2007 Trieste Lectures on

The Proof of the Bloch-Kato Conjecture

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Let k be a field and ℓ a prime number different from char(k). The *Milnor K-theory* of k is defined to be the quotient $K_*^M(k)$ of the tensor algebra of the abelian group $k^\times$ by the ideal generated by elements of the form $\{a, 1-a\}$, $a \in k - \{0,1\}$. The Kummer isomorphism $k^\times/\ell \xrightarrow{\sim} H^1_{\text{ét}}(k, \mu_\ell)$ exends to a (graded) *norm residue homomorphism*

$$k_*^M(k)/\ell \longrightarrow \oplus H^n_{\text{ét}}(k, \mu_\ell^{\otimes n}), \tag{0.1}$$

and Milnor asked in [5] whether this map is an isomorphism for $\ell = 2$. This is true, as was proven by Voevodsky in [MC/2].

The same question for ℓ odd was first formulated by Kazuya Kato in [2, p.608] and is known as the *Bloch-Kato Conjecture.* A proof of this was announced in 1998 by Voevodsky, assuming the existence of what we now call a *Rost variety* (see Lecture 1). Rost produced such a variety in 1998 [R-CL], and the proof that (0.1) is an isomorphism appeared in the 2003 preprint [MC/l] — modulo three assertions. One of these, that the Rost variety has certain properties, was established in [9]. The other two assertions, concerning the motivic cohomology groups $H^{**}(X, \mathbb{Z}/\ell)$, are still unknown. In these lectures we shall prove the Bloch-Kato conjecture by establishing parallel assertions concerning the motivic cohomology groups $H^{**}(X, \mathbb{Z})$.

1 Lecture 1: Overview of the Proof

We begin with a series of reductions.

Lemma 1.1. (Voevodsky [12, 5.2]). *If $K_n^M(k)/\ell \to H^n_{ét}(k, \mu_\ell^{\otimes n})$ is an isomorphism for all fields of characteristic* 0*, then it is an isomorphism for all fields of characteristic $\neq \ell$.*

Proof. By a standard transfer argument, we may assume that k is perfect. Let R be the ring of Witt vectors over k and K its field of fractions. Then the specialization maps are compatible with the norm residue maps in the sense that

$$\begin{array}{ccc} K_n^M(K)/\ell & \xrightarrow{\simeq} & H^n_{\text{ét}}(K, \mu_\ell^{\otimes n}) \\ \downarrow & & \downarrow \\ K_n^M(k)/\ell & \longrightarrow & H^n_{\text{ét}}(k, \mu_\ell^{\otimes n}) \end{array}$$

commutes. Both specialization maps are known to be split surjections; the result follows. □

Now there is a chain complex $\mathbb{Z}(i)$ of étale sheaves and an isomorphism $H^n_{\text{ét}}(X, \mathbb{Z}/\ell(i)) \cong H^n_{\text{ét}}(X, \mu_\ell^{\otimes i})$ for all $n, i \geq 0$; see [4, 10.2] or [MC/2, 6.1]. We have a diagram

$$\begin{array}{ccccccc} K_n^M(k) & \xrightarrow{\ell} & K_n^M(k) & \longrightarrow & K_n^M(k)/\ell & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^n_{\text{ét}}(k, \mathbb{Z}(n)) & \xrightarrow{\ell} & H^n_{\text{ét}}(k, \mathbb{Z}(n)) & \longrightarrow & H^n_{\text{ét}}(k, \mu_\ell^{\otimes n}) & \longrightarrow & H^{n+1}_{\text{ét}}(k, \mathbb{Z}(n)) \end{array}$$

This motivates the following result, whose proof we omit; compare with [11, 7.1].

Theorem 1.2. (Voevodsky [MC/2, 6.10]). *Suppose that $H^{n+1}_{ét}(k, \mathbb{Z}(n)) = 0$ for every field k of characteristic* 0*. Then $K_n^M(k)/\ell \cong H^n_{ét}(k, \mu_\ell^{\otimes n})$.*

We proceed by induction on n, assuming $K_{n-1}^M(k)/\ell \cong H^{n-1}_{\text{ét}}(k, \mu_\ell^{\otimes n})$ for all k.

Proposition 1.3. (Voevodsky [MC/2, p.97]) *Suppose that for every field k and every symbol $\underline{a} = \{a_1, \dots, a_n\}$ in $K_n^M(k)/\ell$ there is a field extension K so that $\underline{a}$ vanishes in $K_n^M(K)/\ell$ and the map $H^{n+1}_{ét}(k, \mathbb{Z}(n)) \to H^{n+1}_{ét}(K, \mathbb{Z}(n))$ is an injection. Then $H^{n+1}_{ét}(k, \mathbb{Z}(n)) = 0$, and hence $K_n^M(k)/\ell \cong H^n_{ét}(k, \mu_\ell^{\otimes n})$, for all k.*

Proof. Fix k. By a transfinite process, we can find an extension field L which has $K_n^M(L)/\ell = 0$, L has no prime-to-ℓ extensions, and such that $H^{n+1}_{\text{ét}}(k,\mathbb{Z}(n))$ embeds in $H^{n+1}_{\text{ét}}(L,\mathbb{Z}(n))$. But for such an L we have $H^{n+1}_{\text{ét}}(L,\mathbb{Z}_{(\ell)}(n)) = 0$ by [MC/2, 5.9 and 6.8]. ☐

Lemma 1.4. (Voevodsky [MC/l, 6.4]) *If* $\{a_1,\dots,a_n\}$ *is a nonzero symbol in* $K_n^M(k)/\ell$, *its image is nonzero in* $H^n_{\acute{e}t}(k,\mu_\ell^{\otimes n})$.

Proof. By a standard transfer argument, we may assume k has no prime-to-ℓ extensions. For $E = k(\gamma)$, $\gamma = \sqrt[\ell]{a_n}$, we have a diagram

$$\begin{array}{ccccccc} K^M_{n-1}(E)/\ell & \xrightarrow{\text{norm}} & K^M_{n-1}(k)/\ell & \xrightarrow{\cup a_n} & K^M_n(k)/\ell & & \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & \\ H^{n-1}_{\text{ét}}(E,\mathbb{Z}/\ell) & \xrightarrow{\text{norm}} & H^{n-1}_{\text{ét}}(k,\mathbb{Z}/\ell) & \xrightarrow{\cup[a_n]} & H^n_{\text{ét}}(k,\mathbb{Z}/\ell) & \longrightarrow & H^n_{\text{ét}}(E,\mathbb{Z}/\ell) \end{array}$$

in which the vertical maps are isomorphisms by induction and the bottom row is exact by [MC/2, 5.2]. Since $\{a_1,\dots,a_n\} \neq 0$, if $\{a_1,\dots,a_n\}$ vanishes in $H^n_{\text{ét}}(k,\mathbb{Z}/\ell)$ then $\{a_1,\dots,a_{n-1}\}$ is the norm of some $s \in K^M_{n-1}(E)/\ell$. But then $\{a_1,\dots,a_n\}$ is the norm of $\{s,a_n\} = \{s,\gamma\}^\ell = 0$ and hence is zero.

We say that "X splits $\underline{a}$" if X is a smooth irreducible projective variety over k such that $\underline{a} = 0$ in $K_n^M(k(X))/\ell$. We write $\check{C}(X)$ for the simplicial scheme $\check{C}(X)_n = X^{n+1}$

$$X \leftleftarrows X\times X \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; X^3 \;\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\; X^4 \;\cdots,$$

whose face maps are given by projections. By [MC/2, 7.3], $H^p_{\text{ét}}(k,\mathbb{Z}(q)) \xrightarrow{\sim} H^p_{\text{ét}}(\check{C}(X),\mathbb{Z}(q))$ is an isomorphism for all p and q. By [MC/2, 6.9(2)], the motivic $H^p_{\text{nis}}(\check{C}(X),\mathbb{Z}/\ell(q))$ is isomorphic to $H^p_{\text{ét}}(\check{C}(X),\mathbb{Z}/\ell(q))$ for all p, q with $p-1 \le q \le n-1$. ☐

Lemma 1.5. (Voevodsky [MC/l, 6.5]) *If* X *splits a nonzero* $\underline{a} \in K_n^M(k)/\ell$, *then there is a nonzero* δ *in* $H^n(\check{C}(X),\mathbb{Z}/\ell(n-1))$.

Proof. By induction, the Bloch-Kato conjecture implies (see [11]) that the Leray spectral sequence for $X_{\text{ét}} \to X_{\text{nis}}$ degenerates to yield the exact sequence for $A = \mathbb{Z}/\ell(n-1)$:

$$\begin{array}{ccccccc}
 & & \delta & \mapsto & \underline{a} & \mapsto & 0 \\
0 & \longrightarrow & H^n_{\text{nis}}(\check{C}(X), A) & \longrightarrow & H^n_{\text{ét}}(\check{C}(X), A) & \longrightarrow & H^0_{\text{nis}}(\check{C}(X), \mathcal{H}^n) \\
 & & & & \cong \uparrow & & \downarrow \text{into} \\
 & & & & H^n_{\text{ét}}(k, A) & \longrightarrow & H^n_{\text{ét}}(k(X), A).
\end{array}$$

Here $\mathcal{H}^n$ is the sheaf associated to $H^n_{\text{ét}}(-, A)$. Now for any simplicial scheme $X_\bullet$, $H^0(X_\bullet, \mathcal{H}^n)$ embeds in $H^0(X_0, \mathcal{H}^n)$. This particular $\mathcal{H}^n$ is a homotopy invariant Nisnevich sheaf with transfers by [4, 6.17 and 22.3], so $H^0(X, \mathcal{H}^n)$ embeds in $H^n_{\text{ét}}(k(X), A)$ by [4, 11.1]. Via a diagram chase, $\underline{a}$ lifts to a nonzero δ in $H^n_{\text{nis}}(\check{C}(X), A)$. □

We will show in Lecture 6 that from the nonzero δ of Lemma 1.5 we can construct something we call a "Rost motive" (this will be defined in 3.4). For this we will need to start with a *Rost variety* X, which is defined in 3.1 and is a variety splitting $\underline{a}$. These varieties were first constructed by Markus Rost in [R-CL]; a proof that they have the defining properties of a Rost variety is published in [9].

When $\ell = 2$, the Rost motive is actually the same as the Rost variety, consided as a motive. Moreover, in that case the Rost variety has a natural interpretation in terms of quadratic forms. Although we do not consider the case $\ell = 2$ in these lectures, Voevodsky's proof in [MC/2] that the norm residue map $K^M_n(k)/2 \to H^n_{\text{ét}}(k, \mathbb{Z}/2)$ is an isomorphism (the "Milnor Conjecture") follows the general lines of our Lectures 1–3.

