

A Brown–Gersten Spectral Sequence for the K -Theory of Varieties with Isolated Singularities

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We produce two tools for computing the K -theory of varieties with isolated singularities: Mayer–Vietoris sequences for open coverings and a Brown–Gersten spectral sequence. © 1989 Academic Press, Inc.

The purpose of this note is to prove the following result about the K -theory of a “quasiprojective scheme” X with “isolated singularities.”

The phrase “quasiprojective scheme” will mean that X is quasiprojective as a scheme over a noetherian ring of finite Krull dimension; actually, it suffices that X have an ample family of line bundles in the sense of [SGA6, (II, 2.2.4)] and that $\dim(X)$ be finite. X has “isolated singularities” if $\text{sing}(X)$ is a finite set of closed points of X . Finally, for every integer n , \mathbf{K}_n denotes the Zariski sheafification of the algebraic K -theory presheaf $U \mapsto K_n(U)$.

MAIN THEOREM. *Let X be a reduced quasiprojective scheme with isolated singularities. Then:*

(1) *For every pair of open subsets U, V of X there is a long exact sequence*

$$\cdots \rightarrow K_{n+1}(U \cap V) \rightarrow K_n(U \cup V) \rightarrow K_n(U) \oplus K_n(V) \rightarrow K_n(U \cap V) \rightarrow \cdots$$

ending in

$$\cdots K_0(U \cap V) \rightarrow K_{-1}(U \cup V) \rightarrow K_{-1}(U) \oplus K_{-1}(V) \rightarrow K_{-1}(U \cap V) \rightarrow 0.$$

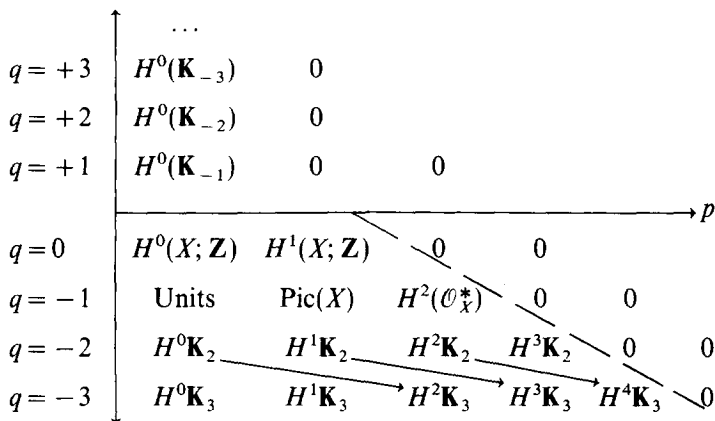
(2) *There is a right half-plane cohomological spectral sequence*

$$E_2^{p,q} = H_{\text{Zar}}^p(X; \mathbf{K}_{-q}) \Rightarrow K_{-p-q}(X).$$

Before delving into the proof of this theorem, we make some general

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comments about the spectral sequence. Because the sheaves \mathbf{K}_n are flasque for $n < 0$ (see [W1]), the nonzero terms of this spectral sequence lie on the positive q -axis and in the fourth quadrant. When X is quasiprojective over a field, the sheaves \mathbf{K}_n have a resolution of length $n + 1$ for $n \geq 0$ (see [PW]), so in the fourth quadrant, $E_2^{pq} = 0$ unless $q \leq 0$, $0 \leq p \leq (-q) + 1$.



When X is regular, we recover the spectral sequence described by Quillen [Q] and by Brown and Gersten [BG]. If X has just one isolated singularity, we recover the spectral sequence of Collino [Coll] (to see this, consult [Gill]). If $\dim(X) = 1$, the spectral sequence degenerates, yielding the short exact sequence of [W2].

