

## A NEGATIVE ANSWER TO A QUESTION OF BASS

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ABSTRACT. We address Bass' question, on whether  $K_n(R) = K_n(R[t])$  implies  $K_n(R) = K_n(R[t_1, t_2])$ . In a companion paper, we establish a positive answer to this question when  $R$  is of finite type over a field of infinite transcendence degree over the rationals. Here we provide an example of an isolated surface singularity over a number field for which the answer to the Bass' question is “no” when  $n = 0$ .

### INTRODUCTION

In 1972, H. Bass posed the following question (see [2], question (VI)<sub>n</sub>):

Does  $K_n(R) = K_n(R[t])$  imply that  $K_n(R) = K_n(R[t_1, t_2])$ ?

Bass' question was inspired by Traverso's theorem [20], from which it follows that  $\text{Pic}(R) = \text{Pic}(R[t])$  implies  $\text{Pic}(R) = \text{Pic}(R[t_1, t_2])$ .

In our companion paper [5], we show that the answer to Bass' question is “yes” for rings of finite type over fields having infinite transcendence degree over  $\mathbb{Q}$ . In this paper, we give an example showing the answer is “no” in general, even when  $n = 0$ . That is, there is a ring  $R$  for which every finitely generated projective module over  $R[t]$  is the extension, up to stable isomorphism, of a projective module over  $R$ , but not every finitely generated projective module over  $R[t_1, t_2]$  is so extended.

Our example is the isolated hypersurface singularity

$$R = F[x, y, z]/(z^2 + y^3 + x^{10} + x^7y),$$

where  $F$  is any algebraic field extension of  $\mathbb{Q}$ . (The proof is given in Theorem 4.1.) This example was first studied by J. Wahl [21].

Our proof that  $R$  indeed gives a negative answer to Bass' question uses what we call *generalized du Bois invariants*,  $b^{p,q}$ , of an isolated singularity in characteristic zero; see (2.8). The (ordinary) du Bois invariants were introduced by Steenbrink [18] using the du Bois complexes  $\underline{\Omega}^p$ ,  $p \geq 0$ . They can equivalently be defined using

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sheaf cohomology in Voevodsky's *cdh* topology thanks to the natural isomorphism (see Lemma 2.1)

$$\mathbb{H}_{\text{zar}}^*(X, \underline{\Omega}_X^p) \cong H_{\text{cdh}}^*(X, \Omega^p).$$

The generalized du Bois invariants are defined as the cohomology of the complex obtained by patching together the du Bois complexes  $\underline{\Omega}^p$  and the higher cotangent complexes used to define André-Quillen homology. The Euler characteristics of these patched together complexes, written  $\chi^p$  for  $p \geq 0$ , turn out to be constant in suitably nice families (see Theorem 2.14). In particular, we prove in Proposition 4.3 that  $\chi^p(R_a)$  is independent of  $a \in F$ , where  $R_a = F[x, y, z]/(z^2 + y^3 + x^{10} + ax^7y)$ . Since the ring  $R_0$  is graded, the values of  $\chi^p(R_0) = \chi^p(R_1)$  are easy to compute, and these computations allow us to prove our assertion about  $R = R_1$ .

**Notation.** Throughout this paper,  $F$  denotes a field of characteristic zero. By “a scheme over  $F$ ” we mean a separated scheme of finite type over  $F$ . We write  $\text{Sch}/F$  for the category of all such schemes. Unless otherwise stated, Hochschild homology and modules of Kähler differentials will be taken relative to  $F$ . That is, we write  $\Omega_X^q$  and  $HH_q(X)$  for  $\Omega_{X/F}^q$  and  $HH_q(X/F)$ .

## 1. ON *cdh*-COHOMOLOGY AND NIL $K$ -THEORY

For any functor  $G$  from rings to an abelian category,  $NG$  is the functor with  $NG(R)$  defined to be the kernel of the map  $G(R[t]) \rightarrow G(R)$  induced by evaluation at  $t = 0$ . Since  $G(R[t]) \rightarrow G(R)$  is split by the canonical map  $G(R) \rightarrow G(R[t])$ , the functor  $NG$  is a summand of the functor  $R \mapsto G(R[t])$ . We define  $N^2G = N(NG)$ .

It is convenient to phrase Bass' question in terms of Bass' Nil groups,  $NK_*(R)$ , as follows:

$$\text{Does } NK_n(R) = 0 \text{ imply that } N^2K_n(R) = 0?$$

Our example uses the following theorem from our companion paper [5]. (The notation in this theorem is discussed below; the particular forms of  $V$  and  $W$  in this theorem reflect extra structure not relevant for this paper.)

**Theorem 1.1** ([5, Theorems 0.1 and 0.7]). *Let  $R$  be a normal domain of dimension 2 that is of finite type over  $\mathbb{Q}$ . Then, letting  $V$  and  $W$  denote the countably infinite-dimensional  $\mathbb{Q}$  vector spaces  $t\mathbb{Q}[t]$  and  $\Omega_{\mathbb{Q}[t]}^1$ , we have:*

- a)  $NK_0(R) \cong H_{\text{cdh}}^1(R, \Omega^1) \otimes_{\mathbb{Q}} V$ .
- b)  $NK_{-1}(R) \cong H_{\text{cdh}}^1(R, \mathcal{O}) \otimes_{\mathbb{Q}} V$ .
- c) *If  $NK_0(R) = 0$ , then  $K_0(R[t_1, t_2]) \cong K_0(R) \oplus (NK_{-1}(R) \otimes_{\mathbb{Q}} W)$ .*

In particular, for  $R$  as in Theorem 1.1, the answer to Bass' question with  $n = 0$  is “no” if and only if  $H_{\text{cdh}}^1(R, \Omega^1) = 0$  and  $H_{\text{cdh}}^1(R, \mathcal{O}) \neq 0$ .

Recall that the *cdh* topology on  $\text{Sch}/F$ , written  $(\text{Sch}/F)_{\text{cdh}}$ , is the Grothendieck topology generated by Nisnevich open covers and abstract blow-up squares [19]. If  $\mathcal{G}$  is a presheaf on  $\text{Sch}/F$ , by  $H_{\text{cdh}}^*(X, \mathcal{G})$ , we mean the *cdh*-sheaf cohomology of the *cdh*-sheafification  $\mathcal{G}$ . For example,  $H_{\text{cdh}}^*(X, \Omega^p)$ , for  $p \geq 0$ , refers to the *cdh*-cohomology of the *cdh*-sheafification of  $Y \mapsto \Omega_Y^p$ . (Of course,  $\Omega_Y^0 = \mathcal{O}_Y$ .) When  $X = \text{Spec } R$  for an  $F$ -algebra  $R$  of finite type over  $F$ , we usually write  $H_{\text{cdh}}^*(R, \mathcal{G})$  for  $H_{\text{cdh}}^*(\text{Spec } R, \mathcal{G})$ .

