

COMPLETE INTERSECTION POINTS ON AFFINE VARIETIES

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Abstract. This paper addresses the following problem: given a commutative ring  $A$ , what is the structure of the set of "CI points," i.e., those maximal ideals generated by  $\dim(A)$  elements? When  $A$  is finitely generated over an algebraically closed field, we conjecture that this set is a countable union of closed subsets of  $\text{Max}(A)$ . When  $A$  is regular of dimension 2 or 3, we verify this conjecture, as well as an analogous set-theoretic conjecture.

§0. Introduction. Given a commutative ring  $A$ , we can ask the question: which ideals are complete intersections? There are variants of this question, depending on just what a complete intersection is taken to be.

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Letting  $v(I)$  denote the minimal number of generators of  $I$ , some possibilities are:

- (a) weak CI:  $v(I) = \text{height}(I)$
- (b) CI:  $I$  is generated by a regular sequence
- (c)  $v(I) = v(I/I^2)$
- (d) stCI:  $I$  is the radical of a CI ideal
- (e) weak stCI:  $I$  is the radical of a weak CI ideal

One way of answering this question is to parametrize a family of ideals by an algebraic variety and ask for the topological structure of the set of complete intersections. For this reason, we focus on two specific families:

(1) The set  $\text{Max}(A)$  of maximal ideals of  $A$ , which is parametrized by  $U = \text{Spec}(A)$ . That is,  $\text{Max}(A) = U(k)$ , the closed points of  $U$ .

(2) The set  $\text{Rad}^r(A)$  of intersections of  $r$  distinct maximal ideals of  $A$ . This set is parametrized by an open subset of the symmetric product  $S^r U = (U \times \cdots \times U) / \Sigma_r$ , where  $\Sigma_r$  is the symmetric group permuting the  $r$  copies of  $U$ .

Here are two conjectures about the set of maximal ideals  $\mathfrak{m}$  of  $A$  which are complete intersections. We will call a subset of a topological space " $\sigma$ -P" if it is the countable union of subsets having the property  $P$  (e.g.,  $P = \text{'closed'}$ ). By  $\text{Sing}(A)$  we mean  $\{\mathfrak{m} : A_{\mathfrak{m}} \text{ is not}$

regular}, and by  $\text{Sing}^r(A)$  we mean  $\{m_1 \cap \dots \cap m_r : \text{some } m_i \text{ is in } \text{Sing}(A)\}$ .

CI conjecture (0.1). Let  $A$  be a finitely generated commutative algebra over an algebraically closed field.

Then:

(1) The following subset of  $\text{Max}(A)$  is  $\sigma$ -closed:

$$\begin{aligned} \{m : v(m) = v(m/m^2)\} &= \{\text{weak CI points } m\} \cup \text{Sing}(A) \\ &= \{\text{CI points } m\} \cup \text{Sing}(A) \end{aligned}$$

(2) The following subset of  $\text{Rad}^r(A)$  is  $\sigma$ -closed:

$$\begin{aligned} \{I : v(I) = v(I/I^2)\} &= \{\text{weak CI ideals } m_1 \cap \dots \cap m_r\} \\ &\quad \cup \text{Sing}^r(A) \\ &= \{\text{CI ideals } I\} \cup \text{Sing}^r(A). \end{aligned}$$

stCI conjecture (0.2). Let  $A$  be a finitely generated commutative algebra over an algebraically closed field.

Then:

(1)  $\{\text{stCI points}\} \cup \text{Sing}(A)$  is a  $\sigma$ -closed subset of  $\text{Max}(A)$ .

(2)  $\{\text{stCI ideals}\} \cup \text{Sing}^r(A)$  is a  $\sigma$ -closed subset of  $\text{Rad}^r(A)$ .

Here is the current status of these conjectures. If  $\dim(A)=1$ ,  $\sigma$ -closed means "either all maximal ideals or only countably many." Both conjectures were verified in [GW1] and [W]. (The hypothesis that  $A$  be reduced is easily removed.) Note that when  $\text{genus}(A)=1$  there are a countably infinite number of stCI points which are not CI points (see [W]).

If  $A$  is regular and either 2- or 3-dimensional, we will verify both conjectures in §3,4 below. We also show that if a maximal ideal is an stCI, then it is also a CI ideal. (We do not know if this is true for finite intersections of maximal ideals.) Our proofs rely heavily on joint work of Murthy with Swan [MS] and Mohan Kumar [MM]. If  $A$  is singular and 2-dimensional, the conjectures are also valid, and a proof is given in [LW].

If  $A$  is regular of dimension 4 or more, there are two partial results.

(i) Varley has shown that both sets {CI points} and {stCI points} are  $\sigma$ -locally closed (=  $\sigma$ -constructible). We give Varley's proof in the appendix below.

(ii) Results of Roitman [R2][R3] imply that the set of maximal ideals  $m$  for which  $[A/m]$  is zero in  $K_0(A)$  is a  $\sigma$ -closed subset of  $\text{Max}(A)$ . This reduces the problem to a question of Murthy: when does  $[A/I]=0$  imply that  $I$  is a CI? We give this reduction in §§3,4 below.

By now the reader may have noticed that this paper is a collage of the results of a large number of people. My role as author is as much that of editor as that of researcher. Here is a table of contents, with credits:

In §1 we discuss weak vs. strong CI's, and show that they agree for locally complete intersections.

(Ed Davis was very helpful in pointing out the subtleties involved here.) We also introduce the concept of projective generation, and discuss its strengths and weaknesses.

In §2 we describe the unstable nature of the CI problem for height 2 locally CI ideals  $I$  in algebras  $A$  which are finitely generated over an arbitrary field. We consider the set  $\underline{SP}_2(A, I) = \{\text{rank 2 projective } A \text{ modules } P \text{ with } \Lambda^2 P = A \text{ mapping onto } I\}$ ;  $I$  is a CI exactly when  $\underline{SP}_2(A, I)$  contains  $A^2$ . We give a number of cases when  $\underline{SP}_2(A, I)$  contains just one element  $P$ , so that  $I$  is a CI exactly when  $P$  is free. For example, this is the case if  $A/A/I$  splits, i.e., if  $I$  is a "rational" maximal ideal. Several of the results in this section were obtained independently by M. Boratyński, and I am grateful to W. van der Kallen for providing me with example (2.2.c).

In §3 we introduce the concept of a Murthy ring in order to discuss the usefulness of the stable invariant  $[A/I]$  in  $K_0(A)$ . Suppose that  $A$  is finitely generated over an algebraically closed field, and let  $m$  be a regular maximal ideal of  $A$ . If  $m$  is a CI (resp., an stCI) then  $[A/m]$  is zero (resp., a torsion element) in  $K_0(A)$ . The converse holds if  $A$  is a Murthy ring. We also prove that  $A$  is a Murthy ring if  $\dim(A)=2$ , or if  $A$  is regular and 3-dimensional. The ideas in this

section are due to Murthy-Swan and to Murthy-Mohankumar.

In §4 we invoke a  $\sigma$ -closedness result of Roitman and prove that the CI and stCI conjectures hold for regular geometric rings which are Murthy rings. By the results of §3, this proves the CI and stCI conjectures for regular rings of dimension 2 and 3. We need the fact that the Chow group of zero-cycles embeds in  $K_0(A)$ , and in this sense our analysis follows [MS, §4] and [MM, §4].

In §5 we study the case in which the ring  $A$  is finitely generated over  $\overline{\mathbb{F}}_p$ , following a suggestion of the referee of [W1]. Since  $A$  has only countably many maximal ideals,  $\sigma$ -closedness results are meaningless. Every maximal ideal (of height =  $\dim(A)$ ) is a weak stCI; the set of stCI points is therefore the set of Cohen-Macaulay points, which is open in  $\text{Max}(A)$ . If  $A$  is the coordinate ring of a smooth affine variety of dimension  $>2$ , we also show that every finite intersection of maximal ideals is a CI; in particular, every maximal ideal is a CI.

In the Appendix we prove Varley's result mentioned above. I am grateful to him for giving permission to include his result here.