The goal of Lecture 3 is to use the Rost motive to show that the map $H^{n+1}_{\text{ét}}(k, \mathbb{Z}(n)) \to H^{n+1}_{\text{ét}}(k(X), \mathbb{Z}(n))$ is an injection, which we saw in Proposition 1.3 will imply the Bloch-Kato conjecture. For this, we need several cohomology operations – and that will be the topic of Lecture 2.

2 Lecture 2: Motivic cohomology operations

Fix a field k of characteristic 0 for simplicity, and consider the category Sm/k of smooth schemes over k. For each coefficient ring R, and each q, there is a chain complex of étale sheaves $R(q) = \mathbb{Z}(q) \otimes R$, constructed in [4, 3.1] and elsewhere; by definition the motivic cohomology groups $H^{p,q}(X, R)$ are the Zariski hypercohomology groups $H^p_{\text{Zar}}(X, R(q))$. We write $H^{p,q}$ for $H^{p,q}(\text{Spec}(k), R)$ when R is understood.

A *cohomology operation* ϕ is a natural transformation from $H^{p',q'}(-, R')$ to $H^{p,q}(-, R)$; it depends upon the integer p', q', p, q and the rings R' and R.

Examples 2.1. (a) If X is connected then $H^{0,0}(X, R) = R$; see [4, 3.1]. Hence any R'-indexed sequence ϕ_j of elements in $H^{p,q}$ with $\phi_0 = 0$, regarded as a function $R' \to H^{pq}$, determines a cohomology operation from $H^{0,0}(X, R')$ to $H^{p,q}(X, R)$. Clearly every such operation has this form.

(b) If t is an indeterminate of bidegree (p', q') then any homogeneous polynomial $f(t) = \Sigma_{i>0} a_i t^i$ of bidegree (p, q), whose coefficients a_i are in $H^{**}(k, R)$, determines a cohomology operation $H^{p',q'}(X, R') \to H^{p,q}(X, R)$ sending λ to $f(\lambda)$.

We will see in 4.6(c) that every cohomology operation $\phi\colon\ H^{2,1}(X, \mathbb{Z}) \to H^{p,q}(X, R)$ is of this form, and that ϕ uniquely determines the polynomial f.

(c) Associated to the exact sequence $0 \to \mathbb{Z}(q) \xrightarrow{\ell} \mathbb{Z}(q) \to \mathbb{Z}/\ell(q) \to 0$ is the *integral Bockstein* $\tilde{\beta}\colon\ H^{p,q}(X, \mathbb{Z}/\ell) \to H^{p+1,q}(X, \mathbb{Z})$ and its reduction modulo ℓ, the *Bockstein* $\beta\colon\ H^{p,q}(X, \mathbb{Z}/\ell) \to H^{p+1,q}(X, \mathbb{Z}/\ell)$.

(d) In [RPO, p. 33], Voevodsky constructed the *reduced power* operations

$$P^i\colon\ H^{p,q}(X, \mathbb{Z}/\ell) \to H^{p+2i(\ell-1),q+i(\ell-1)}(X, \mathbb{Z}/\ell).$$

By [RPO, §9–§10], the P^i satisfy the following axioms when $\ell > 2$:

Axioms for Steenrod Operations 2.2. When $\ell > 2$ the operations P^i satisfy:

1. The operation P^i has bidegree $(2i(\ell-1),\, i(\ell-1))$, and $P^0 x = x$.
2. $P^i x = x^\ell$ if x has bidegree $(2i, i)$.
3. $P^i x = 0$ if x has bidegree (p, q) and $q \le i$, $p < q + i$.
4. The usual Cartan formula $P^n(xy) = \sum P^i(x) P^{n-i}(y)$ holds.
5. The usual Adem relations hold. (These are described in [SE, p.77].)

It follows that the P^i and the Bockstein β generate a bigraded ring, isomorphic to the topological Steenrod Algebra described in [SE]. In particular, every monomial in β and the P^i is a $\mathbb{Z}/\ell$-linear combination of the *admissible* monomials

$$\beta^{\epsilon_0} P^{s_1} \beta^{\epsilon_1} \cdots P^{s_k} \beta^{\epsilon_k}, \quad \epsilon = 0, 1 \text{ and } s_i \geq \ell s_{i+1} + \epsilon_i.$$

The admissible monomials are linearly independent in the (left) $H^{*,*}(k, \mathbb{Z}/\ell)$-module of all cohomology operations by [RPO, 11.5]. We also get operations Q_i in bidegree $(2\ell^i - 1, \ell^i - 1)$ by setting $Q_0 = \beta$ (the Bockstein), $Q_1 = P^1\beta - \beta P^1$ and $Q_{i+1} = [P^{\ell^i}, Q_i]$. The Q_i generate an exterior algebra under composition, and each Q_i is a derivation: $Q_i(xy) = Q_i(x)y + xQ_i(y)$.

2.3. We now turn to a more topological interpretation of motivic cohomology operations. We begin with a quick review of Voevodsky's triangulated category **DM**. Let $\mathbf{Cor} = \mathbf{Cor}(R)$ denote the additive category of finite correspondences between smooth varieties, with coefficients in R; it contains Sm/k and is described in [4, §1]. The category $\mathbf{PST} = \mathbf{PST}(R)$ of *presheaves with transfers* is the category of contravariant R-linear functions from $\mathbf{Cor}(R)$ to R-**mod**; the Yoneda functor $\mathbf{Cor} \to \mathbf{PST}$ is written as $X \mapsto R_{\text{tr}}(X)$; see [4, §2].

A presheaf with transfers which is also a sheaf for the Nisnevich topology on Sm/k is called a *Nisnevich sheaf with transfers.* These form a full subcategory $\mathbf{NST}(R)$ of $\mathbf{PST}(R)$, and sheafification is an exact left adjoint to the inclusion; see [4, §13]. The triangulated category $\mathbf{DM}^{\text{eff}}(k,R)$ of *effective motives* is the localization of $D^-(\mathbf{NST})$ with respect to the class of $\mathbb{A}^1$-weak equivalences, and the motive of X is the class of $R_{\text{tr}}(X)$ in this category. This construction is presented in [4, §14]. In fact, $\mathbf{DM}^{\text{eff}}(k,R)$ is a full subcategory of a larger category $\mathbf{DM}(k,R)$ — but that is irrelevant for us. The key fact for us is that

$$H^{p,q}(X,R) = \text{Hom}_{\mathbf{DM}}(R_{\text{tr}}X, R(q)[p]), \tag{2.3.0}$$

and this is proven in [4, 14.16].

2.3.1. A variant which we shall need is the *motivic homology* $H_{p,q}(X,R)$ of a smooth X, defined as the R-module $\text{Hom}_{\mathbf{DM}}(R(q)[p], R_{\text{tr}}X)$; see [4, 14.7]. When X is smooth projective of dimension d, then by Duality

$$H_{-p,-q}(X,R) \cong \text{Hom}(R, R_{\text{tr}}X(q)[p]) \cong \text{Hom}(R_{\text{tr}}X, R(q+d)[p+2d]) = H^{p+2d,q+d}(X,R).$$

See [4, 16.24]. For $d=0$, when $X=\text{Spec}(k)$, we have $H_{-p,-q}(k,R) \cong H^{p,q}(k,R)$, and in particular $H_{-1,-1}(k,\mathbb{Z}) = H^{1,1}(k,\mathbb{Z}) = k^{\times}$.

2.4. We now introduce Voevodsky's "radditive" model for the fundamental adjunction between the Morel-Voevodsky category [7] of pointed spaces and $\mathbf{DM}$. Let $\mathbf{Rad} = \mathbf{Rad}(Sm/k)$ denote the category of contravariant functors $F\colon Sm/k \to \mathbf{Sets}_*$ (pointed sets) sending ϕ to the point, and disjoint unions to products ("radditive functors"). Using the Yoneda embedding, we can identify Sm/k with a full subcategory of $\mathbf{Rad}$.

Now R-**mod** $\to$ **Sets** induces a forgetful functor $u\colon \mathbf{PST} \to \mathbf{Rad}$. It has a left adjoint R_{tr}, defined by Kan extension from the formula that $Sm/k \to \mathbf{Rad} \to \mathbf{PST}$ is R_{tr}. Thus $R_{\text{tr}}F$ is the quotient of $\oplus_{\alpha\in F(X)} R_{\text{tr}}(X)$ by the relation that for every $f\colon Y \to X$ the copy of $R_{\text{tr}}(X)$ indexed by α is identified with its image in the copy of $R_{\text{tr}}(Y)$ indexed by $f^*(\alpha)$.

A more natural construction involves simplicial objects in **Rad**. There is a canonical resolution $\mathbf{Lres}(F)$ of any simplicial object F in **Rad** by wedges of representable functors and we define $R^L_{\rm tr}(F)$ to be the simplicial object $R_{\rm tr}(\mathbf{Lres}(F))$. This is left adjoint to the simplicial extension of the forgetful functor on the appropriate (global) homotopy category of simplicial presheaves. The details of this machinery are in several of Voevodsky's preprints, including [V07], and there are several variants such as in [8], [6, p.241], and [13].

Definition 2.5. Let U^q denote the simplicial cone of $(\mathbb{A}^q - 0) \to \mathbb{A}^q$, regarded as a simplicial (representable) object in **Rad**. The Lefschetz object $\mathbb{L}^q = \mathbb{L}^q(R)$ is defined to be the simplicial presheaf with transfers $R_{\rm tr}(U^q)$. By [4, 15.3], $\mathbb{L}^q$ is isomorphic to $R(q)[2q]$ in **DM**. As a special case, we have $U^0 = S^0$ and $\mathbb{L}^0 = R$.

Now it is possible to localize the categories $\Delta^{\rm op}\mathbf{Rad}$ and $\Delta^{\rm op}\mathbf{PST}$ with respect to $\mathbb{A}^1$-equivalences. The former yields the Morel-Voevodsky homotopy category of pointed spaces **Ho**, and the latter yields a full subcategory of $\mathbf{DM}^{\rm eff}(k, R)$. Moreover, the pair $(R_{\rm tr}, u)$ form a Quillen adjunction with respect to the $\mathbb{A}^1$-local closed model structures. This is somewhat technical, and the details are in [V07].