2. GENERALIZED DU BOIS INVARIANTS,  $\chi^p$  AND DEFORMATIONS

In this section, we construct invariants of isolated singularities, called the *generalized du Bois invariants*  $b^{p,q} \in \mathbb{N}$ , which for  $q > 0$  coincide with the du Bois invariants introduced by Steenbrink [18]. For isolated singularities that are also local complete intersections, for each fixed  $p$  only a finite number of the integers  $b^{p,q}$  are nonzero. Thus it makes sense to define  $\chi^p := \sum_q (-1)^q b^{p,q}$  in this situation. The main result of this section is Theorem 2.14, that the  $\chi^p$  are invariant under suitably nice deformations.

Recall that we work over a field  $F$  of characteristic zero. Several of the results we quote from here on, in particular, anything involving du Bois complexes have been proved under the assumption that  $F = \mathbb{C}$ . However, flat base change implies that they all remain valid over an arbitrary field  $F$  of characteristic 0.

Fix a scheme  $X$  of finite type over  $F$  and choose a proper simplicial hyperresolution  $\pi : Y_\bullet \rightarrow X$ . Following [7] we fix  $p$  and we consider the  $p$ -th *du Bois complex*

$$\underline{\Omega}_X^p = \mathbb{R}\pi_* \Omega_{Y_\bullet}^p.$$

Du Bois shows in [7] that the assignment  $X \mapsto \underline{\Omega}_X^p$  is natural in  $X$  up to unique isomorphism in the derived category. The relevance for us lies in the fact that the Zariski hypercohomology of the complex  $\underline{\Omega}_X^p$  computes  $H_{\text{cdh}}^*(X, \Omega^p)$ :

**Lemma 2.1.** *Let  $X$  be a scheme of finite type over  $F$  and  $p \geq 0$ . Then there is a natural isomorphism*

$$\mathbb{H}_{\text{zar}}^*(X, \underline{\Omega}_X^p) \cong H_{\text{cdh}}^*(X, \Omega^p).$$

A very similar observation for the  $h$ -topology has been made by Ben Lee [13]; the proof we give here is based upon the proof of [4, 4.1].

*Proof.* Recall that  $H_{\text{cdh}}^*(X, \Omega^p)$  is the Zariski hypercohomology of the complex  $\mathbb{R}a_* a^* \Omega^p|_X$ , where  $a : (\text{Sch}/F)_{\text{cdh}} \rightarrow (\text{Sch}/F)_{\text{zar}}$  is the morphism of sites and  $|_X$  denotes the restriction from the big Zariski site  $(\text{Sch}/F)_{\text{zar}}$  to  $X_{\text{zar}}$ . Let  $Y_\bullet \rightarrow X$  be a simplicial hyperresolution. By [6, 2.5], we have a quasi-isomorphism on  $X_{\text{zar}}$ ,

$$\Omega_{Y_n}^p \xrightarrow{\sim} \mathbb{R}a_* a^* \Omega^p|_{Y_n},$$

since each  $Y_n$  is smooth. Using also [4, 4.3], we have a diagram of equivalences

$$\mathbb{R}a_* a^* \Omega^p|_X \xrightarrow{\sim} \mathbb{R}\pi_*(\mathbb{R}a_* a^* \Omega^p|_{Y_\bullet}) \xleftarrow{\sim} \mathbb{R}\pi_* \Omega_{Y_\bullet}^p = \underline{\Omega}_X^p.$$

Applying  $\mathbb{H}_{\text{zar}}^*(X, -)$  yields  $H_{\text{cdh}}^*(X, \Omega^p) \cong \mathbb{H}_{\text{zar}}^*(X, \underline{\Omega}_X^p)$ . □

**Isolated singularities.** Suppose that  $\text{Sing}(X)$  is an isolated point  $x$ . Choose a good resolution  $\pi : Y \rightarrow X$ , meaning that  $Y$  is smooth,  $\pi$  is proper and an isomorphism away from  $x$ , and  $E = \pi^{-1}(x)_{\text{red}}$  is a normal crossings divisor with smooth components. Then by [7, 4.8, 4.11] we have a distinguished triangle

$$0 \rightarrow \underline{\Omega}_X^p \rightarrow \mathbb{R}\pi_* \Omega_Y^p \oplus \Omega_x^p \rightarrow \mathbb{R}\pi_* \underline{\Omega}_E^p \rightarrow 0.$$

To rewrite this, let  $E_1, \dots, E_t$  denote the (smooth) components of  $E$ , and define

$$(2.2) \quad Y_n = \begin{cases} Y \amalg x_0 & n = 0, \\ \prod_{i_1 < \dots < i_n} E_{i_1} \times_Y \dots \times_Y E_{i_n} & n > 0. \end{cases}$$

By [7, 4.10], the complex  $\underline{\Omega}_E^p$  is quasi-isomorphic to (the total complex of)

$$\underline{\Omega}_{Y_1}^p \rightarrow \underline{\Omega}_{Y_2}^p \rightarrow \dots$$

The maps in this complex are given by the usual alternating sum of restriction maps, since the complex arises from a coskeletal hyperresolution of  $E$ . Generically writing  $\pi : Y_n \rightarrow X$  for the canonical map from  $Y_n$  to  $X$ , we have

$$(2.3) \quad \underline{\Omega}_X^p \simeq \text{Tot} \left( \mathbb{R}\pi_* \underline{\Omega}_{Y_0}^p \rightarrow \mathbb{R}\pi_* \underline{\Omega}_{Y_1}^p \rightarrow \mathbb{R}\pi_* \underline{\Omega}_{Y_2}^p \rightarrow \cdots \right).$$

Now suppose that  $\dim(X) = 2$ . Because  $E_i \times_Y E_j \times_Y E_l = \emptyset$  for distinct  $i, j, l$  and  $\Omega_{E_i \times_Y E_j}^p = 0$  for  $i \neq j$  and  $p > 0$ , (2.3) reduces to  $\underline{\Omega}_X^p \simeq \text{Tot}(\mathbb{R}\pi_* \Omega_Y^p \rightarrow \bigoplus_i \mathbb{R}\pi_* \Omega_{E_i}^p)$  for  $p > 0$ , and

$$\underline{\Omega}_X^0 \simeq \text{Tot} \left( \mathbb{R}\pi_* \mathcal{O}_Y \oplus \mathcal{O}_x \rightarrow \bigoplus_i \mathbb{R}\pi_* \mathcal{O}_{E_i} \rightarrow \bigoplus_{i < j} \mathbb{R}\pi_* \mathcal{O}_{E_i \times_Y E_j} \right).$$