This paper is a revised version of the first half of [W1], which was written in 1980-81. There are two

mathematical differences between this paper and [W1], reflecting the passing of over two years. First, I have introduced the concept of a Murthy ring, and showed that the CI and stCI conjectures both hold for geometric Murthy rings. Secondly, I have extended the results of [W1] to 3-dimensional regular geometric rings; they are Murthy rings. This extension is a straight-forward consequence of [MM].

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### §1. Preliminaries

In this section we set up the translation from the CI problem to a projective module problem. This is a refinement of the translation given by Murthy and Swan [MS] for regular affine surfaces. Most of the results are of the folklore variety, and have their origins in either "Serre's Lemma" in [Se] or in Davis' work [D] on weak vs. strong CI's.

We first need to specify what 'complete intersection' means. We consider an ideal  $I$  in a commutative noetherian ring  $A$ . When  $I$  is generated by a regular sequence, we will call  $I$  a CI (or a strong CI). (The 'CI' is for 'ideal-theoretic complete intersection'.) If  $I$  is the radical of a CI ideal, we will call  $I$  an stCI (or a strong stCI). (The letters stCI are an abbreviation for 'set-theoretic complete intersection'.')

Of greater interest are the related but weaker notions involving  $v(I)$ . Here  $v(M)$  denotes the smallest number of generators of an  $A$ -module  $M$ . If  $v(I) = \text{height}(I)$ , we will call  $I$  a weak CI. Similarly, we will call  $I$  a weak stCI if it is the radical of an ideal  $J$  with  $v(J) = \text{height}(I) (= \text{height}(J))$ .

Note that strong CI's are also weak CI's; the example  $A = \underline{\mathbb{Z}}[x]/(px, x^2)$ ,  $I = pA$  shows that they are not the same notion. If  $A$  is Cohen-Macaulay, it is well-known that the notions of weak CI and strong CI (and also the notions of weak and strong stCI) coincide. Less well-known is the fact that the notions of weak and strong CI also coincide for radical ideals. This fact is due to Ed Davis (see pp.199, 202-3 of [D]; a handy reference for prime ideals is [GS,(13.15)]).

Fortunately, a simple local condition is sufficient to identify the weak and strong notions. If

$I_p$  is a CI for every prime ideal  $p$  containing  $I$ , we will say that  $I$  is locally a CI (or that  $I$  is a locally CI ideal). If in addition  $\text{height}(I_p)=d$  for every  $p$ , we will say that  $I$  is a locally CI ideal of height  $d$ . I am grateful to Ed Davis for pointing out the following folklore result:

Theorem (1.1) (folklore). Let  $I$  be a locally CI ideal of height  $d$  in a noetherian ring  $A$ . Then:

- (a)  $I$  contains a regular sequence of length  $d$ .  
(I.e.,  $\text{grade}(I) = \text{height}(I)$ .)
- (b)  $I$  is a strong CI iff  $I$  is a weak CI.  
(I.e., if  $v(I)=d$ , then  $I$  can be generated by a regular sequence.)
- (c)  $I$  is a strong stCI iff  $I$  is a weak stCI.

Proof. At every prime  $p$  over  $I$  we know that  $I_p$  is a CI, so that by [K, Thm.135] the ideal  $I$  contains a regular sequence of length  $d$ . If  $I$  is the radical of  $J$ , the same is true of  $J$  by [K, p.102,12(c)]. Parts (b), (c) now follow from [K, Thm. 125].

An ideal  $I$  is locally a CI if and only if:

- (i)  $\text{hd}_A(I) < \infty$ , and (ii)  $I/I^2$  is a projective  $A/I$ -module; this is an old result of Ferrand [Fe] and Vasconcelos [V]. The additional hypothesis that  $\text{height}(I_p)=d$  translates into the hypothesis that the rank of the projective module  $I/I^2$  is the constant  $d$ .

Note that  $v(I)$  is either  $v(I/I^2)$  or  $1+v(I/I^2)$ .

Consequently, if  $I/I^2$  is not free, we can never have  $v(I) = d = \text{height}(I)$ .

Question (1.2). If  $I$  is a locally CI ideal of height  $d$ , but the projective  $A/I$ -module  $I/I^2$  is not free, then must  $v(I) = v(I/I^2)$  hold? (It is easy to find examples of locally CI ideals with  $v(I) = 1+v(I/I^2)$  when  $\text{height}(I_p)$  is not constant.)

Our basic approach to the CI question is that of "projective generation": we find a rank  $d$  projective  $A$ -module  $P$  mapping onto  $I$  and study  $P$ . Of course,  $I$  is a CI iff some such  $P$  is free.

Suppose that we are given a surjection  $\phi: P \rightarrow I$ , where  $P$  is a rank  $d$  projective  $A$ -module. The (reduced) Koszul complex  $K(\phi)$  is

$$0 \rightarrow \Lambda^d P \rightarrow \dots \rightarrow \Lambda^2 P \rightarrow P \rightarrow I \xrightarrow{\phi} 0,$$

where  $e_1 \wedge \dots \wedge e_r$  in  $\Lambda^r P$  maps to  $\Sigma(-1)^{i+1} e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge e_r$  in  $\Lambda^{r-1} P$  (cf. [SeM, p.70]). If  $P$  is free with basis  $(e_1, \dots, e_d)$  and  $\phi(e_i) = x_i$ , we write  $(x_1, \dots, x_d)$  for  $\phi$ . The complex  $K(x_1, \dots, x_d)$  is the reduced version of the more familiar Koszul complex associated to the sequence  $(x_1, \dots, x_d)$  of elements of  $A$ .

Lemma (1.3). Let  $I$  be a height  $d$  ideal in a noetherian ring  $A$ , and let  $\phi: P \rightarrow I$  be a mapping of a rank  $d$

projective  $A$ -module  $P$  onto  $I$ . Then  $I$  is locally a CI iff  $K(\phi)$  is exact.

Proof. Since the statement that  $K(\phi)$  is exact is local, we may assume that  $A$  is local,  $P$  is free and either  $I$  is radical or that  $I=A$ . If  $I$  is radical, this is on p.71 of [SeM]. If  $I=A$ , there is an automorphism  $\tau$  of  $P$  so that  $\phi\tau = (1,0,\dots,0)$ ; the natural chain isomorphism  $K(\phi\tau) \cong K(\phi)$  allows us to assume that  $\phi=(1,0,\dots,0)$ , when the result is easy.

The primary difficulty with the projective generation approach is the existence question: for which locally CI ideals  $I$  of height  $d$  does some rank  $d$  projective  $A$ -module  $P$  map onto  $I$ ? (See [Mur,p.209].) If  $d=1$ , then  $P \cong I$  and one studies  $\text{Pic}(A)$ . (This at least is the viewpoint of [W].) If  $d \neq 1$ , it is customary to consider  $P$  with determinant  $\det(P) = A^d$  equal to  $A$ . Accordingly, let  $\underline{\text{SP}}_d(A)$  denote the set of all rank  $d$  projective  $A$ -modules  $P$  with  $\det(P)=A$ , and let  $\underline{\text{SP}}_d(A,I)$  denote the set of all  $P$  in  $\underline{\text{SP}}_d(A)$  mapping onto  $I$ . If  $\underline{\text{SP}}_d(A,I)$  is nonempty, we will say that  $I$  is projectively generated. If  $d=2$ , the following classification is classical:

Proposition (1.4) (Serre [Se][Mur, p.199]). Let  $I$  be a locally CI ideal of height 2. Then there is a

canonical isomorphism

$$\text{Ext}(I, A) \cong \text{Hom}(\Lambda^2(I/I^2), A/I).$$

- (a) If  $\Lambda^2(I/I^2) \neq A/I$  then  $I$  cannot be projectively generated.
- (b) If  $\Lambda^2(I/I^2) = A/I$ , fix an isomorphism  $\text{Ext}(I, A) \cong A/I$ .