The truly novel aspect of our spectral sequence is the fundamental role played by the negative K -group $K_{-1}(X)$, especially when X is quasiprojective over a field. In this case, the edge of the spectral sequence gives a filtration of $K_{-1}(X)$ by quotients of the $H^{p+1}(X; \mathbf{K}_p)$; this filtration is presented in an elementary way in [W1]. More importantly, the induced filtration on $K_0(X)$ has associated graded groups $E_\infty^{p,-p}$ which are subquotients of the $H^p(X; \mathbf{K}_p)$, rather than the quotients which Quillen obtained in [Q] for regular X .

To see the effect of $K_{-1}(X)$ more explicitly, let X be an four-dimensional variety with isolated singularities and let $SK_0(X)$ denote the kernel of

$$(\text{rank}, \det): K_0(X) \rightarrow H^0(X; \mathbf{Z}) \oplus H^1(X; \mathcal{O}_X^*).$$

There is an exact sequence

$$SK_0(X) \longrightarrow H^2(X; \mathbf{K}_2) \xrightarrow{d_2} H^4(X; \mathbf{K}_3) \longrightarrow K_{-1}(X).$$

In this case, the associated graded groups of $K_0(X)$ are, respectively, quotients of $H^4(X; \mathbf{K}_4)$ and $H^3(X; \mathbf{K}_3)$, a subgroup of $H^2(X; \mathbf{K}_2)$, $\text{Pic}(X) = H^1(X; \mathbf{K}_1)$, and $H^0(X; \mathbf{Z})$.

The main technical tool used to prove our result is the Brown–Gersten theorem of [BG]. We shall use the modification for spectra indicated in the footnotes of [BG] (see pp. 280, 282) and sketched by Thomason in [T]. First, we shall define our terms.

By a *fibrant spectrum* we shall mean an Ω -spectrum of fibrant simplicial sets, i.e., a sequence of pointed simplicial sets E_n and weak equivalences $E_n \rightarrow \Omega E_{n+1}$ such that each $E_n \rightarrow *$ is a fibration of simplicial sets. By a *presheaf of fibrant spectra* on a scheme X we mean a contravariant functor F from the poset of open subschemes of X to the category of fibrant spectra. Given such an F and an integer n , the homotopy groups $U \mapsto \pi_n F(U)$ give a presheaf of abelian groups on X , and we let $\pi_n F$ denote the associated sheaf of abelian groups on X .

A presheaf F of fibrant spectra has the *Mayer–Vietoris property* on X if the following square is homotopy cartesian for all open subschemes U and V of X :

$$\begin{array}{ccc} F(U \cup V) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U \cap V). \end{array}$$

Brown and Gersten used the phrase “pseudo-flasque presheaf” in [BG] for a presheaf having the Mayer–Vietoris property. The Mayer–Vietoris property is the analogue (for the homotopy category of presheaves of spectra) of the sheaf axiom for presheaves of abelian groups.

Given a presheaf F of fibrant spectra, there is a presheaf $\mathbb{H}(\cdot; F)$ of fibrant spectra which has the Mayer–Vietoris property, and a universal map $\eta: F \rightarrow \mathbb{H}(\cdot; F)$ from F to presheaves which have the Mayer–Vietoris property (universal in the sense made explicit in the theorem below). In addition, the stalks of F and $\mathbb{H}(\cdot; F)$ are weak equivalent, so that the sheaves $\pi_n F$ and $\pi_n \mathbb{H}(\cdot; F)$ agree. Finally, the natural things that spectral sequences converge to are the homotopy groups of $\mathbb{H}(X; F)$.

To construct $\mathbb{H}(\cdot; F)$, one takes an injective resolution of F to get a cosimplicial presheaf, and then forms the homotopy limit as in [BK]. The presheaf $\mathbb{H}(\cdot; F)$ was constructed in [BG, pp. 227, 281] as a “flasque resolution of F ,” in [T, Sect. 1] using a canonical Godement resolution, and in [Gill, Sect. 2] using Lubkin covers. To see that the presheaf $\mathbb{H}(\cdot; F)$ has the properties claimed in the preceding paragraph, the reader is encouraged to read [T, Sect. 1], especially 1.36, 1.41, and Example 1.49.