Theorem 2.6. *The space $K_a = u\mathbb{L}^a(R)$ represents $H^{2a,a}(-, R)$ in the sense that*

$$H^{2a,a}(X, R) \cong \mathrm{Hom}_{\mathbf{DM}}(R_{tr}X, \mathbb{L}^a) \cong \mathrm{Hom}_{\mathbf{Ho}}(X, K_a)$$

Proof. This is immediate from the adjunction. □

Corollary 2.7. *Motivic cohomology operations from $H^{2a,a}(X, R')$ to $H^{p,q}(X, R)$ are in $1-1$ correspondence with elements of $H^{pq}(K_a, R)$, where $K_a = u\mathbb{L}^a(R')$.*

Proof. Since K_a represents $H^{2a,a}(-, R')$, this is the Yoneda theorem. □

Example 2.8. When $a = 0$, K_0 is the pointed space $uR' = \bigvee_{R'-0} S^0$, and cohomology operations $H^{0,0}(-, R') \to H^{p,q}(-, R)$ correspond to elements of $H^{p,q}(uR', R) = \prod_{R'-0} H^{pq}(k, R)$.

3 Lecture 3: Rost Motives

The material in this lecture is based on [W-ax], which is based on [MC/l].

We begin by giving the definition of a Rost variety. By a ν_{n-1}*-variety* over k we mean a smooth projective variety X of dimension $d = \ell^{n-1} - 1$, with the degree of $s_d(X)$ being $\not\equiv 0 \pmod{\ell^2}$; see [9, 1.20]. Here $s_d(X)$ is the characteristic class of the tangent bundle of X corresponding to the symmetric polynomial $\Sigma\, t_j^d$ in the Chern roots t_j, see [RPO, 14.3].

Definition 3.1. A *Rost variety* for a sequence $\underline{a} = (a_1, \dots, a_n)$ of units of k is a ν_{n-1}-variety X satisfying:

(a) X splits $\underline{a}$, i.e., $\underline{a}$ vanishes in $K_n^M(k(X))/\ell$;

(b) For each $i < n$ there is a ν_i-variety mapping to X;

(c) The following motivic homology sequence is exact (for $R = \mathbb{Z}$):

$$H_{-1,-1}(X \times X) \xrightarrow{\pi_0^* - \pi_1^*} H_{-1,-1}(X) \longrightarrow H_{-1,-1}(k) = k^\times$$

is exact. See 2.3.1 for the definition of $H_{-1,-1}$ and the calculation that $H_{-1,-1}(k) = k^\times$.

As mentioned in Lecture 1, Rost varieties exist for all n, ℓ and $\underline{a}$. When $n = 1$, Spec(L) is a Rost variety for a when $L = k\left(\sqrt[\ell]{a}\right)$. When $n = 2$, the Severi-Brauer variety corresponding to the degree ℓ division algebra with symbol $\underline{a}$ is a Rost variety for $\underline{a}$.

In this lecture we write $\mathfrak{X}$ for the simplicial Čech scheme $\check{C}(X)$ of Lecture 1, and $\Sigma\mathfrak{X}$ for its suspension, i.e., the cone of $\mathfrak{X} \to \mathrm{Spec}(k)$. Note that $H^{p,q}(\mathfrak{X}, \mathbb{Z}) \cong H^{p+1,q}(\Sigma\mathfrak{X}, \mathbb{Z})$ when $p > q$, as $H^{pq}(k, \mathbb{Z}) = 0$ in this range. A transfer argument [W-ax, 2.3] shows that the motivic cohomology groups $H^{**}(\Sigma\mathfrak{X}, \mathbb{Z})$ have exponent ℓ. Hence we have exact sequences

$$0 \to H^{p,q}(\Sigma\mathfrak{X}, \mathbb{Z}) \to H^{p,q}(\Sigma\mathfrak{X}, \mathbb{Z}/\ell) \xrightarrow{\tilde{\beta}} H^{p+1,q}(\Sigma\mathfrak{X}, \mathbb{Z}) \to 0.$$

By 3.1(b), Theorem 3.2 of [MC/2] translates to:

Theorem 3.2. *If $i < n$, the sequences* $\xrightarrow{Q_i} H^{**}(\Sigma\mathfrak{X}, \mathbb{Z}/\ell) \xrightarrow{Q_i}$ *are exact.*

Proposition 3.3. (Voevodsky) *The cohomology operations $Q_0 = \beta$, $Q = Q_{n-2} \cdots Q_0$ and $Q_{n-1}Q$ are injections on $H^{n,n-1}(\mathfrak{X}, \mathbb{Z}/\ell)$, and restrict to injections*

$$H^{n,n-1}(\mathfrak{X}, \mathbb{Z}/\ell) \xrightarrow{\beta} H^{n+1,n-1}(\mathfrak{X}, \mathbb{Z}) \xrightarrow{Q_{n-2}\cdots Q_1} H^{2b+1,b}(\mathfrak{X}, \mathbb{Z}) \xrightarrow{Q_{n-1}} H^{2b\ell+2,b\ell+1}(\mathfrak{X}, \mathbb{Z}),$$

where $b = b/\ell - 1 = 1 + \ell + \cdots + \ell^{n-2}$. In particular, $\mu = Q(\delta) \in H^{2b+1,b}(\mathfrak{X}, \mathbb{Z})$ is nonzero.

Proof. It suffices to show that each $Q_i \ldots Q_0$ is injective on the group $H^{n+1,n-1}(\Sigma\mathfrak{X}, \mathbb{Z}/\ell) \cong H^{n,n-1}(\mathfrak{X}, \mathbb{Z}/\ell)$. This follows from Theorem 3.2 and the observation in [MC/2, 6.9] that

$$H^{p,q}(\Sigma\mathfrak{X}, \mathbb{Z}/\ell) = 0 \text{ when } (p,q) \text{ is in the region } q < n,\ p \leq 1 + q.$$

Indeed, each Q_i is injective on the group $H^{**}(\Sigma\mathfrak{X})$ containing $Q_{i-1} \ldots Q_1 H^{n+2,n}(\Sigma\mathfrak{X})$, because the preceeding term in Theorem 3.2 is zero. Figure 1 shows the case $i = 2$. □

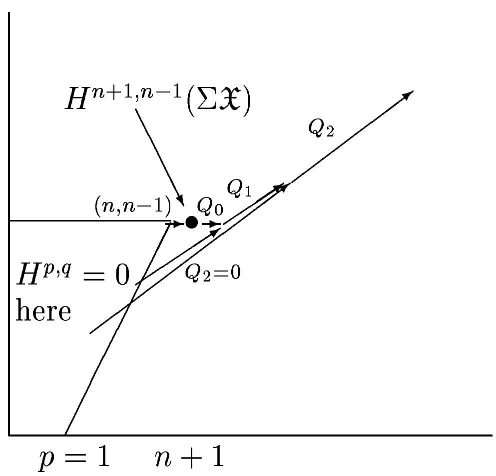

Figure 1: The composition $Q_2 Q_1 Q_0$ is an injection on $H^{n+1,n}(\Sigma\mathfrak{X})$

The goal of Lectures 4-6 is to use $\mu \in H^{2b+1,b}(\mathfrak{X}, \mathbb{Z})$ to construct a summand M of the motive of the Rost variety X satisfying the following axioms.

Rost motives 3.4. A *Rost motive* for $\underline{a}$ is a motive M with coefficients $\mathbb{Z}_{(\ell)}$ satisfying:

(a) M is a direct summand of a Rost variety X, i.e., a Chow motive (X, e).

(b) The transpose $M' = (X, e^t)$ is isomorphic to M via $M' \to X \xrightarrow{p} M$, p being the projection. Here $M' \to X$ is defined as $M' = M^* \otimes \mathbb{L}^d \xrightarrow{p^* \otimes \mathbb{L}^d} X^* \otimes \mathbb{L}^d \cong X$.

(c) There is a motive D related to the evident map $y\colon M \to X \to \mathfrak{X}$ by two distinguished triangles

(3.4.1) $$D \otimes \mathbb{L}^b \to M \xrightarrow{y} \mathfrak{X} \to ,$$

(3.4.2) $$\mathfrak{X} \otimes \mathbb{L}^d \xrightarrow{Dy} M \to D \to .$$

Here Dy is the dual map $\mathfrak{X} \otimes \mathbb{L}^d \to \mathbb{L}^d \xrightarrow{y^*} M^* \otimes \mathbb{L}^d \cong M$, where the final isomorphism comes from axiom (b).

In the rest of this lecture, we will assume that a Rost motive M exists for $\underline{a}$, and use it to verify the Bloch-Kato Conjecture. We will use without comment the standard fact that if $p > q$ then $\mathrm{Hom}_{\mathbf{DM}}(\mathbb{Z}, R_{\mathrm{tr}}(Y)(q)[p]) = 0$ for every smooth Y.

Lemma 3.5. *The structure map $H_{-1,-1}(\mathfrak{X}) \to H_{-1,1}(k) = k^\times$ is injective.*

Proof. (Voevodsky) For all p and $n \geq 2$, $\mathrm{Hom}_{\mathbf{DM}}(\mathbb{Z}, X^p(1)[n]) = 0$ (as $n > 1$). Therefore every row in the spectral sequence

$$E^1_{pq} = \mathrm{Hom}(\mathbb{Z}[q], X^{p+1}(1)) \Longrightarrow \mathrm{Hom}(\mathbb{Z}, \mathfrak{X}(1)[p-q]) = H_{q-p,-1}(\mathfrak{X})$$

is zero below the row $q = -1$. The homology at $(p, q) = (0, -1)$ yields the exact sequence

$$0 \longleftarrow H_{-1,-1}(\mathfrak{X}) \longleftarrow H_{-1,-1}(X) \longleftarrow H_{-1,-1}(X \times X).$$

The result follows from the homology exact sequence 3.1(c) of a Rost variety X. □

Corollary 3.6. *The structure map $H_{-1,-1}(M) \xrightarrow{y} H_{-1,-1}(k) = k^*$ is injective.*

Proof. By axiom 3.4(c), it suffices to observe that $\mathrm{Hom}(\mathbb{Z}, D(b+1)[2b+1]) = 0$. This follows from the vanishing of both $\mathrm{Hom}(\mathbb{Z}, M(b+1)[2b+1])$ and $\mathrm{Hom}(\mathbb{Z}, \mathfrak{X} \otimes \mathbb{L}^{b+d+1})$. □

Lemma 3.7. $H^{2d+1,d+1}(D, \mathbb{Z}) = 0$.