In other words, in the notation of [21],

$$(2.4) \quad \underline{\Omega}_X^0 \simeq \mathbb{R}\pi_* \mathcal{O}_Y(-E) \oplus \mathcal{O}_x \quad \text{and} \quad \underline{\Omega}_X^p \simeq \mathbb{R}\pi_* (\Omega_Y^p(\log E)(-E)), \quad p > 0.$$

**Du Bois invariants.** Suppose for simplicity that  $X = \text{Spec } R$ , where  $R$  is a domain of finite type over  $F$ . For any  $p \geq 0$ , there is a map from the  $p$ -th higher cotangent complex  $\mathcal{L}_X^p$  (see [14, 3.5.4]) to the  $p$ -th du Bois complex  $\underline{\Omega}_X^p$ , obtained by composing the isomorphism  $H_0(\mathcal{L}_X^p) \cong \Omega_X^p$  and the natural map  $\Omega_X^p \rightarrow H^0(\underline{\Omega}_X^p)$ .

**Definition 2.5.** Define the cochain complex  $C_X^p$  of quasi-coherent  $\mathcal{O}_X$ -modules by

$$C_X^p := \text{cone}(\mathcal{L}_X^p \rightarrow \underline{\Omega}_X^p).$$

That is, we have a triangle  $\mathcal{L}_X^p \rightarrow \underline{\Omega}_X^p \rightarrow C_X^p \rightarrow \mathcal{L}_X^p[1]$ .

In the language of [5], the complex  $C_X^p$  gives the homotopy fiber  $\mathcal{F}_{HH}$  of the map from the Hochschild complex of  $X$  to its *cdh*-fibrant replacement:

$$(2.6) \quad \mathbb{H}^i(C_X^p) = H^{i+1-p}(\mathcal{F}_{HH}^{(p)}(X)).$$

Note that the hypercohomology sheaves of  $C_X^p$  are coherent because the Kähler differentials are taken over  $F$ . Using Lemma 2.1, [14, 4.5.13] and [5, Lemma 3.4], we conclude that

$$(2.7) \quad \mathbb{H}^q(C_X^p) = \begin{cases} \mathbb{H}^q(X, \underline{\Omega}^p) & \text{for } q \geq 1, \\ \text{coker}(\Omega_X^p \rightarrow \mathbb{H}^0(X, \underline{\Omega}^p)) & \text{for } q = 0, \\ \text{ker}(\Omega_X^p \rightarrow \mathbb{H}^0(X, \underline{\Omega}^p)) & \text{for } q = -1, \\ D_{-1-q}^{(p)}(X) & \text{for } q \leq -2, \end{cases}$$

where  $D_n^{(p)}$  denotes *André-Quillen homology*. Recall that  $D_n^{(p)}(R) \cong HH_{p+n}^{(p)}(R)$ , where

$$HH_* = \prod_{p \geq 0} HH_*^{(p)}$$

is the Hodge decomposition of Hochschild homology.

If  $X$  has isolated singularities, then each of the hypercohomology modules  $\mathbb{H}^n(C_X^p)$  is of finite length. In this case we may define, following and expanding on Steenbrink’s definition [18], the *generalized du Bois invariants* to be the numbers

$$(2.8) \quad b^{p,q} = b_X^{p,q} = \text{length } \mathbb{H}^q(C_X^p), \quad \text{for } p \geq 0 \text{ and } q \in \mathbb{Z}.$$

**Example 2.8.1.** For  $p = 0$ , we have  $b^{0,q} = 0$  if  $q < 0$ . When  $R$  is a domain,  $b^{0,0} = \text{length}_R(R^+/R)$ , where  $R^+$  is the seminormalization of  $R$ , because  $\mathcal{L}^0 = \mathcal{O}_X$  and  $H_{\text{cdh}}^0(R, \mathcal{O}) = R^+$  by [5, 2.5]. If  $q > 0$ , then (2.4) yields  $b^{0,q} = h^q(\mathcal{O}_Y(-E))$ .

If, moreover,  $X$  is locally a complete intersection, then  $HH_n^{(p)}(R) = 0$  for  $n \gg 0$  (see [8]); hence it follows from (2.7) that  $C_X^p$  is homologically bounded.

**Definition 2.9.** For a local complete intersection  $X \in \text{Sch}/F$  with only isolated singularities, define  $\chi^p(X)$  for  $p \geq 0$  to be the Euler characteristic of  $C_X^p$ :

$$\chi^p(X) := \sum_q (-1)^q b_X^{p,q}.$$

**Lemma 2.10.** *If  $X = \text{Spec}(R)$  for a ring  $R$  that admits a nonnegative grading with  $R_0 = k$ , then  $\sum (-1)^q b_X^{p,q} = 0$  for all  $q$ .*

*Proof.* The cases  $q = -1$ ,  $q = 0$ ,  $q > 0$  follow from (2.7) using the exact sequences

$$(2.11) \quad 0 \rightarrow \text{nil}(R) \rightarrow \text{tors } \Omega_R^1 \rightarrow \text{tors } \Omega_R^2 \rightarrow \text{tors } \Omega_R^3 \rightarrow \dots,$$

$$(2.12) \quad 0 \rightarrow (R^+/R) \rightarrow \Omega_{\text{cdh}}^1(R)/\Omega_R^1 \rightarrow \Omega_{\text{cdh}}^2(R)/\Omega_R^2 \rightarrow \dots,$$

$$(2.13) \quad 0 \rightarrow H_{\text{cdh}}^n(R, \mathcal{O}) \xrightarrow{d} H_{\text{cdh}}^n(R, \Omega^1) \xrightarrow{d} H_{\text{cdh}}^n(R, \Omega^2) \rightarrow \dots, \quad n > 0,$$

respectively, which are established in [5, Example 3.9]. For  $q < -1$  the formula follows from Goodwillie’s Theorem [22, 9.9.1].  $\square$

A key property of  $\chi^p$  is its invariance under deformations of the sort described in the following theorem. In it, we write  $X_s$  for the fiber of  $X$  over a point  $s \in S$ .

