

# SOME SURFACES OF GENERAL TYPE FOR WHICH BLOCH'S CONJECTURE HOLDS

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ABSTRACT. We give many examples of surfaces of general type with  $p_g = 0$  for which Bloch's conjecture holds, for all values of  $K_2 \neq 9$ . Our surfaces are equipped with an involution.

Let  $S$  be a smooth complex projective surface with  $p_g(S) = 0$ . Bloch's conjecture states that the Albanese map  $A_0(S)_0 \rightarrow \text{Alb}(S)$  is an isomorphism, where  $A_0(S)_0$  is the Chow group of 0-cycles of degree 0 on  $S$ . It is known for all surfaces except those of general type (see [BKL]). For a surface  $S$  of general type with  $p_g(S) = 0$  we also have  $q(S) = 0$ , i.e.,  $\text{Alb}(S) = 0$  and the canonical divisor satisfies  $1 \leq K^2 \leq 9$ .

In the decades since this conjecture was formulated, surfaces of general type have become somewhat better understood. Two key developments have been (i) the results of S. Kimura on finite dimensional motives in [Ki] and (ii) the notion of the *transcendental motive*  $t_2(S)$  which was introduced in [KMP]. This includes the theorem that if  $S$  is a surface with  $p_g(S) = q(S) = 0$  then Bloch's conjecture holds for  $S$  iff  $t_2(S) = 0$ ; see Lemma 1.5.

In this paper we give motivic proofs of Bloch's conjecture for several examples of surfaces of general type for each value of  $K^2$  between 1 and 8. This includes some numerical Godeaux surfaces, classical Campedelli surfaces, Keum-Naie surfaces, Burniat surfaces and Inoue's surfaces. All these surfaces carry an involution, and many were previously known to satisfy Bloch's conjecture. We can say nothing about the remaining case  $K^2 = 9$ , because a surface of general type with  $p_g = 0$  and  $K^2 = 9$  has no involution ([DMP, 2.3]).

Bloch's conjecture is satisfied by all surfaces whose minimal models arise as quotients  $C_1 \times C_2/G$  of the product of two curves of genera  $\geq 2$  by the action of a finite group  $G$ . A partial classification of these surfaces has been given in [BCGP] and [BCG, 0.1]; the special case where  $G$  acts freely only occurs when  $K_S^2 = 8$ . We also show in Corollary 7.8 that Bloch's conjecture holds for surfaces with an involution  $\sigma$  for which

$K^2 = 8$  and  $S/\sigma$  is rational. The only known examples with  $K^2 = 7$  are Inoue's surfaces, which satisfy Bloch's conjecture (either by our Theorem 8.1 or Bauer [Bau]), and a new family, recently constructed by Y. Chen in [Ch], who also shows that these surfaces satisfy Bloch's conjecture. Burniat surfaces with  $K^2 = 6$  satisfy Bloch's conjecture (either by our Theorem 9.1 or Bauer-Catanese [BC1, BC2]); this provides also examples with  $K^2 = 3, 4, 5$ . Propositions 9.3 and 9.5 give examples of surfaces with  $K^2 = 6$  which are not Burniat, and which satisfy Bloch's conjecture. Other examples with  $K^2 = 3, 4$  are given in Section 6. Some examples with  $K^2 = 1, 2$  are treated in Sections 4–5.

**Notation.** We will work in the (covariant) category  $\mathcal{M}_{\text{rat}}(k)$  of Chow motives with coefficients in  $\mathbb{Q}$ , where the morphisms from  $h(X)$  to  $h(Y)$  are just the elements of  $A^d(X \times Y)$ ,  $d = \dim(Y)$ . There is a covariant functor  $h$  from smooth projective varieties to  $\mathcal{M}_{\text{rat}}(k)$ , sending a morphism  $f : X \rightarrow Y$  to the map  $f^t : h(X) \rightarrow h(Y)$  determined by the graph  $\Gamma_f$ . Here, and in the rest of this paper,  $A^i(X \times Y)$  denotes the Chow group  $CH^i(X \times Y) \otimes \mathbb{Q}$  of codimension  $i$  cycles modulo rational equivalence, with  $\mathbb{Q}$  coefficients.

