and there is a Mayer-Vietoris sequence for this square starting at K_2.

As in § 7, we deduce that $\tilde{K}_n(R) = 0$ for $n \leqq 1$. (This result is also a consequence of [W ai], 4. 7.)

To determine $\tilde{K}_n(R)$ for $n \geqq 2$ we shall isolate one special case and then proceed by induction.

Proposition 8. 4. *Let $l \subset l_1$ be fields of characteristic zero such that $l\bar{\mathbb{Q}} = l_1\bar{\mathbb{Q}}$, and set $A = l \bigoplus_t l_1[t]$, $B = l_1[t]$ and $I = t l_1[t]$. Assume the* KABI *Conjecture* 0. 1 *holds for A, B and I, and that* $\dim_l(l_1) = b+1$ *is finite. Then for all $n \geqq 2$*

$$K_n(A) \cong K_n(l) \oplus \coprod_{c(b,n)} l \oplus \coprod_{c(b,n-1)} \Omega_l \oplus \cdots \oplus \coprod_{c(b,2)} \Omega_l^{n-2}.$$

Proof. By 5. 3. a) we know that $HC_n(B, I) \cong I \otimes_l \Omega^n_l$. Since $A/I = l$, we have $HC_{n-1}(A, I) \cong \tilde{H}\tilde{C}_{n-1}(A) \cong V_n \oplus I \otimes_l \Omega^{n-1}_l$ and $K_n(A, I) = \tilde{K}_n(A)$. As in the proof of 7. 1, we see that $HC_n(A, I)$ maps onto $HC_n(B, I)$, and that we have an exact diagram:

$$\begin{array}{ccccccc} K_{n+1}(B, I) & \longrightarrow & K_n(A, B, I) & \longrightarrow & \tilde{K}_n(A) & \longrightarrow & K_n(B, I) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (I \otimes \Omega^n_l) & \xrightarrow{\partial} & HC_{n-1}(A, B, I) & \longrightarrow & V_n \oplus (I \otimes \Omega^{n-1}_l) & \longrightarrow & (I \otimes \Omega^{n-1}_l) \longrightarrow 0. \end{array}$$

Since $K_*(B) = K_*(l_1)$, we have $K_*(B, I) = 0$. Since $\partial = 0$, we get

$$\tilde{K}_n(A) \cong K_n(A, B, I) \cong HC_{n-1}(A, B, I) \cong V_n.$$

With this special case in hand, we can now compute the K-theory of

$$R = l \oplus t\left(\prod_{i=1}^{m} l_i\right)[t]$$

using induction on m. Set

$$S = l \oplus t\left(\prod_{i=2}^{m} l_i\right)[t],$$

and let A be as in 8. 4. As in (7. 1. 1), we have a cartesian square

(8. 5. 1)
$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ A \arrow[r] & l. \end{tikzcd}$$

For notational simplicity, write X_n for $K_n(R, S, \tilde{S})$, which is isomorphic to

$$HC_{n-1}(R, S, \tilde{S})$$

by [OW]. From the exact sequences of the ideal $\tilde{S}$, we obtain the following exact diagram, which is the analogue of (7. 1. 2):

(8. 5. 2)
$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & U_n \arrow[d] & 0 \arrow[d] & \\
 & \tilde{K}_{n+1}(S) \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d, "\cong"] & K_n(R, \tilde{S}) \arrow[r] \arrow[d] & \tilde{K}_n(S) \arrow[r] \arrow[d] & X_{n-1} \arrow[d, "\cong"] \\
\tilde{H}\tilde{C}_n(R, \tilde{S}) \arrow[r] \arrow[d] & \tilde{H}\tilde{C}_n(S) \arrow[r] \arrow[d] & X_n \arrow[r] & HC_{n-1}(R, \tilde{S}) \arrow[r] \arrow[d] & \tilde{H}\tilde{C}_{n-1}(S) \arrow[r] \arrow[d] & X_{n-1} \\
W_{n+1} \arrow[r] \arrow[d] & \tilde{S} \otimes \Omega_l^n \arrow[d] & & W_n \arrow[r] \arrow[d] & \tilde{S} \otimes \Omega_l^{n-1} \arrow[d] & \\
0 & 0 & & 0 & 0. &
\end{tikzcd}$$

As in the proof of 7. 1, $HC_{n-1}(R, \tilde{S})$ maps onto $\tilde{S} \otimes \Omega_l^{n-1}$ by 5. 4. 1, and a diagram chase on (8. 5. 2) shows that $W_n \cong \tilde{S} \otimes \Omega_l^{n-1}$ and $U_n = 0$.

Unlike the proof of 7. 1, however, we are not done yet. We need to chase through one more commutative diagram:

(8. 5. 3)

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & & 0 \arrow[d] & L_n \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & \cdot \arrow[r] \arrow[d] & K_n(R, \tilde{S}) \arrow[r] \arrow[d] & \tilde{K}_n(R) \arrow[r] \arrow[d] & \tilde{K}_n(A) \arrow[r] \arrow[d] & K_{n-1}(R, \tilde{S}) \arrow[d] \\
\cdot \arrow[r] \arrow[d] & \tilde{H}\tilde{C}_n(A) \arrow[r] \arrow[d] & HC_{n-1}(R, \tilde{S}) \arrow[r] \arrow[d] & \tilde{H}\tilde{C}_{n-1}(R) \arrow[r] \arrow[d] & \tilde{H}\tilde{C}_{n-1}(A) \arrow[r] \arrow[d] & \cdot \\
M_{n+1} \arrow[r] \arrow[d] & \tilde{A} \otimes_l \Omega_l^n \arrow[r] \arrow[d] & \tilde{S} \otimes_l \Omega_l^{n-1} \arrow[r] \arrow[d] & M_n \arrow[r] \arrow[d] & \tilde{A} \otimes_l \Omega_l^{n-1} \arrow[d] & \\
0 & 0 & 0 & 0 & 0. &
\end{tikzcd}$$

