Complete Intersection Points
on
Affine Surfaces

Charles A. Weibel*
Mathematics Department
Rutgers University
New Brunswick, NJ 08903 USA
December, 1981

Abstract. This paper addresses the following problem: given a commutative ring A, what is the structure of the set of "CI points," i.e., those maximal ideals generated by dim(A) elements? When A is finitely generated over an algebraically closed field, we conjecture that this set is a countable union of closed subsets of Max(A). When A is 2-dimensional and regular, we verify this conjecture, as well as an analogous set-theoretic conjecture.

AMS(MOS) subject classifications (1980) Primary 14C35;

Secondary 13C10, 13D15, 14C05.

Keywords and Phrases. Complete intersection point, SK_0, Chow group of cocycles, algebraic K-theory, Hilbert scheme.

---

*Partially supported by NSF grant

This paper is an assault on the following question: given a commutative noetherian ring A, which ideals I of A are complete intersections?

There are variants of this question, depending on just what a complete intersection is taken to be. Letting $\nu(I)$ denote the minimal number of generators of I, some possibilities are:

(a) weak CI: $\nu(I)$ = height (I)

(b) CI: I is generated by a regular sequence

(c) $\nu(I) = \nu(I/I^2)$

(d) weak stCI: I is the radical of a weak CI ideal

(e) stCI: I is the radical of a CI ideal

One way of answering this question is to parametrize certain ideals by an algebraic variety and ask for the topological structure of the set of complete intersections. For this reason, we focus on the maximal ideals of A, which are parametrized by Max(A), the closed points of Spec(A).

Here are two conjectures about the set of maximal ideals m of A which are complete intersections. By Sing(A) we mean $\{m : A_m$ is not regular$\}$. We will call a subset of a topological space "σ-P" if it is the countable union of subsets having the property P (e.g., P = 'closed').

CI conjecture (0.1). Let A be a finitely generated commutative algebra over an algebraically closed field. Then the following subset of Max(A) is σ-closed:

$$\{m : \nu(m) = \nu(m/m^2)\} = \{\text{weak CI points } m\} \cup \mathrm{Sing}(A)$$
$$= \{\text{CI points } m\} \cup \mathrm{Sing}(A)$$

stCI conjecture (0.2). Let A be a finitely generated commutative algebra over an algebraically closed field. Then Sing(A) $\cup$ {stCI points} is a σ-closed subset of Max(A).

Here is the current stratus of this conjecture. If dim(A)=1, σ-closed means "either all maximal ideals or only countably many" and both conjectures were verified in [GW] and [W2]. (The hypothesis that A be reduced is easily removed.)

If A is regular and 2-dimensional, we will verify both conjectures in §3 below. Murthy and Swan showed in [MS] that {CI points}=Max(A) exactly when $SK_0(A)=0$, and our technique is a modification of theirs.

If A is regular of dimension 3 or more, Varley has shown that both sets {CI points} and {stCI points} are σ-locally closed (= σ-constructible). We give Varley's proof in §4 below.

If A is not regular, little seems known. Alberto [Alb] and Pedrini [P] have constructed a relative Chow group $A^2(A,\mathrm{Sing}(A))$ when Sing(A) is finite. In §5 we construct the obvious generalization of their group. If A is 2-dimensional, we show that there is an exact sequence

(0.3) $SK_1(A) \to SK_1(\mathrm{Sing}(A)) \to A^2(A,\mathrm{Sing}(A)) \to SK_0(A) \to 0$

If m is a nonsingular maximal ideal of A, the class $[m] \in A^2(A,\mathrm{Sing}(A))$ maps to $[A/m] \in SK_0(A)$. In §1 we show that: (1) m is a CI point iff $[A/m] = 0$ and (2) m is an stCI point iff $[A/m]$ is a torsion element of $SK_0(A)$.

Conjecture (0.4). The fibers of $\mathrm{Reg}(A) \to A^2(A,\mathrm{Sing}(A))$ are σ-closed. Here $\mathrm{Reg}(A) = \{m \in \mathrm{Max}(A) : A_m \text{ is regular}\} = \mathrm{Max}(A)-\mathrm{Sing}(A)$.

It is clear that, when Sing(A) is finite, conjecture (0.4) implies the CI conjecture (0.1). If $A^2(A,\mathrm{Sing}(A))$ is either torsion or has a countable torsion subgroup, then (0.4) also implies the stCI conjecture (0.2). To lend credence to conjecture (0.4), we offer two pieces of evidence:

(i) If A is regular, Roitman established (0.4) in [R1] and [R2].

(ii) In §5 we analyze the seminormal surface $x^2 = (y^3+z)z^2$ rather completely, establishing all three conjectures (0.1), (0.2), and (0.4) for this surface.

By now the reader may have noticed that this paper is a collage of the results of a large number of people. My role as author is as much that of editor as that of researcher. Here is a table of contents, with credits:

In §1 we refine some results of Murthy and Swan in [MS], translating the CI problem into a projective module problem when $\dim(A)=2$. When A is finitely generated over an algebraically closed field we show that

(1.9) m is a CI iff $[A/m]=0$ in $SK_0(A)$

(1.10) m is a stCI iff $[A/m]$ is a torsion element of $SK_0(A)$.

Ed Davis was very helpful in pointing out the subtleties of weak vs. strong CI's.

In §2 we describe the unstable nature of the CI problem when A is not finitely generated over an algebraically closed field. We consider the set $\underline{\underline{SP}}_2(A,I)$ = {rank 2 projective A modules P with $\Lambda^2P = A$ mapping onto I}. By (1.8), I is a CI exactly when $\underline{\underline{SP}}_2(A,I)$ contains A^2. We give a number of cases when $\underline{\underline{SP}}_2(A,I)$ contains just one element P, so that I is a CI exactly when P is free. For example, this is the case if $A \to A/I$ splits, i.e., if I is a "rational" maximal ideal. Several of the results in this section were obtained independently by M. Boratýnski, and I am grateful to W. van der Kallen for providing me with example (2.2c).

In §3 we prove that the CI and stCI conjectures hold for smooth affine surfaces. The proof follows Murthy and Swan's analysis in [MS,§4], invoking a σ-closedness result of Roitman. In order to prepare for §5 below, we recall some standard results about the Chow groups of cocycles. Many of these results are due to Grothendieck [G221], Clayborn and Fossum [CF], and Fulton [Fu].

In §4 we prove Varley's result mentioned above. I am grateful to him for his permission to include his result here.

§5 is devoted to defining $A^d(X,Y)$ and proving that the sequence (0.3) is exact. As mentioned above, the definition of $A^d(X,Y)$ is spiritually lifted from [P] and [Alb]. I would like to thank C. Pedrini for focusing my thoughts in this direction. The absolute version of §5 has of course been in circulation for some time, many of the ideas going back at least to Clayborne and Fossum [CF].

§6 is an analysis of a singular example. We verify conjectures (0.1), (0.2) and (0.4) for this case. The strategy is to compute $SK_0(A)$ using classical K-theory techniques, and then to lift elements of $SK_0(A)$ to Max(A). This gives the map $\text{Reg}(A) \to SK_0(A)$ for all points except those lying on two curves, which we handle separately. I have generalized k from a field to a regular noetherian ring, because I have other applications in mind.

One such application is contained in the Appendix. Consider the ring $D = k[X,Y,Z]/(X^2=Y^3+Z^7)$, k a field. This is a 2-dimensional normal domain with a point singularity, so by (0.3) we have $A^2(D,D_+) = SK_0(D)$. There is a birational map $D \to A$ sending X to xz^2, Y to yz^2, and Z to z. It follows that $SK_0(D)$ surjects onto $SK_0(A)$, hence onto Ω_k. Thus if char(k) = 0 and k is big enough, we know that $SK_0(D) \neq 0$. This example is due to Bloch and Murthy. In the appendix, we utilize §6 to show that $NK_0(D)$, $K_{-1}(D)$ and $NK_{-1}(D)$ are nonzero for every field k.

In addition to the above acknowledgements, I would like to thank Queen's University and the University of Genova for their hospitality during part of the research stage. Most of the research was done at the University of Pennsylvania during 1979-80. I would also like to thank the following people for useful discussions about the subject matter: E. Davis, M.R. Gabel, S. Geller, A. Geramita, M.P. Murthy, S. Shatz, R. Swan, W. van der Kallen, and R. Varley.

§1. Preliminaries

In this section, we set up the translation from the CI problem to a projective module problem. This is a refinement of the translation given by Murthy and Swan [MS] for regular affine surfaces. Most of the results are of the folklore variety, and have their origins in either "Serre's Lemma" in [Se] or in Davis' work [D] on weak vs. strong CI's.

We first need to specify what 'complete intersection' means. We consider an ideal I in a commutative noetherian ring A and want to know about $\nu(I)$. Here $\nu(M)$ denotes the smallest number of generators of an A-module M.

If $\nu(I) = \text{height}(I)$, we will call I a weak CI (the 'CI' is for 'ideal-theoretic complete intersection'). When I is generated by a regular sequence, we will call I a strong CI or just a CI. Note that strong CI's are also weak CI's. If I_p is a strong CI (resp., a weak CI) for every prime ideal p containing I, we will say that I is locally a CI (resp., locally a weak CI). Finally, if I is the radical of a CI ideal (resp., of a weak CI ideal), we will call I an stCI (resp., a weak stCI). The letters stCI are an abbreviation for 'set-theoretic complete intersection.'

If A is Cohen-Macaulay, it is well-known that the notions of weak CI and strong CI (and also the notions of weak and strong stCI) coincide. Less well-known is the fact that the notions of weak and strong CI also coincide for radical ideals. This fact is due to Ed Davis (see pp. 199, 202-3 of [D]; a handy reference for prime ideals is [GS, (13.15)]). I am grateful to Ed Davis for pointing out the following folklore result: I is a strong CI if and only if I is both a weak CI and locally a strong CI. We prove this below as (1.1)(a=b), using Koszul complexes.

Recall that if P is a rank n projective module and $\phi:P \to I$ is a mapping of P onto an ideal I, then the (reduced) Koszul complex $K(\phi)$ is

$$0 \to \Lambda^n P \to \cdots \to \Lambda^2 P \to P \to I \xrightarrow{\phi} 0,$$

where $e_1 \wedge \cdots \wedge e_r$ in $\Lambda^r P$ maps to $\Sigma(-1)^{i+1} e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_r$ in $\Lambda^{r-1}P$ (cf. [SeM, p.70]). If T is an automorphism of P, then there is a natural chain isomorphism $T: K(\phi T) \cong K(\phi)$. If P is free with basis $(e_1,\ldots,e_r)$ and $\phi(e_i)=a_i$, we write $(a_1,\ldots,a_r)$ for ϕ. The complex $K(a_1,\ldots,a_r)$ is the reduced version of the more familiar Koszul complex associated to the sequence $(a_1,\ldots,a_r)$ of elements of A.

Theorem (1.1) ("weak CI + locally strong CI = strong CI"). Let I be a height d ideal in a noetherian ring A. The following are equivalent:

(a) I is a strong CI (i.e., I is generated by a regular sequence).

(b) I is locally a strong CI and $\nu(I)$ = height (I) = d.

(c) Some $K(a_1,\dots,a_d)$ is exact.

(d) Whenever $a_1,\dots,a_d$ generate I, $K(a_1,\dots a_d)$ is exact.

Proof. By [SeM, p.71], (c) imples (b). If I is locally generated by a regular sequence then, by [K, Theorem 135], I contains a regular sequence of length d. If $a_1,\dots,a_d$ generate I as well, consider the map $\phi=(a_1,\dots,a_d):A^d \to I$; by [K, Theorem 125] there is an upper triangular matrix T so that ϕT maps $e_1,\dots,e_d$ to a regular sequence in I. By [SeM, p. 70], $K(\phi T) \cong K(\phi)$ is exact. This shows that (b) implies both (a) and (d). As "(a) implies (b)" and "(d) implies (c)" are trivial, we are done.

Corollary (1.2). Let I be a radical ideal in a noetherian ring A. Assume that A_m is Cohen-Macaulay for every maximal ideal m containing I, and that I is locally a CI. Then the following are equivalent:

(a) I is a weak stCI.

(b) I is a strong stCI.