## 1. PRELIMINARIES

Let  $f : X \rightarrow Y$  be a finite morphism between smooth projective varieties of dimension  $d$ . Then  $\Gamma_f^t$  determines a map  $h(Y) \rightarrow h(X)$  and the composition  $\Gamma_f \circ {}^t\Gamma_f$  is  $\deg(f)$  times the identity of  $h(Y)$ .

Y=pX **Lemma 1.1.**  *$f : X \rightarrow Y$  be a finite morphism between smooth projective varieties of dimension  $d$ , and set  $p = ({}^t\Gamma_f \circ \Gamma_f) / \deg(f)$ . Then  $p : h(X) \rightarrow h(X)$  is idempotent, and expresses  $h(Y)$  as a direct summand of  $h(X)$  in  $\mathcal{M}_{\text{rat}}(k)$ .*

*Proof.* We compute:  $p \circ p = {}^t\Gamma_f^t \circ (\Gamma_f \circ {}^t\Gamma_f) \circ \Gamma_f / \deg(f)^2 = p$ . □

ex:Y=Xsigma **Example 1.2.** Suppose that  $\sigma$  is an involution on  $X$  and  $Y = X/\sigma$  is smooth. Then  $p = (1 + \sigma)/2$  is represented by the correspondence  $(\Delta_X + \Gamma_\sigma)/2$ , because  ${}^t\Gamma_f \circ \Gamma_f = \Delta_X + \Gamma_\sigma$ . In particular,  $h(Y) = h(X)^\sigma$ .

More generally, suppose that a finite group  $G$  acts on  $X$  and that  $Y = X/G$  is smooth. Then  $p = (\sum g) / |G|$  is idempotent, and expresses  $h(Y)$  as the direct summand  $h(X)^G$  of  $h(X)$  in  $\mathcal{M}_{\text{rat}}(k)$ .

Let  $\rho(S)$  denote the rank of  $NS(S)$ . The Riemann-Roch Theorem gives a well known formula for  $\rho$ :

rho+K2 **Lemma 1.3.** *If  $p_g(S) = q(S) = 0$ , then  $\rho(S) = 10 - K_S^2$ .*

*Proof.* The topological Euler characteristic of  $S$  is  $\deg(c_2) = 2 + \rho(S)$ . Since the Euler characteristic of  $\mathcal{O}_S$  is 1, the Riemann-Roch Theorem ([Fu, 15.2.2]) yields  $12 = K_S^2 + \deg(c_2)$ . Thus  $\rho(S) = 10 - K_S^2$ .  $\square$

**Remark 1.3.1.** If  $S$  is a minimal surface of general type, then  $K_S^2 > 0$  [BHPV, VII.2.2]. Since  $\rho(S) \geq 1$ , we derive the inequality  $1 \leq K_S^2 \leq 9$ . We also have  $\deg c_2 > 0$ , see [BHPV, VII.1.1]. By Noether's formula  $1 - q(S) + p_g(S)$  is positive, hence if  $p_g(S) = 0$  then also  $q(S) = 0$ .

**The algebraic motive  $h_2^{\text{alg}}(S)$ .** If  $S$  is a surface, the Neron-Severi group  $NS(S)$  determines a summand of the motive of  $S$  (with  $\mathbb{Q}$  coefficients). To construct it, choose an orthogonal basis  $\{E_1, \dots, E_\rho\}$  for the  $\mathbb{Q}$ -vector space  $NS(S)_\mathbb{Q}$ , where the self-intersections  $E_i^2$  are nonzero. Then the correspondences  $\epsilon_i = \frac{[E_i \times E_i]}{(E_i)^2}$  are orthogonal and idempotent, so

$$\pi_2^{\text{alg}}(S) = \sum_{1 \leq i \leq \rho} \frac{[E_i \times E_i]}{(E_i)^2}$$