In diagram (8. 5. 3) we know that the top two rows are exact and that the columns are exact (one by (8. 5. 2) and two by the special case 8. 4). We know that $\tilde{H}\tilde{C}_0(R) = \tilde{R}$ maps onto $\tilde{H}\tilde{C}_0(A) = \tilde{A}$ with kernel $HC_0(R, \tilde{S}) = \tilde{S}$. Therefore in the decomposition of 5. 4 or 5. 7 the summand $\tilde{R} \otimes_l \Omega_l^{n-1}$ of $\tilde{H}\tilde{C}_{n-1}(R)$ maps onto the summand $\tilde{A} \otimes_l \Omega_l^{n-1}$ of $\tilde{H}\tilde{C}_{n-1}(A)$ with kernel the summand $\tilde{S} \otimes_l \Omega_l^{n-1}$ of $HC_{n-1}(R, \tilde{S})$. Hence in (8. 5. 3) $M_n \to \tilde{A} \otimes_l \Omega_l^{n-1}$ is onto for all n. Now a chase shows that $L_n = 0$, that $M_n \cong \tilde{R} \otimes_l \Omega_l^{n-1}$, and therefore that $\tilde{K}_n(R)$ is isomorphic to the summand V_n of $\tilde{H}\tilde{C}_{n-1}(R)$. This completes the proof of Theorem 8. 1.

§ 9. The *K*-theory of cusps

In this section we shall study the K-theory of the cusps $l[t^a, t^b]$ when l is a field of characteristic zero, using our computation of their cyclic homology in § 4. Note that we may assume that a and b are relatively prime, since, if $d = \gcd(a, b)$ and $s = t^d$, then $l[t^a, t^b] \cong l[s^{a/d}, s^{b/d}]$. The conductor ideal from $l[t]$ to $l[t^a, t^b]$, when a and b are relatively prime, is $I = t^c l[t]$, where $c = (a-1)(b-1)$. We shall study the K-theory of $A = l[t^a, t^b]$ via the cartesian square

(9.1)
$$\begin{tikzcd}
A \arrow[r] \arrow[d] & l[t] \arrow[d] \\
A/I \arrow[r] & l[t]/(t^c = 0).
\end{tikzcd}$$

For convenience, we define l-modules V_n by

$$V_n = \begin{cases} 0 & \text{if } n \leqq 1, \\ l \oplus \Omega_l^2 \oplus \Omega_l^4 \oplus \cdots \oplus \Omega_l^{n-2} & \text{if } n \geqq 2,\ n \text{ even}, \\ \Omega_l \oplus \Omega_l^3 \oplus \cdots \oplus \Omega_l^{n-2} & \text{if } n \geqq 3,\ n \text{ odd}. \end{cases}$$

Theorem 9. 2. *Let l be a field of characteristic zero and set $A = l[t^a, t^b]$, where $2 \leqq a < b$ with a and b relatively prime. If the* KABI *Conjecture* 0. 1 *is true for A, $B = l[t]$ and $I = t^c B$, then*

$$K_n(A) \cong K_n(l) \ \oplus\ (B/I) \otimes_l V_n \ \oplus\ (B/A) \otimes_l \Omega_l^n.$$

Remark 9. 2. 1. The "postage stamp problem" in combinatorics states that $\dim(A/I) = \dim(B/A) = c/2$. Using the Mayer-Vietoris sequence for (9. 1) it is easy to show that $K_n(A) = K_n(l)$ for $n < 0$ and that $K_0(A) = K_0(l) \oplus (B/A)$. The calculation

$$K_1(A) = K_1(l) \ \oplus\ (B/A) \otimes_l \Omega_l$$

was performed in [GR] (explicitly for $l[t^2, t^3]$, implicitly for general A). The calculation of $K_n(A)$, $n \geqq 2$, is still open because we do not know if the KABI Conjecture holds. The best we can do is summarized in the following result.

Theorem 9. 3. *Let $A = l[t^a, t^b]$ as in Theorem* 9. 2. *Then*

i) *$\tilde{K}_2(A)$ maps onto $(B/I) \ \oplus\ (B/A) \otimes_l \Omega_l^2$,*

ii) *$\tilde{K}_3(A)$ maps onto $(B/I) \otimes_l \Omega_l \ \oplus\ (B/A) \otimes_l \Omega_l^3$,*

iii) *for all n, the kernel U_n of $v : \tilde{K}_n(A) \to \tilde{H}\tilde{C}_{n-1}(A)$ maps onto $(B/A) \otimes_l \Omega_l^n$.*

The KABI Conjecture predicts that the maps in 9. 3 are isomorphisms. We are able to prove Theorem 9. 3 because we know the KABI Conjecture holds for K_1 by [GW], and because we can establish the following partial result for K_2.

Theorem 9. 4. *Let $A \to B$ be a map of graded $\mathbb{Q}$-algebras and I a homogeneous ideal of both A and B such that $I \cap A_0 = I \cap B_0 = 0$. Then the following map is onto:*

$$v : K_2(A, B, I) \to HC_1(A, B, I).$$

We shall start by proving Theorem 9. 4. When A, B and I are graded, the groups $HC_*(A, I)$, $HC_*(B, I)$, and $HC_*(A, B, I)$ are also graded by the weights of the corresponding cyclic cycles, as described in 1. 1.

Lemma 9. 5. *If $I \cap A_0 = I \cap B_0 = 0$ then, for fixed s and $n > s$, the weight s summands of the maps*

$$HC_*(A, I) \to HC_*(A/I^n, I/I^n) \text{ and } HC_*(A, B, I) \to HC_*(A/I^n, B/I^n, I/I^n)$$

are isomorphisms.

Proof. An element of weight s in $HC_m(A, I)$ is represented by a sum of terms $a_0 \otimes \cdots \otimes a_m$ where some a_i is in I and the sum of the degrees of the a_i is s. In particular, no a_i has degree greater than s. Since the boundary map b in the Connes complex preserves weight, it follows that the Connes complexes for $HC_*(A, I)$ and $HC_*(A/I^n, I/I^n)$ are isomorphic in weight s whenever $n > s$. This establishes the result for $HC_*(A, I)$ and $HC_*(B, I)$. The result for $HC_*(A, B, I)$ may be proven similarly, or may be deduced from the five-lemma.