**Theorem 2.14.** *Suppose  $X \rightarrow S$  is a flat local complete intersection map of affine varieties with  $S$  smooth and such that the singular locus  $X_{\text{sing}}$  of  $X$  is finite and étale over  $S$ . Suppose in addition that one can find a projective map  $\pi : Y \rightarrow X$  which is an isomorphism away from  $X_{\text{sing}}$ , such that  $Y$  is smooth and such that the reduced, irreducible components  $E_1, \dots, E_m$  of  $Y \times_X X_{\text{sing}}$  are smooth over  $S$  and satisfy the property that each*

$$E_{i_1, \dots, i_t} := E_{i_1} \times_Y E_{i_2} \times_Y \dots \times_Y E_{i_t} \rightarrow S$$

*is smooth ( $1 \leq i_1, \dots, i_t \leq m$ ). Then  $\chi^p(X_s)$  is independent of the closed point  $s$ .*

*Suppose in addition that a finite group  $G$  acts on both  $X$  and  $Y$  and that  $\pi$  and  $X \rightarrow S$  are equivariant, where we declare the action of  $G$  on  $S$  to be trivial. Assume that  $X/G \rightarrow S$  is a flat local complete intersection such that  $(X/G)_{\text{sing}}$  is finite and étale over  $S$ . Then  $\chi^p(X_s/G)$  is independent of the closed point  $s \in S$ .*

*Proof.* In analogy with Definition 2.5, we use (2.3) to define a relative version of  $C^p$ :

$$C_{X/S}^p := \text{Tot} \left( \mathcal{L}_{X/S}^p \rightarrow \mathbb{R}\pi_* \underline{\Omega}_{Y_0/S}^p \rightarrow \mathbb{R}\pi_* \underline{\Omega}_{Y_1/S}^p \rightarrow \mathbb{R}\pi_* \underline{\Omega}_{Y_2/S}^p \rightarrow \dots \right),$$

where, as in (2.2),

$$(2.14a) \quad Y_n = \begin{cases} Y \amalg X_{\text{sing}} & n = 0, \\ \coprod_{i_1 < \dots < i_n} E_{i_1, \dots, i_n} & n > 0, \end{cases}$$

and  $\mathcal{L}_{X/S}^p$  is the  $p$ -th cotangent complex for  $X \rightarrow S$ ; the map  $\mathcal{L}_{X/S}^p \rightarrow \mathbb{R}\pi_* \underline{\Omega}_{Y_0/S}^p$  is induced by the composite of the natural maps  $\mathcal{L}_{X/S}^p \rightarrow \Omega_{X/S}^p \rightarrow \pi_* \underline{\Omega}_{Y_0/S}^p$ .

The complex  $C_{X/S}^p$  is a complex of quasi-coherent  $\mathcal{O}_X$ -modules with only finitely many nonzero homology sheaves, each of which is coherent. Moreover, each such homology sheaf is supported on the singular locus of  $X$ , which maps finitely to  $S$ . By restriction of scalars along the affine map  $X \rightarrow S$ , we may therefore regard

$C_{X/S}^p$  as a complex of quasi-coherent  $\mathcal{O}_S$ -modules whose homology is coherent. As such, this complex determines a class  $[C_{X/S}^p]$  in  $G_0(S) = K_0(S)$ . Explicitly, this class is the alternating sum of these homology modules.

For any point  $s \in S$ , let  $j_s : s \rightarrow S$  be the induced map of schemes and let  $j_s^* : K_0(S) \rightarrow K_0(s) \cong \mathbb{Z}$  be the pull-back map in  $K$ -theory. Note that for any  $s$ , the map  $j_s^*$  sends the class of a locally free  $\mathcal{O}_S$ -module to its rank. Consequently, the map  $j_s^* : K_0(S) \rightarrow \mathbb{Z}$  does not depend on the choice of  $s \in S$ . We now prove that for any closed point  $s \in S$ :

$$(2.14b) \quad j_s^*[C_{X/S}^p] = [C_{X_s/s}^p].$$

Since the class  $[C_{X_s/s}^p]$  in  $K_0(s) = \mathbb{Z}$  is  $\chi^p(X_s)$  when  $s \in S$  is a closed point, this will prove the first assertion of the theorem.

Note first of all that if  $\mathcal{F}^\bullet$  is any complex of quasi-coherent  $\mathcal{O}_S$ -modules with bounded, coherent homology, then  $j_s^*[\mathcal{F}^\bullet] = [\mathbb{L}j_s^*\mathcal{F}^\bullet]$ , where  $\mathbb{L}j_s^*$  denotes the left derived functor associated to  $j_s^*$ . For any  $n$ , let  $\tilde{\pi} : Y_n \rightarrow S$  be the structure map, which we are supposing to be smooth and hence flat. Thus  $\tilde{\pi}$  and  $j_s$  are Tor-independent over  $S$ . Consider the pullback diagram

$$\begin{array}{ccc} (Y_n)_s & \xrightarrow{\alpha_s} & Y_n \\ q \downarrow & & \downarrow \tilde{\pi} \\ s & \xrightarrow{j_s} & S. \end{array}$$

By [1, IV.3.1], we have  $\mathbb{L}j_s^*\mathbb{R}\tilde{\pi}_*\Omega_{Y_n/S}^p \simeq \mathbb{R}q_*\mathbb{L}\alpha_s^*\Omega_{Y_n/S}^p$ . Since  $Y_n/S$  is smooth,  $\Omega_{Y_n/S}^p$  is locally free and we have

$$\mathbb{L}\alpha_s^*\Omega_{Y_n/S}^p = \alpha_s^*\Omega_{Y_n/S}^p \cong \Omega_{(Y_n)_s/s}^p.$$

Hence

$$(2.14c) \quad \mathbb{L}j_s^*\mathbb{R}\tilde{\pi}_*\Omega_{Y_n/S}^p \simeq \mathbb{R}q_*\Omega_{(Y_n)_s/s}^p.$$

Similarly, it is a standard property of the cotangent complex that

$$j_s^*\mathcal{L}_{X/S}^p \simeq \mathbb{L}j_s^*\mathcal{L}_{X/S}^p \simeq \mathcal{L}_{X_s/s}^p.$$

Combining these, we get the formula

$$j_s^*[C_{X/S}^p] = \left[ \cdots \rightarrow \mathcal{L}_{X_s/s}^p \rightarrow \mathbb{R}q_*\Omega_{(Y_0)_s/s}^p \rightarrow \mathbb{R}q_*\Omega_{(Y_1)_s/s}^p \rightarrow \cdots \right].$$

Finally, if  $s$  is a closed point, then by (2.3) we have

$$\underline{\Omega}_{X_s}^p \simeq \left( \mathbb{R}q_*\Omega_{(Y_0)_s/s}^p \rightarrow \mathbb{R}q_*\Omega_{(Y_1)_s/s}^p \rightarrow \cdots \right)$$

and hence the formula  $j_s^*[C_{X/S}^p] = [C_{X_s/s}^p]$  of (2.14b), proving the first assertion.