BROWN–GERSTEN THEOREM [BG, p. 283; T, 2.5]. *Let X be a noetherian scheme of finite Krull dimension, and let F be a presheaf of fibrant spectra on X . Then:*

(a) *There is a right half-plane cohomological spectral sequence*

$$E_2^{p,q} = H_{\text{Zar}}^p(X; \pi F_{-q}) \Rightarrow \pi_{-p-q} \mathbb{H}(X; F).$$

(b) *If F has the Mayer–Vietoris property, then the universal map*

$$\eta(X): F(X) \rightarrow \mathbb{H}(X; F)$$

is a weak homotopy equivalence, and the spectral sequence of (a) converges to $\pi_ F(X)$.*

(c) *If G is a connective presheaf having the Mayer–Vietoris property on X , then for every map $F \rightarrow G$ there is a map $\mathbb{H}(; F) \rightarrow G$, unique up to homotopy, such that the following triangle homotopy commutes:*

$$\begin{array}{ccc} F & \longrightarrow & G \\ \eta \downarrow & & \nearrow \\ \mathbb{H}(; F) & & \end{array}$$

It is time for some examples. Given a noetherian separated scheme X , let $\mathbf{M}(X)$ and $\mathbf{P}(X)$ denote the exact categories of coherent modules and of coherent vector bundles on X , respectively. Apply Quillen’s Q -construction $[Q]$ to get the categories $QM(X)$ and $QP(X)$. Feeding these to a friendly infinite loop space machine produces fibrant spectra which we call $G(X)$ and $K^Q(X)$, respectively. These are connective spectra in the sense that $\pi_n = 0$ for $n < 0$, and functorial enough to be presheaves of fibrant spectra on X . We will write G and K^Q for these presheaves.

The presheaf G has the Mayer–Vietoris property, because for all open U in X the fiber of $G(X) \rightarrow G(U)$ is $G(X - U)$ by $[Q, 7.3.2]$. This was the key of the original Brown–Gersten spectral sequences in $[BG, Q]$

In contrast, connectiveness prevents K^Q from having the Mayer–Vietoris property, except when X is regular. For example, let $X = \text{Spec}(k[x, y]/(x^2 - x)(y^2 - y) = 0)$ be the familiar tic-tac-toe curve. Let U_0, U_1 be obtained by removing the horizontal lines $y = 0, y = 1$. Then the cokernel of $K_0(U_0) \oplus K_0(U_1) \rightarrow K_0(U_0 \cap U_1)$ is $K_{-1}(X) = \mathbf{Z}$ instead of $\pi_{-1} K^Q(X) = 0$.

To fix this annoyance, we create the nonconnective fibrant spectrum $K(X)$ whose negative homotopy groups are the negative K -groups of Bass $[B, \text{Chap. XII}]$. The connective cover of $K(X)$ will be $K(X)(0, \dots, \infty) = K^Q(X)$.

If X is affine, the construction of $K(X)$ is well known: see $[Wag]$ or $[PW1]$. If not, we can either cite $[PW2]$ or proceed as follows. The Fundamental Theorem of $[GQ, p. 236]$ (and extended in $[Car]$) states that for every integer n ,

$$K_n(X) \oplus K_n(X[t, t^{-1}]) \cong K_n(X[t]) \oplus K_n(X[t^{-1}]) \oplus K_{n-1}(X).$$

Here we have written $X[t]$, etc., for the schemes $X \times \text{Spec}(\mathbf{Z}[t])$, etc. (A technicality: the Fundamental Theorem of [GQ] only applies to affine schemes. In the Appendix below, we extend this theorem to quasiprojective schemes X such that $\text{sing}(X)$ is contained in an affine open subset. I would like to thank Dan Grayson for pointing out the need for care using this approach.) Thus for $n \leq 0$ we can inductively define $K(X)(n-1, \dots, \infty)$ to be Σ^{-1} of

$$\begin{aligned} &K(X[t])(n, \dots, \infty) \vee K(X[t^{-1}](n, \dots, \infty) \\ &\rightarrow K(X)(n, \dots, \infty) \vee K(X[t, t^{-1}](n, \dots, \infty). \end{aligned}$$