is also an idempotent correspondence. Since  $\{E_i/(E_i)^2\}$  is a dual basis to the  $\{E_i\}$ , it follows from [KMP, 7.2.2] that  $\pi_2^{\text{alg}}(S)$  is independent of the choice of basis. We set  $M_i = (S, \epsilon_i, 0)$  and  $h_2^{\text{alg}}(S) = (S, \pi_2^{\text{alg}}(S)) = \oplus M_i$ . In fact,  $M_i \simeq \mathbb{L}$  for all  $i$  by [KMP, 7.2.3], so we have isomorphisms  $h_2^{\text{alg}}(S) \cong \mathbb{L}^{\oplus \rho}$  and  $H^2(h_2^{\text{alg}}(S)) \cong NS(S)_\mathbb{Q}$ .

**The transcendental motive  $t_2$ .** We also need a description of the transcendental motive  $t_2$  of a surface  $S$ . It is well known that the motive  $h(S)$  has a Chow-Künneth decomposition as  $\sum_0^4 h_i(S)$ , where  $H^i(S) = H^i(h_i(S))$ . The middle factor  $h_2(S)$  further decomposes as  $h_2(S) = h_2^{\text{alg}}(S) \oplus t_2(S)$ ; see [KMP].

The factor  $t_2(S) = (S, \pi_2^{\text{tr}}, 0)$  is called the *transcendental part* of the motive (cf. [KMP, 7.2.3]). This terminology is justified by the following result, implicit in [CG, p. 289], which identifies  $H^2(t_2(S)) = \pi_2^{\text{tr}} H^2(S)$  with the Hodge-theoretic group  $H_{\text{tr}}^2(S)$ .

**Lemma 1.4.** *Under the intersection pairing on  $H^2(S, \mathbb{C})$  the orthogonal complement  $H_{\text{tr}}^2(S)$  of  $NS(S) \otimes \mathbb{C}$  is  $\pi_2^{\text{tr}} H^2(S)$ .*

The following result was established in [KMP, 7.4.9 & 7.6.11].

t2=0.BC

**Lemma 1.5.** *If  $S$  is a smooth projective complex surface, then the following are equivalent:*

- (a)  $t_2(S) = 0$  holds;
- (b)  $p_g(S) = 0$  and the motive  $h(S)$  is finite dimensional;
- (c)  $p_g(S) = 0$  and  $S$  satisfies Bloch's conjecture.

We will often use without comment the fact that  $t_2$  is a birationally invariant functor on the category of smooth 2-dimensional varieties; this is proven in [KMP, 7.8.11].

**lem:t2^G**

**Lemma 1.6.** *If a finite group  $G$  acts on a surface  $S$ , and  $Y$  is a desingularization of  $S/\sigma$ , then  $t_2(Y) = t_2(S)^G$ .*

*Proof.* Because  $t_2$  is a birational invariant, we may blow up  $S$  to assume that  $Y = S/\sigma$ . (See (2.1) below). Since  $h(S) \rightarrow h(Y)$  sends  $t_2(S)$  to  $t_2(Y)$ , and  $h(Y) = h(S)^G$  by Example 1.2, the result follows.  $\square$

Here is the “enough automorphisms” criterion of Inose and Mizukami. Our argument depends upon the motivic Lemma 1.5.

**IMcriterion**

**Lemma 1.7** (Inose–Mizukami [IM, 1.2]). *Suppose that a finite group  $G$  acts on  $S$ , and that  $Y$  is a desingularization of  $S/G$ . If  $\sum g = 0$  in  $\text{End } h(S)$  then  $p_g(Y) = 0$  and Bloch’s conjecture holds for  $Y$ .*

*Proof.* Because  $t_2(S)$  is a birational invariant, we may blow up  $S$  to assume that  $Y = S/G$ . By 1.2,  $h(Y) = h(S)^G$  and  $t_2(Y) = t_2(S)^G$ , so if  $p = (\sum g)/|G| = 0$  in  $\text{End } h(S)$  then  $p = 0$  in  $\text{End } t_2(S)$ , i.e.,  $t_2(Y) = 0$ . By Lemma 1.5, this implies that  $p_g(Y) = 0$  and Bloch’s conjecture holds for  $Y$ .  $\square$