Proof. Set $(p, q) = (2d+1, d+1)$. From (3.4.2) we get an exact sequence

$$H^{0,1}(\mathfrak{X}) \to H^{p,q}(D) \to H^{p,q}(M) \xrightarrow{Dy} H^{p,q}(\mathfrak{X} \otimes \mathbb{L}^d).$$

The first group is 0 because it equals $H^{0,1}(k, \mathbb{Z}) = H^{-1}_{\mathrm{Zar}}(k, \mathcal{O}_X^\times) = 0$. Hence $H^{p,q}(D)$ is the kernel of Dy. But by axiom 3.4(b), for any u in $H_{-1,-1}(M) = \mathrm{Hom}(\mathbb{Z}, M(1)[1])$ the dual of $\mathbb{Z}(-1)[-1] \xrightarrow{u} M \xrightarrow{y} \mathbb{Z}$ is an element of $H^{p,q}(\mathfrak{X} \otimes \mathbb{L}^d)$ represented by:

$$\mathfrak{X} \otimes \mathbb{L}^d \longrightarrow \mathbb{L}^d \xrightarrow{y^*} M^* \otimes \mathbb{L}^d \xrightarrow{u^*} \mathbb{Z}(1)[1] \otimes \mathbb{L}^d = \mathbb{Z}(d+1)[2d+1].$$

Hence $Dy\colon H^{p,q}(M) \to H^{p,q}(\mathfrak{X} \otimes \mathbb{L}^d)$ may be identified with the given structural map $H_{-1,-1}(M) \xrightarrow{y} H_{-1,-1}(k)$, and is an injection by 3.6. Hence $H^{p,q}(D) = \ker(Dy) = 0$. □

Proposition 3.8. $H^{n+1,n}(\mathfrak{X}, \mathbb{Z}) = 0$.

Proof. In the cohomology sequence arising from (3.4.1),

$$H^{2b\ell+1,b\ell+1}(D \otimes \mathbb{L}^b) \to H^{2b\ell+2,b\ell+1}(\mathfrak{X}) \to H^{2b\ell+2,b\ell+1}(M),$$

the first term is $H^{2d+1,d+1}(D)$ because $b\ell = d + b$, and it vanishes by 3.7. The third term is a summand of $H^{2b\ell+2,b\ell+1}(X)$, which is zero because $H^{p,q}(X) = 0$ whenever $p > d + q$ by the Vanishing Theorem [4, 3.6]. Hence $H^{2b\ell+2,b\ell+1}(\mathfrak{X}, \mathbb{Z}) = 0$. By Proposition 3.3, ths implies that $H^{n+1,n}(\mathfrak{X}, \mathbb{Z}) = 0$. □

Theorem 3.9. $H^{n+1}_{\acute{e}t}(k, \mathbb{Z}(n)) \to H^{n+1}_{\acute{e}t}(k(X), \mathbb{Z}(n))$ *is an injection.*

As pointed out in Lecture 1 (Prop. 1.3), this establishes the Bloch-Kato conjecture for n.

Proof. Set $E = \operatorname{Spec} k(X)$, and let $L(n)$ denote the truncation $\tau^{\leq n+1}\mathbb{Z}^{\text{ét}}(n)$ of the complex in **DM** representing étale motivic cohomology. Then $H^{n+1}(E, L(n)) \cong H^{n+1}_{\text{ét}}(E, \mathbb{Z}(n))$ and

$$H^{n+1}(\mathfrak{X}, L(n)) \cong H^{n+1}_{\text{ét}}(\mathfrak{X}, \mathbb{Z}(n)) \cong H^{n+1}_{\text{ét}}(k, \mathbb{Z}(n)).$$

Write $K(n)$ for the cofiber of $\mathbb{Z}(n) \to L(n)$, so that $\mathbb{Z}(n) \to L(n) \to K(n) \to$ is a triangle. Then $\operatorname{Spec}(E) \to \mathfrak{X}$ induces a commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 \overset{3.8}{=} H^{n+1,n}(\mathfrak{X}, \mathbb{Z}) & \longrightarrow & H^{n+1}(\mathfrak{X}, L(n)) & \longrightarrow & H^{n+1}(\mathfrak{X}, K(n)) \\ \downarrow & & \downarrow & & \downarrow \\ 0 = H^{n+1,n}(E, \mathbb{Z}) & \longrightarrow & H^{n+1}(E, L(n)) & \longrightarrow & H^{n+1}(E, K(n)). \end{array}$$

We have to prove that that the middle vertical is an injection. This will follow once we show that the right vertical is an injection. Now $H^{n+1}(X, K(n)) \to H^{n+1}(E, K(n))$ is an injection by [4, 11.1, 13.8, 13.10], and $H^{n+1}(M, K(n))$ is a summand of the former group. It suffices to show that $y\colon M \to \mathfrak{X}$ induces an injection on $H^{n+1}(-, K(n))$. By (3.4.1) it suffices to show that $H^n(D \otimes \mathbb{L}^b, K(n))$ vanishes. By (3.4.2) we have an exact sequence

(3.9.1) $$H^{n-1}(\mathfrak{X} \otimes \mathbb{L}^{b+d}, K(n)) \to H^n(D \otimes \mathbb{L}^b, K(n)) \to H^n(M \otimes \mathbb{L}^b, K(n)).$$

We are deduced to showing the outside terms vanish in (3.9.1). For this we invoke the fact (proven in [MC/2, 6.12]) that $H^*(Y(1), K(n)) = 0$ for every smooth Y. Since M is a summand of X by 3.4(a), this fact implies that $H^n(M \otimes \mathbb{L}^b, K(n)) = 0$ in (3.9.1). It also implies that (for any $m > 1$) the spectral sequence $E^1_{pq} = H^q(X^p \otimes \mathbb{L}^m, K(n)) \Rightarrow H^{p+1}(\mathfrak{X} \otimes \mathbb{L}^m, K(n))$ collapses, yielding $H^{n-1}(\mathfrak{X} \otimes \mathbb{L}^m, K(n)) = 0$. Thus the left term in (3.9.1) also vanishes. $\square$

Remark 3.10. Markus Rost has proposed a construction of M in [R-BC]. He shows that the element μ of Proposition 3.3 determines an equivalence class of summands (X, e) of the Chow motive of X. By construction $M = (X, e)$ satisfies axioms 3.4(a,b). I do not know if it satisfies axiom 3.4(c).

4 Lecture 4: Symmetric products of motives

This lecture summarizes the main points of [V07]. If X is a normal quasiprojective variety over a field of characteristic 0, the symmetric product $S^mX = X^m/\Sigma_m$ is also normal (where Σ_m is the symmetric group). More generally, if G is a subgroup of Σ_m then $S^GX = X^m/G$ determines a functor from the category **Norm** of normal quasiprojective varieties to itself. If X_+ denotes the disjoint union of X and $\mathrm{Spec}(X)$ then there is a natural split sequence of pointed objects (which extends to simplicial objects as well):

$$S^{m-1}(X_+) \to S^m(X_+) \to (S^mX)_+. \tag{4.1}$$

It is easy to see that $\widetilde{S}^m(X_+) = (S^mX)_+$ is a functor on $\mathbf{Norm}_+$, and that it extends to finite correspondences, giving us a functor S^m_{tr} from **Cor** — or even **Cor**(**Norm**) — to itself, characterized by the formula:

$$S^m_{\mathrm{tr}}(R_{\mathrm{tr}}X) = R_{\mathrm{tr}}\widetilde{S}^m(X_+) = R_{\mathrm{tr}}(S^mX).$$

This functor extends to simplicial objects and commutes with direct sums.

Lemma 4.2. *For any (simplicial) normal X, Y we have*

$$S^m_{tr}(R_{tr}X \oplus R_{tr}Y) \cong \bigoplus_{i+j=m} S^i_{tr}(R_{tr}X) \otimes S^j_{tr}(R_{tr}Y).$$

Proof. Immediate from $R_{\mathrm{tr}}(X \coprod Y) = R_{\mathrm{tr}}X \oplus R_{\mathrm{tr}}Y$ and $S^m(X \coprod Y) = \coprod S^i(X) \times S^j(Y)$. □

Examples 4.3. (a) Since $S^m(S^0) \cong \{0, 1, \dots, m\}$ we have $\widetilde{S}^m(S^0) = S^0$ and $S^m_{\mathrm{tr}}(R) \cong R$.

(b) Since $R_{\mathrm{tr}}(\mathbb{P}^1) \cong R \oplus \mathbb{L}^1$ and $S^m\mathbb{P}^1 \cong \mathbb{P}^m$, Lemma 4.2 yields $S^m_{\mathrm{tr}}(\mathbb{L}^1) \cong \mathbb{L}^m$.

We write $S^\infty(X_+)$ for the colimit of the pointed spaces $S^n(X_+)$. From (4.1) one gets:

Proposition 4.4. *For any simplicial object $V_\bullet$ of $\mathbf{Norm}_+$ there is an isomorphism*

$$R_{tr}(S^\infty V_\bullet) \cong \bigoplus_{m=0}^{\infty} S^m_{tr} R_{tr}(V_\bullet).$$

The following result is a translation of the Suslin-Voevodsky result [10] that finite correspondences of degree $m \geq 0$ from X to Y correspond to morphisms from X to $S^m(Y)$, together with the fact that a connected simplicial H-space has a homotopy inverse.

Theorem 4.5. *Let $V_\bullet$ be a simplicial object of* $\mathbf{Norm}_+$. *If the simplicial sets* $\mathrm{Hom}(X, V_\bullet)$ *are connected for all X, then the morphism $S^\infty(V_\bullet) \to u\mathbb{Z}_{tr}(V_\bullet)$ is a global weak equivalence of spaces (functors on Sm/k).*

Examples 4.6. (a) When V is S^0 (which is *not* connected), the morphism in 4.5 is $\mathbb{N} \to \mathbb{Z}$.

(b) If $n \geq 1$, 4.4 and 4.5 yield $K_n = u\mathbb{Z}_{\mathrm{tr}}(V_\bullet) \cong S^\infty(V_\bullet)$ and $R_{\mathrm{tr}}(K_n) \cong \oplus_{m=1}^{\infty} S^m_{\mathrm{tr}}(\mathbb{L}^n)$.