Thinking of a spectrum as a sequence of spaces, we can arrange that the N th spaces of $K(X)(n, \dots, \infty)$ are identical for $N \geq 0$, $n \leq -N$. The spectrum $K(X)$ is obtained by passing to the colimit as $n \rightarrow -\infty$.

We are going to show that, when X has isolated singularities, both the presheaf K and its (-2) -connected cover $K(-1, \dots, \infty)$ have the Mayer-Vietoris property. The Main Theorem will be an immediate consequence of the Brown-Gersten theorem. One of our tools will be the elementary but useful

REDUCTION LEMMA. *Let $F_0 \rightarrow F_1 \rightarrow F_2$ be maps of presheaves such that $F_0(U) \rightarrow F_1(U) \rightarrow F_2(U)$ is a fibration sequence for every open U in X . (We will call this a fibration sequence of presheaves.) If two of the three presheaves have the Mayer-Vietoris property on X , so does the third.*

Here is another useful but elementary trick for producing presheaves with the Mayer-Vietoris property:

COBASE-CHANGE LEMMA. *Let $j: Y \rightarrow X$ be a scheme map, and let F be a presheaf on Y . The presheaf j_*F on X sends U to $F(j^{-1}(U))$. If F has the Mayer-Vietoris property on Y then j_*F has the Mayer-Vietoris property on X . Conversely, if j is an inclusion and j_*F has the Mayer-Vietoris property on X , then F has the Mayer-Vietoris property on Y .*

The third of our tricks will be a slight modification of the usual localization sequence in [GQ, Ger]. The usual localization theorem identifies the fiber of $K(X) \rightarrow K(U)$ when U is an open subset of X such that (i) U is affine and (ii) $X - U$ is locally cut out by a nonzero divisor.

MODIFIED LOCALIZATION THEOREM. *Let X be a quasiprojective scheme, and let Y be a closed subscheme of X missing $\text{sing}(X)$.*

(a) *There is a long exact sequence*

$$\dots \rightarrow G_n(Y) \rightarrow K_n(X) \rightarrow K_n(X - Y) \rightarrow G_{n-1}(Y) \rightarrow \dots$$

part of which is $\dots \rightarrow G_0(Y) \rightarrow K_0(X) \rightarrow K_0(X - Y) \rightarrow 0$.

(b) *The following sequences are fibrations of fibrant spectra:*

$$\begin{aligned} G(Y) &\rightarrow K(X) \rightarrow K(X - Y) \\ G(Y) &\rightarrow K(X)(n, \dots, \infty) \rightarrow K(X - Y)(n, \dots, \infty) \quad (\text{all } n < 0) \\ G(Y) &\rightarrow K^Q(X) \rightarrow K^Q(X - Y). \end{aligned}$$

(c) *Write $i^Y: Y \rightarrow X$ and $j: (X - Y) \rightarrow X$ for the canonical immersions.*

Then

$$\begin{aligned} i_*^Y G &\rightarrow K \rightarrow j_* K \\ i_*^Y G &\rightarrow K^Q \rightarrow j_* K^Q \end{aligned}$$

are fibrations of presheaves on X . In particular, K has the Mayer–Vietoris property on X if and only if K has the Mayer–Vietoris property on $X - Y$.