## 2. INVOLUTIONS ON SURFACES

**sec:involutions**

Let  $S$  be a smooth projective surface with an involution  $\sigma$ . The fixed locus consists of a 1-dimensional part  $D$  (a union of smooth curves, possibly empty) and  $k \geq 0$  isolated fixed points  $\{P_1, \dots, P_k\}$ . The images  $Q_i$  of the  $P_i$  are nodes on the quotient surface  $S/\sigma$ , and  $S/\sigma$  is smooth elsewhere. To resolve these singularities, let  $X$  denote the blow-up of  $S$  at the set of isolated fixed points;  $\sigma$  lifts to an involution on  $X$  (which we will still call  $\sigma$ ), and the quotient  $Y = X/\sigma$  is a desingularization of  $S/\sigma$ . The images  $C_1, \dots, C_k$  in  $Y$  of the exceptional divisors of  $X$  are disjoint nodal curves, i.e., smooth rational curves with self-intersection  $-2$ . In summary, we have a commutative diagram

$$(2.1) \quad \begin{array}{ccc} X & \xrightarrow{h} & S \\ \downarrow \pi & & \downarrow f \\ Y & \xrightarrow{g} & S/\sigma. \end{array}$$

**square:S-sigma**

The image  $f_*D$  is a smooth curve on  $S/\sigma$ , disjoint from the singular points  $Q_i$ , and its proper transform  $B' = g^*(f_*D)$  in  $Y$  is smooth and disjoint from the curves  $C_i$ . It follows that  $\pi : X \rightarrow Y$  is a double cover with smooth branch locus  $B = B' + \sum C_i$ . As such,  $\pi$  is determined by a line bundle  $L$  on  $Y$  such that  $2L \equiv B$ .

**2L2=D2-k** **Lemma 2.2.**  $2L^2 = D^2 - k$ .

*Proof.* Because the curves  $C_i$  on  $X$  have  $C_i^2 = -2$  and are disjoint from  $B' = g^*(f_*D)$  we have  $(B')^2 = 2D^2$ . Since  $2L \equiv B' + \sum C_i$  we have  $4L^2 = (B')^2 + \sum C_i^2 = 2D^2 - 2k$ .  $\square$

**1(2K+L)** **Remark 2.2.1.** If  $S$  is a minimal surface of general type and  $p_g(S) = 0$ , it is proven in [CCM, 3.3] that  $k \geq 4$  and the linear system  $2K_Y + L$  has dimension  $l(2K_Y + L) = (K_S^2 + 4 - k)/2$ .

Since  $t_2(-)$  is a birational invariant for smooth projective surfaces, the maps  $h : X \rightarrow S$  and  $\pi : X \rightarrow Y$  induce a morphism

$$\theta : t_2(S) \cong t_2(X) \rightarrow t_2(Y).$$

By Lemma 1.1 and Example 1.2,  $\theta$  is the projection onto the direct summand  $t_2(S)^\sigma$  of  $t_2(S)$ , and  $A_0(Y)$  is the direct summand of  $A_0(X)$  fixed by  $\sigma$ .

**old nice thm** **Proposition 2.3.** *Let  $S$  be a smooth surface  $q(S) = 0$ . If  $\sigma$  is an involution on  $S$  then:*

- (i)  $t_2(S) \cong t_2(Y) \iff \bar{\sigma} = +1$  in  $\text{End}_{\mathcal{M}_{\text{rat}}}(t_2(S))$
- (ii)  $t_2(Y) = 0 \iff \bar{\sigma} = -1$  in  $\text{End}_{\mathcal{M}_{\text{rat}}}(t_2(S))$ .