We shall also need a decomposition of K-theory by weights. For this, we introduce the automorphism λ of a graded $\mathbb{Q}$-algebra A which is multiplication by 2^s on A_s. This makes $\tilde{K}_*(A)$, $HC_*(A)$, etc. into $\mathbb{Q}[\lambda]$-modules. It is easy to see that λ acts as multiplication by 2^s on the weight s summand of $HC_*(A)$, $HC_*(A, I)$ and $HC_*(A, B, I)$, so that these $\mathbb{Q}$-modules are the direct sum of eigenspaces. It also induces a primary decomposition on the torsion submodules of the $\mathbb{Q}[\lambda]$-modules $\tilde{K}_*(A)$, $K_*(A, I)$ and $K_*(A, B, I)$.

Proof of Theorem 9. 4. By 9. 5, [GW] and Goodwillie's Theorem ([G]) for the nilpotent ideal I/I^n, the following is an exact diagram of $\mathbb{Q}[\lambda]$-modules:

$$\begin{array}{ccccc}
K_2(A, B, I) & \longrightarrow & K_2(A/I^n, B/I^n, I/I^n) & \longrightarrow & K_1(A, B, I^n) \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
HC_1(A, B, I) & \longrightarrow & HC_1(A/I^n, B/I^n, I/I^n) & \longrightarrow & HC_0(A, B, I^n).
\end{array}$$

We claim that $T = K_2(A, B, I)/\ker(v)$ is a direct sum of eigenspaces for λ with eigenvalues 2^s, $s \geqq 0$. Indeed, T is a submodule of $HC_1(A, B, I)$, hence a torsion $\mathbb{Q}[\lambda]$-module, hence a direct sum of primary components T_s associated to the maximal ideals $(\lambda - 2^s)\,\mathbb{Q}[\lambda]$. But each T_s must be annihilated by $(\lambda - 2^s)$, i.e., an eigenspace.

When $n > s$, the weight s summand of $HC_1(A, B, I)$ injects into its right-hand neighbour, and a diagram chase shows that this summand is T_s. Hence

$$T \cong HC_1(A, B, I),$$

i.e., v is onto.

Remark. The groups $K_*(A, B, I)$ in 9. 4 are continuous modules over the ring $W(\mathbb{Q})$ of Witt vectors over $\mathbb{Q}$, and λ corresponds to multiplication by the Witt vector $(1 - 2t)$. Hence $K_*(A, B, I)$ is also a direct sum of eigenspaces for λ with eigenvalues 2^s by the general theory of continuous $W(\mathbb{Q})$-modules.

9. 6. Our next step is to perform some relative calculations for the ideal I in A and $B = l[t]$. We start by using Goodwillie's Theorem and 5. 4 applied to 1. 10 to find that:

$$\tilde{K}_{n+1}(B/I) \cong \tilde{H}\tilde{C}_n(B/I) \cong (\tilde{B}/I) \otimes_l V_n \quad \oplus \quad (\tilde{B}/I) \otimes_l \Omega_l^n.$$

Note that $\tilde{K}_{n+1}(B/I)$ is isomorphic to $K_n(B, I)$. On the other hand, we know $\tilde{H}\tilde{C}_n(B) = \tilde{B} \otimes_l \Omega_l^n$ by 5. 3. a), and can easily deduce that

$$HC_n(B, I) \cong (\tilde{B}/I) \otimes_l V_{n+1} \quad \oplus \quad (I \otimes_l \Omega_l^n).$$

Thus we have the following exact diagram, where we have set $B = l[t]$, $X_n = K_n(A, B, I)$ and $Y_n = HC_{n-1}(A, B, I)$:

(9. 6. 1)

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & & N_n \arrow[r] \arrow[d] & \tilde{B}/I \otimes_l \Omega_l^n \arrow[d] & \\
 & K_{n+1}(B, I) \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & K_n(A, I) \arrow[r] \arrow[d] & K_n(B, I) \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
\cdot \arrow[r] \arrow[d] & HC_n(B, I) \arrow[r] \arrow[d] & Y_n \arrow[r] & HC_{n-1}(A, I) \arrow[r] \arrow[d] & HC_{n-1}(B, I) \arrow[r] \arrow[d] & Y_{n-1} \\
Q_{n+1} \arrow[r] \arrow[d] & I \otimes_l \Omega_l^n \arrow[d] & & Q_n \arrow[r] \arrow[d] & I \otimes_l \Omega_l^{n-1} \arrow[d] & \\
0 & 0 & & 0 & 0. &
\end{tikzcd}$$

Lemma 9. 6. 2. *In diagram* (9. 6. 1), *the maps* $Q_n \to I \otimes_l \Omega_l^{n-1}$ *are onto for all n. If* $X_{n-1} \to Y_{n-1}$ *is an injection and* $X_n \to Y_n$ *is onto, then* $Q_n \cong I \otimes_l \Omega_l^{n-1}$.

Proof. Let $A_0 = \mathbb{Q}[t^a, t^b]$ *and* $I_0 = t^c \mathbb{Q}[t]$, so that $A = A_0 \otimes_{\mathbb{Q}} l$ and $I = A_0 \otimes_{\mathbb{Q}} l$. By 5. 4, we know that $HC_n(A, I)$ has a natural decomposition as $M_n \oplus (I \otimes_l \Omega_l^n)$, where

$$M_n = \coprod_{i=0}^{n-1} HC_{n-i}(A_0, I_0) \otimes_{\mathbb{Q}} \Omega_l^i.$$

The map $HC_n(A, I) \to HC_n(B, I)$ maps M_n to $(\tilde{B}/I) \otimes_l V_{n+1}$ and maps $I \otimes_l \Omega_l^n$ isomorphically onto $I \otimes_l \Omega_l^n$. Hence $Q_{n+1} \to I \otimes_l \Omega_l^n$ is onto. The rest of the lemma is a formal diagram chase on (9. 6. 1).