Proof. Let $\mathbf{H}(X)$ denote the exact subcategory of those modules in $\mathbf{M}(X)$ which have finite resolutions by objects of $\mathbf{P}(X)$. Then $\mathbf{M}(Y)$ is a subcategory of $\mathbf{H}(X)$ by [SGA6], and this induces $G(Y) \rightarrow K(X)$. Next, note that the map $K_0(X) \rightarrow K_0(X - Y)$ is onto by [B, pp. 436, 499]. Replacing X by $X[t, t^{-1}]$ and arguing inductively, we see that the fibers of $K(X) \rightarrow K(X - Y)$ and $K^Q(X) \rightarrow K^Q(X - Y)$ are the same. But M. Levine [L] has proven that $G(Y) \rightarrow K^Q(X) \rightarrow K^Q(X - Y)$ is a fibration. (We give a quick proof in the Appendix below.) The theorem follows immediately from this, given the Reduction and Cobase-Change lemmas.

REDUCTION TO THE SEMILOCAL AFFINE CASE. Let X be a quasiprojective scheme such that $\text{sing}(X)$ is contained in an affine open subscheme V . Then K has the Mayer–Vietoris property on X if and only if K has the Mayer–Vietoris property on V . This is immediate from version (c) of the Modified Localization Theorem above. In fact, there is a ring R such that $\text{Spec}(R)$ is the intersection of all the open subsets of $\text{sing}(X)$. Taking the filtered colimit of the corresponding fibrations yields the fibration sequence

$$\text{colim}_{\phi = Y \cap \text{sing}(X)} (i_*^Y G) \rightarrow K \rightarrow j_* K$$

of presheaves on X , where $j: \text{Spec}(R) \rightarrow X$ is the canonical immersion. Hence K has the Mayer–Vietoris property on X iff K has the Mayer–Vietoris property on $\text{Spec}(R)$. If X has isolated singularities, then R

is a semilocal noetherian ring. With this reduction, we may assume that $X = \text{Spec}(R)$ is semilocal. Here is the simplest case:

COLLINO'S THEOREM [Coll]. *Let X be a quasiprojective scheme with exactly one singularity. Then K has the Mayer–Vietoris property on X , and there is a fourth quadrant spectral sequence*

$$E_2^{p,q} = H_{\text{Zar}}^{p,q}(X; \mathbf{K}_{-q}) \Rightarrow K_{-p-q}(X).$$

If X is quasiprojective over a field, then the spectral sequence lies in the octant $p \geq 0, p + q \leq 0$.

Proof. We will show that $K, K(-1, \dots, \infty)$, and $K^Q = K(0, \dots, \infty)$ all have the Mayer–Vietoris property on X ; the Brown–Gersten theorem gives the spectral sequence. By the above reduction, we can assume that $X = \text{Spec}(R)$ for a local ring R . In order to verify the Mayer–Vietoris property on X , choose open subsets U and V . If $U = X$ or if $V = X$ there is nothing to prove. Otherwise $U \cup V$ is regular, and the Mayer–Vietoris property for K (and its covers) follows from the Mayer–Vietoris property for $G \cong K$ on $U \cup V$.

We now make the following inductive assumption: if Z is a quasiprojective scheme with at most n isolated singular points, then the presheaves K and $K(-1, \dots, \infty)$ have the Mayer–Vietoris property on Z . The case $n = 0$ is the original Brown–Gersten theorem of [BG, Q]; the case $n = 1$ is Collino’s theorem. We consider $X = \text{Spec}(R)$, where R is a semilocal noetherian ring with $n + 1$ isolated singularities. Choose x in $\text{sing}(X)$ and set $Y = \text{sing}(X) - \{x\}$.

If $\dim(R) = 0$, X is a disjoint union of local schemes, each containing exactly one point, and K trivially has the Mayer–Vietoris property. Otherwise, since R is a reduced ring, we can use the Chinese Remainder Theorem to choose nonzero divisors s, t in R satisfying

$$\begin{aligned} s &\equiv 0 \pmod{(x)}, & s &\equiv 1 \pmod{(y)}, & \text{all } y \text{ in } Y \\ t &\equiv 1 \pmod{(x)}, & t &\equiv 0 \pmod{(y)}, & \text{all } y \text{ in } Y. \end{aligned}$$

Given an open set U of X and a nonzero divisor r of R , write U_r for $U \cap \text{Spec}(R[r^{-1}])$ and i_r for $\text{Spec}(R[r^{-1}]) \rightarrow \text{Spec}(R)$.