(c) The pointed space $K_1 = u\mathbb{L}^1$ represents $H^{2,1}(-, \mathbb{Z})$, where $\mathbb{L}^1 = \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1/\mathbb{A}^1 - 0)$. Since $S^m_{\mathrm{tr}}(\mathbb{L}^1) \cong \mathbb{L}^m$ by 4.3(b), Proposition 4.4 yields:

$$R_{\mathrm{tr}}(K_1) \simeq R_{\mathrm{tr}} S^\infty(\mathbb{A}^1/\mathbb{A}^1 - 0) \simeq \oplus \mathbb{L}^m.$$

So cohomology operations $H^{2,1}(X, \mathbb{Z}) \to H^{p,q}(X, R)$ are classified by the elements of

$$H^{p,q}(K_1, R) \cong \mathrm{Hom}_{\mathbf{DM}}(R_{\mathrm{tr}} K_1, R(q)[p]) \cong \prod_{m=1}^{\infty} H^{p-2m, q-m}(k, R).$$

These correspond to homogeneous polynomials $f(t) = \Sigma a_i t^i$ of bidegree (p, q) in $H^{**}(k, R)[t]$ with $a_0 = 0$ and bidegree$(t) = (2, 1)$, as described in Example 2.1(b). The operations $x \mapsto f(x)$ are nontrivial on $x \in H^{2,1}(\mathbb{P}^N, R)$ for large N, since $H^{*,*}(\mathbb{P}^N, R) = H^{*,*}(k, R)[x]/(x^{N+1})$; see [4, 15.5]

Now suppose that R is either $\mathbb{Z}_{(\ell)}$ or $\mathbb{Z}/\ell$, so that $(\ell - 1)!$ is a unit of R. If $m < \ell$, the symmetrizing idempotent $e = (\Sigma\sigma)/m!$ of $R\,[\Sigma_m]$ acts on $R_{\mathrm{tr}}(X^m)$ and it is easy to see that the canonical map $R_{\mathrm{tr}}(X^m) \to S^m_{\mathrm{tr}}(R_{\mathrm{tr}} X) = R_{\mathrm{tr}}(S^m X)$ induces an isomorphism

(4.7)
$$S^m_{\mathrm{tr}}(R_{\mathrm{tr}} X) \cong e \cdot R_{\mathrm{tr}}(X^m), \qquad m < \ell.$$

Example 4.7.1. Fix $m < \ell$. If the interchange τ on $T \otimes T$ is equivalent to the identity (e.g., $T = \mathbb{L}^a[2b]$); then $S^m_{\mathrm{tr}}(T) \cong T^{\otimes m}$. If $\tau \simeq -1$ (e.g., $T = \mathbb{L}^a[2b+1]$), then $S^m_{\mathrm{tr}}(T) \cong 0$.

We will now describe $S^m_{\mathrm{tr}}(M)$ in terms of S^ℓ_{tr}. If G is any subgroup of Σ_m, the wreath product

$$G \wr \Sigma_n = G^n \rtimes \Sigma_n$$

acts on $\{1, \dots, mn\}$ by decomposing it into n blocks of m elements, with G acting on the blocks and Σ_n permuting the blocks. Thus $G \wr \Sigma_n \subset \Sigma_{mn}$. It is easy to see that

$$S^n(S^G(X_+)) = S^{G \wr \Sigma_n}(X_+).$$

Similarly, if H is a subgroup of Σ_n and we embed $\Sigma_m \times \Sigma_n$ in Σ_{m+n} then $S^{G\times H}(X_+) = S^G(X_+) \times S^H(X_+)$ and $S^{G\times H}_{\mathrm{tr}}(R_{\mathrm{tr}}X) = S^G_{\mathrm{tr}}(R_{\mathrm{tr}}X) \otimes S^H_{\mathrm{tr}}(R_{\mathrm{tr}}X)$.

Proposition 4.8. *If $m = m_0 + m_1\ell + \cdots + m_r\ell^r$ with $0 \le m_i < \ell$, the subgroup*

$$G = \Sigma_{m_0} \times (\Sigma_\ell \wr \Sigma_{m_1}) \times ((\Sigma_\ell \wr \Sigma_\ell) \wr \Sigma_{m_2}) \cdots \times ((\Sigma_\ell{}^{\wr r}) \wr \Sigma_{m_r})$$

of Σ_m contains a Sylow ℓ-subgroup of Σ_m. If $R = \mathbb{Z}_{(\ell)}$ or $\mathbb{Z}/\ell$ then for every simplicial V and $M = R_{tr}(V)$, $S^m_{tr}(M)$ is a direct summand of

$$S^G_{tr}(M) = (S^{m_0}_{tr}M) \otimes S^{m_1}_{tr}(S^\ell_{tr}M) \otimes S^{m_2}_{tr}(S^\ell_{tr}(S^\ell_{tr}M)) \otimes \cdots \otimes S^{m_r}_{tr}((S^\ell_{tr})^r M).$$

Proof. (Voevodsky, [V07]) The display is $S^G_{\mathrm{tr}}(M)$ by the above remarks, and the map π from $S^G(V) = V^m/G$ to $S^mV = V^m/\Sigma_m$ is finite of degree $d = [\Sigma_m : G]$. It is well known (and easy to check) that G contains a Sylow ℓ-subgroup of Σ_m, so $\ell \nmid d$. The transpose π^t is a finite correspondence, and the composition $\pi \circ \pi^t$ is multiplication by d on $R_{\mathrm{tr}}(\widetilde{S}^m V) = S^m_{\mathrm{tr}}(M)$. □

Theorem 4.9. *When $R = \mathbb{Z}/\ell$, $S^\ell_{tr}(\mathbb{L}^n)$ is $\mathbb{A}^1$-equivalent to*

$$\mathbb{L}^{n\ell} \oplus \bigoplus_{i=1}^{n-1} \left\{ \mathbb{L}^{n+i(\ell-1)} \oplus \mathbb{L}^{n+i(\ell-1)}[1] \right\}.$$

Proof. (Sketch) Let C be the cyclic group of order ℓ and $G = C \rtimes (\mathbb{Z}/\ell)^\times \subseteq \Sigma_\ell$. Using the methods of [RPO], Voevodsky [V07] computes $R_{\mathrm{tr}}(V-0)/C$, where V is the direct sum of n copies of the reduced regular representation $\mathbb{A}^{\ell-1}$ of C. Next, he observes that $S^C_{\mathrm{tr}}\mathbb{L}^n$ is $\mathbb{A}^1$-equivalent to $\mathbb{L}^n \otimes R_{\mathrm{tr}}(V-0)/C[1]$. Taking $(\mathbb{Z}/\ell)^\times$-invariants, it follows that $S^G_{\mathrm{tr}}(\mathbb{L}^n)$ is $\mathbb{A}^1$-equivalent to the motive displayed in 4.9. Since $[\Sigma_\ell\colon\ G] = (\ell-2)!$, $S^\ell_{\mathrm{tr}}(\mathbb{L}^n)$ is a summand. Using the computation of $B\mu_\ell$ and $B\Sigma_\ell$ in [RPO], one shows that each summand of $S^G_{\mathrm{tr}}(\mathbb{L}^n)$ belongs to $S^\ell_{\mathrm{tr}}(\mathbb{L}^n)$. ∎

Corollary 4.10. *When $R = \mathbb{Z}/\ell$ and $a > 0$, $S^\ell_{tr}(\mathbb{L}^a[b])[1] \to S^\ell_{tr}(\mathbb{L}^a[b+1])$ is a split injection for all b, and we have:*

$$S^\ell_{tr}(\mathbb{L}^a[1]) = \bigoplus_{i=1}^{a} \{\mathbb{L}^{a+i(\ell-1)}[1] \oplus \mathbb{L}^{a+i(\ell-1)}[2]\};$$

$$S^\ell_{tr}(\mathbb{L}^a[b]) = S^\ell_{tr}(\mathbb{L}^a[1])[b-1] \oplus \bigoplus_{i=1}^{k} \{\mathbb{L}^{a\ell}[2i\ell+1] \oplus \mathbb{L}^{a\ell}[2i\ell+2]\}, \quad b = 2k+1;$$

$$S^\ell_{tr}(\mathbb{L}^a[b]) = S^\ell_{tr}(\mathbb{L}^a[b-1])[1] \oplus \mathbb{L}^{a\ell}[b\ell], \quad b \geq 2 \text{ even.}$$

Proof. Set $T = \mathbb{L}^a[b]$. Voevodsky shows in [V07] that the cone of $(S^\ell_{\mathrm{tr}}T)[1] \to S^\ell_{\mathrm{tr}}(T[1])$ is : $T^{\otimes\ell}[2]$ for b even, and $T^{\otimes\ell}[\ell]$ for b odd. In the odd case, the boundary map is zero for weight reasons. In the even case, the boundary map is an element of $\mathrm{Hom}(T^{\otimes\ell}, S^\ell_{\mathrm{tr}}T) = \mathbb{Z}/\ell$. Using the topological realization functor, the topological calculations of Cartan [1] show that the boundary map is also zero. The result now follows by induction on b. ∎

Remark 4.10.1. The above formulas are incorrect for $a = 0$, where $\mathbb{L}^0 = R$; here we have $S^\ell_{tr}(R[1]) = 0$, and $S^\ell_{tr}(R[2]) \cong R[2\ell]$.

A *proper Tate motive* is a direct sum of motives of the form $\mathbb{L}^a[b]$ with $b \geq 0$. The category of proper Tate motives over a field R is idempotent complete, and closed in **DM** under $\otimes$.

Theorem 4.11. *When $R = \mathbb{Z}/\ell$, $S^\infty_{tr}(\mathbb{L}^n)$ is a proper Tate motive. For each a there are only finitely many terms of weight a.*

Proof. Combining 4.4, 4.8, 4.9 and 4.10 yields the theorem. □

Proposition 4.12. (Pure Künneth formula) *Let X and Y be pointed simplicial schemes such that $R_{tr}(Y)$ is a direct sum of motives $R(q_\alpha)[p_\alpha]$. Assume that for each q there are only finitely many α with $q_\alpha = q$. Then the Künneth homomorphism is an isomorphism:*

$$H^{**}(X,R) \otimes_{H^{**}(k,R)} H^{**}(Y,R) \to H^{**}(X \times Y, R).$$

Proof. By (2.3), $H^{n,i}(X \times Y, R) = \mathrm{Hom}_{\mathbf{DM}}(R_{\mathrm{tr}}(X \times Y), R(i)[n])$. Now $R_{\mathrm{tr}}(X \times Y)$ is the direct sum of the $R^{\mathrm{tr}}(X)(q_\alpha)[p_\alpha]$, and we claim that

$$\mathrm{Hom}(R_{\mathrm{tr}}(X)(q)[p], R(i)[n]) = \begin{cases} H^{n-p,i-q}(X,R) & \text{if } q \le i; \\ 0 & \text{if } q > i. \end{cases}$$

The case $X = \mathrm{Spec}(k)$ shows that $H^{**}(Y,R)$ is a free $H^{**}(k,R)$-module on finitely many generators γ_α in bidegrees (p_α, q_α), and the result follows.