Proof. Clearly (a) implies (b). If I is the radical of $J=(a_1,\dots,a_d)A$, d=height(I), we have to show that I is an stCI. Locally J_m is a strong CI, since A_m is Cohen-Macaulay whenever $J \subset m$. By (1.1), J is generated by a regular sequence, so its radical I is a strong stCI.

Corollary (1.3). Let I be a height 2 ideal in a noetherian ring A. Then I is a strong CI if and only Ext(I,A) contains an exact sequence

$$\alpha: 0 \to A \xrightarrow[(u,v)]{} A^2 \xrightarrow[(x,y)]{} I \to 0.$$

Proof. Given such a sequence, we must have $(-y,x) = c(u,v)$ for some c in A. This implies that the height 2 ideal $I=(x,y)A$ is contained in cA, which means that c is a unit of A. As $(u,v) = c^{-1}(-y,x)$, the extension $c^{-1}\alpha$ is the Koszul resolution $K(x,y)$ of I. Done by (1.1).

Motivated by (1.3), we fix a height 2 ideal which is locally a (strong) CI, and consider Ext(I,A). We will adapt the notation that the extension $\alpha \in \mathrm{Ext}(I,A)$ is

$$0 \to A \xrightarrow[\theta_\alpha]{} P_\alpha \xrightarrow[\phi_\alpha]{} I \to 0$$

Proposition (1.4) (Serre [Se][Mur, p.199]). Let I be a height 2 ideal which is locally a CI. Then

(a) I/I^2 is a projective A/I-module of rank 2, and is free if I is a CI.

(b) There is a canonical isomorphism

$$\mathrm{Ext}(I,A) \cong \mathrm{Hom}(\Lambda^2(I/I^2),A)$$

and if $\Lambda^2(I/I^2) \ncong A/I$ then no P_α is a projective A-module.

(c) If $\Lambda^2(I/I^2) \cong A/I$, fix an isomorphism $\mathrm{Ext}(I,A) \cong A/I$. Then P_α is projective if and only if α is a unit of A/I.

(d) $\nu(I)$ is either $\nu(I/I^2)$ or $1+\nu(I/I^2)$.

We remark that any ideal I is locally a CI if and only if (i) I/I^2 is a projective A/I-module and (ii) $\mathrm{hd}_A(I)<\infty$; this is an old result of Ferrand [Fe] and Vasconcelos [V].

Question (1.5). If I is locally a CI, unmixed of height d, but the projective A/I-module I/I^2 is not free, then must $\nu(I) = \nu(I/I^2)$ hold? (It is easy to find examples where $\nu(I) = 1+\nu(I/I^2)$ when I/I^2 does not have constant rank, but if I is unmixed the rank of I/I^2 is the constant d.)

Lemma (1.6). Let I be a height 2 ideal, locally a CI, in a noetherian ring A. Let $\phi: P \to I$ be a map of a rank 2 projective A-module P onto I. Then

(a) $K(\phi)$ is exact.

(b) For α in $\mathrm{Ext}(I,A)$ we have $\Lambda^2(P_\alpha) \cong A$, and $K(\phi_\alpha)$ is an extension of I by A of the form $c\alpha$ for some unit c of A.

Proof (cf. [MS, (4.4)]. If we localize at a maximal ideal m containing I, $K(\phi)_m \cong K(a_1, a_2)_m$ is exact by (1.1d). If $\phi = \phi_\alpha: P_\alpha \to I$, this yields $A = \ker(\phi_\alpha) \cong \Lambda^2(P_\alpha)$. If we fix an isomorphism $A \cong \Lambda^2(P_\alpha)$, then the map $d: \Lambda^2(P_\alpha) \to P$ sends $1 \in A$ to $c\theta_\alpha(1)$ for some unit c of A. It follows from the definition of $c\alpha$ in $\mathrm{Ext}(I,A)$ that $K(\phi_\alpha) = c\alpha$.

Notation (1.7). $\underline{\underline{SP}}_n(A)$ denotes the set of all rank n projective A-modules P with $\det(P) = \Lambda^n(P) = A$. The subgroup $SK_0(A)$ of $K_0(A)$ is the set of all $n-[P]$, where n is an integer and P belongs to $\underline{\underline{SP}}_n(A)$. We can also describe $SK_0(A)$ externally as the subgroup of all $[P]-[Q]$ with $\mathrm{rank}(P) = \mathrm{rank}(Q)$ and $\det(P) = \det(Q)$.

If I is locally a CI of height 2, let $\underline{\underline{SP}}_2(A,I)$ denote the set of all P in $\underline{\underline{SP}}_2(A)$ mapping onto I. By (1.6), when A is noetherian, it is also the set of isomorphism classes of the P_α, α in the set units(A/I) . By Schanuel's lemma, the element $[A/I] = 2-[P]$ of $SK_0(A)$ is independent of the choice of P in $\underline{\underline{SP}}_2(A,I)$. Thus $I \mapsto [A/I]$ gives a well-defined set map from the set of height 2 ideals which are locally a CI to $SK_0(A)$.

Proposition (1.8) Let I be a height 2 ideal, locally a CI, in a noetherian ring A. Assume that I/I^2 is free. Then

(a) I is a strong CI if and only if I is a weak CI, i.e., if and only if $\nu(I) = 2$.

(b) I is a (strong) CI if and only if A^2 is in the set $\underline{\underline{SP}}_2(A,I)$.

(c) If $\nu(I) = 2$, then $[A/I] = 0$ in $SK_0(A)$.

(d) If $\nu(I/I^2) = 2$, then $\nu(I)$ is either 2 or 3.

Proof. Parts (a), (b), (c) follow from (1.1) and (1.3); part (d) is just (1.4d). If A^2 is in $\underline{\underline{SP}}_2(A,I)$, then $[A/I] = 2-[A^2] = 0$, whence part (c). Note that this proposition is trivial if either A is Cohen-Macaulay or if I is a radical ideal.

Theorem (1.9) (Murthy-Swan [MS, (4.5)]). Let A be a finitely generated 2-dimensional k-algebra, where k is an algebraically closed field. Let I be a height 2 ideal of A which is locally a CI. Then $[A/I] = 0$ iff I is a CI; if $[A/I] \neq 0$ then $\nu(I) = 3$.

Proof. By Theorem 1 of [MS], the map $\underline{\underline{SP}}_2(A) \to SK_0(A)$ is an isomorphism. If $[A/I] = 0 = 2-[A^2]$, we must therefore have A^2 in $\underline{\underline{SP}}_2(A,I)$. Since A/I is semilocal, the projective A/I-module I/I^2 is free. The result now follows from (1.8).

Theorem (1.10). Let A be a finitely generated 2-dimensional k-algebra, where k is an algebraically closed field. Let m be a height 2 maximal ideal with A_m regular. Then the following are equivalent:

(a) m is an stCI.

(b) m is a weak stCI.

(c) $[A/m]$ is a torsion element of $SK_0(A)$.

Proof. Corollary (1.2) shows that (a) and (b) are equivalent. If I is an m-primary CI ideal, then by (1.8c) we have $[A/I] = 0$. By devissage, we also have $[A/I] = n[A/m]$, where $n = \text{length}(A/I)$. This shows that (a) implies (c).

Now suppose that $n[A/m] = 0$ in $SK_0(A)$. Let $a_1, a_2 \in A$ generate mA_m, and let I be the m-primary component of the ideal $(a_1, a_2^n)A$. Then I is locally a CI and $\text{length}(A/I) = n$. Hence $[A/I] = n[A/m] = 0$ in $SK_0(A)$. If k is algebraically closed, (1.9) implies that I is a CI, so that (c) implies (a).

Remark (1.11). If we replace m by $m_1 \cap \cdots \cap m_r = I$, then the above goes through, *mutatis mutandis*, to show that the following are equivalent:

(a) I is an stCI.

(b) I is a weak stCI.

(c) There are integers $n_1, \ldots, n_r \geq 1$ with $\Sigma n_i[A/m_i] = 0$ in $SK_0(A)$.

§2. Unstable Theory

The goal of this section is to describe the situation when A is not finitely generated over an algebraically closed field. For example, let A be the coordinate ring of a curve over $\underline{\mathbb{Z}}$, a surface over $\underline{\mathbb{R}}$, or a surface over $\underline{\mathbb{F}}_p$. We associate to every regular maximal ideal m a unique projective module $\chi(m)$, and show in (2.3), (2.5) that m is a CI point if and only if $\chi(m)$ is free. Several of the results in this section have also been obtained independently by M. Boratynski [BorO].

Recall from (1.7) that the set $\underline{SP}_2(A,I) = \{$all P in $SP_2(A,I)$ mapping onto I$\}$ is a quotient set of the group units(A/I) of units of A/I. The set $\underline{SP}_2(A,I)$ is generally much smaller than units(A/I). In Theorem (1.9) for example, $\underline{SP}_2(A,I)$ was a one-element set and units(A/I) was infinite. The following result gives a much better upper bound on $\underline{SP}_2(A,I)$; it has also been obtained by Boratynski [BorO].

Proposition (2.1). Assume $\mathrm{Ext}(I,A) \cong A/I$. Then $P_\alpha \cong P_{\alpha\beta^2 c}$ for α,β units of A/I and for c a unit of A. Consequently, there is an onto set map

$$G(I) = \frac{\text{units}(A/I)}{\text{units}^2(A/I) \cdot \text{units}(A)} \twoheadrightarrow SP_2(A,I).$$

Proof. Recall that for $\alpha \in A/I$, $c \in A$, the extension $c\alpha$ is constructed via pushout (qv. [Mac, p. 66]):

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, dashed, "\pi_\alpha"'] \arrow[d, dashed, "c"'] & P_\alpha \arrow[r, dashed] \arrow[d] & I \arrow[r, dashed] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, dashed] & P_{c\alpha} \arrow[r, dashed] & I \arrow[r, dashed] & 0.
\end{tikzcd}$$

If c is a unit of A, then $P_{c\alpha} \cong P_\alpha$ is immediate. In general, $P_{c\alpha} \cong P_\alpha \oplus A/(\pi_\alpha,-c)A$. It follows from Theorem 4 of [MS] that $P_{c\alpha} \cong P_\alpha$ if $c=b^2$. Thus given $b \in A$ mapping to $\beta \in A/I$, we have $P_{\alpha\beta^2 c} = P_{\alpha b^2 c} \cong P_{\alpha c} \cong P_\alpha$.

Examples (2.2). Let $A = \underline{\underline{R}}[x,y,z]/(x^2+y^2+z^2=1)$.

(a) If I is primary for a maximal ideal of A it is easy to see that $G(I)=1$. If $\nu(I/I^2)=2$, so that I is locally a CI, there is a unique projective A-module mapping onto I. If this projective is not free, then $\nu(I)=3$.

For example, let $P = \{(a,b,c) \in A^3 : ax+by+cz=0\}$. It is well known that P is not free. In [Gab, p. 139], Gabel gives a map of P onto a height 2 primary ideal I with $\nu(I/I^2)=2$. It follows that $\nu(I)=3$. This is the same as Boratyński's proof, cited in [Gab].)

(b) On the other hand, projection onto the first coordinate of P gives a surjection $P \to I=(x,y)A$ whose kernel is generated by $(0,z,-y) \in P$. Since $G(I)$ is a group with two elements, we have $\underline{\underline{SP}}_2(A,I) = \{A^2, P\}$.