Let  $\alpha$  be the extension  $0 \rightarrow A \rightarrow P \rightarrow I \rightarrow 0$ . Then  $\alpha$  is a unit of  $A/I$  if and only if the  $A$ -module  $P$  is projective. In particular,  $I$  is projectively generated.

Example (1.5) (Failure of projective generation). Let  $C$  be any smooth projective curve of genus  $> 2$  over  $\underline{C}$ , and set  $A = \underline{C}[x, y, z]$ . For every point  $P$  on  $C$  there is a height 2 prime ideal  $I$  of  $A$  (necessarily a local CI) with  $\text{spec}(A/I) = C - P$ . For all but a finite number of  $P$ , the ideal  $I$  is not a CI. (See [Abh, p.102].) Since all projective  $A$ -modules are free, it follows that no rank 2 projective  $A$ -module maps onto  $I$ , i.e., that  $I$  cannot be "projectively generated."

If  $d > 3$ , little is known. We cite two results:

- [Bor1, Thm.3] Let  $I$  be a height  $d$  locally CI ideal with  $I/I^2$  free. Then  $\underline{SP}_d(A, J) \neq \emptyset$  for some ideal  $J \subseteq I$  such that  $\text{radical}(J) = \text{radical}(I)$ . (The projective module constructed by Boratyński may be seen to have trivial determinant by inspection of [Su].)

- [MM,Thm.(3.1)] If  $A$  is a  $d$ -dimensional affine domain over an algebraically closed field  $k$ , and  $I$  is a height  $d$  locally CI ideal, then  $\underline{SP}_d(A,I) \neq \emptyset$  if either (i)  $\text{char}(k) \nmid (d-1)!$  or (ii)  $A$  is normal and  $I$  is radical.

## §2. Unstable Theory

The goal of this section is to describe the situation for height 2 locally CI ideals in a finitely generated 2-dimensional algebra over a field  $k$  which is not algebraically closed. We try to determine when the nonempty set  $\underline{SP}_2(A,I) = \{ \text{all } P \text{ in } \underline{SP}_2(A,I) \text{ mapping onto } I \}$  contains only one element, which we denote  $\chi(I)$ . Thus  $I$  is a CI point if and only if  $\chi(I)$  is free. For example, this will be the case if  $I$  is a regular maximal ideal of  $A$  with  $A/I \cong k$ . Several of the results in this section have also been obtained independently by M. Boratyński [Bor].

Our first result is of a general nature; it has also been obtained by Boratyński [Bor]. Recall that  $\underline{SP}_2(A)$  is the set of all rank 2 projective  $A$ -modules  $P$  with  $\Lambda^2(P) = A$ , and that  $\underline{SP}_2(A,I)$  is the subset of all  $P$  in  $\underline{SP}_2(A)$  mapping onto  $I$ .

Proposition (2.1). Let  $I$  be a height 2 ideal in a noetherian ring  $A$ . Assume that  $I$  is locally a CI and that  $\Lambda^2(I/I^2)$  is free, so that  $\text{Ext}(I,A) \cong A/I$ . Write

$P(\alpha)$  for the middle module of the extension  $\alpha$ ;  $P(\alpha)$  is projective iff  $\alpha$  is a unit of  $A/I$ .

(a) Let  $\text{Epi}(\underline{SP}_2(A), I)$  denote the set of epimorphisms  $\phi: P \rightarrow I$  with  $P$  in  $\underline{SP}_2(A)$ , modulo isomorphisms of  $P$ . Then  $\phi \mapsto K(\phi)$  gives a set bijection

$$\text{Epi}(\underline{SP}_2(A), I) \cong \text{units}(A/I) / \text{units}(A).$$

(b)  $P(\alpha) \cong P(\alpha\beta^2c)$  whenever  $\alpha, \beta$  are units of  $A/I$  and  $c$  is a unit of  $A$ .

(c) There is an onto set map

$$G(I) = \frac{\text{units}(A/I)}{\text{units}^2(A/I) \cdot \text{units}(A)} \longrightarrow \underline{SP}_2(A, I).$$

Proof. The class of  $K(\phi)$  in  $\text{Ext}(I, A)$  is not well-defined because it depends on the choice of the isomorphism  $\Lambda^2 P \cong A$ . A different isomorphism will change the class from  $\alpha$  to  $c\alpha$ , where  $c$  is a unit of  $A$ . Hence the map in (a) is well-defined and injective. To see that the map in (a) is onto, assume that

$$\alpha: 0 \rightarrow A \xrightarrow{\theta} P \xrightarrow{\phi} I \rightarrow 0$$

is an extension with  $P = P(\alpha)$  projective. We compare  $\alpha$  with  $K(\phi)$ , which is exact by (1.3). The comparison yields an isomorphism of  $\Lambda^2 P = \ker(\phi)$  with  $A \cong \theta(A)$ , and with this choice of  $\Lambda^2 P \cong A$  we have  $K(\phi) = \alpha$ . To prove part (b), recall that for  $\alpha \in A/I$ ,  $c \in A$ , the extension  $c\alpha$  is constructed via pushout (qv. [Mac, p. 66]):

$$\begin{array}{ccccccc}
 0 & \rightarrow & A & \xrightarrow{\quad} & P(\alpha) & \longrightarrow & I \longrightarrow 0 \\
 & & c \downarrow & & \pi_\alpha \downarrow & & \parallel \\
 0 & \rightarrow & A & \longrightarrow & P(c\alpha) & \longrightarrow & I \longrightarrow 0.
 \end{array}$$

If  $c$  is a unit of  $A$ , then  $P(c\alpha) \cong P(\alpha)$  is immediate. In general,  $P(c\alpha) \cong P(\alpha) \oplus A/(\pi_\alpha, -c)A$ . It follows from [MS, Thm.4] that  $P(c\alpha) \cong P(\alpha)$  if  $c=b^2$ . Thus given  $b \in A$  mapping to  $\beta \in A/I$ , we have  $P(\alpha\beta^2c) = P(\alpha b^2c) \cong P(\alpha c) \cong P(\alpha)$ . Finally, (a) and (b) imply (c).

It follows that the set  $\underline{SP}_2(A, I)$  is generally much smaller than  $\text{units}(A/I)$ . If  $A = \underline{C}[x, y]$ , for example,  $\underline{SP}_2(A, I)$  is a one-element set and  $\text{units}(A/I)$  is infinite. The following examples show the extent to which  $G(I)$  reflects the size of  $\underline{SP}_2(A, I)$ .

Examples (2.2). Let  $A = \underline{R}[x, y, z]/(x^2+y^2+z^2=1)$ .

(a) If  $I$  is primary for a maximal ideal of  $A$  it is easy to see that  $G(I)=1$ . If  $v(I/I^2)=2$ , so that  $I$  is locally a CI, there is a unique projective  $A$ -module mapping onto  $I$ . If this projective is not free, then  $v(I)=3$ .

For example, let  $P = \{(a, b, c) \in A^3: ax+by+cz=0\}$ . It is well-known that  $P$  is not free. In [Gab, p. 139], Gabel gives a map of  $P$  onto a height 2 primary ideal  $I$  with  $v(I/I^2)=2$ . It follows that  $v(I)=3$ . This is the same as Boratynski's proof [Bor], cited in [Gab].)

(b) On the other hand, projection onto the first coordinate of  $P$  gives a surjection  $P \rightarrow I=(x, y)A$  whose

kernel is generated by  $(0, z, -y) \in P$ . Since  $G(I)$  is a group with two elements, we have  $\underline{SP}_2(A, I) = \{A^2, P\}$ .