Lemma 9. 6. 3. *In diagram* (9. 6. 1), *the maps* $N_n \to \tilde{B}/I \otimes_l \Omega_l^n$ *are onto. If in addition* $X_n \to Y_n$ *is injective, then* $N_n \cong \tilde{B}/I \otimes_l \Omega_l^n$.

Proof. We first prove by induction on n that the composite

$$K_n(A, I) \to K_n(B, I) \cong \tilde{K}_{n+1}(B/I) \cong \tilde{H}\tilde{C}_n(B/I)$$

maps N_n onto the summand $\tilde{B}/I \otimes_l \Omega_l^n$ of $\tilde{H}\tilde{C}_n(B/I)$. Since $K_0(A, I) \cong K_0(B, I)$, this is true for $n = 0$. Assume N_n maps onto the summand $\tilde{B}/I \otimes_l \Omega_l^n$. Given ω in this summand,

choose $x \in N_n$, $y \in \tilde{K}_{n+1}(B/I)$ and $z \in HH_n(B/I; B/I)$ mapping to ω. Note that the Dennis trace map maps y to $B(\omega) = BI(z)$ in $HH_{n+1}(B/I; B/I)$. For $\beta \in l^* = K_1(l)$, the cup product $x \cup \beta \in K_{n+1}(A, I)$ maps to $y \cup \beta \in K_{n+1}(B, I)$ and, via the Dennis trace map, to the shuffle product $BI(z) \# (\beta^{-1} d\beta)$. Since $\beta^{-1} d\beta$ is closed, this equals $BI(z \# \beta^{-1} d\beta)$ by 5. 5. 2. Thus $x \cup \beta$ maps to $v(y \cup \beta) = I(\beta^{-1} z \# d\beta)$ in the summand $B/I \otimes_l \Omega_l^{n+1}$ of $\tilde{H}\tilde{C}_{n+1}(B/I)$. On the other hand, $x \cup \beta \in N_n$ because the Dennis trace map kills x, hence $x \cup \beta$. This proves the inductive step. The last sentence of the lemma follows from 9. 6. 2 and a diagram chase on (9. 6. 1).

Corollary 9. 6. 4. *$K_1(A, I) \cong (\tilde{B}/I) \otimes_l \Omega_l$, and the map from $K_1(A, I)$ to $HC_0(A, I)$ is zero. The image of $v: K_2(A, I) \to HC_1(A, I)$ is $HC_1(A_0, I_0) \otimes l$, and the kernel of v maps onto $(\tilde{B}/I) \otimes_l \Omega_l^2$. If the* KABI *Conjecture* 0. 1 *holds for this A, B and I, then $K_n(A, I) \cong (\tilde{B}/I \otimes_l \Omega_l^n) \oplus M_{n-1}$ for all n.*

9. 7. We now consider the map $v: \tilde{K}_n(A) \to \tilde{H}\tilde{C}_{n-1}(A)$. The following diagram is exact for all n:

(9. 7. 1)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
 & N_n \arrow[d] & U_n \arrow[d] & & N_{n-1} \arrow[r] \arrow[d] & U_{n-1} \arrow[d] & \\
\cdot \arrow[r] \arrow[d, "\cong"] & K_n(A, I) \arrow[r] \arrow[d] & \tilde{K}_n(A) \arrow[r] \arrow[d, "v"] & \cdot \arrow[r] \arrow[d, "\cong"] & K_{n-1}(A, I) \arrow[r] \arrow[d] & \tilde{K}_{n-1}(A) \arrow[r] \arrow[d] & \cdot \arrow[d, "\cong"] \\
\cdot \arrow[r] & HC_{n-1}(A, I) \arrow[r] \arrow[d] & \tilde{H}\tilde{C}_{n-1}(A) \arrow[r] \arrow[d] & \cdot \arrow[r] & HC_{n-2}(A, I) \arrow[r] & \cdot \arrow[r] & \cdot \\
 & Q_n \arrow[r] \arrow[d] & W_n \arrow[d] & & & & \\
 & 0 & 0. & & & &
\end{tikzcd}$$

In 5. 8 we found that $\tilde{H}\tilde{C}_{n-1}(A) \cong (B/I) \otimes_l V_n \oplus (\tilde{A} \otimes_l \Omega_l^{n-1})$.

Lemma 9. 7. 2. *In diagram* (9. 7. 1), *the image of $v: \tilde{K}_2(A) \to \tilde{H}\tilde{C}_1(A)$ is*

$$B/I \otimes_l V_2 \cong B/I,$$

and $W_2 \cong \tilde{A} \otimes_l \Omega_l$; W_n contains $\tilde{A} \otimes_l \Omega_l^{n-1}$ in general. If the KABI *Conjecture* 0. 1 *holds for A, B and I, then for all n the image of $v: \tilde{K}_n(A) \to \tilde{H}\tilde{C}_{n-1}(A)$ is $B/I \otimes_l V_n$ and $W_n \cong \tilde{A} \otimes_l \Omega_l^{n-1}$.*

Proof. From the commutative diagram

$$\begin{array}{ccccc}
\tilde{K}_n(A) & \xrightarrow{\ v\ } & \tilde{H}\tilde{C}_{n-1}(A) & \longrightarrow & \tilde{A}\otimes_l \Omega_l^{n-1} \\
\downarrow & & \downarrow & & \downarrow \text{injection} \\
0=\tilde{K}_n(B) & \longrightarrow & \tilde{H}\tilde{C}_{n-1}(B) & \xrightarrow{\ \cong\ } & \tilde{B}\otimes_l \Omega_l^{n-1}
\end{array}$$

we see that the map $\tilde{K}_n(A) \to \tilde{A}\otimes_l \Omega_l^{n-1}$ is zero, so W_n contains $\tilde{A}\otimes_l \Omega_l^{n-1}$. To see equality, a chase of diagram (9. 7. 1) shows that it is enough to see that the map $\tilde{H}\tilde{C}_{n-1}(A) \to N_{n-1} \to \tilde{B}/I\otimes_l \Omega_l^{n-1}$ sends $(B/I)\otimes_l V_n$ to zero. But $B/I\otimes_l V_n$ vanishes in $\tilde{H}\tilde{C}_{n-1}(B)$ by 5. 4. 1, and $\tilde{B}/I\otimes_l \Omega_l^{n-1}$ injects into $K_{n-1}(B, I)$ by 9. 6. The assertion follows from a comparison of (9. 7. 1) with the corresponding diagram for B and I.