Here  $\bar{\sigma}$  is the endomorphism of  $t_2(S)$  induced by  $\sigma$

*Proof.* Since we have  $t_2(Y) = t_2(S)^\sigma$  by Lemma 1.6, the projection onto  $t_2(Y)$  is given by the idempotent endomorphism  $e = (\bar{\sigma} + 1)/2$  of  $t_2(S)$ . Since  $t_2(S) \cong t_2(Y)$  is equivalent to  $e = 1$ , and  $t_2(Y) = 0$  is equivalent to  $e = 0$ , the result follows.  $\square$

**Remark 2.3.1.** If  $S$  is a smooth minimal surface of general type with  $p_g(S) = 0$  and an involution  $\sigma$ , such that the minimal model  $W$  of  $Y$  is either an Enriques surface, a rational surface or a surface of Kodaira dimension equal to 1 then by [GP2] we have  $t_2(Y) = 0$ .

**def:bidouble** **Definition 2.4.** A *bidouble cover*  $V \xrightarrow{f} X$  between smooth projective surfaces is a finite flat Galois morphism with Galois group  $\mathbb{Z}/2 \times \mathbb{Z}/2$ . By [Pa91], in order to define  $f$  it is enough to give smooth divisors  $D_1, D_2, D_3$  in  $X$  with pairwise transverse intersections and no common intersections, and line bundles  $L_1, L_2, L_3$  such that  $2L_i \equiv D_j + D_k$  for

each permutation  $(i, j, k)$  of  $(1, 2, 3)$ . We will frequently use the fact that every nontrivial element  $\sigma$  of  $G$  is an involution on  $V$ .

**3-involutions**

**Theorem 2.5.** *Let  $S$  be surface of general type with  $p_g(S) = 0$  which is the smooth minimal model of a bidouble cover of a surface. Let  $Y_i$  denote the desingularization of  $S/\sigma_i$ , where  $\sigma_1, \sigma_2, \sigma_3$  are the non-trivial involutions of  $S$  associated to the bidouble cover. If  $t_2(Y_i) = 0$  for  $i = 1, 2, 3$  then  $t_2(S) = 0$ .*

*Proof.* By Lemma 1.5,  $t_2(S)^\sigma = t_2(Y_i) = 0$ . Thus each  $\sigma_i$  acts as multiplication by  $-1$  on  $t_2(S)$ . But since  $\sigma_1\sigma_2 = \sigma_3$ ,  $\sigma_3$  must act as multiplication by  $(-1)^2 = +1$ . Since  $+1 = -1$  on  $t_2(S)$ ,  $t_2(S) = 0$ .  $\square$

**Remark 2.5.1.** Theorem 2.5 is the special case where  $\sigma_1, \sigma_2$  and  $\sigma_3$  act as  $-1$  of Lemma 1.7 (the “enough automorphisms” criterion of Inose and Mizukami). (Compare Proposition 1.3 in [Bau].)

### 3. COMPOSED INVOLUTIONS

If  $S$  is a smooth minimal surface of general type, then the linear system  $|2K_S|$  determines the bicanonical map  $\Phi_2 : S \rightarrow \mathbb{P}^N$ , where  $N = \dim(2K_S)$ . We say that  $\Phi_2$  is *composed* with an involution  $\sigma$  if  $\Phi_2$  factors through the map  $S \rightarrow S/\sigma$ .

Recall that  $k$  denotes the number of isolated fixed points, and that  $Y$  is a resolution of  $S/\sigma$ , as in (2.1).

**sigma-composed**

**Theorem 3.1.** *Let  $S$  be a smooth minimal surface of general type with  $p_g(S) = 0$ , supporting an involution  $\sigma$ . Then*

(1)  $\Phi_2$  is composed with  $\sigma$  iff  $k = K_S^2 + 4$ .

If the bicanonical map  $\Phi_2$  is composed with  $\sigma$  then

(2)  $S/\sigma$  is either rational or it is birational to an Enriques surface.

(3)  $-4 \leq K_Y^2 \leq 0$ , and  $K_Y^2 = 0$  iff  $K_Y$  is numerically zero.