(c) The following example, due to van der Kallen, shows that the group  $G(I)$  does not act naturally on the set  $\underline{SP}_2(A, I)$ . (Cf. [vdK, (4.13)].) Let  $I = (x^2 + xy, z)A$  and let the Koszul complex  $K(x^2 + xy, z)$  correspond to 1 in  $A/I$ , so that  $P(1) = A^2$ .  $G(I)$  is the group  $(\underline{Z}/2) \times (\underline{Z}/2) \times (\underline{Z}/2)$ , generated by  $y$ ,  $2x + y = \alpha$ , and  $2x + 1 = \beta$ . The projective modules  $P(y)$  and  $P(\alpha)$  are defined by the unimodular rows

$$\begin{aligned} (-y, x^2 + xy, z) &\sim (0, 1, 0) \text{ and} \\ (-\alpha, x^2 + xy, z) &\sim (-\alpha, x^2, z) \sim (0, 1, 0) \end{aligned}$$

respectively (the last equivalence is because one entry is a square). Thus  $P(1), P(y), P(\alpha)$  are all free. On the other hand,  $P(\alpha y) \cong P(-\alpha y)$  is defined by the unimodular row

$$(2xy + y^2, x^2 + xy, z).$$

$P(\alpha y)$  determines a vector bundle  $\xi$  on  $S^2$  which is given by the clutching function

$$f(x, y) = (2xy + y^2, x^2 + xy)$$

on the circle  $x^2 + y^2 = 1, z = 0$ . Now  $\xi$  is orientable; under the isomorphism

$$c_1: \{SO_2\text{-bundles on } S^2\} \cong \pi_1(SO_2) = \underline{Z}$$

(q.v. [H, p. 86]), the bundle  $\xi$  corresponds to

$c_1(\xi) = \pm 2$ . As  $\xi$  is not free, neither is  $P(\alpha y)$ . If  $G(I)$  acted on  $\underline{SP}_2(A, I)$ , though, we would have had  $P(\alpha y) \cong P(y) \cong A^2$ , so  $G(I)$  cannot act.

Incidentally, the other four projective modules  $P(\beta)$ ,  $P(\alpha\beta)$ ,  $P(\beta y)$  and  $P(\alpha\beta y)$  all give rise to the tangent bundle  $\tau$  on  $S^2$ , which has  $c_1(\tau) = 1$ . I do not know if they are distinct  $A$ -modules or if they are isomorphic.

Theorem (2.3). Let  $I$  be any height 2 local CI in a 2-dimensional noetherian ring  $A$ . Assume that, for every maximal ideal  $m$  containing  $I$ , one of the following holds:

- (a)  $\text{Units}(A/m)$  is a 2-divisible abelian group,
- (b)  $A$  is a  $k$ -algebra with  $k \rightarrow A/m$  an isomorphism.

Assume moreover that:

- If  $\text{char}(A/m) = 2$  then  $I_m = mA_m$ , and
- There is at most one maximal ideal  $m$  containing  $I$  for which (a) fails.

Then there is a unique rank 2 projective module  $P$  mapping onto  $I$ , with  $\Lambda^2 P = A$ . If  $P \neq A^2$  then  $v(I) = 3$ ; if  $P = A^2$  then  $I$  is a CI.

Proof. The primary decomposition of  $I$  is  $I = \bigcap q_i$ , where  $q_i$  is primary for  $m_i$ , and  $A/I = \Pi A/q_i$ . Thus  $\text{units}(A/I)$  is an extension of  $\Pi \text{units}(A/m_i)$  by

$\Pi(1+m_1/q_1)$ . The latter group is 2-divisible, since we have assumed that  $q_1 \neq m_1$  only if  $\text{char}(A/m) \neq 2$ . Hence  $\text{units}(A/I)/\text{units}^2(A/I)$  equals zero if (a) occurs for all  $m_1$ , and equals  $\text{units}(k)/\text{units}^2(k)$  if (a) fails for exactly one  $m_1$ . In the latter case, these units come from A. In any event,  $G(I) = 1$ , so that  $\underline{SP}_2(A, I)$  is a 1-element set by (2.1). The result now follows from (1.1.b).

Corollary (2.4). Let A be a finitely generated k-algebra of dimension 2, where k is a field. Let  $\text{Reg}(A, k)$  denote the set of maximal ideals m for which  $A_m$  is a regular ring and  $k \rightarrow A/m$  is an isomorphism. Then there is a well-defined set map

$$\chi: \text{Reg}(A, k) \rightarrow \underline{SP}_2(A) = \left\{ \begin{array}{l} \text{rank 2 projective A-} \\ \text{modules P with } \Lambda^2 P = A \end{array} \right\}.$$

If  $\chi(m) \neq A^2$ , then  $v(m) = 3$ ; if  $\chi(m) = A^2$ , then m is a CI.

Proof. The map  $\chi$  is well-defined by case (b) of (2.3).

Corollary (2.5). Let A be a finitely generated k-algebra of dimension 2, where k is either algebraically closed, real closed or perfect of characteristic 2. Let  $\text{Reg}(A)$  denote the set of maximal

ideals  $m$  of  $A$  with  $A_m$  regular. Then there is a well-defined set map

$$\chi: \text{Reg}(A) \rightarrow \underline{SP}_2(A).$$

If  $\chi(m) \neq A^2$ , then  $v(m) = 3$ ; if  $\chi(m) = A^2$ , then  $m$  is a CI.

Proof. In each case,  $A/m$  is a finite extension of  $k$ , and these are all of type (a) or (b) in Theorem (2.3).

Example (2.6). In order to illustrate the interplay between the various ideas involved, we analyze the maximal ideals of the UFD  $A = \underline{R}[x, y, z]/(y^2 + z^2 = x^3 - x)$ .

Every complex maximal ideal  $m$  of  $A$  (i.e., with  $A/m = \underline{C}$ ) is a CI: every one contains either  $x - \alpha$  or  $x^2 + bx + c$ , and every complex maximal ideal of  $A/(x - \alpha) = \underline{R}[y, z]/(y^2 + z^2 = \text{const.})$  and  $A/(x^2 + bx + c) = \underline{C}[u, v]/(uv = \text{const.})$  is known to be principal. (See e.g., [GW1].)

To analyze the real maximal ideals of  $A$ , note that  $\text{spec}(A)$  is the surface of revolution obtained by rotating an elliptic curve about the  $x$ -axis. Thus the real points of  $\text{Spec}(A)$  consist in a bounded component, topologically  $S^2$ , and an unbounded component which resembles a bell. We know that  $v(m) = 3$  for the real points  $m$  on the bounded component by [GW, p. 215]. In fact  $[A/m] \neq 0$  in  $SK_0(A)$ , as the map  $SK_0(A) \rightarrow KO(S^2) =$

$\underline{\mathbb{Z}}/2$  maps  $[A/\mathfrak{m}]$  to the nonzero element. (In fact,  $SK_0(A) = \underline{\mathbb{Z}}/2.$ )

On the other hand, if  $\mathfrak{m}$  is a real point on the bell (the unbounded component), I do not know if  $v(\mathfrak{m})$  is 2 or 3. Since by (2.5) there is a unique rank 2 projective  $A$ -module  $P$  mapping onto  $\mathfrak{m}$ . We have  $v(\mathfrak{m}) = 2$  iff  $P = A^2$ , but I do not know how to decide this. Note that  $[A/\mathfrak{m}] = 0$  in  $SK_0(A)$  since there is a line in  $\underline{A}^3$  meeting the real points of  $\text{Spec}(A)$  exactly once, transversely. In the spirit of the CI conjecture (0.1), my guess is that there are countably many circles  $x=\text{constant}$  on which  $v(\mathfrak{m}) = 2$ , and that  $v(\mathfrak{m}) = 3$  on the rest of the bell.

### §3. Stable theory and Murthy rings.

When  $I$  is a locally CI ideal of height  $d$ , the "Euler characteristic"  $[A/I]$  makes sense as an element of  $K_0(A)$  because  $\text{hd}_A(A/I)=d$ . If  $\phi:P \rightarrow I$  is onto, exactness of  $K(\phi)$  yields the formula

$$[A/I] = [A] + \sum (-1)^i [\wedge^i P].$$

It is easy to calculate that  $[A/I]=0$  if  $I$  is a CI, and we might hope that the converse is true.

Definition (3.1). We will call a  $d$ -dimensional noetherian ring  $A$  a Murthy ring if the following are equivalent for all locally CI ideals  $I$  of height  $d$  in  $A$

which can be projectively generated:

- (a)  $[A/I]=0$  in  $K_0(A)$ ;
- (b)  $I$  is a CI, i.e.,  $v(I)=d$ .