Proof of Theorem 9. 2. To prove 9. 2, we follow the outline given in Section 6, computing $\tilde{K}_n(A)$ from $K_n(A, I)$. Using 9. 6 and a diagram chase on (9. 7. 1), we find an exact sequence

$$0 \to I\otimes_l \Omega_l^{n-1} \to W_n \to \tilde{B}/I\otimes_l \Omega_l^{n-1} \to U_{n-1} \to 0.$$

Evidently, the cokernel of $I\otimes_l \Omega_l^{n-1} \to W_n$ is $\tilde{A}/I\otimes_l \Omega_l^{n-1}$, so $U_{n-1} \cong (B/A)\otimes_l \Omega_l^{n-1}$. Note that $\dim_l(B/A) = c/2$ by Remark 9. 2. 1. It follows from 9. 7. 2 that

$$\tilde{K}_n(A) \cong B/I\otimes_l V_n \oplus B/A\otimes_l \Omega_l^n,$$

proving Theorem 9. 2.

Proof of Theorem 9. 3. We proved in 9. 7. 2 that the image of

$$v : \tilde{K}_2(A) \to \tilde{H}\tilde{C}_1(A)$$

is B/I. Multiplicativity of the Dennis trace map D [Igusa] implies that the image of $\tilde{K}_3(A) \to \tilde{H}\tilde{C}_2(A)$ contains $B/I\otimes_l \Omega_l$, as follows. Since $I : HH_1(A) \to \tilde{H}\tilde{C}_1(A)$ is onto, we can write elements in the subgroup B/I of $\tilde{H}\tilde{C}_1(A)$ as $I(z)$. Choose $x \in \tilde{K}_2(A)$ so that $v(x) = I(z)$, and hence $D(x) = BI(z)$ in $HH_2(A; A)$. Then for $\beta \in l^* = K_1(l)$, the cup product $x \cup \beta$ in $\tilde{K}_3(A)$ maps to the shuffle product $BI(z) \# \beta^{-1}d\beta$ in $HH_3(A; A)$. Since $\beta^{-1}d\beta$ is closed, this equals $BI(\beta^{-1}z \# d\beta)$ by 5. 5. 2. Thus $v(x\cup\beta) = I(\beta^{-1}z \# d\beta)$ in the summand $B/I\otimes_l \Omega_l$ of $\tilde{H}\tilde{C}_2(A) \cong (B/I\otimes_l \Omega_l) \oplus (\tilde{A}\otimes_l \Omega_l^2)$. Since every term in this summand is a sum of such terms, the image of $\tilde{K}_3(A) \to \tilde{H}\tilde{C}_2(A)$ contains $B/I\otimes_l \Omega_l$. By 9. 7. 2, it is precisely the image, and the cokernel is $W_3 \cong \tilde{A}\otimes_l \Omega_l^2$ by 9. 6. 2.

To finish the proof of 9. 3, it now suffices to prove part (iii), i.e., that the group U_n of (9. 7. 1) maps onto $B/A\otimes_l \Omega_l^n$. From (9. 7. 1) for $n+1$ (and the corresponding diagram for B) we extract a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & Q_{n+1} & \longrightarrow & W_{n+1} & \longrightarrow & N_n & \longrightarrow & U_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I\otimes_l \Omega_l^n & \longrightarrow & \tilde{A}\otimes \Omega^n & \longrightarrow & \tilde{B}/I\otimes_l \Omega^n & \longrightarrow & B/A\otimes_l \Omega_l^n & \longrightarrow & 0
\end{array}$$

with the top row exact. The bottom row is exact by inspection. The verticals are onto by 9. 6. 2, 9. 7. 2 and 9. 6. 3. Hence the induced map from U_n to $B/A \otimes_l \Omega_l^n$ is onto, proving part (iii) and hence 9. 3.

Porism[3]) 9. 8. *Let $A_{\mathbb{Q}} \subset B_{\mathbb{Q}}$ be a finite extension of commutative graded $\mathbb{Q}$-algebras such that the augmentation ideal of $B_{\mathbb{Q}}/I_{\mathbb{Q}}$ is nilpotent, where $I_{\mathbb{Q}}$ is the conductor from $B_{\mathbb{Q}}$ to $A_{\mathbb{Q}}$. Let l be a field of characteristic zero and set $A = A_{\mathbb{Q}} \otimes_{\mathbb{Q}} l$, $B = B_{\mathbb{Q}} \otimes_{\mathbb{Q}} l$. Suppose that $\tilde{K}_*(B) = 0$, i.e. that $K_*(B) \cong K_*(B_0)$. Then the kernel of $\tilde{K}_n(A) \to \tilde{H}\tilde{C}_{n-1}^{\mathbb{Q}}(A)$ maps onto $B/A \otimes_l \Omega_l^n$.*

Proof. Set $I = I_{\mathbb{Q}} \otimes_{\mathbb{Q}} l$. Since B/I is extended, we know from 5. 4. 1 and 9. 6 that $\tilde{B}/I \otimes_l \Omega_l^n$ is a natural $\mathbb{Q}$-submodule of $\tilde{H}\tilde{C}_n(B/I) \cong K_n(B, I)$ which is in the kernel of $v_n : K_n(B, I) \to HC_{n-1}(B, I)$. Similarly, $I \otimes_l \Omega_l^n$ is a natural submodule of $HC_n(B, I)$ and of $\operatorname{coker}(v_{n+1})$. Since (9. 6. 1) remains exact if we use $\ker(v_n)$ and $\operatorname{coker}(v_n)$, the proofs of 9. 6. 2 and 9. 6. 3 show that images of $Q_{n+1} \to \operatorname{coker}(v_{n+1})$ and $N_n \to \ker(v_n)$ contain $I \otimes_l \Omega_l^n$ and $B/I \otimes_l \Omega_l^n$, respectively.