To verify the claim, we may suppose that $p = 0$. Suppose first that $q \le i$. By the Cancellation Theorem [4, 16.25] we have $\mathrm{Hom}(M(q), R(i)) = \mathrm{Hom}(M, R(i-q))$ for any M in **DM**. In particular, $\mathrm{Hom}(R_{\mathrm{tr}}(X)(q), R(i)[n]) = \mathrm{Hom}(R_{\mathrm{tr}}(X), R(i-q)[n]) = H^{n,i-q}(X,R)$. Similarly, the case when $q > i$ reduces to the case $i = 0$, $q > 0$. Here $R_{\mathrm{tr}}(X)(q)$ is a summand of $R_{\mathrm{tr}}(X \times \mathbb{P}^q)$ and $H^{p,0}(-,R) = H^p_{\mathrm{Zar}}(-,R)$, so the result follows from $H^*_{\mathrm{Zar}}(X,R) \cong H^*_{\mathrm{Zar}}(X \times \mathbb{P}^q, R)$; see [RPO, 3.5]. □

Recall from 2.5 that $K_n = u\mathbb{L}^n$ represents $H^{2n,n}(-,\mathbb{Z})$, and that $\mathrm{char}(k) = 0$.

Corollary 4.13. *For all $n > 0$ the Künneth maps are isomorphisms:*

$$H^{**}(K_n, \mathbb{Z}/\ell) \otimes_{H^{**}} \cdots \otimes_{H^{**}} H^{**}(K_n, \mathbb{Z}/\ell) \xrightarrow{\simeq} H^{**}(K_n \times \cdots \times K_n, \mathbb{Z}/\ell).$$

This replaces the unproven "Lemma 2.3" in [MC/l]. Note that 4.13 is equivalent to: $\tilde{H}^{**}(K_n, \mathbb{Z}/\ell) \otimes_{H^{**}} \cdots \otimes_{H^{**}} \tilde{H}^{**}(K_n, \mathbb{Z}/\ell) \xrightarrow{\simeq} \tilde{H}^{**}(K_n \wedge \cdots \wedge K_n, \mathbb{Z}/\ell)$.

5 Lecture 5: Uniqueness of βP^n

The purpose of this lecture is to prove Theorem 5.1, which replaces the unproven "Theorem 2.1" in [MC/l]. As before, k is a field of characteristic 0, and P^n is the cohomology operation of bidegree $(2n(\ell-1), n(\ell-1))$ referred to in Lecture 2. (The n of this lecture is unrelated to the n in the norm residue homomorphism of the Bloch-Kato conjecture.)

Note that the Bockstein β is a derivation, βP^n is additive and commutes with simplicial suspension Σ by [RPO]. Thus by axiom 2.2(2), if $y \in H^{2n,n}(X, \mathbb{Z}/\ell)$ then $P^n(y) = y^\ell$ and:

$$\beta P^n(\Sigma y) = \Sigma\beta(P^n y) = \Sigma\beta(y^\ell) = \ell\Sigma\beta(y) = 0.$$

Thus βP^n satisfies (1) and (2) of the following uniqueness theorem.

Theorem 5.1. *Let* $\phi\colon H^{2n+1,n}(-,\mathbb{Z}) \to H^{2n\ell+2,n\ell}(-,\mathbb{Z}/\ell)$ *be a cohomology operation such that for all* X *and all* $x \in H^{2n+1,n}(X,\mathbb{Z})$*:*

1. $\phi(bx) = b\phi(x)$ *for* $b \in \mathbb{Z}$*;*

2. *If* $x = \Sigma y$ *for* $y \in H^{2n,n}(X,\mathbb{Z})$ *then* $\phi(x) = 0$.

Then ϕ *is a multiple of* βP^n.

Remark 5.1.1. In topology, βP^n is the image of a cohomology operation $H^{2n+1,n}(-,\mathbb{Z}) \to H^{2n\ell+2,n\ell}(-,\mathbb{Z})$. We will see the relevance of this in the next Lecture.

We saw in Lecture 2 (2.6) that the cohomology operations described in Theorem 5.1 correspond to elements of $H^{**}(BK_n, \mathbb{Z}/\ell)$. We saw in Lecture 4 (see 4.8) that a complete description of this cohomology is possible, but it is messy for $n > 1$. To cut down on the bookkeeping in the proof of 5.1, it is useful to introduce the notion of *scalar weight.*

The multiplicative action of the monoid $(\mathbb{Z}, \times)$ on the sheaf of $\mathbb{Z}$-modules $K_n = u\mathbb{L}^n$ and $BK_n = u\mathbb{L}^n[1]$ defines an action of $\mathbb{Z}/\ell^\times$ on $H^{**}(K_n, \mathbb{Z}/\ell)$ and $H^{**}(BK_n, \mathbb{Z}/\ell)$, at least when $n > 0$. The induced representations of $(\mathbb{Z}/\ell)^\times$ decompose the cohomology into the direct sum of its isotypical pieces, corresponding to the irreducible representations $\mathbb{Z}/\ell, \mu_\ell, \mu_\ell^{\otimes 2}, \dots, \mu_\ell^{\otimes \ell-2}$. We will say that the isotypical piece corresponding to $\mu_\ell^{\otimes s}$ has *scalar weight* s; an element $x \in H^{**}(K_n, \mathbb{Z}/\ell)$ has scalar weight s if $a \cdot x = a^s x$ for all $a \in \mathbb{Z}/\ell$.

Recall from 4.6(b) that $K_n \simeq S^\infty(V_n)$, and that $R_{\mathrm{tr}}(K_n) \simeq \oplus R_{\mathrm{tr}}(\widetilde{S}^i V_n) = \oplus S^i_{\mathrm{tr}}(\mathbb{L}^n)$.

Theorem 5.2. *Under the decomposition* $H^{**}(K_n, \mathbb{Z}/\ell) = \oplus H^{**}(\widetilde{S}^i V_n, \mathbb{Z}/\ell)$*, the summands* $H^{**}(\widetilde{S}^i V_n, \mathbb{Z}/\ell) \cong \mathrm{Hom}_{\mathbf{DM}}(S^i_{tr}\mathbb{L}^n_R, R(*)[*])$ *have scalar weight* $i \mod (\ell-1)$.

Lemma 5.3. *If* $R(q)[p]$ *is a summand of* $S^m_{tr}(\mathbb{L}^n)$*, and* $m \equiv s \mod (\ell-1)$ *for* $0 \le s < \ell-1$*, then:*

(a) $q \ge ns$*, with equality iff* $m < \ell$

(b) $q \ge n(\ell-1)$ *if* $s = 0$*, with equality iff* $m = \ell - 1$.

(c) $p \ge 2q \ge 2n$

Proof. Recall from Theorem 4.16 that $R(q)[p] = \mathbb{L}^q[b]$ for $b \ge 0$, so $p = 2q + b$. Hence (a) and (b) imply (c). If $m < \ell$ then $S^m_{\mathrm{tr}}(\mathbb{L}^n) = \mathbb{L}^{mn}$ and $q = mn$ by 4.7.1. This yields the 'if' parts. To prove the 'only if' parts of (1) and (2), suppose that $m \ge \ell$ and write $m = \sum m_i \ell^i$, noting that $\sum m_i > m_0$, $\sum m_i \equiv m \mod (\ell-1)$. We also have $q \ge (\sum m_i)n + (\ell-1)$ by Proposition 4.8. Since $\sum m_i \ge s$, we have $q > ns$. If $s = 0$ then $\sum m_i \ge \ell - 1$ and we have $q \ge (n+1)(\ell-1)$. □

Remark 5.3.1. These are the equations (2.6), (2.7) and (2.8) of [MC/l], strengthened to inequalities when $m \ge \ell$.

We now turn to the cohomology of $K_n \wedge \dots \wedge K_n$ in scalar weight 1. The following presentation is due to Voevodsky and is taken from [MC/l].

Lemma 5.4. *The scalar weight $s = 1$ part of $H^{p,q}(K_n^{\wedge r}, \mathbb{Z}/\ell)$ vanishes if $q < n\ell$ and $r \geq 2$, and also if $q = n\ell$ and $p < 2n\ell$.*

Proof. ([MC/l, 2.7 and 2.8]) By 4.13, 4.11 and 5.2 it suffices to consider $x_1 \otimes \cdots \otimes x_r$ where the x_i are in $\operatorname{Hom}(S^{a_i}_{\mathrm{tr}}\mathbb{L}^n, R(q_i)[p_i])$, $\Sigma p_i = p$, $\Sigma q_i = q$ and $\Sigma a_i \equiv 1 \mod (\ell - 1)$. If $q < n\ell$ then $a_i \not\equiv 0$ by 5.3(b) and we must have $\Sigma a_i \geq \ell$, which is excluded by 5.3(a) as $q \geq n\Sigma a_i$. This establishes the case $q < n\ell$. When $q = n\ell$, the vanishing comes from 5.3(c). □

We now analyze the motivic cohomology $H^{2n\ell+2,n\ell}(BK_n, \mathbb{Z}/\ell)$, where BK_n is the simplicial classifying space $[r] \mapsto (K_n)^r$.

$$* \Longleftarrow K_n \Lleftarrow K_n \times K_n \cdots$$

There is a standard spectral sequence for the cohomology of a simplicial space with coefficients in $\mathbb{Z}/\ell(n\ell)$; for BK_n it has

$$E_1^{r,s} = \widetilde{H}^{s,n\ell}(K_n^{\wedge r}, \mathbb{Z}/\ell) \Longrightarrow H^{r+s,n\ell}(BK_n, \mathbb{Z}/\ell). \tag{5.5}$$

For $n \geq 1$, the spectral sequence converges because $K_n^{\wedge r}$ is nr-fold T-connected so that $E_1^{r,s} = 0$ for $r > \ell$. This in turn follows from the fact [RPO, 3.7] that K_n is n-fold T-connected, together the Künneth formula 4.13. (This is the argument of [MC/l, 2.6].)

The relevant part of the spectral sequence looks like this, using Lemma 5.4:

	0			
	0	$H^{2n\ell+1,n\ell}(K_n) \to$		
$s = 2n\ell$	0	$H^{2n\ell,n\ell}(K_n) \ \to$	$H^{2n\ell,n\ell}(K_n \wedge K_n) \to$	$H^{2n\ell,n\ell}(K_n^{\wedge 3}) \to$
	0	0	(nothing in scalar weight 1 below here)	

Recall from [RPO, 3.7] that $H^{2n,n}(K_n, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$ on the fundamental class α.