The phrase ' $I$  can be projectively generated' means that there is a rank  $d$  projective  $A$ -module  $P$  mapping onto  $I$  with  $\det(P)=A$ . Note that (b) trivially implies (a).

All 1-dimensional rings are Murthy rings, but even 2-dimensional algebras over  $\mathbb{R}$  can fail to be Murthy rings (see [Gab] or example (2.2.a) above). A blend of several questions that Murthy has asked yields the question: if  $A$  is finitely generated over an algebraically closed field, is  $A$  a Murthy ring? We prove in this section that the answer is yes if  $d=2$  or if  $A$  is regular and  $d=3$ . The question is open for  $d>4$ .

Theorem (3.2) (Murthy-Swan [MS, (4.5)]). Let  $A$  be a finitely generated 2-dimensional algebra over an algebraically closed field.

- (a)  $A$  is a Murthy ring.
- (b) Suppose that  $I$  is a locally CI ideal of height 2 in  $A$ . Then  $[A/I] = 0$  iff  $I$  is a CI iff  $v(I)=2$ ;  
if  $[A/I] \neq 0$  then  $v(I)=3$ .

Proof. Every height 2 locally CI ideal in  $A$  is projectively generated; this follows from (1.4.b).

Hence (a)=(b). Given  $I$ , choose  $P$  in  $\underline{SP}_2(A, I)$ , so that  $[A/I] = 2 - [P]$ . Then  $[A/I] = 0$  if and only if  $[P] = [A^2]$ . By [MS, Thm.1] or [Su], this holds iff  $P = A^2$ , i.e., iff  $v(I) = 2$ .

Theorem (3.3) (Murthy-Mohankumar). Let  $A$  be a finitely generated regular 3-dimensional algebra over an algebraically closed field  $k$ .

(a)  $A$  is a Murthy ring.

(b) Let  $I$  be a locally CI ideal of height 2. If  $\text{char}(k) = 2$ , assume in addition either that  $I$  is radical or that  $I$  is projectively generated. Then  $[A/I] = 0$  iff  $I$  is a CI iff  $v(I) = 3$ ;  
if  $[A/I] \neq 0$  then  $v(I) = 4$ .

Proof. By [MM, (3.1)], (a) and (b) are equivalent. The proof of Lemma (4.2) of [MM] applies to give a rank 3 projective  $A$ -module  $F$  mapping onto  $I$  with  $0 = C_1(F) = C_2(F)$ . That is,  $\Lambda^3(F) = A$  and  $[\Lambda^2(F)] = 6 - [F]$  in  $K_0(A)$ . From the Koszul resolution we see that

$$[A/I] = [\Lambda^2 F] - [F] = 2(3 - [F])$$

in the torsion-free subgroup  $F^3 K_0(A) \cong A^3(\text{spec}(A))$  of  $K_0(A)$ . Thus if  $[A/I] = 0$  then  $F$  is stably free, hence free by [Su]. But if  $F$  is free we have that  $v(I) = 3$ , i.e.,  $I$  is a CI.

In the remainder of this section, we consider the set-theoretic analogue of Theorems (3.2) and (3.3). When  $I$  is a maximal ideal we can replace ' $[A/I]=0$ ' by ' $[A/I]$  is a torsion element', but otherwise the description is a little awkward.

Proposition (3.4). Let  $A$  be a  $d$ -dimensional noetherian ring, and let  $I$  be a radical ideal which is locally a CI of height  $d$  (that is,  $A_m$  is regular for every maximal ideal  $m$  containing  $I$ ).

If  $I$  is a stCI, then in  $K_0(A)$ :

- (a) If  $I$  is a maximal ideal then  $[A/I]$  is a torsion element.
- (b) If  $I = m_1 \cap \cdots \cap m_r$ , then there are positive integers  $n_1, \dots, n_r$  with  $\sum n_i [A/m_i] = 0$  in  $K_0(A)$ .

Proof. If  $J$  is a CI ideal whose radical is  $I = m_1 \cap \cdots \cap m_r$ , then by devissage we have  $0 = [A/J] = \sum n_i [A/m_i]$ , where  $n_i$  is the length of  $A/J$  localized at the prime  $m_i$ .

Theorem (3.5). Let  $A$  be a Murthy ring of dimension  $d$ .

- (1) The following are equivalent for a regular maximal ideal  $I$  of  $A$ :
  - (a)  $I$  is an stCI.

(b)  $I$  is a weak stCI.

(c)  $[A/I]$  is a torsion element of  $K_0(A)$ .

(2) Let  $I = m_1 \cap \cdots \cap m_r$  be an intersection of regular maximal ideals  $m_i$ . Then the following are equivalent:

(a)  $I$  is an stCI.

(b)  $I$  is a weak stCI.

(c) There are integers  $n_1, \dots, n_r \geq 1$  with

$$\sum n_i [A/m_i] = 0 \text{ in } K_0(A).$$

Proof. We only supply the proof of part (1), since mutatis mutandis the proof of part (2) is the same. Since  $A_I$  is regular,  $I$  is locally a CI, and (a)=(b) by (1.1). By (3.4) we have that (a) implies (c). To see that (c) implies (a), suppose that  $n[A/I] = 0$  in  $K_0(A)$ . Let  $r$  be a number divisible by both  $n$  and  $(d-1)!$ , and choose  $a_1, \dots, a_d$  in  $I$  so that  $(a_1, \dots, a_d)A_{m_i} = I_{m_i}$  for  $i=1, \dots, r$ . By [Bor1, Thm.3], the  $m$ -primary ideal  $J = (a_2, \dots, a_d)A + I^r$  is projectively generated.  $J$  is locally a CI and  $\text{length}(A/I) = r$ , so in  $K_0(A)$  we have  $[A/J] = r[A/I] = 0$ . Since  $A$  is a Murthy ring,  $J$  is a CI.

Corollary (3.6). Let  $A$  be a finitely generated algebra  $A$  over an algebraically closed field. Suppose that either  $\dim(A)=2$  or  $A$  is regular and  $\dim(A)=3$ . Then the conclusions (1) and (2) of (3.5) hold for  $A$ .

#### §4. Nonsingular Affine Varieties

The main results of this section are Theorems (4.1) and (4.2), which verify the CI and stCI conjectures for regular rings of dimension  $d=2,3$ , and more generally for geometric Murthy rings. This reduces the problem for  $d>4$  to determining which rings are Murthy rings (see (3.1)). Recall that a subset of a topological space is called  $\sigma$ -closed if it is the countable union of closed subsets. For example, a  $\sigma$ -closed subset of  $\text{Spec}(A)$  has the form  $\bigcup V(I_n)$  for a sequence of ideals  $I_n$  of  $A$ .

Theorem (4.1). Let  $A$  be a regular ring of dimension  $d = 2$  or  $3$ , finitely generated over an algebraically closed field. Then

- (a) If a maximal ideal is an stCI, it is a CI.
- (b) The set of CI points of  $\text{Max}(A)$  is  $\sigma$ -closed.
- (c) The set of radical ideals  $I = m_1 \cap \dots \cap m_r$  which are CI ideals is a  $\sigma$ -closed subset of  $\text{Rad}^r(A)$ .
- (d) The set of radical ideals  $I = m_1 \cap \dots \cap m_r$  which are stCI ideals is a  $\sigma$ -closed subset of  $\text{Rad}^r(A)$ .

Proof.  $A$  is a Murthy ring by (3.2) or (3.3). Moreover  $A_0(\text{Spec}(A))$  is torsion-free for  $d=3$  by [MM, (1.1)]. Theorem (4.1) is now a consequence of resolution of singularities and:

Theorem (4.2). Let  $A$  be the coordinate ring of an affine open subset  $U$  of a smooth projective variety over an algebraically closed field. Assume that  $A$  is a Murthy ring and that  $\dim(A) = d > 2$ . Then:

- (a) If a maximal ideal is an stCI, it is a CI.
- (b) The set of CI points of  $\text{Max}(A)$  is  $\sigma$ -closed.
- (c) The set of radical ideals  $I = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_r$  which are CI ideals is a  $\sigma$ -closed subset of  $\text{Rad}^r(A)$ .
- (d) The set of radical ideals  $I = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_r$  which are stCI ideals is a  $\sigma$ -closed subset of  $\text{Rad}^r(A)$ .