Suppose now that a finite group  $G$  acts on  $X$  and  $Y$  as in the statement of the theorem. Let  $Y_n$  be as in (2.14a) above; then  $G$  acts on  $Y_n \rightarrow S$  and hence on  $\underline{\Omega}_{Y_n/S}^p$  and  $\mathbb{R}\tilde{\pi}_*\underline{\Omega}_{Y_n/S}^p$  for all  $n$ . For each  $s \in S$ , the group  $G$  acts also on  $\underline{\Omega}_{(Y_n)_s}^p$ .

Since  $G$  is a finite group and we are in characteristic 0, taking  $G$ -invariants is exact. This implies the key property we will need, proven in [7, 5.12], namely that

$$\underline{\Omega}_{(Y_n)_s/G}^p \simeq (\underline{\Omega}_{(Y_n)_s}^p)^G \simeq (\Omega_{(Y_n)_s}^p)^G.$$

Since taking  $G$ -invariants also commutes with  $\mathbb{R}q_*$ , this property implies that

$$(2.14d) \quad \mathbb{R}q_*(\underline{\Omega}_{(Y_n)_s/G}^p) \simeq \mathbb{R}q_*((\Omega_{(Y_n)_s}^p)^G) \simeq (\mathbb{R}q_*\Omega_{(Y_n)_s}^p)^G.$$

Define the analogue  $D_{X/S}^p$  of  $C_{X/S}^p$  by

$$D_{X/S}^p = \left( \mathcal{L}_{(X/G)/S}^p \rightarrow (\mathbb{R}\pi_*\Omega_{Y_0/S}^p)^G \rightarrow (\mathbb{R}\pi_*\Omega_{Y_1/S}^p)^G \rightarrow \dots \right).$$

Now taking  $G$ -invariants commutes with  $\mathbb{L}j_s^*$ . Using (2.14c) and (2.14d), we have

$$\mathbb{L}j_s^*((\mathbb{R}\tilde{\pi}_*\Omega_{Y_i/S}^p)^G) \simeq (\mathbb{L}j_s^*(\mathbb{R}\tilde{\pi}_*\Omega_{Y_i/S}^p))^G \simeq (\mathbb{R}q_*\Omega_{(Y_i)_s}^p)^G \simeq \mathbb{R}q_*(\underline{\Omega}_{(Y_i)_s/G}^p).$$

Finally, observe that a similar argument as that used to prove (2.3) shows that

$$\underline{\Omega}_{X_s/G}^p \simeq \left( \mathbb{R}q_*\underline{\Omega}_{(Y_0)_s/G}^p \rightarrow \mathbb{R}q_*\underline{\Omega}_{(Y_1)_s/G}^p \rightarrow \dots \right).$$

Indeed,  $X_s/G, Y_s/G$ , and the  $(E_i)_s/G$  satisfy the same hypotheses as do  $X, Y$ , and the  $E_i$ , except for smoothness, so that the results in [7, 4.8, 4.10, 4.11] apply. It follows that

$$j_s^*[D_{X/S}^p] \simeq [C_{(X_s/G)}^p].$$

Since the class of  $[C_{X_s/G}^p]$  in  $K_0(s) = \mathbb{Z}$  is  $\chi^p(X_s/G)$ , it is independent of  $s$ . □

### 3. ISOLATED (HYPER)SURFACE SINGULARITIES

In this section we consider the du Bois invariants of a two-dimensional isolated hypersurface singularity  $X$ . That is,  $X = \text{Spec } R$  where  $R = F[x, y, z]/(f(x, y, z))$  and  $\Omega_{R/F}^3 \cong R/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$  is supported at the origin (*i.e.*, the unique singular point  $x_0$  is defined by the maximal ideal  $(x, y, z)$ ). The analytic analogues of some of our results are due to Steenbrink and may be found in Wahl's paper [21].

We will need the following well known calculation of  $\Omega_R^p = \Omega_{R/F}^p$ . Recall that the *Tjurina number*  $\tau$  is:

$$\tau = \text{length}_R \left( R / \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \right).$$

**Lemma 3.1.** *Let  $X = \text{Spec } R$  be a 2-dimensional isolated hypersurface singularity. Then each of the following  $R$ -modules has length equal to  $\tau$ :*

$$\Omega_R^3 \cong \text{Ext}_R^1(\Omega_R^1, R) \cong \text{Ext}_R^2(\Omega_R^2, R), \quad \text{tors}(\Omega_R^2) \cong \text{Ext}_R^1(\Omega_R^2, R).$$

*Proof.* Write  $R = P/f$ , where  $P = F[x, y, z]$ , and consider the complex  $\mathcal{K}$  of free  $R$ -modules, whose boundary maps are induced by exterior multiplication with  $df$ , indexed with  $R$  in degree 0:

$$\mathcal{K}: \quad 0 \rightarrow R \xrightarrow{\wedge df} \Omega_P^1 \otimes_P R \xrightarrow{\wedge df} \Omega_P^2 \otimes_P R \xrightarrow{\wedge df} \Omega_P^3 \otimes_P R \rightarrow 0.$$

By [16, p. 326], the  $n$ -th cohomology of the complex  $\mathcal{K}$  is the torsion submodule of  $\Omega_R^n$ . In the isolated singularity case considered here, it follows from Lebelt's results [12] (see also [15, Prop. 1]) that  $\Omega_R^n$  is a torsionfree module for  $n \leq 1$ . In particular, we have the free resolutions:

$$\begin{aligned} 0 \rightarrow R \xrightarrow{\wedge df} \Omega_P^1 \otimes_P R \rightarrow \Omega_R^1 \rightarrow 0, \\ 0 \rightarrow R \xrightarrow{\wedge df} \Omega_P^1 \otimes_P R \xrightarrow{\wedge df} \Omega_P^2 \otimes_P R \rightarrow \Omega_R^2 \rightarrow 0. \end{aligned}$$

Moreover the perfect pairing  $\Omega_P^p \otimes_P \Omega_P^{3-p} \rightarrow \Omega_P^3 \cong P$  induces a perfect pairing  $\mathcal{K}^p \otimes_R \mathcal{K}^{3-p} \rightarrow \mathcal{K}^3 \cong R$ . From this we get an isomorphism of complexes  $\text{Hom}_R(\mathcal{K}, R)[-3] \cong \mathcal{K}$ . It follows that

$$\begin{aligned} \text{Ext}_R^1(\Omega_R^1, R) &= H^3(\mathcal{K}) = \Omega_R^3 \cong R / \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right), \\ \text{Ext}_R^1(\Omega_R^2, R) &= H^2(\mathcal{K}) = \text{tors } \Omega_R^2. \end{aligned}$$

By definition, the length of the first of these modules is  $\tau$ ; by [15, Thm. 3], the second module also has length  $\tau$ . □

Recall the definition of the (generalized) du Bois invariants  $b^{p,q}$  from (2.8).