Example 5.5.1. $\gamma = \{(\alpha\otimes 1 + 1\otimes\alpha)^\ell - \alpha^\ell\otimes 1 - 1\otimes\alpha^\ell\}/\ell = \alpha^{\ell-1}\otimes\alpha + \cdots + \alpha\otimes\alpha^{\ell-1}$ is an element of $H^{2n\ell,n\ell}(K_n\wedge K_n, \mathbb{Z}/\ell)$. A calculation shows that $d_1^{2,2n\ell}$ maps γ to zero in $H^{2n\ell,n\ell}(K_n\wedge K_n\wedge K_n, \mathbb{Z}/\ell)$. (Formally this follows from $\delta(\alpha^\ell) = -\ell\gamma$ and $\delta^2 = 0$).

Lemma 5.6. ([MC/l, 2.9]) *Let D_r denote the subset of elements of scalar weight one in $H^{2n\ell,n\ell}(K_n^{\wedge r}, \mathbb{Z}/\ell)$. Then D_r is the $\mathbb{Z}/\ell$-vector space generated by monomials of the form $\alpha^{i_1}\wedge\ldots\wedge\alpha^{i_r}$, where $i_1+\cdots+i_r = \ell$ and each $i_j > 0$.*

Proof. The monomials are linearly independent by the Künneth formula 4.13. Lemma 5.4 implies that D_r is generated by elements of the form $x_1\otimes\cdots\otimes x_r$ whre the $x_i \in \mathrm{Hom}(S^{a_i}\mathbb{L}^n, R(q_i)[p_i])$. By 5.3(b), at most one x_i can have scalar weight 0, and only if $r = 2$.

If $r = 2$ and $a_1 \equiv 0$ then $a_2 \equiv 1$. Then by 3.4(a, b) we must have $q_1 = n(\ell-1)$ and $q_2 = n$, and $p_i = 2q_i$. By 3.4(b) and (4.7) we must have $x_1 = \alpha^{\ell-1}$ and $x_2 = \alpha$. Thus we are reduced to the case in which all $a_i \not\equiv 0 \mod (\ell-1)$. By 5.3(a, c) and $q = n\ell$ we must have $\Sigma a_i = \ell$, $q_i = na_i$ and $p_i = 2q_i$. Since $S^{a_i}_{\mathrm{tr}}(\mathbb{L}^n) = \mathbb{L}^{na_i}$ by (4.7) we must have $x_i = \alpha^{a_i}$ up to scalars. □

Proof of Theorem 5.1: (Voevodsky) We regard ϕ as an element of $H^{2n\ell+2,n\ell}(BK_n, \mathbb{Z}/\ell)$. Condition 5.1(1) says that ϕ has scalar weight one. Condition 5.1(2) says that ϕ (like βP^n) is in the kernel of the map

$$H^{2n\ell+2,n\ell}(BK_n, \mathbb{Z}\ell) \to H^{2n\ell+2,n\ell}(\Sigma K_n, \mathbb{Z}\ell) = H^{2n\ell+1,n\ell}(K_n, \mathbb{Z}/\ell)$$

defined by the inclusion of ΣK_n as the degree one part of BK_n. That is, ϕ and βP^n lie in the kernel of the edge map in the spectral sequence (5.5).

Voevodsky observes in [MC/l] that, by a formal calculation of Lazard [3, 1.21], the kernel of $E_1^{2,2n\ell} \to E_1^{3,2n\ell}$ is $\mathbb{Z}/\ell$ in scalar weight one, on the cycle γ displayed in 5.5.1. Since βP^n represents a nonzero element of $H^{2n\ell+2,n\ell}(BK_n, \mathbb{Z}/\ell)$ by [1] and [RPO], it follows that every such element ϕ must be a multiple of βP^n. □

6 Lecture 6

In this lecture, we will construct a Rost motive for $\underline{a}$ in the sense of Definition 3.4. As we saw in Lecture 3, this suffices to verify the Bloch-Kato conjecture.

Let X be a Rost variety (Definition 3.1), and write $\mathfrak{X}$ for the simplicial Čech variety associated to X. In 1.5, we produced a nonzero $\delta \in H^{n,n-1}(\mathfrak{X}, \mathbb{Z}/\ell)$, and used it in Proposition 3.3 to construct a nonzero element μ of $H^{2b+1,b}(\mathfrak{X}, \mathbb{Z})$. Now any $z \in H^{2b+1,b}(\mathfrak{X})$ can be interpreted as a map $\mathfrak{X} \to \mathbb{L}^b[1]$ in **DM**; tensoring with $\mathfrak{X}$, and using $\mathfrak{X} \otimes \mathfrak{X} \cong \mathfrak{X}$ yields a map $\mathfrak{X} \to \mathfrak{X} \otimes \mathbb{L}^b[1]$, which we also call z, and fit into a triangle

$$\mathfrak{X} \otimes \mathbb{L}^b \xrightarrow{x} A \xrightarrow{y} \mathfrak{X} \xrightarrow{z} \mathfrak{X} \otimes \mathbb{L}^b[1]. \tag{6.1}$$

Applying $\Sigma_{i-1} \subset \Sigma_i$ to $A^{\otimes i}$, we get a corestriction map $S^{i-1}(A) \otimes A \to S^i(A)$. There is also a transfer map $\mathrm{tr}\colon S^i(A) \to S^{i-1}(A) \otimes A$, induced by the endomorphism

$$a_1 \otimes \cdots \otimes a_i \longmapsto \Sigma(\cdots \otimes \hat{a}_j \otimes \cdots) \otimes a_j$$

of $A^{\otimes i}$. Now $S^i(A) \cong \mathfrak{X} \otimes S^i(A)$. Composing tr with $1 \otimes y$ yields a map $u\colon S^i(A) \to S^{i-1}(A)$; composing $1 \otimes x$ with corestriction yields a map $v\colon S^{i-1}(A) \otimes \mathbb{L}^b \to S^i(A)$.

Lemma 6.2. *If $i < \ell$ and $1/(\ell-1)! \in R$, the maps u and v fit into triangles*

(a) $S^{i-1}(A) \otimes \mathbb{L}^b \xrightarrow{v} S^i(A) \xrightarrow{S^i y} \mathfrak{X} \xrightarrow{s} S^{i-1}(A) \otimes \mathbb{L}^b[1]$.

(b) $\mathfrak{X} \otimes \mathbb{L}^{bi} \xrightarrow{S^i x} S^i(A) \xrightarrow{u} S^{i-1}(A) \xrightarrow{r} \mathfrak{X} \otimes \mathbb{L}^{bi}[1]$.

Proof. This is proven in [MC/l, 3.1] using the slice filtration on $A^{\otimes i}$. □

Setting $D = S^{\ell-2}(A)$ and $M = S^{\ell-1}(A)$, we see that M satisfies property 3.4(c) of a Rost motive. The composition of s and $r \otimes 1$ yields a map (for $i = \ell - 1$)

$$\phi(z)\colon \mathfrak{X} \xrightarrow{s} D \otimes \mathbb{L}^b[1] \xrightarrow{r \otimes 1} \mathfrak{X} \otimes \mathbb{L}^{b\ell}[2] \to \mathbb{L}^{b\ell}[2]$$

i.e., an element of $H^{2b\ell+2,b\ell}(\mathfrak{X}, \mathbb{Z}_{(\ell)})$. Consider the function $z \mapsto \phi(z)$.

Proposition 6.3. (Voevodsky) *The function* $\phi\colon H^{2b+1,b}(\mathfrak{X},\mathbb{Z}) \to H^{2b\ell+2,b\ell}(\mathfrak{X},\mathbb{Z}_{(\ell)})$ *extends to a cohomology operation* $\phi\colon H^{2b+1,b}(-,\mathbb{Z}) \to H^{2b\ell+2,b\ell}(-,\mathbb{Z}_{(\ell)})$ *satisfying*

(a) $\phi(az) = a^\ell\phi(z)$ *for* $a \in \mathbb{Z}$*;*

(b) $\phi(\Sigma y) = 0$ *for* $y \in H^{2b,b}(-,\mathbb{Z})$*.*

Proof. This is the content of 3.2, 3.5 and 3.6 of [MC/l]. □

Corollary 6.4. *The mod–*ℓ *reduction* $\bar\phi$ *of* ϕ*, regarded as a motivic cohomology operation* $H^{2b+1,b}(-,\mathbb{Z}) \to H^{2b\ell+2,b\ell}(-,\mathbb{Z}/\ell)$*, is a multiple of* βP^b*.*

Proof. Combine Theorem 5.1 and Proposition 6.3. □

Remark 6.4.1. It is easy to show that $\bar\phi \neq 0$, so that $\bar\phi(x) = c\beta P^b(\bar x)$ for a nonzero $c \in \mathbb{Z}/\ell$.

Lemma 6.5. *There are maps* $\lambda\colon \mathbb{Z}_{tr}(X) \to S^{\ell-1}(A)$ *such that the inclusion* $\iota\colon X \to \mathfrak{X}$ *factors in* **DM** *as:*

$$\mathbb{Z}_{tr}(X) \xrightarrow{\lambda} S^{\ell-1}(A) \xrightarrow{S^{\ell-1}y} \mathfrak{X}.$$

Proof. (Voevodsky) [MC/l, 5.11] Applying $\mathrm{Hom}_{\mathbf{DM}}(\mathbb{Z}_{\mathrm{tr}}X,-)$ to the triangle (6.1) defining A yields the exact sequence

$$\mathrm{Hom}(X,A) \xrightarrow{y} \mathrm{Hom}(X,\mathfrak{X}) \xrightarrow{z} \mathrm{Hom}(X,\mathbb{L}^b[1]) = 0;$$

the group on the right vanishes since it equals $H^{2b+1,b}(X,\mathbb{Z}) = 0$. Hence ι factors through some $\lambda_1\colon \mathbb{Z}_{\mathrm{tr}}X \to A$. Similarly, triangle 6.2(b) yields exact sequences

$$\mathrm{Hom}(X,S^i(A)) \xrightarrow{u} \mathrm{Hom}(X,S^{i-1}(A)) \to \mathrm{Hom}(X,\mathfrak{X}\otimes\mathbb{L}^{bi}[1]) = 0.$$

The group on the right is $H^{2bi+1,bi}(X,\mathbb{Z}) = 0$. By induction, there are maps $\lambda_i\colon \mathbb{Z}_{\mathrm{tr}}(X) \to S^i(A)$ for $i \le \ell-1$ such that $\lambda_{i-1} = u\lambda_i$. By the construction of u, $yu^i = S^iy\colon S^i(A) \to \mathfrak{X}$. □

Recall from 2.3.1 that $\mathrm{Hom}_{\mathbf{DM}}(\mathbb{L}^d, \mathbb{Z}_{\mathrm{tr}}(X)) = H_{2d,d}(X,\mathbb{Z}) \cong H^0(X,\mathbb{Z})$ by duality, so there is a fundamental class $\tau\colon \mathbb{L}^d \to \mathbb{Z}_{\mathrm{tr}}(X)$. Since $\mathfrak{X} \otimes X \cong X$, we may also view τ as a map from $\mathfrak{X} \otimes \mathbb{L}^d$ to $\mathbb{Z}_{\mathrm{tr}}(X)$.