We will give the proof of Theorem (4.2) after (4.5) below. The idea of the proof is this: map  $\text{Max}(A)$  into the Chow group  $A^d(U) = A_0(U)$  of zero-cycles on  $U = \text{Spec}(A)$ . Those points mapping to zero are the CI points, and those points mapping to torsion elements are the stCI points. These sets are  $\sigma$ -closed by results of Roitman. Note that  $v(I)=d$  for a CI ideal in  $\text{Rad}^r(A)$ , while  $v(I)=d+1$  for every other ideal in  $\text{Rad}^r(A)$ .

(4.3). We will use the following facts about the Chow groups  $A^d(U)$  of codimension  $d$  cocycles on a noetherian scheme  $U$ :

- (i)  $A^d(U)$  is generated by the classes  $[x]$  of points  $x$  in  $U^d$ , the set of codimension  $d$  points of  $U$ . For the

precise definition, we refer the reader to [CF] (where the notation  $W_d(F)$  was used), to [MS] or to [0,7.5.14]. If  $U$  is unmixed,  $A^d(U)$  agrees with the Chow homology groups of [G225] and [Fu].

(ii) If  $U$  is an open subset of  $X$ , there is an exact sequence

$$\begin{array}{c} | \\ \text{closed} \\ y \in X^d - U \end{array} \underline{Z} \rightarrow A^d(X) \xrightarrow{\text{res}} A^d(U) \rightarrow 0.$$

(See for example [G225,p.41],[MS, p.138]).

(iii) When  $U$  is smooth and quasiprojective of dimension  $d$ , there is a natural map  $A^d(U) \rightarrow K_0(U)$  sending  $[x]$  to  $[\mathcal{O}_{X,x}/\mathfrak{m}_x]$ . The kernel of this map is annihilated by  $(d-1)!$ . (See [G225,p.64].)

(iv) Let  $X$  be a nonsingular projective variety of dimension  $d$  over an algebraically closed field. Then  $A^d(X)$  is the group considered by Roitman. More precisely, if  $a, b$  are codimension  $d$  cocycles ("zero-cycles"), then the following are equivalent:

(a)  $a-b=0$  in  $A^d(X)$ .

(b) There is a codimension  $d$  cocycle  $T$  on  $X \times \underline{P}^1$  such that  $i_0^*(T)=a$ ,  $i_\infty^*(T)=b$ . Here  $i_y: X \rightarrow X \times \underline{P}^1$  is the map  $x \mapsto (x, y)$ .

(c) There exists an integer  $i$  and a morphism  $T: \underline{P}^1 \rightarrow S^i X \times S^i X$  for which  $Z(T(0))=a$  and  $Z(T(\infty))=b$ . Here  $S^i X$  is the symmetric product,  $S^i X = X^i / \Sigma_i$ , and  $Z: S^i X \times S^j X \rightarrow Z^n X$  is the map  $Z(a_1, a_2) = a_1 - a_2$ .

(The equivalence of (a) and (b) is given in [MS,p.134]; the equivalence of (b) and (c) is implicit in [Mum] and is proven in [R1,p.557].)

(v) Let  $X$  be a smooth projective variety of dimension  $d$  over an algebraically closed field. Then  $A^d(X)$  is a divisible group [R2], and the map  $A^d(X) \rightarrow \text{Alb}(X)$  to the Albanese variety  $\text{Alb}(X)$  induces an isomorphism on torsion. (See [R3][Milne].)

Proposition (4.4) (Roitman). Let  $U$  be an affine open subset of a smooth projective variety  $X$  over an algebraically closed field. If  $\dim(U)=d>2$  then  $A^d(U)$  is a torsion-free divisible abelian group, and the map  $A^d(U) \rightarrow K_0(U)$  is an injection.

Proof. (See [BMS][MM,(1.1)].) This is an easy consequence of (4.3) and the fact that  $\text{Alb}(X-U) \rightarrow \text{Alb}(X)$  is onto if  $d>2$ .

Theorem (4.5) (Roitman). Let  $X$  be a nonsingular projective variety of dimension  $d$  over an algebraically closed field  $k$ . Let  $Y$  be a closed subvariety of  $X$  and set  $U = X-Y$ . Let the composite:

$$\gamma: X^{m+n}(k) \xrightarrow{\text{cycle}} A^d(X) \xrightarrow{\text{res}} A^d(U)$$

be the map  $\gamma(a_1, \dots, a_m, b_1, \dots, b_n) = \sum a_i - \sum b_j$ . Then

$\gamma^{-1}(0)$  is a  $\sigma$ -closed subset of  $X^{m+n}(k)$  containing  $Y^{m+n}(k)$ .

Proof. Consider the sets  $W_{r,s} \subset Y^{r+s}(k) \times X^{m+n}(k)$ :

$$W_{r,s} = \{(y_1, y_2, x_1, x_2) : \text{cycle}(x_1 - x_2) = \text{cycle}(y_1 - y_2) \text{ in } A^d(X)\}.$$

By Roitman's Lemma 3 of [R2, p.577], each  $W_{r,s}$  is  $\sigma$ -closed (the assumption that  $\text{char}(k) = 0$  is not used). Hence the projection

$$W = \{(x_1, x_2) \in X^{m+n}(k) : \text{cycle}(x_1 - x_2) = \text{cycle}(y_1 - y_2) \text{ in } A^d(X) \text{ for some } (y_1, y_2) \in Y^r \times Y^s\}$$

is a  $\sigma$ -closed subset of  $X^{m+n}(k)$  containing  $Y^{m+n}(k)$ . It follows from the exact sequence (4.3.11) that  $W = \gamma^{-1}(0)$ , proving the theorem.

Proof of Theorem (4.2). Since  $A$  is a Murthy ring, part (a) follows from (4.4) and (3.5.1). Now choose a nonsingular projective variety  $X$  and a closed subvariety  $Y$  such that  $\text{Spec}(A) = U = X - Y$ . The set map  $\gamma_{10} | U : U(k) \subset X(k) \xrightarrow{Y} A^d(U)$  sends  $x \in U$  to the cocycle  $[x]$  of  $A^d(U)$ . The map  $A^d(U) \rightarrow K_0(A)$  sends  $[x]$  to  $[A/m_x]$ , and is an injection by (4.4). Since  $A$  is a Murthy ring, we have

$$\begin{aligned} \{\text{CI points}\} &= \{x \in U(k) : v(m_x) = d\} \\ &= \{x \in U(k) : \gamma_{10} | U(x) = 0\}. \end{aligned}$$

By Theorem (4.5), with  $m=1$  and  $n=0$ , this is a  $\sigma$ -closed subset of  $U(k)$ . To prove (c) and (d), consider the set map

$$U^r(k) \subset X^r(k) \xrightarrow{\text{diag}} X^{n_1}(k) \times \cdots \times X^{n_r}(k) \xrightarrow{\gamma} A^d(U)$$

which sends  $(x_1, \dots, x_r)$  to  $\Sigma_{n_i}[x_i]$ . By Theorem (4.5), the inverse image of 0 is  $\sigma$ -closed. Hence for every  $r$ -tuple of integers  $(n_1, \dots, n_r)$  the set  $\{(x_1, \dots, x_r) : \Sigma_{n_i}[x_i] = 0 \text{ in } A^d(U)\}$  is a  $\sigma$ -closed subset of  $U^r(k)$ . Restricting to distinct  $r$ -tuples, and modding out by the action of  $\Sigma_r$ , we see that

$$\{(x_1, \dots, x_r) : \text{all } x_i \text{ distinct, } \Sigma_{n_i}[x_i] = 0 \text{ in } A^d(U)\}$$

is a  $\sigma$ -closed subset of  $\text{Rad}^r(A)$ . The result now follows from (3.5).