**Proposition 3.2.** *Let  $X = \text{Spec } R$  be a 2-dimensional isolated hypersurface singularity. Then the following hold:*

- (a)  $b^{p,q} = 0$  unless  $p + q \in \{1, 2\}$ .
- (b)  $b^{1-q,q} = b^{2-q,q} = \tau$  for all  $q < 0$ .
- (c)  $b^{0,2} = 0$  and  $b^{0,1} = -\chi^0$ .

*Proof.* To prove (a), note that for  $q > 0$ , it is a particular case of a general statement for isolated singularities proved by Steenbrink in [18, Thm. 1], since  $b^{p,q}$  is the length of  $\mathbb{H}^q(X, \underline{\Omega}_X^p)$  by (2.7). In our case, Steenbrink’s result is immediate from Grauert-Riemenschneider vanishing [9, Satz 2.3] and from the fact, proved in [6, Prop. 2.6], that for an affine surface  $X$ ,

$$(3.3) \quad H_{\text{cdh}}^2(X, \Omega^p) = 0 \quad (p \geq 0).$$

If  $q = 0$  and  $p > 2$ , (a) holds since then  $a^*\Omega^p = 0$ . If  $q = p = 0$ , it holds since  $R$  is normal, hence seminormal. For  $q = -1$ , (a) holds because  $\Omega_R^p = 0$  for  $p > 3$  and  $R$  and  $\Omega_R^1$  are torsionfree; see [5, Lemma 5.6 and Remark 5.6.1]. For  $q \leq -2$ , we have

$$(3.4) \quad H_q(C_X^p) = D_{-1-q}^{(p)}(R) = HH_{p-q-1}^{(p)}(R) = \text{tors}(\Omega_R^{p+q+1}),$$

which is zero unless  $p + q \in \{1, 2\}$ , by a result of Michler [16].

Assertion (b) follows from (3.4) and the fact that the kernel of  $\Omega_R^n \rightarrow H_{\text{cdh}}^0(X, \Omega^n)$  is  $\text{tors}(\Omega_R^n)$  (see [5, Lemma 5.6 and Remark 5.6.1]), using [15, Thm. 3] (see Lemma 3.1).

For assertion (c), the vanishing of  $b^{0,2}$  is a particular case of (3.3). The other assertion follows from part (a) and the definition (see 2.9) of  $\chi^0$ . □

**Proposition 3.5.** *Let  $X = \text{Spec } R$  be a 2-dimensional isolated hypersurface singularity. Further let  $\pi : Y \rightarrow X$  be a good resolution,  $E$  the exceptional divisor,  $E_1, \dots, E_r$  its reduced irreducible components,  $g_i$  the genus of  $E_i$ , and  $l$  the number of loops in the incidence graph. Put  $g = \sum_i g_i$  and  $p_g = \text{length}_R H^1(Y, \mathcal{O}_Y)$ .*

- (a) *The map  $H_{\text{cdh}}^n(X, \mathcal{O}) \rightarrow H_{\text{cdh}}^n(Y, \mathcal{O}) = H^n(Y, \mathcal{O})$  is an isomorphism for  $n \neq 1$ , and an injection for  $n = 1$ . We have*

$$b^{0,1} = p_g - g - l.$$

- In particular,  $H_{\text{cdh}}^1(X, \mathcal{O}) \rightarrow H^1(Y, \mathcal{O})$  is an isomorphism  $\iff g = l = 0$ .*
- (b)  $H_{\text{cdh}}^n(X, \Omega^2) \cong H^n(Y, \Omega^2)$  for  $n \geq 0$ . In particular,  $H_{\text{cdh}}^n(X, \Omega^2) = 0$  for  $n \geq 1$ .
- (c)  $\text{Ext}_R^i(H^0(Y, \Omega^2), R) \cong H^i(Y, \mathcal{O}_Y)$ . In particular,  $\text{Ext}_R^2(H^0(Y, \Omega^2), R) = 0$ .
- (d)  $b^{1,0} \leq \tau$ .
- (e)  $b^{2,0} = \tau - p_g$  and  $\chi^2 = -p_g$ .

*Proof.* To prove (a), observe that  $R$  is normal and  $Y \rightarrow X$  is projective, so that  $R = H^0_{\text{cdh}}(X, \mathcal{O}) = H^0(Y, \mathcal{O})$  by Zariski's Main Theorem (and [5, Proposition 2.5]). Since  $Y \rightarrow X$  has fibers of dimension at most 1 and  $X$  is affine,

$$(3.6) \quad H^2(Y, \mathcal{F}) = H^0(X, \mathbb{R}^2\pi_*\mathcal{F}) = 0$$

for all coherent sheaves  $\mathcal{F}$ . In particular,  $H^2(Y, \mathcal{O}) = 0$ . Similarly,  $H^2_{\text{cdh}}(X, \mathcal{O}) = 0$  by [3, Theorem 6.1]. Since  $\text{Sing } X = \{x_0\}$ , we have a blow-up square

$$(3.7) \quad \begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow \\ x_0 & \longrightarrow & X. \end{array}$$

From the Mayer-Vietoris sequence associated to this square, we extract the short exact sequence

$$0 \rightarrow H^1_{\text{cdh}}(X, \mathcal{O}) \rightarrow H^1(Y, \mathcal{O}) \rightarrow H^1_{\text{cdh}}(E, \mathcal{O}) \rightarrow 0.$$

Hence  $b^{0,1} = \text{length}_R H^1(Y, \mathcal{O}) - \text{length}_R H^1_{\text{cdh}}(E, \mathcal{O})$ . Applying descent to the cover  $\coprod_i E_i \rightarrow E$ , we obtain  $\text{length}_R H^1_{\text{cdh}}(E, \mathcal{O}) = l + g$ .

For (b), the isomorphisms  $H^n_{\text{cdh}}(X, \Omega^2) \cong H^n(Y, \Omega^2)$  follow from the Mayer-Vietoris sequence associated to the square (3.7). By Grauert-Riemenschneider vanishing [9, Satz 2.3],  $\mathbb{R}\pi_*\Omega^2_Y \simeq \pi_*\Omega^2_Y$ , so  $H^n(Y, \Omega^2) = H^0(X, R^n\pi_*\Omega^2_Y)$  vanishes for  $n > 0$  because  $X$  is affine.