Similarly, $\tilde{A} \otimes_l \Omega_l^{n-1}$ is a natural submodule of $\tilde{H}\tilde{C}_{n-1}(A)$; the proof of 9. 7. 2 shows that it is also a submodule of the cokernel W_n of $\tilde{K}_n(A) \to \tilde{H}\tilde{C}_{n-1}(A)$. (The nilpotence of $\tilde{A}/I$ and $\tilde{B}/I$ is used to obtain the isomorphisms in (9. 7. 1).) The proof of 9. 3 (iii) now applies to establish the porism.

Appendix. Analytic isomorphisms for cyclic homology

Since the spectral sequences in the body of this paper require graded rings, and we are interested in the cyclic homology of more general rings, we need some tools to pass from one to the other. Our main tool is the use of analytic isomorphisms, and this appendix develops that tool.

If s is a central element of a ring A, and $i : A \to B$ is a ring map with $i(s)$ a central element of B and $A/s^n A \cong B/s^n B$ for all n, we shall call i an *analytic isomorphism along* s (because the s-adic completions $\hat{A}$ and $\hat{B}$ are isomorphic). For example, if $sA + tA = A$ for some central t then $A \to A[t^{-1}]$ and $A \to \hat{A}$ are analytic isomorphisms along s. Our main result may now be stated.

Theorem A. 1. *Let $A \to B$ be an analytic isomorphism of flat k-algebras along $s \in A$. Assume B is flat over A. Then there are long exact sequences:*

$$\cdots \xrightarrow{\partial} HH_n(A; A) \to HH_n(B; B) \oplus HH_n(A_s; A_s) \to HH_n(B_s; B_s) \xrightarrow{\partial} HH_{n-1}(A; A) \to \cdots;$$

$$\cdots \xrightarrow{\partial} HC_n(A) \to HC_n(B) \oplus HC_n(A_s) \to HC_n(B_s) \xrightarrow{\partial} HC_{n-1}(A) \to \cdots.$$

Here A_s and B_s denote $A[s^{-1}]$ and $B[s^{-1}]$, respectively.

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[3]) A *porism* is a deduction which follows from the proof of a preceding result. It comes from the Greek word *porisma*, which is often translated as "corollary".

Corollary A. 2. *Let $s, t \in A$ be central elements with $sA + tA = A$, and write A_s for $A[s^{-1}]$, etc. Then there are long exact sequences:*

$$\cdots \xrightarrow{\partial} HH_n(A; A) \to HH_n(A_s; A_s) \oplus HH_n(A_t; A_t) \to HH_n(A_{st}; A_{st}) \xrightarrow{\partial} HH_{n-1}(A; A) \to \cdots;$$
$$\cdots \xrightarrow{\partial} HC_n(A) \to HC_n(A_s) \oplus HC_n(A_t) \to HC_n(A_{st}) \xrightarrow{\partial} HC_{n-1}(A) \to \cdots.$$

Remark. This corollary has been noted by many other people, such as Jon Bloch, J.-L. Brylinski [Bry] and J.-L. Loday. Using it as the key step, Loday pointed out that "[Hochschild and] cyclic homology can be extended to algebraic varieties (and schemes) by sheafifying the complexes and taking the hyperhomology." [L survey], 3. 4.

Remark. Some hypothesis similar to the flatness of B over A is necessary in Theorem A. 1. For example, let $B = k[s]$ and $A = B \oplus M$, where M is an s-torsion, s-divisible B-module such as $k[s, s^{-1}]/k[s]$. Then the projection $A \to B$ is an analytic isomorphism along s, and $A[s^{-1}] \cong B[s^{-1}]$. The conclusion of Theorem A. 1 fails in this case because A and B have different HH_* and HC_* groups (e.g., when $* = 0$).

To begin the proof of A. 1, set $A^e = A \otimes A^{\mathrm{op}}$. Since A is flat over k we have

$$HH_*(A; A) = \mathrm{Tor}_*^{A^e}(A, A).$$

This is a module over the center C of A, and we have the following folklore result.

Proposition A. 3. *Let C be the center of a flat k-algebra A. Then for every s in C*

$$HH_*(A; A) \otimes_C C[s^{-1}] \cong HH_*(A; A[s^{-1}]) \cong HH_*(A[s^{-1}]; A[s^{-1}]).$$

Proof. Since A is an (A^e, C)-bimodule, the first isomorphism is [BX], § 6. 5, Prop. 7. Setting $B = (A[s^{-1}])^e = N$ in [BX], § 6. 6, Prop. 8, we obtain the second isomorphism as follows:

$$\begin{aligned} HH_*(A[s^{-1}]; A[s^{-1}]) &= \mathrm{Tor}_*^B(A[s^{-1}], A[s^{-1}]) = \mathrm{Tor}_*^B(A[s^{-1}], B \otimes_{A^e} A) \\ &\cong \mathrm{Tor}_*^{A^e}(A[s^{-1}] \otimes_B B, A) = HH_*(A; A[s^{-1}]). \end{aligned}$$

Remark. This is a generalization of the well-known fact that, for A commutative, we have $\Omega^{\bullet}_{A/k} \otimes_A A[s^{-1}] = \Omega^{\bullet}_{A[s^{-1}]/k}$.

Lemma A. 4. *Let $A \to B$ be an analytic isomorphism along s, and let I denote the kernel of $B \otimes_A B \to B$. Assume that B is a flat A-algebra. Then*

a) *$B[s^{-1}]/B$ is naturally isomorphic to $A[s^{-1}]/A$,*

b) $\mathrm{ann}_A(s^n) \cong \mathrm{ann}_B(s^n)$ *for all n,*

c) *left multiplication by s is an automorphism of I.*

Proof. Since $A/s^n A \cong B/s^n B$ for each n, we have

$$A_s/A = \varinjlim (A/sA \xrightarrow{s} A/s^2 A \xrightarrow{s} A/s^3 A \longrightarrow \cdots) \cong B_s/B.$$

To see part b), we apply $\otimes_A B$ to the exact sequence

$$0 \longrightarrow \operatorname{ann}_A(s^n) \longrightarrow A \xrightarrow{s^n} A \longrightarrow A/s^n A \longrightarrow 0,$$

and use the fact that, if $Ns^n = 0$, then $N \otimes_A B \cong N$.