*Proof.* This is proven in [CCM, 3.6(iv) and 3.7(iv,v)].  $\square$

**Corollary 3.2.** *Let  $S$  be a minimal surface of general type with  $p_g(S) = 0$  and let  $X$  be the image of the bicanonical map  $\Phi_2$ . If  $\Phi_2 : S \rightarrow X$  is a bidouble cover, then  $S$  satisfies Bloch’s conjecture.*

*Proof.* The assumption that the image  $X$  of the bicanonical map of  $S$  is a surface and that  $\Phi_2$  is a morphism imply that  $K_S^2 \geq 2$  (see [MP02, 2.1]) and that the degree of  $\Phi_2$  is  $\leq 8$ . If  $\Phi_2$  is a bidouble cover then all three nontrivial involutions  $\sigma_i$  of  $S$  (see 2.4) are composed with the bicanonical map. By Theorem 3.1,  $t_2(S/\sigma_i) = 0$  so Theorem 2.5 implies that  $t_2(S) = 0$ . That is,  $S$  satisfies Bloch’s conjecture.  $\square$

Recall that  $\rho(S)$  denotes the rank of the Néron-Severi group  $NS(S)$ .

**D<sup>2</sup> bound**

**Corollary 3.3.** *Let  $S$  be a minimal complex surface of general type with  $p_g(S) = 0$  and an involution  $\sigma$ . Let  $D$  be the 1-dimensional part of the fixed locus  $S^\sigma$ . Then*

(1)  $\sigma$  acts as the identity on  $H^2(S, \mathbb{Q})$  iff  $D^2 = K_S^2 - 8$ .

(2) If the bicanonical map is composed with  $\sigma$  and  $X, Y$  are as in diagram (2.1), then  $D^2 = K_S^2 + 2K_Y^2$  and

$$K_S^2 - 8 \leq D^2 \leq K_S^2.$$

When  $D^2 = K_S^2$ ,  $S/\sigma$  is birational to an Enriques surface.

*Proof.* Since  $p_g(S) = 0$  we have  $H^2(S, \mathbb{Q}) \cong NS(S)_\mathbb{Q}$ . Let  $t$  denote the trace of  $\sigma$  on  $H^2(S, \mathbb{Q})$ ; it is at most  $\rho(S) = \dim H^2(S, \mathbb{Q})$ . By [DMP, 4.2],  $t = 2 - D^2$ . Since  $\rho(S) = 10 - K_S^2$  by 1.3, we deduce that  $D^2 \geq K_S^2 - 8$ , with equality iff  $t = \rho(S)$ , i.e., iff  $\sigma$  acts as the identity on  $H^2(S, \mathbb{Q})$ .

Assume now that  $\sigma$  is composed with the bicanonical map. By Theorem 3.1,  $\sigma$  has  $k = K_S^2 + 4$  fixed points. By [CCM, 3.3(ii) and 3.6(iii)], the line bundle  $L$  introduced after (2.1) satisfies  $L^2 + 2 = -K_Y \cdot L = +K_Y^2$ . We also have  $2L^2 = D^2 - k$ , by Lemma 2.2. Therefore

$$D^2 = K_S^2 + 2K_Y^2.$$

Since  $-4 \leq K_Y^2 \leq 0$  by Theorem 3.1, we get  $K_S^2 - 8 \leq D^2 \leq K_S^2$ . Theorem 3.1 also implies that if  $K_Y^2 = 0$  then (since  $Y$  is not rational) the minimal model of  $Y$  is an Enriques surface, i.e.,  $2K_Y = 0$ .  $\square$

**rem:rhoY**

**Remark 3.3.1.** From Lemma 1.3,  $\rho(X) = \rho(S) + k$  equals  $10 - K_S^2 + k$ . If the bicanonical map is composed with  $\sigma$  then  $k = K_S^2 + 4$  by 3.1, so we get  $\rho(X) = 14$ . Since  $Y = X/\sigma$ , we have  $p_g(Y) = q(Y) = 0$ ,  $NS(Y)_\mathbb{Q} = NS(X)_\mathbb{Q}^\sigma$  and hence  $\rho(Y) \leq \rho(X)$ . Since  $K_Y^2 \leq 0$  and  $\rho(Y) = 10 - K_Y^2$  we get the bounds

$$10 \leq \rho(Y) \leq 14.$$

The equality  $\rho(Y) = 14 = \rho(X)$  corresponds to the case  $K_Y^2 = -4$ , in which case  $D^2 = K_S^2 - 8$ .