LEMMA. *For each U , the following squares are homotopy cartesian:*

$$\begin{array}{ccc} K(U) & \longrightarrow & K(U_s) & & K(U)(-1, \dots, \infty) & \longrightarrow & K(U_s)(-1, \dots, \infty) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K(U_t) & \longrightarrow & K(U_{st}) & & K(U_t)(-1, \dots, \infty) & \longrightarrow & K(U_{st})(-1, \dots, \infty). \end{array}$$

Proof. It is enough to prove that the left square is homotopy cartesian, because the fact that U_{st} is regular implies that $K_n(U_{st}) = 0$ for all $n \leq -1$. If $U \neq X$, then U has fewer singular points than X , and K has the Mayer–Vietoris property on U by induction. If $U = X$, the map $R \rightarrow R[t^{-1}]$ is an analytic isomorphism along $\{s^n\}$, so the square is homotopy cartesian by [W3].

As an immediate corollary, we obtain a fibration of presheaves on X :

$$K \rightarrow (i_s)_* K \vee (i_t)_* K \rightarrow (i_{st})_* K.$$

By induction, K has the Mayer–Vietoris property on X_s, X_t , and X_{st} . Therefore $(i_s)_* K$, $(i_t)_* K$, and $(i_{st})_* K$ have the Mayer–Vietoris property on X . By the Reduction Lemma, K has the Mayer–Vietoris property on X . The same argument shows that $K(-1, \dots, \infty)$ also has the Mayer–Vietoris property on X . By the Brown–Gersten theorem, we have finished the proof of the Main Theorem.

Conjectures. Let X be a quasiprojective scheme with a closed nonempty singular set. We conjecture that:

(1) K has the Mayer–Vietoris property on X . The Brown–Gersten theorem would then imply the existence of a spectral sequence computing the K -theory of X .

(2) We have $H^p(X; \mathbf{K}_q) = 0$ for $p \geq 2 + q + \dim(\text{sing}(X))$. For $q \geq 0$, this would follow from Gersten’s conjecture [Q] about the K -theory of regular local rings.

(3) If Z is a closed subset of X locally cut out by a nonzero divisor, let $\mathbf{H}_Z(X)$ denote the subcategory of modules in $\mathbf{H}(X)$ with support in Z . The fiber of $K(X) \rightarrow K(X - Z)$ should be $K(\mathbf{H}_Z(X))$. This point was raised in [Ger]. The usual localization theorem states that this is true if $X - Z$ is affine. In the Appendix below we show that it is true when X has isolated singularities, or more generally when there is a neighbourhood U of $\text{sing}(X)$ such that $U - Z$ is affine.

(4) If $\dim(\text{sing}(X)) = d$, K is $-(d + 2)$ connected. In particular, $K_n(X) = 0$ if $n \leq -(d + 2)$. Since for reduced X we have $d < \dim(X)$, this is an improvement on the guess given in [W3, Question 2.9]. In particular, we conjecture that when X has isolated singularities we have $K = K(-1, \dots, \infty)$. If this holds, then we can apply the usual Whitehead lemma to deduce that there is a homotopy equivalence $K(X) \simeq \mathbb{H}(X; K)$, and not just a weak homotopy equivalence.