(c) The following example, due to van der Kallen, shows that the group $G(I)$ does not act naturally on the set $\underline{\underline{SP}}_2(A,I)$. Let $I = (x^2+xy,z)A$ and let the Koszul complex $K(x^2+xy,z)$ correspond to 1 in A/I, so that $P_1 = A^2$. $G(I)$ is the group

$(\underline{\underline{Z}}/2)\times(\underline{\underline{Z}}/2)\times(\underline{\underline{Z}}/2)$, generated by y, $2x+y=\alpha$, and $2x+1=\beta$. The projective modules P_y and P_α are defined by the unimodular rows

$$(-y,x^2+xy,z) \sim (0,1,0) \text{ and}$$
$$(-\ \alpha,x^2+xy,z) \sim (-\alpha,x^2,z) \sim (0,1,0)$$

respectively (the last equivalence is because one entry is a square). Thus P_1,P_y,P_α are all free. On the other hand, $P_{\alpha y} \cong P_{-\alpha y}$ is defined by the unimodular row

$$(2xy+y^2,x^2+xy,z).$$

$P_{\alpha y}$ determines a vector bundle ξ on S^2 which is given by the clutching function

$$f(x,y) = (2xy+y^2,x^2+xy)$$

on the circle $x^2+y^2=1$, $z=0$. Now ξ is orientable; under the isomorphism

$$c_1:\{SO_2\text{-bundles on } S^2\} \cong \pi_1(SO_2) = \underline{\underline{Z}}$$

(q.v. [H, p. 86]), the bundle ξ corresponds to $c_1(\xi) = \pm 2$. As ξ is not free, neither is $P_{\alpha y}$. If $G(I)$ acted on $\underline{\underline{SP}}_2(A,I)$, though, we would have had $P_{\alpha y} \cong P_y \cong A^2$, so $G(I)$ cannot act.

Incidentally, the other four projective modules P_β, $P_{\alpha\beta}$, $P_{\beta y}$ and $P_{\alpha\beta y}$ all give rise to the tangent bundle τ on S^2, which has $c_1(\tau) = 1$. I do not know if they are distinct A-modules or if they are isomorphic.

Theorem (2.3). Let I be any height 2 local CI in a 2-dimensional noetherian ring A. Assume that, for every maximal ideal m containing I, one of the following holds:

(a) A/m is algebraically closed of characteristic $\neq 2$,

(b) A/m is perfect of characteristic 2, and $I_m = mA_m$,

(c) A contains a field k with $k \to A/m$ an isomorphism, and: if char(k)=2 then $I_m = mA_m$.

Assume moreover that (a) or (b) holds for all but at most one maximal ideal m containing I. Then there is a unique rank 2 projective module P mapping onto I, with $\Lambda^2 P = A$. If $P \neq A^2$ then $\nu(I) = 3$; if $P = A^2$ then I is a CI.

Proof. The primary decomposition of I is $I = \bigcap q_i$, where q_i is primary for m_i, and $A/I = \Pi A/q_i$. Thus units(A/I) is an extension of Π units(A/m_i) by $\Pi(1+m_i/q_i)$. The latter group is 2-divisible, since we have assumed that $q_i \neq m_i$ only if char(A/m) $\neq 2$. If A/m is algebraically closed or perfect of characteristic 2, units(A/m) is also 2-divisible. Hence units(A/I)/units^2(A/I) is zero if (c) does not occur, and units(k)/units^2(k) if (c) does occur. In the latter case, these units come from A. In any event, G(I) = 1, so that $\underline{SP}_2(A,I)$ is a 1-element set by (2.1). The result now follows from (1.8).

Corollary (2.4). Let A be a finitely generated k-algebra of dimension 2, where k is a field. Let Reg(A,k) denote the set of maximal ideals m for which A_m is a regular ring and $k \to A/m$ is an isomorphism. Then there is a well-defined set map

$$\chi:\mathrm{Reg}(A,k) \to \underline{SP}_2(A) = \left\{\begin{matrix}\text{rank 2 projective A-}\\ \text{modules P with } \Lambda^2 P = A\end{matrix}\right\} .$$

If $\chi(m) \neq A^2$, then $\nu(m) = 3$; if $\chi(m) = A^2$, then m is a CI.

Proof. The map χ is well-defined by case (c) of (2.3).

Corollary (2.5). Let A be a finitely generated k-algebra of dimension 2, where k is either $\underline{R}$, perfect of characteristic 2, or algebraically closed. Let Reg(A) denote the set of maximal ideals m of A with A_m regular. Then there is a well-defined set map

$$\chi:\mathrm{Reg}(A) \to \underline{SP}_2(A).$$

If $\chi(m) \neq A^2$, then $\nu(m) = 3$; if $\chi(m) = A^2$, then m is a CI.

Proof. In each case, A/m is a finite extension of k, and these are all of type (a), (b), or (c) in Theorem (2.3).

Example (2.6). Let $A = \mathbb{R}[x,y,z]/(y^2+z^2=x^3-x)$. Spec($A$) is the surface of revolution obtained by rotating an elliptic curve about the x-axis. Projecting onto the x-axis, we see that Spec($A\otimes\mathbb{C}$) is birationally $\mathbb{P}^1\times\mathbb{P}^1$, hence that $SK_0(A\otimes\mathbb{C}) = 0$ by a result of Murthy (or by Theorem 3 of [MS]). By Theorem 2 of [MS], every maximal ideal of $A\otimes\mathbb{C}$ is a CI. Similarly, every complex maximal ideal m of A (i.e., $A/m = \mathbb{C}$) is a CI: every complex maximal ideal of $A/(x-\alpha) = \mathbb{R}[y,z]/(y^2+z^2=\text{const.})$ and $A/(x^2+bx+c) = \mathbb{C}[u,v]/(uv=\text{const.})$ is known to be principal.

The real points of Spec(A) consist in a bounded component, topologically S^2, and an unbounded component which resembles a bell. We know that $\nu(m) = 3$ for the real points m on the bounded component by [GW, p. 215]. In fact $[A/m] \neq 0$ in $SK_0(A)$, as the map $SK_0(A) \to KO(S^2) = \mathbb{Z}/2$ maps $[A/m]$ to the nonzero element.

On the other hand, if m is a real point on the bell (the unbounded component), I do not know if $\nu(m)$ is 2 or 3. Since there is a line in $\mathbb{A}^3$ meeting the real points of Spec(A) exactly once, transversely, we have $[A/m] \neq 0$ in $SK_0(A)$. Incidentally, this shows that $SK_0(S^2) = \mathbb{Z}/2$ by [MS, (4.2)]. A is also a UFD, so by (2.5) there is a unique rank 2 projective A-module P mapping onto m. We have $\nu(m) = 2$ iff $P = A^2$, but I do not know how to decide this. In the spirit of the CI conjecture (0.1), my guess is that there are countably many circles x=constant on which $\nu(m) = 2$, and that $\nu(m) = 3$ on the rest of the bell.

§3. Nonsingular Affine Surfaces

In this section, we use the Chow groups of zero-cycles on affine surfaces to prove Theorem (3.1). Recall that a subset of a topological space is called σ-closed if it is the countable union of closed subsets. For example, a σ-closed subset of Spec(A) has the form $\bigcup V(I_n)$ for a sequence of ideals I_n of A.

Theorem (3.1). Let A be a 2-dimensional regular ring, finitely generated over an algebraically closed field. Then

(a) The set of CI points of Max(A) is σ-closed. (Recall that the weak and strong CI notions of complete intersection points agree.)

(b) The set of stCI points of Max(A) is σ-closed. (Recall that the weak and strong stCI notions of set-theoretic complete intersection agree by (1.2).)

The idea of the proof is this: map Max(A) into the Chow group $A^2(U) = A_0(U)$ of zero-cycles on $U = \mathrm{Spec}(A)$. Those points mapping to zero are the CI points, and those points mapping to torsion elements are the stCI points. These sets are σ-closed by results of Roitman.

We first recall the definition of the Chow group $A^d(X)$ of codimension d cocycles from [CF] (where the notation $W_d(X)$ was used). This is a variant of the Chow homology groups defined by Fulton in [Fu]. For a noetherian scheme X, let X^d denote the set of all points x of X of codimension d, and let k(x) denote the residue field at x. Define the "cycle map"

$$(3.2)\quad \mathrm{cycle}^d(X): \coprod_{x \in X^{d-1}} k(x)^* \longrightarrow \coprod_{y \in X^d} \underline{\mathbb{Z}} \text{ (on generator } [y])$$

as follows. If $y \notin V(x)$, the component $k(x)^* \to \underline{\mathbb{Z}}[y]$ is zero. If $y \in V(x)$, $R = \mathcal{O}_y/p_x$ is a 1-dimensional local domain with k(x) as quotient field; the map $k(x)^* \to \underline{\mathbb{Z}}[y]$ sends $r/s \in k(x)^*$ $(r,s \in R)$ to

$$\left(\mathrm{length}(R/r)-\mathrm{length}(R/s)\right)[y].$$

The target $\coprod \underline{\mathbb{Z}}$ of "cycle^d" is called the group $Z^d(X)$ of <u>codimension d cocycles</u>. The cokernel of "cycle^d" is the <u>Chow group</u> $A^d(X)$. If X is unmixed, $A^d(X)$ agrees with Fulton's $A_{n-d}(X)$, where $n = \dim(X)$.

<u>Lemma (3.3)</u>(folklore). Let X be a nonsingular projective variety of dimension n over an algebraically closed field. Let a,b be codimension n cocycles ("zero-cycles"). The following are equivalent:

(a) $\mathrm{Cycle}^n(a) = \mathrm{cycle}^n(b)$ in $A^n(X)$.

(b) There is a codimension n cocycle T on $X \times \underline{P}^1$ such that $i_0^*(T)=a$, $i_\infty^*(T)=b$. Here $i_y: X \to X \times \underline{P}^1$ is the map $x \mapsto (x,y)$.

(c) There exists an integer i and a morphism $T:P^1 \to S^iX\times S^iX$ for which $Z(T(0))=a$ and $Z(T(\infty)=b$. Here S^iX is the symmetric product, $S^iX = X^i/\Sigma_i$, and $Z:S^iX\times S^jX \to Z^nX$ is the map $Z(a_1,a_2) = a_1-a_2$.

Proof. The equivalence of (a) and (b) is given on p. 134 of [MS], and the equivalence of (b) and (c) is proven on p. 557 of [R1] (and is implicit in [Mum] at least).

Lemma (3.4) (Grothendieck). Assume that either X is Spec(A), A a 2-dimensional regular ring, or that X is a nonsingular quasi-projective surface over an algebraically closed field. Then $A^2(X) \cong SK_0(X)$ and $A^1(X) \cong \mathrm{Pic}(X)$.

Proof. If X is quasiprojective, the result is proven by Grothendieck in [G225, p. 64]. (Recall that $SK_0(X)$ is the kernel of $(\mathrm{rank}, \det):K_0(X) \to H^0(X;\underline{Z})\oplus\mathrm{Pic}(X)$.) If X is Spec(A), the result is proven as (4.2)(c,d) of [MS]. (The assumption that A be finitely generated over an algebraically closed field was only used to establish (4.2e) of [MS].)

Scholium(3.5). If X is any noetherian scheme, let $\underline{M}^d$ denote the abelian category of coherent sheaves on X whose support has codimension $\geq$ d. Then

$$K_0(\underline{M}^{d-1}/\underline{M}^{d+1}) \cong A^d(X) \oplus \coprod_{x\in X^{d-1}} \underline{Z} .$$

This is proven in §2 of [CF], in §7.5 of [Q] and as (5.6) below. The filtration of $\underline{M}$ gives a filtration for $K_0\underline{M}$ whose associated graded ring $Gr(K_0\underline{M})$ is a quotient of $\oplus A^d(X)$. This follows from either [G225] or [Q, §7.5]. In particular, if X is nonsingular and 2-dimensional, then $A^1(X)=Pic(X)$ and there is a surjection $A^2(X) \to SK_0(X)$.

Theorem (3.6)(Roitman). Let X be a nonsingular projective surface over an algebraically closed field k. Let Y be a closed subvariety of X and set U = X-Y. Let

$$\gamma:X^{m+n}(k) \xrightarrow[\text{cycle}]{} A^2(X) \to A^2(U)$$

be the map $\gamma(a_1,\dots,a_m,b_1,\dots,b_n) = \Sigma a_i-\Sigma b_j$. Then $\gamma^{-1}(0)$ is a σ-closed subset of $X^{m+n}(k)$ containing $Y^{m+n}(k)$.