**Example 3.4** ( $K_S^2 = 8$ ). Let  $S$  be a minimal surface of general type with  $p_g(S) = 0$  and  $K_S^2 = 8$ , having an involution  $\sigma$ . Then  $\rho(S) = 10 - K_S^2 = 2$  (Lemma 1.3), and the Hodge Index Theorem implies that the involution  $\sigma$  acts as the identity on  $H^2(S, \mathbb{Q})$ . By Corollary 3.3,  $D^2 = K_S^2 - 8 = 0$ .

The following facts are established in [DMP, 4.4]. The number  $k$  of the fixed points is one of 4,6,8,10,12. The bicanonical map is composed with  $\sigma$  iff  $k = 12$  and in that case  $Y = X/\sigma$  is rational (see 3.1). If

$k = 10$  then  $Y$  is a rational surface and  $\rho(Y) = \rho(X) = 12$ . If  $k = 8$  the Kodaira dimension of  $Y$  is 1. Finally, if  $k = 4, 6$  the quotient surface  $Y$  is of general type.

Carlos Rito has analyzed the situation where  $\Phi_2$  is *not* composed with  $\sigma$ . We extract the following result from [Ri2, Thm.2]. Given a surface  $S$  with involution, we construct a resolution  $Y \rightarrow S/\sigma$  and  $X \rightarrow Y$  with smooth branch locus  $B$  as in (2.1); by blowing down curves on  $Y$ , we obtain a minimal resolution  $W$  of  $S/\sigma$ . Let  $\bar{B}$  denote the image of  $B$  under the proper map  $Y \rightarrow W$ .

not-composed

**Theorem 3.5.** [Rito] *Let  $S$  be a smooth minimal surface of general type with  $p_g(S) = 0$ , and  $\sigma$  an involution on  $S$  with  $k$  isolated fixed points. Suppose that the bicanonical map is not composed with  $\sigma$ . Let  $W$  be a minimal model of the resolution  $Y$  of  $S/\sigma$  and let  $\bar{B}$  be as above. If  $W$  has Kodaira dimension 2, then one of the following holds:*

- (1)  $K_S^2 = 2K_W^2$ , and  $\bar{B}$  is a disjoint union of 4 nodal curves;
- (2)  $4 \leq K_S^2 \leq 8$ ,  $K_W^2 = 1$ ,  $\bar{B}^2 = -12$ ,  $K_W \cdot \bar{B} = 2$ ,  
 $k = K_S^2$  and  $\bar{B}$  has at most one double point;
- (3)  $6 \leq K_S^2 \leq 8$ ,  $K_W^2 = 2$ ,  $\bar{B}^2 = -12$ ,  $K_W \cdot \bar{B} = 2$ ,  
 $k = K_S^2 - 2$  and  $\bar{B}$  is smooth.

S/s if K2=3

**Corollary 3.6.** *Let  $S$  be a smooth minimal surface of general type with  $p_g(S) = 0$ , having a nontrivial involution  $\sigma$ . If  $K_S^2 = 3$  then the resolution  $Y$  of  $S/\sigma$  is not of general type. In particular,  $S/\sigma$  satisfies Bloch's conjecture.*

*Proof.* If the bicanonical map of  $S$  is composed with  $\sigma$ , then Theorem 3.1(2) states that  $Y$  is either rational or its minimal model is an Enriques surface. If the bicanonical map of  $S$  is *not* composed with  $\sigma$ , and  $Y$  is of general type (has Kodaira dimension 2), then Theorem 3.5 shows that  $K_S^2$  cannot be 3.  $\square$

**Example 3.7.** Here are some examples in which the minimal model  $W$  of  $S/\sigma$  is of general type, and the divisorial part  $D$  of  $S^\sigma$  satisfies  $D = 0$ . Let  $k$  be the number of isolated fixed points of  $\sigma$ .