Proposition 6.6. *The composition $\mathfrak{X} \otimes \mathbb{L}^d \xrightarrow{\tau} \mathbb{Z}_{tr}(X) \xrightarrow{\lambda} S^{\ell-1}(A)$ is not divisible by ℓ.*

Proof. (Voevodsky [MC/l, 5.12]) By the definition of ϕ in terms of the map s of 6.2(a), the restriction $S^{\ell-1}y^*(\phi)$ of ϕ to $S^{\ell-1}(A)$ is zero. By 6.4.1, βP^b also vanishes on $S^{\ell-1}(A)$. Since the Q_i anticommute we have $Q_i(\mu) = 0$ for $i \le n-2$.

Consider the element $\alpha = Q_{n-1}(\mu) \in H^{b\ell+2,b\ell}(\mathfrak{X}, \mathbb{Z}/\ell)$. By 3.3, $\alpha \neq 0$, and $Q_{n-1}(\alpha) = 0$ as $Q_i^2 = 0$. By the definition of Q_{n-1} we have

$$\alpha = Q_{n-1}(\mu) = Q_{n-2}(P^{\ell^{n-2}}(\mu)) = \cdots = \beta P^b(\mu),$$

so $(S^{\ell-1}y)^*(\alpha) = \beta P^n((S^{\ell-1}y)^*\mu) = 0$ in $H^{b\ell+2,b\ell}(S^{\ell-1}(A), R)$. By the Motivic Degree Theorem of [MC/l, 4.4], applied to the factorization in Lemma 6.5, the existence of $\alpha \neq 0$ implies the mod–ℓ reduction of the map $\lambda\tau : \mathfrak{X} \otimes \mathbb{L}^d \to S^{\ell-1}(A)$ is nonzero. □

Because $\mu\colon \mathfrak{X} \to \mathfrak{X} \otimes \mathbb{L}^b[1]$ is a map between Tate objects, it is self dual ($\mu = \mu^* \otimes \mathbb{L}^b$ under the identification of $\mathfrak{X}$ with $\mathfrak{X}^*$). It follows that $A \cong A^* \otimes \mathbb{L}^b$. Since $S^i(M) \cong (S^i M)^*$ for every M we also have $S^i(A) \cong S^i(A)^* \otimes \mathbb{L}^{bi}$. (See [MC/l, 5.7].) For the map λ of 6.5, we write $D\lambda$ for the dual map

$$D\lambda\colon\ S^{\ell-1}(A) \cong S^{\ell-1}(A)^* \otimes \mathbb{L}^d \xrightarrow{\lambda^*\otimes 1} \mathbb{Z}_{\mathrm{tr}}(X)^* \otimes \mathbb{L}^d \cong \mathbb{Z}_{\mathrm{tr}}(X).$$

Theorem 6.7. *The composition $\lambda \circ D\lambda$ is an isomorphism on $S^{\ell-1}(A)$ (with coefficients $\mathbb{Z}_{(\ell)}$ or $\mathbb{Z}/\ell$), and there is an integer $c \not\equiv 0 \pmod{\ell}$ so that the following diagram commutes:*

$$\begin{array}{ccc} S^{\ell-1}(A) & \xrightarrow{\lambda\circ D\lambda} & S^{\ell-1}(A) \\ {\scriptstyle S^{\ell-1}y}\downarrow & & \downarrow{\scriptstyle S^{\ell-1}y} \\ \mathfrak{X} & \xrightarrow{c} & \mathfrak{X}. \end{array}$$

In particular, $S^{\ell-1}(A)$ is a direct summand of $R_{tr}(X)$ for $R = \mathbb{Z}_{(\ell)}$ or $\mathbb{Z}/\ell$.

Proof. (Voevodsky [MC/l, 5.15]) From triangle 6.2(b) we have an exact sequence

$$\begin{array}{ccccc} \mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d,\mathfrak{X}\otimes\mathbb{L}^d) & \xrightarrow{S^{\ell-1}x} & \mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d,S^{\ell-1}A) & \xrightarrow{u} & \mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d,S^{\ell-2}A)=0. \\ c & \mapsto & \lambda\tau\not\equiv 0 \pmod{\ell} & & \end{array}$$

The fact that the right side is zero follows from the exact sequences of 6.2(a),

$$\mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d,S^{i-1}A\otimes\mathbb{L}^b)\xrightarrow{v}\mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d,S^iA)\to\mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d,\mathfrak{X}),$$

because the outer terms vanish — the right because maps between Tate objects cannot decrease weight, and the left by induction on i. Hence the map $\lambda\tau$ of Proposition 6.6 lifts to an element c of $\mathbb{Z}=\mathrm{Hom}(\mathfrak{X}\otimes\mathbb{L}^d,\mathfrak{X}\otimes\mathbb{L}^d)$. Since $\lambda\tau\not\equiv 0 \pmod{\ell}$ by 6.6, $c\not\equiv 0 \pmod{\ell}$. Dualizing $\lambda\tau=(S^{\ell-1}x)c$ yields the left square in the following diagram, since $S^{\ell-1}y$ is dual to $S^{\ell-1}x$ and ι is dual to τ: $\mathfrak{X}\otimes\mathbb{L}^d\to\mathbb{Z}_{\mathrm{tr}}(X)$, so $\iota\circ D\lambda$ is dual to $\lambda\tau$.

$$\begin{array}{ccccc} S^{\ell-1}(A) & \xrightarrow{D\lambda} & \mathbb{Z}_{\mathrm{tr}}(X) & \xrightarrow{\lambda} & S^{\ell-1}(A) \\ {\scriptstyle S^{\ell-1}y}\downarrow & & {\scriptstyle\iota}\downarrow & \swarrow{\scriptstyle S^{\ell-1}y} & \\ \mathfrak{X} & \xrightarrow{c} & \mathfrak{X}. & & \end{array}$$

The right triangle commutes by Lemma 6.5. □

Corollary 6.8. *When $R=\mathbb{Z}_{(\ell)}$, the maps λ and $D\lambda$ make $M=S^{\ell-1}(A)$ into a direct summand of $R_{tr}(X)$, and the following composition is an isomorphism:*

$$M\cong M^*\otimes\mathbb{L}^d\xrightarrow{\lambda^*}R_{tr}(X)^*\otimes\mathbb{L}^d\cong R_{tr}(X)\xrightarrow{\lambda}M.$$

Indeed, this is just a restatement of Theorem 6.7 in the form of axioms 3.4(a,b) of Lecture 3. Since axiom 3.4(c) holds by Lemma 6.2, M is a Rost motive. We saw in Lecture 3 that the Bloch-Kato conjecture follows form the existence of a Rost motive, so we are done.

References

[MC/2] V. Voevodsky, Motivic Cohomology with $\mathbb{Z}/2$ coefficients, *Publ. Math. IHES* 98 (2003), 1–57.

[MC/l] V. Voevodsky, On Motivic Cohomology with $\mathbb{Z}/l$ coefficients, *Preprint*, 2003. Available at http://www.math.uiuc. edu/K-theory/0639/.

[RPO] V. Voevodsky, Reduced Power operations in Motivic Cohomology, *Publ. Math. IHES* 98 (2003), 1–57.

[R-CL] M. Rost, Chain lemma for splitting fields of symbols, Preprint, 1998. Available at http://www.math. uni-bielefeld.de/~rost/chain-lemma.html

[R-BC] M. Rost, On the Basic Correspondence of a Splitting Variety, Preprint, 2006. Available at http://www.math. uni-bielefeld.de/~rost/chain-lemma.html

[SE] N. Steenrod and D. Epstein, *Cohomology Operations*, Annals of Math Studies 50, Princeton University Press, 1962.

[V07] V. Voevodsky, Operations in motivic cohomology II, under construction. The numbering in these lectures refers to the version dated October, 2006.

[W-ax] C. Weibel, Axioms for the Norm Residue Isomorphism. Preprint, 2006. Available at http://www.math.uiuc. edu/K-theory/0809/.

[1] H. Cartan, Sur les groupes d'Eilenberg Mac Lane, II, *Proc. Nat. Acad. Sci.* (USA) 40 (1954), 704–707.

[2] K. Kato, A generalization of local class field theory by using K-groups, *J. Fac. Sci. U. Tokyo* 27 (1980), 603–683.

[3] M. Lazard, Lois de groupes et analyseurs, *Ann. Sci. ENS* (Paris) 72 (1955), 299–400.

[4] C. Mazza, V. Voevodsky and C. Weibel, *Lecture Notes on Motivic Cohomology*, Clay Monographs in Math. **2**, AMS, 2006.

[5] J. Milnor, Algebraic K-theory and quadratic forms, *Inv. Math.* 9 (1970), 318–344.

[6] F. Morel, On the motivic π_0 of the sphere spectrum, pp. 219–260 in NATO ASI Series II, vol. 131, Kluwer, 2004.

[7] F. Morel and V. Voevodsky, $\mathbb{A}^1$-homotopy theory of schemes, *Publ. Math. IHES* 90 (1999), 45–143.

[8] J. Riou, Théorie homotopique des S-schémas, Memoire de DEA, 2002. Available at http:// www.math.jussieu.fr/∼riou/dea/

[9] A. Suslin and S. Joukhovitski, Norm Varieties, *J. Pure Appl. Alg.* 206 (2006), 245–276.

[10] A. Suslin and V. Voevodsky, Singular homology of abstract algebraic varieties, *Invent. Math.* 123 (1996), 61–94.

[11] A. Suslin and V. Voevodsky, Bloch-Kato conjecture and motivic cohomology with finite coefficients, pp. 117–189 in *The arithmetic and geometry of algebraic cycles (Banff, 1998)*, NATO ASI Ser. C, vol. 548, Kluwer, Dordrecht, 2000.

[12] V. Voevodsky, The Milnor Conjecture, Unpublished preprint, 1996. Available at http://www.math.uiuc. edu/K-theory/0170/.

[13] C. Weibel, A Road Map of Motivic Homotopy and Homology Theory, pp. 385–392 in NATO ASI Series II, vol. 131, Kluwer, 2004.