Proof. Consider the sets

$$W_{r,s} \subset Y^{r+s}(k)\times X^{m+n}(k),$$
$$W_{r,s} = \{(y_1,y_2,x_1,x_2):\text{cycle}(x_1-x_2) = \text{cycle}(y_1-y_2) \text{ in } A^2(X)\}.$$

By Roitman's Lemma 3 of [R2, p.577], each $W_{r,s}$ is σ-closed (the assumption that char(k) = 0 is not used). Hence the projection

$$W = \{(x_1,x_2)\in X^{m+n}(k):\text{cycle}(x_1-x_2) = \text{cycle}(y_1-y_2) \text{ in } A^2(X) \text{ for some } (y_1,y_2)\in Y^r\times Y^s\}$$

is a σ-closed subset of $X^{m+n}(k)$. On the other hand, there is an exact sequence

$$\coprod_{\substack{\text{closed}\\ y\in Y}} \underline{Z} \to A(X) \to A(U) \to 0$$

(see for example [MS, p.138]). It follows that $W = \gamma^{-1}(0)$, proving the theorem.

Proof of Theorem (3.1). (a) Choose a nonsingular projective variety X and a closed subvariety Y such that Spec(A)=U=X-Y. The set map

$$\gamma_{10}|U : U(k) \subset X(k) \xrightarrow{\gamma} A^2(U) \cong SK_0(A)$$

sends $x \in \mathrm{Spec}(A)$ to the cocycle [x] of $A^2(U)$, which maps to $[A/m_x]$ in $SK_0(A)$. Hence $\gamma_{10}|U$ is the map of §1. By Theorem (1.9), $\nu(m_x)=2$ if and only if $\gamma_{10}|U$ sends x to zero. By Theorem (3.6), with m=1 and n=0,

$$\{\text{CI points}\} = \{x \in U(k) : \nu(m_x)=2\} = W - Y(k)$$

is a σ-closed subset of U(k). This proves part (a).

To prove part (b), we invoke (1.10). For each n we have to show that $\{x \in U(k) : n[A/m_x]=0 \text{ in } SK_0(A)\}$ is a σ-closed set. The set map

$$\gamma_{n0}|U^n\Delta : U(k) \xrightarrow{\Delta} U^n(k) \subset X^n(k) \xrightarrow{\gamma} A^2(X) \cong SK_0(A)$$

sends x to $n[A/m_x]$. By Theorem (3.6)

$$\{x \in U^n(k) : \gamma_{n0}|U(x)=0 \text{ in } SK_0(A)\} = W \cap U^n(k)$$

is a σ-closed subset of $U^n(k)$. Hence

$$\{\text{stCI points} = \{x \in U(k) : n[A/m_x] = \gamma_{n0}|U^n\Delta(x)=0\}$$

is a σ-closed subset of U(k). Done.

§4. Higher Dimensions

In this section we give the following result of R. Varley, which partially answers Conjectures (0.1) and (0.2). Recall (e.g., from [Mat]) that a locally closed set is the intersection of an open and a closed set, while a constructible set is a finite union of locally closed sets. Thus the properties "σ-locally closed" and "σ-constructible" of a set S are the same; they mean that S is the countable union of locally closed sets.

Theorem (4.1) (R. Varley). Let X be a nonsingular projective variety over an algebraically closed field, and let A be the coordinate ring of an affine open subset. Then

(a) The set of CI points of Max(A) is σ-constructible.

(b The set of stCI points of Max(A) is σ-constructible.

Varley's proof uses the Hilbert schemes $\mathrm{Hilb}^f(X)$, constructed in [G221]. Choose a finitely generated graded k-algebra S with X=Proj(S), so that closed subschemes of X have the form Proj(S/I). If $f \in \underline{Q}[t]$, $\mathrm{Hilb}^f(X)$ is a subvariety of $G(f(N),S_N)$, the Grassmannian variety of codimension f(N) subspaces V of S_N for N>>0 ; Y=Proj(S/I) corresponds to I_N and V corresponds to Proj(S/SV).

Lemma (4.2). Given polynomials $f_1,\dots,f_d,g\in\underline{\underline{Q}}[t]$, let Z be the subset of $\mathrm{Hilb}^{f_1}(X)\times\cdots\times\mathrm{Hilb}^{f_d}(X)$ on which the function $\cap: Z\to\mathrm{Hilb}^{g}(X)$ is defined. This function takes the point $(Y_1,\dots,Y_d)$, $Y_i=\mathrm{Proj}(S/I_i)$, to the point $Y_1\cap\cdots\cap Y_d=\mathrm{Proj}(S/(I_1+\cdots+I_d))$. Then the set Z is locally closed.

Proof. The map

$$G(f_1(N),S_N)\times\cdots\times G(f_d(N),S_N)\to\underline{\underline{N}}$$
$$(V_1.\dots,V_d)\mapsto\mathrm{codim}(V_1+\cdots+V_d,S_N)$$

is lower semicontinuous: $Z_1=\{(V_1,\dots,V_d):\mathrm{codim}(V_1+\dots+V_d,S_N)=g(N)\}$ is locally closed. Let $\Sigma:Z_1\to G(g(N),S_N)$ be the map $\Sigma(V_1,\dots,V_d)=V_1+\cdots+V_N$. Then Σ is algebraic, and $Z=\Sigma^{-1}(\mathrm{Hilb}^g(X))\cap\Pi\mathrm{Hilb}^{f_i}(X)$ is locally closed.

Lemma (4.3). Let U=Spec(A) be an affine open subset of a nonsingular variety X, and let $f\in\underline{\underline{Q}}[t]$ have degree dim(X)-1. Then the set of all divisors D in $\mathrm{Hilb}^f(X)$ which are principal on U is a σ-closed subset of $\mathrm{Hilb}^f(X)$.

Proof. The kernel K of $\mathrm{Pic}(X)\to\mathrm{Pic}(U)$ is the countable subgroup generated by the classes of the components of X-U. That is, K is σ-closed. The map $[\ ]:\mathrm{Hilb}^f(X)\to\mathrm{Pic}(X)$ is a morphism of algebraic varieties, so $[\ ]^{-1}(K)$ is σ-closed. But $[\ ]^{-1}(K)$ is the set of all D with $[D]\in K$, i.e., with D principal on U.

Proof of Theorem (4.1). We use the notation developed above. Elements a of A correspond to divisors D of X which are principal on U=Spec(A). This is a one-to-countably-many correspondence. If $I=(a_1,\dots,a_d)A$ is an ideal of A, then there are divisors $D_1,\dots,D_d$ on X, principal on U, with $U\cap(D_1\cap\cdots\cap D_d)=\mathrm{Spec}(A/I)$. If $\dim(A/I)=0$ then the subscheme $D_1\cap\cdots\cap D_d$ is the disjoint union of the closed subschemes Spec(A/I) and $Y=(X-U)\cap(D_1\cap\cdots\cap D_d)$, so the Hilbert polynomial of $D_1\cap\cdots\cap D_d$ is the sum of the length of A/I and the Hilbert polynomial of Y.

Fix $d=\dim(X)$, $n\in\underline{N}$, $g\in\underline{Q}[t]$. Fix polynomials $f_1,\dots,f_d$ in $\underline{Q}[t]$ of degree d-1. We define the subset $Z=Z(f_1,\dots,f_d,g,n)$ of $\mathrm{Hilb}^{f_1}(X)\times\cdots\times\mathrm{Hilb}^{f_d}(X)$ to be the set of those d-tuples $(D_1,\dots,D_d)$ of divisors, each principal on U, such that the Hilbert polynomials of $D_1\cap\cdots\cap D_d$ and $(X-U)\cap(D_1\cap\cdots\cap D_d)$ are g and g-n, respectively. By (4.2) and (4.3), the set Z is locally closed. The map $Z\to\mathrm{Hilb}^n(U)$ given by $(D_1,\cdots,D_d)\mapsto U\cap(D_1\cap\cdots\cap D_d)$ is well defined and algebraic by the definition of Z, and its image is constructible. ($\mathrm{Hilb}^n(U)$ is the open subvariety of $\mathrm{Hilb}^n(X)$ of subschemes of X with support in U, i.e., of those Spec(A/I) with length(A/I)=n.)

In case (a) we have n=1 and $\mathrm{Hilb}^1(U)=U$. A maximal ideal x of U is generated by d elements just in case x lies in the image of some $Z(f_1,\dots,f_d,g,1)$. Hence the set of weak CI(=strong CI) points is the union of the constructible images of the countably many Z's.

In case (b), consider the map $\mathrm{Hilb}^n(U) \to S^nU$ taking $Y=\mathrm{Spec}(A/I)$ to the support of Y, i.e., to the sum of the primes over I (with the multiplicity of x equal to $\mathrm{length}(A_x/I_x)$). Here S^nU denotes the n^{th} symmetric product $(U\times\cdots\times U)/\Sigma_n$. This map is algebraic, and is defined on p. 26 of [G221]. The image $\overline{Z}$ of $Z(f_1,\ldots,f_d,g,n) \to \mathrm{Hilb}^n(U) \to S^nU$ is therefore constructible. The intersection $\overline{Z}\cap\Delta U$ is a constructible subset of U; it consists of those primes x in U which are radicals of ideals $(a_1,\ldots,a_d)A=I$, with $\mathrm{length}(A/I)=n$ and $\mathrm{Spec}(A/I)=U\cap(D_1\cap\cdots\cap D_d)$ for some $(D_1,\cdots,D_d)\in Z(f_1,\ldots,f_d,g,n)$. The subset of stCI points of U(k) is the union of the sets $\overline{Z}\cap\Delta U$ over all of the (countably many) choices of $f_1,\ldots f_d,g$, and n. (Note that strong stCI=weak stCI.) This shows that the set of stCI points is σ-closed in $U(k)=\mathrm{Max}(A)$.

§5. Relative Chow Groups

When A is 2-dimensional but not regular, the techniques of §3 and §4 do not apply, because the absolute Chow group is not well-behaved. The results (1.9)(1.10) still hold: the inverse image of zero under the map $\mathrm{Reg}(A) \to SK_0(A)$ is the set of CI maximal ideals of A, and the inverse image of the torsion subgroup of $SK_0(A)$ is the set of stCI maximal ideals of A.

In this section we introduce relative Chow groups $A^d(X,Y)$ for closed subschemes Y of noetherian schemes X, and factor the above map as

$$\mathrm{Reg}(A) \to A^2\big(\mathrm{Spec}(A), \mathrm{Sing}(A)\big) \to SK_0(A).$$

In the introduction we conjectured that the fibers of the first map were σ-closed. The purpose of this section is to analyze the second map. We show (following Pedrini [P]) that this map is an isomorphism when A is normal, and describe the kernel in the general case. Our main result is the following, in which we have written $A^d(A,I)$ for $A^d\big(\mathrm{Spec}(A), \mathrm{Spec}(A/I)\big)$.

Theorem (5.1). Let A be a 2-dimensional noetherian ring, and let I be an ideal of A properly containing every minimal prime ideal of A. Suppose that $\mathrm{Sing}(A) \subseteq V(I)$ (i.e., that A_p is regular whenever $I \not\subseteq p$). Then there is an exact sequence

$$SK_1(A) \to SK_1(A/I) \to A^2(A,I) \to SK_0(A) \to 0.$$

Corollary (5.2) (Pedrini [P, Theorem 3]). Let A be a 2-dimensional noetherian domain with only finitely many singular points. Then

$$SK_0(A) \cong A^2(\mathrm{Spec}(A),\ \mathrm{Sing}(A)).$$

Proof of (5.2). We take I to be the intersection of the singular primes. As $\dim(A/I)=0$ we have $SK_1(A/I)=0$.

Remark. If Spec(A) is 'nearly' rational in characteristic p, it seems plausible that $A^2(\mathrm{Spec}(A),\ \mathrm{Sing}(A))$ should be a p-group. By (5.1), every point of Max(A) would then be a set-theoretic complete intersection.

We also recover the following result, which is due to Murthy (using another method). It shows that blowing up singularities (or monoidal transformations) preserves some information about SK_0.

Corollary (5.3) (Murthy). Let $A \to B$ be a birational map of 2-dimensional k-algebras, both localizations of finitely generated algebras over the field k. Then $f^*: SK_0(A) \to SK_0(B)$ is onto.

Proof. 'Birational' means that some Spec(B)-V(J) is regular and isomorphic to some Spec(A)-V(I), for $I \subset A$, $J \subset B$. By (5.9) below, the map $A^2(A,I) \to A^2(B,J)$ is onto. The corollary now follows from (5.1).