Remark (4.7). On a similar note, we can describe which ideals  $I$  with  $\text{length}(A/I)=r$  are CI or stCI. We use the Hilbert scheme  $\text{Hilb}^r(U)$  of all ideals  $I$  of  $A$  with  $\text{length}(A/I)=r$  to parametrize this problem; it is a quasiprojective variety. (See [G221].) Let  $\text{LCI}^r(A)$  denote the set of locally CI ideals of length  $r$ ; it forms an open subset of  $\text{Hilb}^r(U)$ . Let  $\text{PG}^r(A)$  denote the set of all projectively generated ideals  $I$ ; this is a subset of  $\text{LCI}^r(A)$  (and may equal  $\text{LCI}^r(A)$ ).

Under the hypotheses of Theorem (4.2), the CI ideals  $I$  form a  $\sigma$ -closed subset of  $\text{PG}^r(A)$ , and those locally CI ideals which are stCI form a  $\sigma$ -closed subset

of  $\text{LCI}^{\Gamma}(A)$ . (To prove this, use the algebraic map  $\text{Hilb}^{\Gamma}(U) \rightarrow S^{\Gamma}(U)$  and the proof of (4.5).)

§5. The case  $k = \overline{\mathbb{F}}_p$ .

When  $k$  is countable,  $\sigma$ -closedness means nothing, because a finitely generated  $k$ -algebra has only countably many maximal ideals. In this section we prove two structure theorems about CI and stCI points when  $k$  is algebraic over a finite field, following the suggestions of the referee of [W1]. Recall that a height  $d$  ideal  $I$  is a weak stCI just in case  $I$  is the radical of an ideal of the form  $(x_1, \dots, x_d)A$ .

Theorem (5.1). Let  $k$  be an algebraic extension of a finite field, and let  $A$  be any finitely generated  $d$ -dimensional  $k$ -algebra. Then:

- (1)(a) Every maximal ideal of height  $d$  is a weak stCI.
- (b) A height  $d$  maximal ideal  $m$  of  $A$  is an stCI if and only if  $A_m$  is Cohen-Macaulay.
- (c) {stCI points} is an open subset of  $\text{Max}(A)$ .
- (2)(a) Every radical ideal of height  $d$  is a weak stCI.
- (b) A height  $d$  radical ideal  $m_1 \cap \dots \cap m_r$  of  $A$  is an stCI if and only if each  $A_{m_i}$  is Cohen-Macaulay.
- (c) {stCI points} is an open subset of  $\text{Rad}^{\Gamma}(A)$ .

Proof. It is enough to prove part (2). Part (c) follows from (a) and (b) because the set of  $m$  such that

$A_m$  is Cohen-Macaulay is an open subset of  $\text{Max}(A)$ . (See [EGA IV, (6.11.3)].) If  $m_1 \cap \dots \cap m_r$  is the radical of a weak CI ideal  $I$  (i.e.,  $v(I)=d$ ), and each  $A_{m_i}$  is Cohen-Macaulay, then  $I$  is a CI by [K, Thm.125]. Hence (a) implies (b). We now prove (a) by induction on  $d$ . The case  $d=1$  follows from Proposition (5.2) below. If  $d>1$ , let  $a_1$  be an element of  $I$  contained in no minimal prime ideal of  $A$ . By induction, there are  $a_2, \dots, a_d$  in  $I$  such that  $I/a_1$  is the radical of  $(a_2, \dots, a_d)A/a_1A$ . But then  $I$  is the radical of  $(a_1, \dots, a_d)A$ .

Proposition (5.2) (folklore). Let  $k$  be an algebraic extension of a finite field, and let  $A$  be a 1-dimensional reduced ring, finitely generated over  $k$ . Then every height 1 radical ideal of  $A$  is a stCI.

Proof. (Cf. [DM, (2.2)], [W, (3.1)].) We will show that  $I = m_1 \cap \dots \cap m_r$  is an stCI. Let  $x$  be a nonzerodivisor of  $A$  contained in  $I$ , and let  $J$  be the intersection of all the  $m_i$ -primary components of  $xA$ ;  $J$  is a locally CI ideal of  $A$  with radical  $I$ . We will show that some  $J^n$  is a CI. Since  $A$  is a Murthy ring, it is enough to show that  $\text{Pic}(A)$  is a torsion group. If  $A$  is regular, this fact is classical. In general, let  $B$  be the integral closure of  $A$ , and let  $\mathcal{C}$  be the conductor from  $B$  to  $A$ . The fact that  $\text{Pic}(A)$  is torsion follows from

exactness of the sequence (see [B,p.482]):

$$\text{units}(B/\mathcal{C}) \rightarrow \text{Pic}(A) \rightarrow \text{Pic}(B).$$

Theorem (5.3). Let  $A$  be the coordinate ring of an affine open subset  $U$  of a smooth projective variety  $X$  over  $k = \overline{\mathbb{F}}_p$ . If  $\dim(X) > 2$  then:

- (a) The Chow group  $A_0(U)$  is zero.
- (b) Every maximal ideal of  $A$  is a CI.
- (c) Every finite intersection of maximal ideals of  $A$  is a CI.

Proof. Since  $A_0(X)$  is generated by the images of  $\text{Pic}(C) \rightarrow A_0(X)$  for curves  $C$  on  $X$ , it follows that  $A_0(X)$  - and its quotient  $A_0(U)$  - is a torsion group. But  $A_0(U)$  is torsionfree by (4.4), whence  $A_0(U) = 0$ . (Cf. [MS, §5].)

When  $d=2$  or  $3$ ,  $A$  is a Murthy ring, and we can deduce (b),(c) either from (4.1.a) and (5.1.1.a), or from part (a) and (4.4). We are now reduced to proving that  $I = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_r$  is a CI when  $d > 3$ . We claim that there is an element  $a_1$  of  $I$  and a smooth projective subvariety  $Y$  of dimension  $d-1$  in  $X$  such that  $\text{Spec}(A/a_1A) = Y \cap U$ . By induction on  $d$ ,  $I/a_1A$  is generated by a regular sequence  $(a_2, \dots, a_d)$ , so that  $I$  is generated by the regular sequence  $(a_1, a_2, \dots, a_d)$ .

It remains to establish the claim. Let  $X$  be embedded in  $\mathbb{P}^n$  as  $\text{Proj}(S)$ ,  $S=k[x_0, \dots, x_n]$ , in such a way that  $U$  is the locus of  $x_0 \neq 0$ . Choose  $t$  large enough that  $I^2$  is generated by elements  $h_i/x_0^t$  with the  $h_i$  in  $S_t$ , and that the only common zeros of the  $h_i$  are the points of  $V(I)$ . Choose an element  $s$  in  $S_{t+1}$  such that  $s/x_0^{t+1}$  lies in  $I$  but not in  $I^2$ . We now cite Swan's version of Bertini's theorem from [Sw] [MM, p.586], applied to  $X-V(I)$ , the vector bundle  $E = \mathcal{O}(t) = \tilde{S}(t)$  (which is generated by global sections), and the section  $(s, 1)$  of  $E(1) \oplus \mathcal{O}^*$ :

For any generic  $v$  in the subspace of  $S_{t+1}$  generated by the  $h_i x_j$ , the scheme  $V(s+v)$  of zeros of the section  $s+v$  of  $E(1)$  is a smooth irreducible subvariety of codimension 1 in  $X-V(I)$ .

We let  $a_1$  be  $(s+v)/x_0^{t+1}$  and let  $Y$  be the subvariety  $V(s+v)$  of  $X$ . Note that smoothness of  $Y$  at  $V(I)$  follows from the fact that  $a_1$  is in  $I$  but not  $I^2$ .