Let λ denote left multiplication by s. To see that λ is an automorphism of I, chase the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & I \arrow[r, "\lambda"] \arrow[d] & I \arrow[d] & & \\
0 \arrow[r] & \operatorname{ann}(s) \otimes_A B \arrow[r] \arrow[d, equal] & B \otimes_A B \arrow[r, "s \otimes 1"] \arrow[d] & B \otimes_A B \arrow[r] \arrow[d] & (B/sB) \otimes_A B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{ann}(s) \arrow[r] & B \arrow[r, "s"] & B \arrow[r] & B/sB \arrow[r] & 0.
\end{tikzcd}$$

Corollary A. 5. *Let $A \to B$ be an analytic isomorphism along s, with A flat over k and B flat over A. Then*

$$HH_*(A; A[s^{-1}]/A) \cong HH_*(B; B[s^{-1}]/B)$$

and

$$HH_*(A; \operatorname{ann}(s^n)) \cong HH_*(B; \operatorname{ann}(s^n)) \quad \textit{for all } n.$$

Proof. Let N denote either $\operatorname{ann}(s^n)$ or $B/s^n B$. By A. 4. c) λ_*^n is an automorphism of $\operatorname{Tor}_*^{B^e}(I, N)$; by [BX], § 4. 4, Cor. to Prop. 6, λ_*^n is zero. Hence $\operatorname{Tor}_*^{B^e}(I, N) = 0$, and therefore

$$HH_*(B; N) = \operatorname{Tor}_*^{B^e}(B, N) \cong \operatorname{Tor}_*^{B^e}(B \otimes_A B, N).$$

On the other hand, $B \otimes_A B = A \otimes_{A^e} B^e$ and B^e is flat over A^e, so by [BX], § 4, Ex. 8, there are isomorphisms

$$\operatorname{Tor}_*^{B^e}(B \otimes_A B, N) \cong \operatorname{Tor}_*^{A^e}(A, N) = HH_*(A; N).$$

This shows that $HH_*(A; \operatorname{ann}(s^n)) = HH_*(B; \operatorname{ann}(s^n))$ and that

$$\begin{aligned} HH_*(A; A[s^{-1}]/A) &= \varinjlim HH_*(A; A/s^n A) \\ &\cong \varinjlim HH_*(B; B/s^n B) = HH_*(B; B[s^{-1}]/B). \end{aligned}$$

Proof of Theorem A. 1. Let $A \to B$ be an analytic isomorphism along s, with A flat over k and B flat over A, and set $I = \bigcup \operatorname{ann}_A(s^n) = \bigcup \operatorname{ann}_B(s^n)$. Let $C_\cdot$ (resp. $D_\cdot$) denote the mapping cylinder of the map of chain complexes which is

$$A \otimes A^{\otimes m} \to B \otimes B^{\otimes m} \quad (\text{resp. } (A/I) \otimes A^{\otimes m} \to (B/I) \otimes B^{\otimes m})$$

in degree m. From the diagram of chain complexes

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
I \otimes A^{\otimes *} \arrow[r] \arrow[d] & I \otimes B^{\otimes *} \arrow[d] & \\
A \otimes A^{\otimes *} \arrow[r] \arrow[d] & B \otimes B^{\otimes *} \arrow[r] \arrow[d] & C_{\bullet} \arrow[d] \\
A/I \otimes A^{\otimes *} \arrow[r] \arrow[d] & B/I \otimes B^{\otimes *} \arrow[r] \arrow[d] & D_{\bullet} \\
0 & 0 &
\end{tikzcd}$$

and the fact that the top arrow is a homology isomorphism by A. 5, we see that $C_{\bullet} \to D_{\bullet}$ is a homology isomorphism.

Now let $E_{\bullet}$ denote the mapping cylinder of the map

$$A[s^{-1}] \otimes A^{\otimes *} \to B[s^{-1}] \otimes B^{\otimes *}$$

of chain complexes. From the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
(A/I) \otimes A^{\otimes *} \arrow[r] \arrow[d] & (B/I) \otimes B^{\otimes *} \arrow[r] \arrow[d] & D_{\bullet} \arrow[d] \\
A[s^{-1}] \otimes A^{\otimes *} \arrow[r] \arrow[d] & B[s^{-1}] \otimes B^{\otimes *} \arrow[r] \arrow[d] & E_{\bullet} \\
(A[s^{-1}]/A) \otimes A^{\otimes *} \arrow[r] \arrow[d] & (B[s^{-1}]/B) \otimes B^{\otimes *} \arrow[d] & \\
0 & 0 &
\end{tikzcd}$$

and the fact that the bottom arrow is a homology isomorphism (again by A. 5), we deduce that $D_{\cdot} \to E_{\cdot}$ is also a homology isomorphism. Finally, we deduce from

$$\begin{array}{ccccc} A \otimes A^{\otimes *} & \longrightarrow & B \otimes B^{\otimes *} & \longrightarrow & C_{\cdot} \\ \downarrow & & \downarrow & & \downarrow \\ A[s^{-1}] \otimes A^{\otimes *} & \longrightarrow & B[s^{-1}] \otimes B^{\otimes *} & \longrightarrow & E_{\cdot} \end{array}$$

that there is a long exact sequence on Hochschild homology

$$\cdots \xrightarrow{\partial} HH_n(A; A) \to HH_n(B; B) \oplus HH_n(A; A[s^{-1}]) \to HH_n(B; B[s^{-1}])$$
$$\xrightarrow{\partial} HH_{n-1}(A; A) \to \cdots.$$

The HH_* part of Theorem A. 1 follows from this and A. 3. The HC_* part of Theorem A. 1 follows from the HH_* part and the Connes-Gysin sequence relating HC_* to HH_*.