Corollary (5.4). Let A be a 2-dimensional integrally closed domain, finitely generated over an algebraically closed field k. Then

(a) The set of CI points in Reg(A)=Max(A)-Sing(A) is the inverse image of zero under the restricted cycle map $[\ ]:\mathrm{Reg}:(A) \to A^2(A,\mathrm{Sing}(A))$. Hence Conjecture (0.4) implies the CI conjecture (0.1).

(b) The set of stCI points in Reg(A) is the set of all m with [m] a torsion element of $A^2(A,\mathrm{Sing}(A))$. If the torsion subgroup of $A^2(A,\mathrm{Sing}(A))$ is countable, then Conjecture (0.4) implies the stCI conjecture (0.2).

Proof. Sing(A) is a finite set of points, so $A^2(A,\mathrm{Sing}(A))=SK_0(A)$, and (5.4) is just a restatement of (1.9) and (1.10).

Definition (5.5). Let Y be a closed subscheme of a noetherian scheme X. The cycle map (3.2) induces the restricted cycle map

$$\mathrm{cycle}^d(X,Y): \coprod_{\substack{x \in X^{d-1} \\ V(x)\cap Y=0}} k(x)^* \dashrightarrow \coprod_{\substack{y \in X^{d} \\ V(y)\cap Y=0}} \mathbb{Z}.$$

This map is well-defined because $k(x)^* \to \underline{\underline{Z}}[y]$ is nonzero only when $V(y) \subseteq V(x)$. The cokernel of $\mathrm{cycle}^d(X,Y)$ is denoted $A^d(X,Y)$. Note that $A^d(X,\phi)=A^d(X)$, $A^d(X,X)=0$, and that $A^d(X,Y)=\mathrm{colim}\{A^d(V):V$ is a closed subscheme of X disjoint from $V\}$.

We now shift directions slightly, in order to give an alternative definition of the relative groups $A^d(X,Y)$. By $\underline{\underline{M}}_Y(X)$, we will mean the category of coherent sheaves $\mathcal{F}$ on X with $\mathrm{Supp}(\mathcal{F})$ disjoint from Y. By $\underline{\underline{M}}_Y^d(X)$, we will mean the subcategory of sheaves in $\underline{\underline{M}}_Y(X)$ whose support is of codimension at least d. If $d < e$, we will write $\underline{\underline{M}}_Y^{d/e}(X)$ for the quotient of the abelian category $\underline{\underline{M}}_Y^d(X)$ by its Serre subcategory $\underline{\underline{M}}_Y^e(X)$.

<u>Proposition (5.6)</u>(cf. [CF,§2]). There is a canonical decomposition

$$K_0(\underline{\underline{M}}_Y^{d-1/d+1}(X)) = A^d(X,Y) \oplus \coprod_{\substack{x \in X^{d-1} \\ V(x) \cap Y=0}} \underline{\underline{Z}}.$$

Proof. As in [B, VIII.5] [Q, p. 131] there is an equivalence of categories

$$\underline{M}_Y^{d/d+1}(X) \cong \coprod_{\substack{x \in X^d \\ V(x) \in Y=0}} \bigcup_n \underline{M}(\mathcal{O}_{X,x}/m_x^n)$$

and therefore isomorphisms

$$K_*(\underline{M}_Y^{d/d+1}(X)) \cong \coprod_{\substack{x \in X^d \\ V(x) \cap Y=0}} K_*(k(x)).$$

We now consider the sequence $\underline{M}_Y^{d/d+1} \to \underline{M}_Y^{d-1/d+1} \to \underline{M}_Y^{d-1/d}$ of abelian categories. From either [B, p. 427] or [Q, p. 113] there is an exact sequence of abelian groups

$$\coprod_{\substack{x \in X^{d-1} \\ V(x) \cap Y=0}} k(x)^* \xrightarrow{\partial} \coprod_{\substack{y \in X^d \\ V(y) \cap Y=0}} \underline{Z} \to K_0(\underline{M}_Y^{d-1/d+1}) \to \coprod_{\substack{x \in X^{d-1} \\ V(x) \cap Y=0}} \underline{Z} \to 0.$$

We can split the right-hand map by sending $[x] \in \coprod \underline{Z}$ to the class $[\mathcal{O}_{V(x)}] \in K_0$. The left-hand map is the restricted cycle map $\mathrm{cycle}^d(X,Y)$, by the definition of ∂ on p. 427 of [B] (see also [Q, p. 135]), so its cokernel is $A^d(X,Y)$. Done.

Remark (5.7) (cf. [CF, p. 235], [Q, p. 131], [Alb, §2]). The category $\underline{M}_Y(X)$ is filtered by the $\underline{M}_Y^d$, giving a filtration on $K_*(\underline{M}_Y(X))$. The associated spectral sequence

$$E_1^{pq} = \coprod_{\substack{x \in X^p \\ V(x) \cap Y = 0}} K_{-p-q}k(x) \Rightarrow K_{-p-q}\underline{\underline{M}}_Y(X)$$

has $E_2^{p,-p} = A^p(X,Y)$. Consequently, the associated graded group of $K_0\underline{\underline{M}}_Y(X)$ is a homomorphic image of $\coprod A^d(X,Y)$. In [Alb,(5.6)], it is shown that $A^d(X,x_0) \to Gr_d\big(K_0\underline{\underline{M}}_{x_0}(X)\big)$ is an isomorphism modulo torsion, at least when X is an irreducible quasiprojective variety with $Sing(X)=\{x_0\}$

<u>Corollary (5.8)</u>([CF,(5.2)][Fu,(1.9)]). Let $f:X_2 \to X_1$ be a morphism, and let $Y_i \subset X_i$ be closed subschemes with $f(Y_2) \subseteq Y_1$. If the induced morphism $X_2 - f^{-1}(Y_1) \to X_1 - Y_1$ is flat, then there is a "Gysin map"

$$f^*: A^d(X_1,Y_1) \to A^d(X_2,Y_2).$$

<u>Proof</u>. If $\mathcal{F}$ is a coherent sheaf on X_1 whose support's closure is in $X_1 - Y_1$, $f^*\mathcal{F}$ is a coherent sheaf on X_2 whose support's closure is in $X_2 - f^{-1}(Y_1)$. As f is flat on the support, the codimension in X_2 of $supp(f^*\mathcal{F})$ is at least the codimension in X_1 of $supp(\mathcal{F})$. We therefore have a commutative diagram of categories

$$\begin{tikzcd}
\underline{\underline{M}}_{Y_1}^{d/d+1}(X_1) \arrow[r] \arrow[d] & \underline{\underline{M}}_{Y_1}^{d-1/d+1}(X_1) \arrow[d] \\
\underline{\underline{M}}_{Y_2}^{d/d+1}(X_2) \arrow[r] & \underline{\underline{M}}_{Y_2}^{d-1/d+1}(X_2).
\end{tikzcd}$$

Applying K_0 and invoking (5.6) gives the result.

<u>Detail</u>. More explicitly, let $\mathcal{F}$ be a coherent sheaf of $\underline{M}^d_Y(X)$. We define a cycle $Z^d(\mathcal{F})$ in $\underline{\coprod}\,\underline{Z} = K_0\underline{M}_Y^{d/d+1}(X)$ as in [Fu,(1.1)], whose component in the x-summand is the length of $\mathcal{F}_x$ as an Artinian module over $\mathcal{O}_{X,x}$. We then have

$$f^*[V(x)] = \ulcorner Z^d(f^{-1}(\mathcal{O}_{V(x)}))\urcorner .$$

Note that $A^d(X,Y)$ is <u>not</u> a function of $(X-Y)$ alone: the requirement that $V(x)\cap Y=0$ depends on Y. However, we do have:

<u>Lemma (5.9)</u>. Let $X_2 \to X_1$ induce an isomorphism $(X_2-Y_2) \cong (X_1-Y_1)$ Set $n=\dim(X_i-Y_i)$. Then

(i) $A^n(X_1,Y_1) \to A^n(X_2,Y_2)$ is onto

(ii) If $X_2 \to X_1$ is a closed map, then $A^d(X_1,Y_1) \cong A^d(X_2,Y_2)$ for every d.

<u>Proof</u>. If $X_2 \to X_1$ is closed and $(X_2-Y_2)=(X_1-Y_1)$, then the cycle maps are identical, so (ii) holds. At any rate, $A^n(X_i,Y_i)$ is a quotient group of the free abelian group on the set of closed codimension n points of $X_2-Y_2=X_1-Y_1$. It follows that the map $A^n(X_1,Y_1) \to A^n(X_2,Y_2)$ is a quotient map, hence onto.

<u>Lemma (5.10)</u>(cf.[CF, (6.5)]). If Y is a closed subscheme of X, define the scheme X_Y to be the intersection of all the open sets U in X containing Y. For example, if $X=\mathrm{Spec}(A)$ and $Y=V(I)$ then $X_Y=\mathrm{Spec}(S^{-1}A)$ for $S=1+I$. Then for every d there is an exact sequence

$$A^d(X,Y) \to A^d(X) \to A^d(X_Y) \to 0.$$

Proof. If V is a closed subscheme of X with $V \cap Y = \phi$, the following sequence is exact by [Fu, (1.9)]:

$$A^d(V) \to A^d(X) \to A^d(X-V) \to 0.$$

The direct colimit of this sequence over all V is the sequence of the lemma, and is exact as colim is an exact functor.

Proof of Theorem (5.1). When X-Spec(A), Y-$V(I)$, the objects of $\underline{M}_Y(X)$ are finitely generated A-modules M with $M_p=0$ whenever $I \subseteq p$. If Sing$(A) \subseteq V(I)$, it follows that each object has finite homological dimension over A. In the notation of [B], $\underline{M}_Y(X)=\underline{H}_S(A)$ for the multiplicative set $S=1+I$, and S is "regular for A." We now replace A by $A/\text{nil}(A)$; this changes neither the hypotheses nor the conclusions of (5.1), as $SK_i(A)=SK_i(A/\text{nil}(A))$ by [B, p. 449], and $A^d(A,I)=A^d(A/\text{nil}A, I/\text{nil}A)$ by (5.9). The assumption that I contains every minimal prime ideal of A implies that $\underline{M}_Y(X)=\underline{M}^1_Y(X)$ and that S consists of nonzerodivisors. By [B, p. 506], there is an epimorphism of exact sequences

$$\begin{tikzcd}
K_1(A) \arrow[r] \arrow[d] & K_1(A_S) \arrow[r] \arrow[d] & K_0\underline{M}_Y(X) \arrow[r] \arrow[d] & \tilde{K}_0(A)) \arrow[r] \arrow[d] & \tilde{K}_0(A_S) \arrow[r] \arrow[d] & 0 \\
U(A) \arrow[r] & U(A_S) \arrow[r] & \text{Pic}(A,S) \arrow[r] & \text{Pic}(A) \arrow[r] & \text{Pic}(A_S) \arrow[r] & 0.
\end{tikzcd}$$

By [B, p. 144], Pic(A,S) is $\underline{\amalg}\ \underline{Z}$, the sum being over all height one p with $p+I=A$. Thus Pic$(A,S)=K_0\underline{M}^{1/2}_Y(X)$, and the middle vertical map is the natural map. When A is 2-dimensional, the (exact) kernel sequence is

$$SK_1(A) \to SK_1(A_S) \to A^2(X,Y) \to SK_0(A) \to SK_0(A_S) \to 0.$$

By [B,p. 449], $SK_1(A_S)=SK_1(A/I)$ and $SK_0(A_S) \subseteq SK_0(A/I)$. But $\dim(A/I)=1$, so $SK_0(A/I)=0$ by [B,p. 468]. Hence $SK_0(A_S)=0$. With these replacements, the above kernel sequence becomes the sequence to be proven exact. Done.

§6. A Singular Example

In this section we analyze the CI and stCI maximal ideals of the seminormal surface $x^2=(y^3+z)z^2$. We verify Conjectures (0.1), (0.2) and (0.4) for this case by explicitly constructing the map from $\mathrm{Reg}(A)$ to $A^2(A,\mathrm{Sing}(A)) \cong SK_0(A) \cong \Omega_k$.

Because most of the results hold in much greater generality, we have stated the intermediate results in the context of height 2 ideals. As an application, we can replace k by $k[t,t^{-1}]$ to study $K_{-1}(A)$, an analysis we defer to the Appendix.