To prove (c), recall that  $\omega_X \cong \mathcal{O}_X[2]$  because  $X$  is an affine hypersurface. For any bounded complex of quasi-coherent sheaves  $\mathcal{F}^\bullet$  on  $Y$ , Grothendieck-Serre duality gives a quasi-isomorphism:

$$\mathbb{R}\pi_*\mathbb{R}\text{Hom}_Y(\mathcal{F}^\bullet, \Omega^2_Y) \simeq \mathbb{R}\text{Hom}_X(\mathbb{R}\pi_*\mathcal{F}^\bullet, \mathcal{O}_X).$$

Taking  $\mathcal{F}^\bullet = \Omega^p_Y$  and using the duality pairing on  $Y$ ,

$$\mathbb{R}\text{Hom}_Y(\Omega^p_Y, \Omega^2_Y) \simeq \text{Hom}_Y(\Omega^p_Y, \Omega^2_Y) \cong \Omega^{2-p}_Y,$$

we get a spectral sequence

$$(3.8) \quad \text{Ext}^i_R(H^j(Y, \Omega^p), R) \Rightarrow H^{i-j}(Y, \Omega^{2-p}).$$

Taking  $p = 2$  and using Grauert-Riemenschneider vanishing [9, Satz 2.3], which gives  $H^j(Y, \Omega^2) = 0$  for  $j > 0$ , we obtain the conclusion of (c):

$$\text{Ext}^i_R(H^0(Y, \Omega^2), R) \cong H^i(Y, \mathcal{O}_Y).$$

In particular, by (3.6),  $\text{Ext}^2_R(H^0(Y, \Omega^2), R) = 0$ .

To prove (d), recall that  $b^{1,0}$  is the length of the  $R$ -module  $L = \mathbb{H}^0(C^1_X)$ . Since  $b^{1,-1} = 0$  by Proposition 3.2, it follows from (2.7) that we have an exact sequence

$$(3.9) \quad 0 \rightarrow \Omega^1_R \rightarrow H^0_{\text{cdh}}(X, \Omega^1) \rightarrow L \rightarrow 0.$$

From (3.9) we get the exact sequence

$$(3.10) \quad \text{Ext}^1_R(\Omega^1_R, R) \rightarrow \text{Ext}^2_R(L, R) \rightarrow \text{Ext}^2_R(H^0_{\text{cdh}}(X, \Omega^1), R).$$

From the spectral sequence (3.8) with  $p = 1$ , we have an exact sequence

$$\text{Hom}_R(H^1(Y, \Omega^1), R) \xrightarrow{d_2} \text{Ext}^2_R(H^0(Y, \Omega^1), R) \rightarrow H^2(Y, \Omega^1).$$

Since the  $R$ -module  $H^1(Y, \Omega^1)$  is supported at  $x_0$ ,  $\text{Hom}_R(H^1(Y, \Omega^1), R) = 0$ . The right side also vanishes, by (3.6), so we get  $\text{Ext}^2_R(H^0(Y, \Omega^1), R) = 0$ .

By part (a), the map  $H_{\text{cdh}}^0(X, \Omega^1) \rightarrow H^0(Y, \Omega^1)$  is injective, so the map

$$\text{Ext}_R^2(H^0(Y, \Omega^1), R) \rightarrow \text{Ext}_R^2(H_{\text{cdh}}^0(X, \Omega^1), R)$$

is surjective and hence

$$\text{Ext}_R^2(H_{\text{cdh}}^0(X, \Omega^1), R) = 0.$$

From (3.10) we get that  $\text{Ext}_R^1(\Omega_R^1, R) \rightarrow \text{Ext}_R^2(L, R)$  is surjective and hence

$$\begin{aligned} b^{1,0} = \text{length}_R(L) &= \text{length}_R(\text{Ext}_R^2(L, R)) \\ &\leq \text{length}_R(\text{Ext}_R^1(\Omega_R^1, R)) \\ &= \tau, \text{ by Lemma 3.1.} \end{aligned}$$

To prove (e), define finite length  $R$ -modules  $N$  and  $M$  so that

$$(3.11) \quad 0 \rightarrow N \rightarrow \Omega_R^2 \rightarrow H^0(Y, \Omega^2) \rightarrow M \rightarrow 0$$

is exact. By definition (2.8) and the fact that  $R$  is Gorenstein, we get

$$(3.12) \quad b^{2,0} = \text{length}_R(M) = \text{length}_R(\text{Ext}_R^2(M, R)).$$

Because  $N$  has finite length,  $\text{Ext}^i(N, R) = 0$  for  $i < 2$ , and hence there are isomorphisms

$$\text{Ext}^i(\Omega_R^2/N, R) \xrightarrow{\cong} \text{Ext}^i(\Omega_R^2, R) \quad (i < 2).$$

Using this together with part (c) and (3.11), we get an exact sequence

$$0 \rightarrow H^1(Y, \mathcal{O}_Y) \rightarrow \text{Ext}_R^1(\Omega_R^2, R) \rightarrow \text{Ext}_R^2(M, R) \rightarrow 0.$$

Using this sequence, and taking into account Lemma 3.1 and (3.12), we get

$$b^{2,0} = \tau - \text{length}_R H^1(Y, \mathcal{O}) = \tau - p_g.$$

By Proposition 3.2(a,b), this yields  $\chi^2 = b^{2,0} - \tau = -p_g$ . □

#### 4. WAHL'S EXAMPLE

Using the general results of the preceding sections, we can now prove:

**Theorem 4.1.** *Let  $F$  be a field of characteristic 0 and*

$$R = F[x, y, z]/(z^2 + y^3 + x^{10} + ax^7y),$$

*for any nonzero  $a \in F$ . Then  $b^{0,1} = 1$  and  $b^{1,1} = 0$ . That is,*

- (a)  $H_{\text{cdh}}^1(R, \mathcal{O}) \cong F$  and
- (b)  $H_{\text{cdh}}^1(R, \Omega_{R/F}^1) = 0$ .

*In particular, if  $F$  is an algebraic field extension of  $\mathbb{Q}$ , then  $R$  gives a negative answer to Bass' question for  $n = 0$ :*

$$K_0(R) = K_0(R[t]) \text{ but } K_0(R[t_1, t_2]) \cong K_0(R) \oplus \text{st}F[s, t].$$

*Remark 4.1.1.* The *cdh* cohomology groups in question may also be computed using an explicit description of a resolution of singularities, together with the self-intersection numbers of the exceptional components. For the surface in Theorem 4.1 for all values of  $a$  (including 0), the resolution data was checked for us by Liz Sell and is displayed in Figure 1.