#### §Appendix. Higher Dimensions

In this appendix we give the following result of R. Varley, which partially answers Conjectures (0.1) and (0.2). Recall (e.g., from [Mat]) that a locally closed set is the intersection of an open and a closed set, while a constructible set is a finite union of locally closed sets. Thus the properties " $\sigma$ -locally

closed" and " $\sigma$ -constructible" of a set  $S$  are the same; they mean that  $S$  is the countable union of locally closed sets.

Theorem (A.1)(R. Varley). Let  $X$  be a nonsingular projective variety over an algebraically closed field, and let  $A$  be the coordinate ring of an affine open subset. Then

- (a) The set of CI points of  $\text{Max}(A)$  is  $\sigma$ -constructible.
- (b) The set of stCI points of  $\text{Max}(A)$  is  $\sigma$ -constructible.

Varley's proof uses the Hilbert schemes  $\text{Hilb}^f(X)$ , constructed in [G221]. Let  $G(i, V)$  denote the Grassmannian variety of codimension  $i$  subspaces of a vector space  $V$ , and choose a finitely generated graded  $k$ -algebra  $S$  with  $X = \text{Proj}(S)$ . If  $f \in \mathbb{Q}[t]$ ,  $\text{Hilb}^f(X)$  is a subvariety of  $G(f(N), S_N)$  for  $N \gg 0$ ;  $Y = \text{Proj}(S/I)$  corresponds to  $I_N$  and a subspace  $V$  of  $S_N$  corresponds to  $\text{Proj}(S/SV)$ .

Lemma (A.2). Given polynomials  $f_1, \dots, f_d, g$  in  $\mathbb{Q}[t]$ , let  $Z$  be the subset of  $\text{Hilb}^{f_1}(X) \times \dots \times \text{Hilb}^{f_d}(X)$  on which the function  $\bigcap : Z \rightarrow \text{Hilb}^g(X)$  is defined. This function takes the point  $(Y_1, \dots, Y_d)$ ,  $Y_i = \text{Proj}(S/I_i)$ , to

the point  $Y_1 \cap \cdots \cap Y_d = \text{Proj}(S/(I_1 + \cdots + I_d))$ . Then the set  $Z$  is locally closed.

Proof. The "codimension" map

$$G(f_1(N), S_N) \times \cdots \times G(f_d(N), S_N) \rightarrow \underline{N} \text{ (the natural numbers)}$$

$$(V_1, \dots, V_d) \mapsto \text{codim}(V_1 + \cdots + V_d, S_N)$$

is lower semicontinuous. In particular the set

$$Z_1 = \{(V_1, \dots, V_d) : \text{codim}(V_1 + \cdots + V_d, S_N) = g(N)\}$$

is locally closed. Let  $\Sigma : Z_1 \rightarrow G(g(N), S_N)$  be the map

given by  $\Sigma(V_1, \dots, V_d) = V_1 + \cdots + V_d$ . Then  $\Sigma$  is

algebraic, and the set  $Z = \Sigma^{-1}(\text{Hilb}^g(X)) \cap \text{Hilb}^f(X)$  is locally closed.

Lemma (A.3). Let  $U = \text{Spec}(A)$  be an affine open subset of a nonsingular variety  $X$ , and let  $f \in \underline{Q}[t]$  have degree  $\dim(X) - 1$ . Then the set of all divisors  $D$  in  $\text{Hilb}^f(X)$  which are principal on  $U$  is a  $\sigma$ -closed subset of  $\text{Hilb}^f(X)$ .

Proof. The kernel  $K$  of  $\text{Pic}(X) \rightarrow \text{Pic}(U)$  is the countable subgroup generated by the classes of the components of  $X - U$ . That is,  $K$  is  $\sigma$ -closed. The map  $[\ ] : \text{Hilb}^f(X) \rightarrow \text{Pic}(X)$  is a morphism of algebraic varieties, so  $[\ ]^{-1}(K)$  is  $\sigma$ -closed. But  $[\ ]^{-1}(K)$  is the set of all  $D$  with  $[D] \in K$ , i.e., with  $D$  principal on  $U$ .

Proof of Theorem (A.1). We use the notation developed above. Elements  $a$  of  $A$  correspond to divisors  $D$  of  $X$  which are principal on  $U = \text{Spec}(A)$ . This is a one-to-countably-many correspondence. If  $I = (a_1, \dots, a_d)A$  is an ideal of  $A$ , then there are divisors  $D_1, \dots, D_d$  on  $X$ , principal on  $U$ , with  $U \cap (D_1 \wedge \dots \wedge D_d) = \text{Spec}(A/I)$ . If  $\dim(A/I) = 0$  then the subscheme  $D_1 \wedge \dots \wedge D_d$  is the disjoint union of the closed subschemes  $\text{Spec}(A/I)$  and  $Y = (X - U) \cap (D_1 \wedge \dots \wedge D_d)$ , so the Hilbert polynomial of  $D_1 \wedge \dots \wedge D_d$  is the sum of the length of  $A/I$  and the Hilbert polynomial of  $Y$ .

Fix  $d = \dim(X)$ ,  $n \in \mathbb{N}$ ,  $g \in \mathbb{Q}[t]$ . Fix polynomials  $f_1, \dots, f_d$  in  $\mathbb{Q}[t]$  of degree  $d-1$ . We define the subset  $Z = Z(f_1, \dots, f_d, g, n)$  of  $\text{Hilb}^f(X) \times \dots \times \text{Hilb}^f(X)$  to be the set of those  $d$ -tuples  $(D_1, \dots, D_d)$  of divisors, principal on  $U$ , such that the Hilbert polynomials of  $D_1 \wedge \dots \wedge D_d$  and  $(X - U) \cap (D_1 \wedge \dots \wedge D_d)$  are  $g$  and  $g-n$ , respectively. By (A.2) and (A.3), the set  $Z$  is locally closed. The map  $Z \rightarrow \text{Hilb}^n(U)$  given by  $(D_1, \dots, D_d) \mapsto U \cap (D_1 \wedge \dots \wedge D_d)$  is well defined and algebraic by the definition of  $Z$ , and its image is constructible. ( $\text{Hilb}^n(U)$  is the open subvariety of  $\text{Hilb}^n(X)$  of subschemes of  $X$  with support in  $U$ , i.e., of those  $\text{Spec}(A/I)$  with  $\text{length}(A/I) = n$ .)

In case (a) we have  $n=1$  and  $\text{Hilb}^1(U) = U$ . A maximal ideal  $\mathfrak{x}$  of  $U$  is generated by  $d$  elements just in case  $\mathfrak{x}$

lies in the image of some  $Z(f_1, \dots, f_d, g, 1)$ . Hence the set of weak CI (=strong CI) points is the union of the constructible images of the countably many  $Z$ 's.

In case (b), consider the map  $\text{Hilb}^n(U) \rightarrow S^n U$  taking  $Y = \text{Spec}(A/I)$  to the support of  $Y$ , i.e., to the sum of the primes over  $I$  (with the multiplicity of  $x$  equal to  $\text{length}(A_x/I_x)$ ). Here  $S^n U$  denotes the  $n^{\text{th}}$  symmetric product  $(U \times \dots \times U) / \Sigma_n$ . This map is algebraic, and is defined on p. 26 of [G221]. The image  $Z$  of  $Z(f_1, \dots, f_d, g, n) \rightarrow \text{Hilb}^n(U) \rightarrow S^n U$  is therefore constructible. The intersection  $\bar{Z} \cap \Delta U$  is a constructible subset of  $U$ ; it consists of those primes  $x$  in  $U$  which are radicals of ideals  $(a_1, \dots, a_d)A = I$ , with  $\text{length}(A/I) = n$  and  $\text{Spec}(A/I) = U \cap (D_1 \cap \dots \cap D_d)$  for some  $(D_1, \dots, D_d) \in Z(f_1, \dots, f_d, g, n)$ . The subset of stCI points of  $U(k)$  is the union of the sets  $\bar{Z} \cap \Delta U$  over all of the (countably many) choices of  $f_1, \dots, f_d, g$ , and  $n$ . (Note that strong stCI = weak stCI.) This shows that the set of stCI points is  $\sigma$ -closed in  $U(k) = \text{Max}(A)$ .

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