Notation (6.1). Let k be a commutative noetherian ring. Set $A=k[x,y,z]/(x^2=(y^3+z)z^2)$, $V=\mathrm{Spec}(A)$. We define

$$V(k) = \{(x,y,z)\in k^3 : x^2=(y^3+z)z^2\},$$

$$\Gamma_0 = \{(0,y,0)\in V(k)\},$$

$$\Gamma_t = \{(x,y,z)\in V(k) : z=-ty^3\},\quad 0\neq t\in k,$$

$$\Gamma_\infty = \{(x,0,z)\in V(k)\}.$$

Note that $0=(0,0,0)\in\Gamma_t$ for every t (including $t=\infty$).

For $p=(\alpha,\beta,\gamma)\in V(k)$ we have $A/m_p \cong k$ for the ideal $m_p=(x-\alpha,y-\beta,z-\gamma)A$.

When k is a field, the following is true. A is 2-dimensional, and V(k) is the set of 'rational' maximal ideals. If k is algebraically closed, then $V(k)=\mathrm{Max}(A)$. $\mathrm{Sing}(A)=V(z)$ and $\Gamma_0=\mathrm{Sing}(A)\cap V(k)$. Every point p of V(k) lies on exactly one $\Gamma_t, t\in k\cup\{\infty\}$, except for $p=0$. A is seminormal with integral closure $k[x/z,y]$ and conductor ideal $(z,x)A$.

Theorem (6.2). Let k be either $\overline{\mathbb{Q}}$ or an algebraically closed field of characteristic $\neq 0$. Then $SK_0(A)=0$ and:

(a) Every regular maximal ideal is a CI point, i.e., $\nu(m_p)=\nu(m_p/m_p^2)=2$ for $p \notin \Gamma_0$.

(b) For $p \in \mathrm{Sing}(A)=\Gamma_0$ we have $\nu(m_p)=\nu(m_p/m_p^2)=3$.

(c) For every height 2 ideal I we have $\nu(I)=\nu(I/I^2)$.

Theorem (6.3). Let k be an algebraically closed field of characteristic zero. Then:

(a) {CI points}={regular stCI points}$=\bigcup\{\Gamma_t : 0\neq t\in\overline{\mathbb{Q}}$ or $t=\infty\}-\{0\}$

(b) $\nu(m_p)=\nu(m_p/m_p^2)$ for $p\in\bigcup\{\Gamma_t : t\in\overline{\mathbb{Q}}$ or $t=\infty\}$. For all other p in $V(k)$ we have $\nu(m_p)=3$, $\nu(m_p/m_p2)=2$.

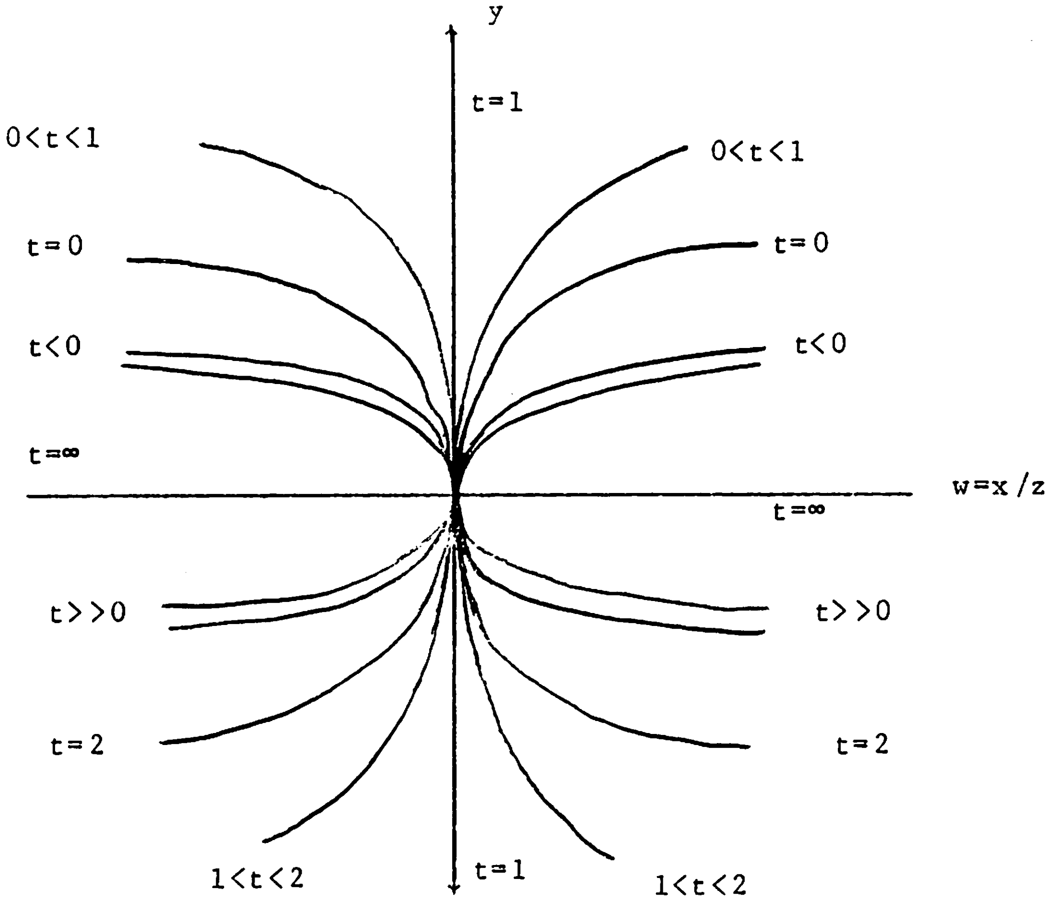
(c) The CI and stCI conjectures (0.1) and (0.2) hold for A, e.g., {CI points}$\cup$Sing(A) is a σ-closed subset of $\mathrm{Max}(A)=V(k)$. If $k=\mathbb{C}$, this set is a dense set of Lesbesque measure zero in the metric topology on $V(\mathbb{C})$.

(d) Conjecture (0.4) holds for A. Specifically, $A^2(V,\mathrm{Sing}(V)) \cong SK_0(A) \cong \Omega_k$, and for every $\omega\neq 0$ in Ω_k the fiber $\{p\in \mathrm{Max}(A)-\mathrm{Sing}(A):[p]=\omega\}$ is a countable set.

For any noetherian ring k, set $w=x/z$. Then $x=wz$ and $z=w^2-y^3$, so that A $k[w,y]$. The conductor ideal from $k[w,y]$ to A is $(x,z)A=zk[w,y]$. Setting $C=k[w,y]/(z=w^2-y^3=0)$, we have the conductor square

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[d] & k[w,y] \arrow[d] \\ k[y] \arrow[r] & C \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathrm{Spec}(C) \arrow[r, phantom, "\subset"] \arrow[d] & \mathbb{A}^2_k \arrow[d] \\ \mathbb{A}^1_k \arrow[r, phantom, "\subset"] & V \end{tikzcd}$$

Thus we may think of $V=\mathrm{Spec}(A)$ as being obtained from the affine plane $\underset{\sim}{A}^2=\mathrm{Spec}(k[w,y])$ by folding the cusp $\mathrm{Spec}(C)$ over onto itself. This allows us to illustrate $V(\underset{\sim}{R})$ as the points of the (w,y)-plane in Figure 1, with the line Γ_0 unfolded into the cusp $w^2=y^3$. Note that in this representation we have $\Gamma_t=\{(w,y):w^2=(1-t)y^3\}$.



<u>Figure 1</u>. The curves Γ_t on $V(\underset{\sim}{R})$ in the (w,y)-plane.

Lemma (6.4). Let k be a noetherian ring and let $p=(\alpha,\beta,\gamma)$ be a point in $V(k)$.

(a) If γ is not a unit of k, then $\nu(m_p)=\nu(m_p/m_p^2)=3$.

(b) If ν is a unit of k, then m_p is locally a height 2 CI with $m_p/m_p^2 \cong k^2$, and $\nu(m_p)$ is either 2 or 3.

(c) If γ is a unit of k, there is a unique rank 2 projective A-module P mapping onto m_p, and P satisfies $\Lambda^2 P=A$. The ideal m_p is a CI if and only if $P=A^2$.

Proof. It is clear that $\text{height}(m_p)=2$ and that $\nu(m_p) \leq 3$ for every $p \in V(k)$. If $\gamma=0$, it is easy to see that $\nu(m_p/m_p^2)=3$, and consequently that $\nu(m_p)=3$. If γ is not a unit of k, choose a ring map $k \to \ell$ sending γ to 0. Then $\nu(m_p \otimes \ell)=3$, so again $\nu(m_p)=3$. If γ is a unit of k, then modulo m_p^2 we have

$$z-\gamma \equiv \gamma^{-2}(x^2-\alpha^2) - (y^3-\beta^3).$$

Thus $\nu(m_p/m_p^2)=2$, and so m_p is locally generated by the regular sequence $(x-\alpha, y-\beta)$. Thus m_p is locally a CI. Since the projective k-module m_p/m_p^2 is 2-generated, it is free. By (1.4), $\text{Ext}(m_p,A) \cong k$. By (2.1), (1.7) and (1.8b), there is a unique P in $\underline{\underline{SP}}_2(A,I)$, and m_p is a CI if and only if P is free.

Lemma (6.5). For every commutative regular ring A, there is an isomorphism $SK_1(C) \cong SK_0(A)$, and a surjection $SK_1(C) \to \Omega_k$. If $1/6 \in k$ or if k is a perfect field of characteristic 2 or 3, then $SK_1(C) \cong \Omega_k$. If k is any field with char(k)=3, the kernel of $SK_1(C) \to \Omega_k$ is k/k^3; if char(k)=2, the kernel contains k/k^4.

Proof. From the conductor square above and the fact that k is regular, we easily deduce the isomorphism $SK_1(C) \cong SK_0(A)$ (see, e.g., [B,IX.5.3, XII.4.3 and XII.5.4]).

If $1/6 \in k$, Krusemeyer [Kr] showed that $SK_1(C) \cong \Omega_k$. In general, it follows from the square

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & k[v=w/y] \arrow[d] \\
k \arrow[r] & k[\varepsilon]/(\varepsilon^2=0)
\end{tikzcd}$$

that there is a surjection $SK_1(C) \to K_2(k[\varepsilon],\varepsilon)$ with kernel Φ (see [GlrW,(1.1)]). By [GlrW, p.7], Φ is k/k^3 or k/k^4 if k is a field of characteristic 3 or 2 (if k is perfect, $\Phi=0$ by [GR2] also). Finally, there is a surjection $K_2(k[\varepsilon],\varepsilon) \to \Omega_k$ which is an isomorphism if $1/2 \in k$ or if k is a perfect field by [vdK].

Proof of Theorem (6.2). Under the hypotheses on k, (6.5) states that $SK_0(A)=0$, so part (a) follows from (1.9). Part (b) is (6.4a), and part (c) is immediate.

To prove Theorem (6.3), we need to describe the composite set map:

$$V(k)-\Gamma_0 \to SK_0(A) \cong SK_1(C) \to \Omega_k.$$

To do this we first lift $dt/t \in \Omega_k$ to a Mennicke symbol in $SK_1(C)$, and then lift the Mennicke symbol to height 2 ideals m_p of A. This direct approach handles all p except those lying on Γ_1 and Γ_∞, which we can consider separately.

We need an explicit representation of elements of $SK_1(C)$ from [GR1]. It is proven there that the Mennicke symbol

$$\begin{bmatrix} (1-sv)v^2 \\ (1-(1-t)s^2v^2 \end{bmatrix} = \begin{bmatrix} y-sw \\ 1-(1-t)s^2y \end{bmatrix}$$

in $SK_1(C)$ maps to $sdt/t \in \Omega_k$ (where $v=w/y$). The following matrix over $k[w,y]$ is a lift of a representative in $Sl_2(C)$ of the above Mennicke symbol:

$$M(s,t) = \begin{pmatrix} 1-(1-t)s^2y & y-sw \\ t^{-1}s^4(1-t)^3(y+sw) & 1+(1-t)s^2y+t^{-1}s^4(1-t)^2y^2 \end{pmatrix}.$$

Note that $\det(M(s,t))=1+t^{-1}s^6(1-t)^3z$. In order to find an ideal $I(s,t)$ with $[A/I] \in SK_0(A)$ corresponding to $[M] \in SK_1(C)$, we need the following result. The idea for the proof is due to Murthy.