APPENDIX: LOCALIZATION THEOREMS

The usual localization theorems in [GQ, Ger] identify the fiber of $K(X) \rightarrow K(U)$ when U is an open subset of X such that (i) U is affine and (ii) $X - U$ is locally cut out by a nonzero divisor. We need to extend these for the purposes of this paper, in particular to remove the hypothesis that U is affine. The key result is

MIXED LOCALIZATION THEOREM (Levine [L]). *Let \mathbf{A} be a Serre subcategory of an abelian category \mathbf{M} . Let \mathbf{H} be an exact subcategory of \mathbf{M} containing \mathbf{A} . Then \mathbf{H}/\mathbf{A} is an exact subcategory of \mathbf{M}/\mathbf{A} , and $B\mathbf{Q}\mathbf{A} \rightarrow B\mathbf{Q}\mathbf{H} \rightarrow B\mathbf{Q}\mathbf{H}/\mathbf{A}$ is a fibration. In particular, there is a long exact sequence*

$$\dots K_n(\mathbf{A}) \rightarrow K_n(\mathbf{H}) \rightarrow K_n(\mathbf{H}/\mathbf{A}) \rightarrow \dots$$

part of which is

$$\dots \rightarrow K_0(\mathbf{A}) \rightarrow K_0(\mathbf{H}) \rightarrow K_0(\mathbf{H}/\mathbf{A}) \rightarrow 0.$$

Proof. Quillen’s argument of [Q, pp.113–116] for $\mathbf{A} \rightarrow \mathbf{M} \rightarrow \mathbf{M}/\mathbf{A}$ applies verbatim to the subsequence $\mathbf{A} \rightarrow \mathbf{H} \rightarrow \mathbf{H}/\mathbf{A}$.

COROLLARY 1. *Let X be a quasiprojective scheme, and let Y be a closed subscheme of X missing $\text{sing}(X)$. Then there is a fibration of fibrant spectra*

$$G(Y) \rightarrow K^{\mathcal{Q}}(X) \rightarrow K^{\mathcal{Q}}(X - Y)$$

and a long exact sequence

$$\dots \rightarrow G_n(Y) \rightarrow K_n(X) \rightarrow K_n(X - Y) \rightarrow G_{n-1}(Y) \rightarrow \dots$$

part of which is $\dots \rightarrow G_0(Y) \rightarrow K_0(X) \rightarrow K_0(X - Y) \rightarrow 0$.

Proof. Take $\mathbf{H} = \mathbf{H}(X)$ inside $\mathbf{M}(X)$, $\mathbf{A} = \mathbf{M}(Y)$, and note that $\mathbf{H}(X)/\mathbf{M}(Y) = \mathbf{H}(X - Y)$. The zeroth space of $K^{\mathcal{Q}}(X)$ is $\Omega B\mathbf{Q}\mathbf{H}(X)$. Note that the hypothesis that X be a “quasiprojective scheme” guarantees that the category $\mathbf{H}(X)$ is closed under extensions, and that it contains $\mathbf{M}(Y)$.

THEOREM. *Let X be a quasiprojective scheme and Z a closed subset of X locally cut out by a nonzero divisor. Let $\mathbf{H}_Z(X)$ denote the subcategory of $\mathbf{H}(X)$ of modules with support in Z . Assume that one of the following holds:*

- (a) X has isolated singularities,
- (b) $\overline{\text{Sing}(X)}$ is contained in an affine subset of X , or
- (c) there is a neighbourhood U of $\overline{\text{sing}(X)}$ such that $U - Z$ is affine.

Then $\Omega B\mathbf{Q}\mathbf{H}_Z(X)$ is the fiber of $\Omega B\mathbf{Q}\mathbf{H}(X) \rightarrow \Omega B\mathbf{Q}\mathbf{H}(X - Z)$.