A. 6. The following is an application of the above ideas to the cyclic homology of an affine plane curve, i.e., of the ring $A = k[x, y]/fk[x, y]$, where $fk[x, y]$ is a radical ideal. The ideal $I = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) A$ is supported at finitely many maximal ideals $m_1, \ldots, m_r$, and we can choose a nonzerodivisor $s \in \bigcap m_i$ supported at

$$\{m_i\} \cup \{n_1, \ldots, n_l\}.$$

The map $A \to \prod A_{m_i} \times \prod A_{n_j} = B$ is an analytic isomorphism along s; both the k-algebra $A[s^{-1}]$ and its localizations are smooth. Since $\Omega_A^2 \cong A/I$ on generator $dx\,dy$, $\Omega_A^3 = 0$ and the Ω_A^i are sheaves, it follows that $\Omega_R^i = HH_i(R; R) = 0$ for $i \geqq 2$ when R is A_{n_j}, $A[s^{-1}]$ or a localization of these rings. It follows from A. 1 that

$$HH_i(A; A) \cong \bigoplus_j HH_i(A_{m_j}; A_{m_j}), \qquad i \geqq 2,$$
$$\Omega_A^2 \cong \oplus\, \Omega_{A_{m_j}}^2 = \Omega_B^2.$$

It also follows from A. 1 that there is a short exact sequence of de Rham chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & A[s^{-1}] \oplus B & \longrightarrow & B[s^{-1}] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_A & \longrightarrow & \Omega_{A[s^{-1}]} \oplus \Omega_B & \longrightarrow & \Omega_{B[s^{-1}]} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_A^2 & \longrightarrow & \Omega_B^2 & \longrightarrow & 0 & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \cdots & & \cdots & & \cdots & & \end{array}$$

Hence there is a long exact sequence

$$\begin{aligned} 0 &\to H^0_{DR}(A) \to H^0_{DR}(A[s^{-1}]) \oplus H^0_{DR}(B) \to H^0_{DR}(B[s^{-1}]) \\ &\to H^1_{DR}(A) \to H^1_{DR}(A[s^{-1}]) \oplus H^1_{DR}(B) \to H^1_{DR}(B[s^{-1}]) \\ &\to H^2_{DR}(A) \longrightarrow 0 \oplus H^2_{DR}(B) \to 0. \end{aligned} \tag{A. 6. 1}$$

For simplicity, let us now assume that A is graded with $A_0 = k$. That is, there are weights on x and y such that $f(x, y)$ is homogeneous. (The examples $f(x, y) = xy$ and $f(x, y) = x^a - y^b$ occur in this paper.) In this case $B = A_m$ and we shall establish

Proposition A. 6. 2. *Let $A = k[x, y]/(f)$ be reduced and graded with $A_0 = k$, a field of characteristic zero. Then*

(a) $HC_0(A) = A$ *and* $HC_0(A_m) = A_m$,

(b) $HC_{2n}(A) \cong HC_{2n}(A_m)$ *for* $n \neq 0$,

(c) *there is a short exact sequence for all* $n \geqq 0$

$$0 \to HC_{2n+1}(A) \to HC_{2n+1}(A_m) \to H^1_{DR}(A_m) \to 0.$$

Proof. Part (a) is trivial. It is known that $H^0_{DR}(A) = H^0_{DR}(A_m) = k$. Since A is graded, $H^i_{DR}(A) = 0$ for $i \neq 0$. It follows from (A. 6. 1) that

$$H^1_{DR}(A_m) \oplus H^1_{DR}(A[s^{-1}]) \cong H^1_{DR}(A_m[s^{-1}]),$$
$$H^0_{DR}(A[s^{-1}]) \cong H^0_{DR}(A_m[s^{-1}]).$$

If R is any smooth k-algebra with $\Omega^2_{R/k} = 0$, we have $HC_{2n}(R) \cong H^0_{DR}(R)$ for $n \neq 0$ and $HC_{2n+1}(R) \cong H^1_{DR}(R)$ for all $n \geqq 0$. Applying this remark to $A[s^{-1}]$ and $A_m[s^{-1}]$, we deduce

$$H^1_{DR}(A_m) \oplus HC_{2n+1}(A[s^{-1}]) \cong HC_{2n+1}(A_m[s^{-1}]), \qquad n \geqq 0,$$

and

$$HC_{2n}(A[s^{-1}]) \cong HC_{2n}(A_m[s^{-1}]), \qquad n \neq 0.$$

From A. 1 we get the diagram for $n \neq 0$

$$\begin{tikzcd} HC_{2n}(A) \arrow[r] \arrow[d] & HC_{2n}(A_m) \oplus HC_{2n}(A[s^{-1}]) \arrow[r] \arrow[d] & HC_{2n}(A_m[s^{-1}]) \\ HH_{2n+1}(A; A) \arrow[r, "\cong"] & HH_{2n+1}(A_m; A_m). & \end{tikzcd}$$

Since A is graded, the kernel of the left vertical arrow is the summand $HC_{2n}(k)$, proving part (b) by a diagram chase. Since $HC_{2n}(A)$ injects into $HC_{2n}(A_m)$ even for $n=0$, the long exact sequence in A. 1 breaks up into short exact sequences

$$0 \to HC_{2n+1}(A) \to HC_{2n+1}(A_m) \oplus HC_{2n+1}(A[s^{-1}]) \to HC_{2n+1}(A_m[s^{-1}]) \to 0,$$

and we deduce part (c) from this.

Remark A. 7. Using the same ideas, but requiring more patience, it is not hard to work out the cyclic homology of the m-adic completion $\hat{A}_m$ from that of A. In this way the cyclic homology of many other curves can be deduced from the cyclic homology of the curves discussed in this paper. For example, consider $A=k[x, y]/(y^2=x^3-x)$. If $\hat{A}$ is the completion at $m=(x, y)A$, then $A \to \hat{A}$ is an analytic isomorphism along x, and there is also an analytic isomorphism $k[u, v]/(uv=0) \to \hat{A}$ along $s=u+v$. Consequently, the cyclic homology of A can be deduced from that of $k[u, v]/(uv=0)$. We hope to pursue this topic in a subsequent paper.

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Department of Mathematics, Rutgers University, New Brunswick, NJ 08903 USA

Department of Mathematics, Southwest Missouri State University, Springfield, MO 65804, USA

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