For this result, we forget our convention (6.1) about A. Note that when A is the coordinate ring of an affine surface, A/J has finite length and is supported away from $V(\underline{b})$.

Theorem (6.6). Let

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A/\underline{b} \arrow[r] & B/\underline{b} \end{tikzcd}$$

be a pullback square of rings, A a subring of B. Set

$$M = \begin{pmatrix} \bar{x} & \bar{y} \\ \bar{z} & \bar{w} \end{pmatrix} \in Sl_2(B/\underline{b}).$$

Assume that (x, y) is a B-sequence mapping to $(\bar{x}, \bar{y})$, and let P be the rank 2 projective A-module obtained by patching B^2 and $(A/\underline{b})^2$ together via M. Then:

(a) The map $\partial : SK_1(B/\underline{b}) \to SK_0(A)$ sends [M] to 2-[P].

(b) There is an element f of $Hom_A(P, A)$ which extends to $(x, y) \in Hom_B(B^2, B)$ via $P \to B^2$.

(c) J=f(P) is a height 2 ideal of A, locally a CI, with $J+\underline{b}=A$, and $[A/J] = 2-[P]$ in $SK_0(A)$.

Proof. Part (a) follows from the construction (up to sign) of ∂ on p. 28 of [Mil]. We also have

$$P = \left\{ \left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \end{bmatrix} \right) \in B^2 \times (A/\underline{b})^2 : \begin{pmatrix} \bar{x} & \bar{y} \\ \bar{z} & \bar{w} \end{pmatrix} \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \end{bmatrix} \right\}.$$

The following elements of P map to a basis of $(A/\underline{b})^2$:

$$e_1 = \left(\begin{bmatrix} w \\ -z \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \quad \text{and} \quad e_2 = \left(\begin{bmatrix} -y \\ x \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

Define $f \in Hom_A(P, A)$ by the formula

$$f\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \end{bmatrix} \right) = (x b_1 + y b_2, \bar{a}_1) \in A \subset B \times (A/\underline{b}).$$

It is clear that $\mathrm{Hom}(P,A) \to \mathrm{Hom}(B^2,B)$ maps f to (x,y), and that $\mathrm{Hom}(P,A) \to \mathrm{Hom}((A/\underline{b})^2,(A/\underline{b}))$ maps f to $(1,0)$. In particular, $f(e_1) \equiv 1 \bmod \underline{b}$, so that $f(P)+\underline{b}=A$. Since Λ^2P is obtained by patching B and $A/\underline{b}$ together via $\det(M)=1$, we have $\Lambda^2P=A$. We will show that $J=f(P)$ is a height 2 ideal of A which is locally a CI. Using (1.4), this will complete the proof of Theorem (6.6).

The element $f(e_1)$ is in J, and maps into the height 2 ideal $(x,y)B$ of B. As $c=1-f(e_1)$ is in the conductor, the map $A[c^{-1}] \to B[c^{-1}]$ is onto, hence an isomorphism. By part (b), JB is contained in $(x,y)B$. On the other hand,

$$xc = f\left(\begin{bmatrix} c \\ 0 \end{bmatrix},0\right) \quad , \quad yc = f\left(\begin{bmatrix} 0 \\ c \end{bmatrix},0\right)$$

are in J, so $J[c^{-1}] \cong (x,y)B[c^{-1}]$. This shows that $J[c^{-1}]$ is a CI of height 2, and hence that J_m is a CI of height 2 in A_m for every maximal ideal m of A containing J. Done with (6.6).

Remark. If A were not a subring of B, (a) and (b) would still hold, and f(P) would be a height 2 ideal, locally a weak CI. The difficulty is that $(x,y)A[c^{-1}] = J[c^{-1}]$ may consist of zerodivisors, even though (x,y) is a regular sequence in $B[c^{-1}]$.

Proposition (6.7). Let t be a unit of a regular ring k, and let $A=k[x,y,z]/(x^2=(y^3+z)z^2)$. Then

$$I(s,t) = (1+(t-1)s^2y,\ t+s^6(1-t)^3z,\ sx-yz)A$$

is a height 2 ideal of A, locally a CI. Under the surjection $SK_0(A) \to \Omega_k$, the element $[A/I(s,t)]$ maps to sdt/t.

Proof. Without loss of generality, we can assume that $k=\mathbb{Z}[s,t,t^{-1}]$, since both $I(s,t)$ and sdt/t are defined in this generic case. The element $[\overline{M}(s,t)]$ of $SK_1(C)$ maps to sdt/t, and the first row of $M(s,t)$ is a regular sequence in $k[w,y]$. By (6.6), $[\overline{M}(s,t)]$ corresponds to $[A/J]$ in $SK_0(A)$, where

$$J=(1+(t-1)s^2y,y-sw)P.$$

As in the proof of (6.6), J contains $f(e_1)=\det(M(s,t))=1+t^{-1}s^6(1-t)^3z$ and $z(y-sw)=yz-sx$. J also contains $(1+(t-1)s^2y)\underline{b}$; as $J+\underline{b}=A$, $1+(t-1)s^2y \in J$ as well. This shows that $I(s,t) \subseteq J$. But in the generic case it is easy to see that $I(s,t)$ is the kernel of a surjection

$$A \to k[s^{-1},(1-t)^{-1}] = \mathbb{Z}[s,s^{-1},t,(t-t^2)^{-1}].$$

Hence $I(s,t)$ is a height 2 prime ideal contained in the height 2 ideal J. Thus $I(s,t)=J$ in the generic case, hence always.

Remark. It follows for every ring k that $A/I=k[s^{-1},(1-t)^{-1}]$. (If s or 1-t is nilpotent then I=A.)

Remark (6.8). When t is not a unit of k, the definition of $I=I(st,t)$ still makes sense. Since $I(st,t)+tA=A$, either $I_p=A_p$ or t is a unit of A_p for every prime ideal p of A. Hence $I(st,t)$ is a height 2 ideal of A, locally a CI. Since $\Omega_k \otimes k[t^{-1}]=\Omega_{k[t^{-1}]}$, it follows, at least when t is a nonzerodivisor of k, that the element $[A/I(st,t)]$ of $SK_0(A)$ maps under $SK_0(A) \to \Omega_k$ to the differential sdt.

When $t=1$, $I(s,t)=A$. When s,t, and $1-t$ are all units of k we have $m_p=I(s,t)$ for

$$p=(\alpha,\beta,\gamma) = \left(\frac{-t}{s^9(1-t)^4}, \frac{1}{s^2(1-t)}, \frac{t}{s^6(1-t)^3}\right).$$

Conversely, if α,β,γ are units of k then we can solve for s and t to get

$$m_p = I\left(\frac{\beta^5}{\alpha\gamma^3}, \frac{-\gamma}{\beta^3}\right).$$

Corollary (6.9). Let t be such that t and $1-t$ are units of k. Then the ideals $I(s,t)$ with s a unit of k are exactly the ideals m_p with $p=(\alpha,\beta,\gamma)\in\Gamma_t$ and γ a unit of k.

Proof. It $p\in\Gamma_t$ then $\gamma=-t\beta^3$ and $\alpha^2=(1-t)\beta^3$. If γ is a unit, then, so are α and β. Now apply the formulas of (6.8).

Corollary (6.10). Let k be a field, $t\in k-\{0,1\}$. Then the image of

$$\Gamma_t - \{0\} \to SK_0(A) \to \Omega_k$$

is: $\{0\}$ if $dt=0$, and disjoint from $\{0\}$ if $dt\neq 0$ in Ω_k.

Proof. Combine (6.7), (6.8) and (6.9).

Proposition (6.11). Let k be a field, and consider $p=(\varepsilon^3,0,\varepsilon^2)\in\Gamma_\infty$, $\varepsilon\neq 0$. Then in $SK_0(A)$ the class $[A/m_p]$ is zero--unless char(k)=3 and $\varepsilon\notin k^3$, when it is nonzero.

Proof. We shall show that under $SK_0(A)\cong SK_1(C)$ the class $[A/m_p]$ corresponds to the Mennicke symbol

$$\begin{bmatrix} y \\ \varepsilon - w \end{bmatrix}$$

By [GlrW, p.7], this Mennicke symbol is zero, unless char(k)=3 and $\varepsilon\notin k^3$, when it is nonzero. The Mennicke symbol lifts to the matrix

$$M = \begin{pmatrix} \varepsilon - w & y \\ -y^2 & (\varepsilon + w) \end{pmatrix}.$$

By (6.6), $[\overline{M}]$ corresponds to $[A/J]$, where $J=(\varepsilon-w,y)P$. As in (6.7), J contains $y\underline{b}$ and $J+\underline{b}=A$, so $y\in J$. J also contains $\det(M)=\varepsilon^2-z$ and $z(\varepsilon-w)=\varepsilon z-x$. It follows that $J=m_p=(x-\varepsilon^3,y,z-\varepsilon^2)$, as desired.

Remark. Let P be the rank 2 projective A-module obtained by patching via $\overline{M}\in Sl_2(C)$. It follows from (6.4c) that $\nu(m_p)=2$ if and only if $P=\chi(m_p)$ is free.

Proposition (6.12). Let k be a field, and consider $p=(0,\beta,-\beta^3)\in\Gamma_1$, $\beta\neq 0$. In $SK_0(A)$ the class $[A/m_p]$ is zero unless char(k)=2. If char(k)=2, $[A/m_p]$ is in the kernel of $SK_0(A)\to\Omega_k$, and maps to zero in $K_2(k[\varepsilon],\varepsilon)$ if and only if $\beta\in k^2$.

Proof. Consider the Mennicke symbol

$$\begin{bmatrix} w \\ 1-\beta^{-1}y \end{bmatrix}$$

in $SK_1(C)$. It lifts to the matrix

$$M=\begin{pmatrix} \beta^{-1}(\beta-y) & w \\ -\beta^{-3}w & \beta^{-2}(\beta^2+\beta y+y^2) \end{pmatrix}.$$

By (6.6), [M] corresponds to [A/J], where $J=(\beta-y,w)P$. As before, J contains $\beta-y$, $\beta^3\det(M)=\beta^3-z$, and $zw-x$, i.e., $J=m_p$. We will show that under $SK_1(C)\to K_2(k[\varepsilon],\varepsilon)$ the above Mennicke symbol maps to $\langle\varepsilon,-\beta^{-1}\varepsilon\rangle$. This element is nonzero only when char(k)=2 and $\beta\notin k^2$. Invoking (6.5) will then finish the proof.

The argument on p. 4 of [GlrW] applies verbatim to $J=(y,w)C=v^2k[v]$, $C\subset k[v=w/y]$, with $\langle\bar{x},\bar{b}\rangle$ replaced by $\langle\bar{v},-\beta^{-1}\bar{v}\rangle$. It shows that the element

$$\begin{bmatrix} +\beta^{-1}v^3 \\ 1-\beta^{-1}v^2 \end{bmatrix}\quad\begin{bmatrix} \beta^{-2}v^4 \\ 1+\beta^{-1}v^2 \end{bmatrix}=\begin{bmatrix} \beta^{-1}w \\ 1-\beta^{-1}y \end{bmatrix}\quad\begin{bmatrix} \beta^{-2}y^2 \\ 1+\beta^{-1}y \end{bmatrix}\quad\begin{bmatrix} w \\ 1-\beta^{-1}y \end{bmatrix}$$

of $SK_1(C,J) \cong SK_1(C)$ maps to $\partial\langle\varepsilon,-\beta^{-1}\varepsilon\rangle$ in $SK_1(k[v],J)$. As $\partial : K_2(k[\varepsilon],\varepsilon) \cong SK_1(k[v],J)$, we are done.

Theorem (6.13). Let k be any field. The subset of $V(k)-\Gamma_0$ mapping to $0 \in \Omega_k$ under

$$V(k)-\Gamma_0 \to \mathrm{Reg}(A) \to SK_0(A) \to \Omega_k$$

is $\bigcup\Gamma_t$, the union being taken over $t=\infty$ and all t in k with $dt=0$ in Ω_k.