Proof. Each hypothesis implies that there is a closed subset C of X which is locally cut out by a nonzero divisor, which misses $\text{sing}(X)$, and is such that $X - C - Z$ is affine. Hence the right-hand column of the following diagram is a fibration by [GQ]:

$$\begin{array}{ccccc}
 \Omega BQM_{C \cap Z}(X) & \longrightarrow & \Omega BQH_Z(X) & \longrightarrow & \Omega BQH_{Z-C}(X-C) \\
 \downarrow & & \downarrow & & \downarrow \\
 \Omega BQM_C(X) & \longrightarrow & \Omega BQH(X) & \longrightarrow & \Omega BQH(X-C) \\
 \downarrow & & \downarrow & & \downarrow \\
 \Omega BQM_{C-Z}(X-Z) & \longrightarrow & \Omega BQH(X-Z) & \longrightarrow & \Omega BQH(X-C-Z).
 \end{array}$$

The bottom two rows are fibrations by the above corollary. The left-hand column is a fibration by [Q], since $M_C(X)$ is an abelian category and $M_{C-Z}(X) = M_C(X)/M_{C \cap Z}(X)$. The similar identity of abelian categories $M_{Z-C}(X-C) = M_Z(X)/M_{C \cap Z}(X)$ induces the identity of exact categories $H_{Z-C}(X-C) = H_Z(X)/M_{C \cap Z}(X)$, so the top row is exact by the Mixed Localization Theorem above. Hence the middle column is a fibration as claimed.

COROLLARY (Fundamental Theorem of K -theory). *Let X be a quasiprojective scheme whose singular locus is contained in an affine open subset of the form $X - C$ for some divisor C . Then there is an isomorphism for all n :*

$$K_n(X[t, t^{-1}]) \cong K_n(X) \oplus NK_n(X) \oplus NK_n(X) \oplus K_{n-1}(X).$$

Here $X[t, t^{-1}]$ means $X \times \text{Spec}(\mathbf{Z}[t, t^{-1}])$ and $NK_n(X)$ is the cokernel of $K_n(X) \rightarrow K_n(X[t])$.

Proof. There is a map of fibrations

$$\begin{array}{ccccc}
 \Omega BQH_{X \times 0}(X \times \mathbb{P}^1) & \longrightarrow & \Omega BQH(X \times \mathbb{P}^1) & \longrightarrow & \Omega BQH(X[t^{-1}]) \\
 \parallel & & \downarrow & & \downarrow \\
 \Omega BQH_n(X[t]) & \longrightarrow & \Omega BQH(X[t]) & \longrightarrow & \Omega BQH(X[t, t^{-1}]).
 \end{array}$$

The left-hand category $H_n(X[t])$ is equivalent to the category $\text{Nil}(X)$ of coherent vector bundles and nilpotent endomorphisms, as one can see by modifying the proof for the affine case in [GQ, p. 236]. In fact, the proof of the Fundamental Theorem in [GQ] applies directly to show that $K_n \text{Nil}(X) \cong NK_{n+1}(X) \oplus K_n(X)$, and that there is a short exact sequence

$$0 \rightarrow K_n(X) \oplus NK_n(X) \oplus NK_n(X) \rightarrow K_n(X[t, t^{-1}]) \xrightarrow{\alpha} K_{n-1}(X) \rightarrow 0.$$

If α is in $K_{n-1}(X)$ and t is in $K_1(\mathbf{Z}[t, t^{-1}])$, then Waldhausen’s product

yields an element $\{t, \alpha\}$ in $K_n(X[t, t^{-1}])$; since the above sequence is a sequence of $K_*(X)$ -modules, we have $\partial\{t, \alpha\} = \{(\partial t), \alpha\} = \alpha$. Hence ∂ is split by the product with t , proving the theorem.

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Note added in proof. The conjectures, made in early 1984, have been mostly settled. Conjectures (1) and (3) were correct; this was proven in 1988 by Thomason and Trobaugh, “Higher algebraic K -theory of schemes and of derived categories.” As noted, Conjecture (2) for $q \geq 0$ is only open for those X for which Gersten’s conjecture is open. Conjecture (2) for $q < 0$ and Conjecture (4) are false; counterexamples were given in 1985 by Les Reid in “ N -dimensional rings with an isolated singular point having nonzero K_{-N} ,” *K-theory* 1 (1987), 197–206.

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