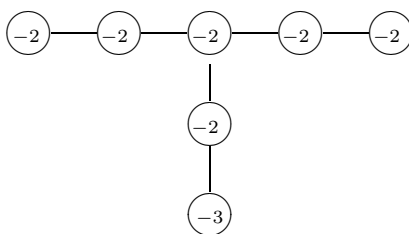


FIGURE 1. The resolution graph for  $z^2 + y^3 + x^{10} + ax^7y$

The proof we shall give here will be a straightforward application of the invariance of  $\chi^p$  (Theorem 2.14), applied to the specific example:

$$(4.2) \quad X = \text{Spec } F[x, y, z, t]/(z^2 + y^3 + x^{10} + tx^7y).$$

Consider the map  $X \rightarrow S = \text{Spec } F[t]$  induced by the obvious inclusion of rings, and write  $X_s$  for the fiber over  $s \in S$ . When  $s$  is the point  $t = a$  we have  $X_s = \text{Spec}(R)$  for the ring  $R$  in Theorem 4.1.

**Proposition 4.3.** *Let  $X$  be the affine variety of (4.2). Then the integer  $\chi^p(X_s)$  is independent of the choice of closed point  $s \in S$ .*

*Proof.* Since the value of  $\chi^p$  does not change upon passing to a finite extension, we may assume that  $F$  contains a primitive 30-th root of unity. Put

$$\tilde{X} = \text{Spec } F[u, v, w, t]/(u^{30} + v^{30} + w^{30} + tu^{21}v^{10}).$$

Let  $G = \mu_3 \times \mu_{10} \times \mu_{15}$  act on  $\tilde{X}$  by scalar multiplication on the variables  $x, y, z$  so that the assignment  $x = u^3, y = v^{10}$  and  $z = w^{15}$  identifies  $X$  with  $\tilde{X}/G$ .

The map  $X \rightarrow S$  is a flat local complete intersection whose singular locus is defined by  $x = y = z = 0$  and hence maps isomorphically onto  $S$ . The singular locus of  $\tilde{X}$  is defined by  $u = v = w = 0$  and hence also maps isomorphically onto  $S$ . Let  $\tilde{Y}$  be the blowup of  $\tilde{X}$  along its singular locus. Then

$$\tilde{Y} = \text{Proj} \left( \frac{F[t, u, v, w, A, B, C]}{(A^{30} + B^{30} + C^{30} + tuB^{10}A^{20}, uB - vA, uC - wA, vC - wB)} \right),$$

where  $t, u, v, w$  have degree 0 and  $A, B, C$  have degree 1. Direct calculations show that  $\tilde{Y} \rightarrow S$  is smooth and the fiber of  $\tilde{Y} \rightarrow \tilde{X}$  over  $\tilde{X}_{\text{sing}}$  is

$$\tilde{E} = \text{Proj } F[t, A, B, C]/(A^{30} + B^{30} + C^{30}) \cong S \times E_0,$$

where  $E_0$  is a smooth curve. We see that all the hypotheses of Theorem 2.14 are satisfied. □

Since  $X_0$  is quasi-homogeneous, its du Bois invariants may be computed, as shown in the following example. These calculations and the above results lead to the proof below of Theorem 4.1.

**Example 4.4.** The surface  $X_0 = \text{Spec } F[x, y, z]/(z^2 + y^3 + x^{10})$  is discussed by Wahl in [21, 4.4]. Elementary calculations, described in [21, 4.3], give that  $\tau = 1 \cdot 2 \cdot 9 = 18$ ,  $g = 0$  and

$$p_g = \dim \left( F[x, y, z]/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \right)_{\leq 2} = 1,$$

where  $f = z^2 + y^3 + x^{10}$ . Moreover, as with any isolated normal surface singularity defined by a nonnegatively graded ring, we have  $l = 0$  by [17, Theorem 2.3.1].

(Or, one may see that  $l = 0$  from the graph of Figure 1.) Using Lemma 2.10 and Proposition 3.5(a,e), this yields

$$b^{1,1} = b^{0,1} = p_g - g - l = 1, \quad b^{1,0} = b^{2,0} = \tau - p_g = 17.$$

By Proposition 3.2(a),  $\chi^0 = -b^{0,1} = -1$ ,  $\chi^1 = b^{1,0} - b^{1,1} = 16$ ,  $\chi^2 = -1$ .

*Proof of Theorem 4.1.* By Theorem 2.14,  $\chi^p(X_s)$  does not depend on  $s$ , and we write  $\chi^p = \chi_s^p$ . By Proposition 3.2(c),  $b^{0,1} = -\chi^0$  is also independent of  $s$ . For the choice  $s = 0$ , we have  $b_0^{0,1} = 1$  by [21, 4.4] (see Example 4.4). This proves assertion (a). To compute  $b^{1,1}$  when  $a \neq 0$ , we use the calculation of  $\tau(X_a)$  given in [21, 4.4]:

$$(4.5) \quad \tau(X_a) = \begin{cases} 18 & a = 0, \\ 16 & a \neq 0. \end{cases}$$

By Proposition 3.5(d),

$$(4.6) \quad b^{1,0}(X_a) \leq \tau(X_a) = 16 \quad \text{for all } a \neq 0.$$

By the invariance of  $\chi^1$  (see Proposition 4.3), Example 4.4 and (4.6), we have

$$\begin{aligned} 16 = \chi^1 &= b^{1,0}(X_a) - b^{1,1}(X_a) \\ &\leq 16 - b^{1,1}(X_a) \end{aligned}$$

for any  $a \neq 0$ , and hence  $0 = b^{1,1}(X_a) = \dim_F H_{\text{cdh}}^1(X_a, \Omega^1)$ .

The final assertion follows from Theorem 1.1. □

*Remark 4.7.* We conclude with a few remarks.

- (a) In (4.5) of the proof, we refer to the calculation of the Tjurina numbers  $\tau$  stated by Wahl in [21, 4.4]. These can be checked directly using the Tjurina function of the SINGULAR library `sing.lib` ([11], [10]).
- (b) Steenbrink uses analytic methods to define an invariant  $\alpha$  and proves that  $b^{1,1} = p_g - g - l - \alpha$ ; see [21, (1.9.1)]. Comparing with Proposition 3.2(a), and using GAGA, we see that  $\alpha = b^{0,1} - b^{1,1}$ . It is this invariant that is computed by Wahl in [21, 4.4].
- (c) If  $R_F = F[x, y, z]/(z^2 + y^3 + x^{10})$  and  $F$  is not algebraic over  $\mathbb{Q}$ , then  $NK_0(R_F)$  is nonzero. Indeed,  $NK_0(R_F) \cong \Omega_{F/\mathbb{Q}}^1 \otimes_F tF[t]$ . This follows from [5, (7.4)], which says that

$$NK_0(R_F) \cong NK_0(R_{\mathbb{Q}}) \otimes_{\mathbb{Q}} F \oplus NK_{-1}(R_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \Omega_{F/\mathbb{Q}}^1,$$

since  $NK_0(R_{\mathbb{Q}}) = 0$  and  $NK_{-1}(R_F) \cong tF[t]$  by Theorems 1.1(b) and 4.1(a).

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