Proof. Combine (6.10), (6.11) and (6.12).

Proof of Theorem (6.3). By (6.5) and (6.13), the map $\mathrm{Reg}(A) \to SK_0(A)$ has $\bigcup\{\Gamma_t : 0 \neq t \in \overline{\mathbb{Q}} \text{ or } t=\infty\}-\{0\}$ as the inverse image of zero. As $SK_0(A)$ is torsion free (being a $\mathbb{Q}$-vector space), part (a) follows from (1.9) and (1.10). Parts (b) and (c) follow from this and (6.4). To see part (d), fix $0 \neq \omega = u\,dv/v$ in Ω_k. If $s\,dt/t=\omega$ then t must be algebraic over $\mathbb{Q}(v)$, but not in $\overline{\mathbb{Q}}$. For each such t we have $s=(t/v)/(dt/dv)$. Thus the set of pairs (s,t) with $s\,dt/t=\omega$ corresponds to the countable set $\overline{\mathbb{Q}(v)} - \overline{\mathbb{Q}}$. By (6.7) and (6.9), this set is in 1-1 correspondence with the set of primes m_p mapping to ω under $\mathrm{Reg}(A) \to SK_0(A) \cong \Omega_k$. Finally, since $SK_1(A/(x,z)A)=SK_1(k[y])=0$, Theorem (5.1) gives the isomorphism of $A^2(V,\mathrm{Sing}(V))=A^2(A,(x,z)A)$ with $SK_0(A)$. Done.

Appendix: Bloch and Murthy's Example

The ring $A=k[x,y,z]$, $x^2=(y^3+)z^2$, of §6 first came to my attention in November 1979, when Bloch and Murthy showed that $K_0(D)\neq \underline{Z}$ (and hence that $NK_0(D)\neq 0$) for the normal surface $D=\underline{C}[X,Y,Z]/(X^2=Y^3+Z^7)$. In fact, $K_{-1}(D)\neq 0$ as well. Their technique was to relate the K-theory of D and A. In this appendix we prove

Theorem (A.1). Let k be any field, and let $D=k[X,Y,Z]/(X^2=Y^3+Z^7)$. Spec(D) is a normal surface whose only singularity is at the origin. There are surjections

(S) $SK_0(D)\to\Omega_k$;
(N) $NK_0(D)\to N\Omega_k=(\Omega_k\otimes tk[t]dt)\oplus(k[t]dt)$;
(L) $K_{-1}(D)\to NL\Omega_k\cong k$;
(NL) $NK_{-1}(D)\to NL\Omega_k\cong uk[u]$.

In particular, $NK_0(D)$ and $K_{-1}(D)$ are always nonzero.

The interest in this example is that it is a 2-dimensional normal domain with $NK_0\neq 0$, providing a minimal example of a normal ring D with projective $D[t]$-modules which are not even stably extended from D. Theorem (A.1) constitutes an improvement of the results of [W].

Our proof of (A.1.S) follows Bloch and Murthy's, as communicated to me by Swan. The proof of the rest of (A.1) is similar in spirit, and allows us to explicitly construct the projective modules in question.

Recall from [B, p.658][W, p.179] that for a functor F from rings to abelian groups: $NF(R)=\ker(F(R[t])\to F(R))$ and $LF(R) = \operatorname{coker}\left[F(R[t])\oplus F(R[t^{-1}])\to F(R[t,t^{-1}])\right]$. For example, LK_0 is K_{-1}.

Exercise (A.2). For $F(R) = \Omega_R$ we have: $N\Omega_R = (\Omega_R\otimes tR[t])\oplus R[t]dt$; $L\Omega_R \cong R$, generated by the image of dt/t; $NL\Omega_R \cong uR[u]$; $L^2\Omega_R=0$.

Proposition (A.3). Let k be a regular ring containing 1/6, and let $A = k[x,y,z]$, $x^2=(y^3+z)z^2$. Then A is seminormal, and

(S) $SK_0(A) \cong \Omega_k$;

(N) $NK_0(A) \cong N\Omega_k$;

(L) $K_{-1}(A) \cong L\Omega_k \cong k$;

(NL) $NK_{-1}(A) \cong NL\Omega_k$.

Proof. We appeal to the conductor squares of (6.5). The square for A yields $N^iL^j\mathrm{Pic}(A)=0$ and isomorphisms $N^iL^jSK_1(C) \cong N^iL^jSK_0(A) \cong N^iL^j\tilde{K}_0(A)$ for $i,j \geq 0$. If $1/6 \in k$, the argument of (6.5) for the conductor square of C shows that $N^iL^jSK_1(C) \cong N^iL^jK_2(k[\varepsilon],\varepsilon) \cong N^iL^j\Omega_k$ (excision holds by [Kr]). Done.

Remark. The same proof shows that for any regular ring k we have surjections $N^iL^jSK_0(A) \to N^iL^j\Omega_k$.

Proposition (A.4). (a) The elements $\lceil A[t]/I(tf\beta,\beta)\rceil$ and $\lceil A[t]/I(tf,t)\rceil$ of $SK_0(A[t])$ represent classes in $NK_0(A)$ which map to the differentials $tfd\beta$ and fdt in $N\Omega_k$. ($\beta \in k-0$, $f \in k[t]$).

(b) The elements $\lceil A[t,t^{-1}]/I(\alpha,t)\rceil$ of $SK_0(A[t,t^{-1}])$ ($\alpha \in k$) represent classes in $K_{-1}(A)$ which map to dt/t in $L\Omega_k$.

(c) When $\alpha \in k[u]$, the element $\lceil A[t,t^{-1},u]/I(u\alpha,t)\rceil$ of $SK_0(A[t,t^{-1},u])$ represents a class in $NK_{-1}(A)$ which maps to $\alpha(udt/t)$ in $NL\Omega_k$.

Proof. All are elements of the appropriate SK_0 group by Proposition (6.7) and Remark (6.8). By (6.7), they map to the differentials $tfd\beta$, fdt in $\Omega_{k[t]}$, $\alpha dt/t$ in $\Omega_{k[t,t^{-1}]}$ and $\alpha(udt/t)$ in $\Omega_{k[t,t^{-1},u]}$, respectively. These are in $N\Omega_k$, $L\Omega_k$ and $NL\Omega_k$ by (A.2).

Proof of Theorem (A.2). Map D to A by sending X to xz^2, Y to yz^2, and Z to z. Since $D[Z^{-1}]=A[z^{-1}]$, $\mathrm{Spec}(A) \to \mathrm{Spec}(D)$ is a birational isomorphism. By (6.6.c), none of the ideals I described in (A.4) meet the exceptional locus $V(z)$. It follows that the classes $\lceil A[t]/I\rceil$ of $SK_0(A[t])$, etc., come from classes $\lceil D[t]/I\cap D[t]\rceil$ of $SK_0(D[t])$, etc. By subtracting off $\lceil D[t]/(t,I\cap D[t])\rceil \in SK_0(D)$ if necessary (etc.), we obtain an element of $NK_0(D) \subseteq SK_0(D[t])$ which maps to $\lceil A[t]/I\rceil \in NK_0(A)$ (etc.). By (A.2) and (A.4), we have bodily lifted generators of Ω_k, $N\Omega_k$, $L\Omega_k$ and $NL\Omega_k$ back to $SK_0(D)$, $NK_0(D)$, $K_{-1}(D)$, and $NK_{-1}(D)$ as desired. Done.

REFERENCES

[Alb] C. Alberto, Quillen's K-theory and algebraic cycles on almost nonsingular varieties, preprint (1980).

[B] H. Bass, Algebraic K-theory, Benjamin, New York, 1968.

[Bor0] M. Boratýnski, unpublished communications.

[Bor1] M. Boratýnski, A note on the set-theoretical complete intersection ideals, J. Alg. 54(1978), 1-5.

[CF] L. Clayborn and R. Fossum, Generalization of the notion of class group, Ill. J. Math. 12(1968), 228-253.

[D] E.D. Davis, Ideals of the principal class, R-sequences, and a certain monoidal transformation, Pac. J. Math. 20 (1967), 197-205.

[Fe] D. Ferrand, Suites régulière et intersection complète, C.R. Acad. Sci.(Paris) 264(1967), A427-428.

[Fu] W. Fulton, Rational equivalence on singular varieties, Publ. Math. IHES (Paris) No. 45(1975), 147-167.

[Gab] M.R. Gabel, Algebraic vector fields on S^2, J. Pure Appl. Alg. 11(1977), 29-40.

[GlrW] S.C. Geller and C.A. Weibel, K_2 measures excision for K_1, Proc. AMS 80(1980), 1-9.

[GR1] S.C. Geller and L.G. Roberts, The relationship between the Picard groups and SK_1 of some algebraic curves, J. Alg. 55(1978), 213-230.

[GR2] S.C. Geller and L.G. Roberts, Kähler differentials and excision for curves, J. Pure Appl. Alg. 17(1980), 85-112.

[GS] A.V. Geramita and C. Small, Introduction to Homological Methods in Commutative Rings, Queen's Papers in Pure and Applied Math. No. 43, Queen's University, Kingston, 1976.

[GW] A.V. Geramita and C.A. Weibel, Ideals with trivial conormal bundle, Can. J. Math. 32(1980), 210-218.

[G221] A. Grothendieck, Techniques de construction et théorèmes d'existence en géométric algébrique IV: Les schemas de Hilbert, Sem. Bourbaki 13(1960/61), No. 221, Paris, 1961. Also in Fondements de la Géométrie Algébrique, Paris, 1962.

[G225] A. Grothendieck, Classes de faisceaux et théorèmes de Riemann-Roch, Lecture Notes in Math. No. 225(SGA6), Springer-Verlag, 1971.

[H] D. Husemoller, *Fibre Bundles*, McGraw-Hill, 1966.

[K] Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.

[Kr] M. Kursemeyer, Fundamental groups, algebraic K-theory, and a problem of Abhyankar, *Inv. Math* 19(1973), 15-47.

[Mac] S. MacLane, *Homology,* Springer-Verlag, 1971.

[Mat] H. Matsumura, *Commutative Algebra*, Benjamin, New York, 1970.

[Mil] J. Milnor, *Introduction to Algebraic K-theory*, Annals of Math. Studies No. 72, Princeton University Press, Princeton, 1971.

[Mum] D. Mumford, Rational equivalence of 0-cycles on surfaces, *J. Math. Kyoto Univ*. 9(1969), 195-204.

[Mur] M.P. Murthy, Complete intersections, Queen's Papers in Pure and Applied Math. No. 42, Queen's University, Kingston, 1975.

[MS] M.P. Murthy and R.G. Swan, Vector bundles over affine surfaces, *Inv. Math*. 36(1976), 125-165.

[P] C. Pedrini, Vector bundles over singular affine surfaces, preprint (1979) (to appear in *Boll. Un. Mat. Ital*.).

[Q] D. Quillen, Higher algebraic K-theory:I, Lecture Notes in Math. No. 341, Springer-Verlag, 1973.

[R1] A.A. Roitman, On Γ-equivalence of zero-dimensional cycles, *Math. USSR Sbornik* 15(1971), 555-567.

[R2] A.A. Roitman, Rational equivalence of zero-cycles, *Math. USSR Sbornik* 18(1972), 571-588.

[Se] J.-P. Serre, Sur les modules projectifs, *Sem. Dubreil-Pisot* 14(1960/61), Paris, 1963.

[SeM] J.-P. Serre, *Algèbre Locale: Multiplicités*, Lecture Notes in Math. No. 11, Springer, 1975.

[Su] A.A. Suslin, On stably free modules, *Mat. Sbornik* 102(1977) = *Math. USSR Sbornik* 31(1977), 479-491.

[Sw] R.G. Swan, A cancellation theorem for projective modules in the metastable range, Inv. Math. 27(1974), 23-43.

[V] W. Vasconcelos, Ideals generated by R-sequences, J. Alg. 6(1967), 309-316.

[vdK] W.L. van der Kallen, Le K_2 des nombres duaux, C.R. Acad. Sci.(Paris) 273(1971), A1204-1207.

[W] C.A. Weibel, K-theory and analytic isomorphisms, Inv. Math. 61(1980), 177-197.

[W2] C.A.Weibel, Set-theoretic complete intersection points on on curves, preprint, 1981.