1) The Barlow surface  $W$  in [Bar] is of general type with  $K_W^2 = 1$  and  $p_g(W) = 0$ . It is the minimal model of  $S/\sigma$ , where  $S$  is a Catanese surface with  $K_S^2 = 2$ , by factoring with an involution  $\sigma$  which has only 4 isolated fixed points. Hence  $D = 0$  so that  $\bar{B} = \sum_{1 \leq i \leq k} C_i$  is the disjoint union of 4 nodal curves. Therefore  $W$  satisfies condition (1) of Theorem 3.5. In [Bar] it is proved that  $t_2(W) = 0$ .

2) Another example comes from [MP01a, Ex.4.3]; cf. [DMP, 4.4(i)]. Using Pardini's method for the group  $G = (\mathbb{Z}/2)^4$  (as in Lemma 7.6

below), one first constructs two smooth  $G$ -covers  $C_1, C_2$  of  $\mathbb{P}^1$  of genus 5 as in [MP01a, Ex. 4.3]. The authors choose a subgroup  $\Gamma$  of  $G \times G$  with  $|\Gamma| = 16$  acting freely on  $C_1 \times C_2$ ; the quotient  $S = (C_1 \times C_2)/\Gamma$  is the desired minimal surface of general type with  $K_S^2 = 8$  and  $p_g(S) = q(S) = 0$ .

For this  $S$ , it is shown in *loc. cit.* that the bicanonical map is birational, hence not composed with any involution. The construction of  $S$  starts with a basis of  $G$  and constructs the  $G$ -covers  $C_i \rightarrow \mathbb{P}^1$  so that any of the 11 other nontrivial elements of  $G$  acts freely on both curves, and acts on  $S$  (via the diagonal action) with  $D = 0$  and  $k = K_S \cdot D + 4 = 4$  fixed points.

Since  $D = 0$  and  $k = 4$ , a minimal model  $W$  of  $S/\sigma$  has  $K_W^2 = 4$  by [DMP, 4.4(i)]. Thus  $W$  satisfies condition (1) of Theorem 3.5. In fact, the branch locus of  $X \rightarrow Y$  is the disjoint union of the 4 lines over the fixed points of  $\sigma$ , so  $\bar{B}$  is the disjoint union of the 4 nodal curves  $C_i$ .

Since the motives  $h(C_i)$  are finite dimensional, so are  $h(C_1 \times C_2)$  and  $h(S) = h(C_1 \times_G C_2)$ . Hence  $t_2(S)$  is also finite dimensional. As  $p_g(S) = 0$ , Lemma 1.5 implies that  $S$  satisfies Bloch's conjecture.

#### 4. NUMERICAL GODEAUX SURFACES ( $K^2 = 1$ )

sec:k2=1

In the next few sections, we will give examples of complex surfaces  $S$  of general type, having an involution  $\sigma$ , with  $p_g(S) = 0$  and  $3 \leq K_S^2 \leq 7$ , for which Bloch's conjecture holds. In this section and the next we deal with the cases  $K_S^2 = 1, 2$ .

Complex surfaces of general type with  $p_g = q = 0$  and  $K_S^2 = 1$  are called *numerical Godeaux surfaces*. Examples of Godeaux surfaces are obtained as quotients of a Catanese surface under the action of a finite group. (A Catanese surface is a minimal surface  $V$  of general type with  $p_g(V) = 0$  and  $K_V^2 = 2$ ).

Barlow gave two such examples of numerical Godeaux surfaces  $S$  in the 1980's, both with  $\pi_1(S) = \mathbb{Z}/5$  and  $\pi_1(S) = \{1\}$ , and showed in [Bar] that Bloch's conjecture holds for these  $S$ .

By a *typical Godeaux surface* we mean the quotient surface of the Fermat quintic in  $\mathbb{P}^3$  by a cyclic group of order 5. It has  $\rho(S) = 9$  and  $K_S^2 = 1$ . These surfaces were first shown to satisfy Bloch's conjecture by Inose-Mizukami [IM, Thm. 1]. From a motivic viewpoint,  $S$  satisfies Bloch's conjecture because it has a finite dimensional motive; see [GP1].

We now turn to numerical Godeaux surfaces with an involution  $\sigma$ . These were classified in [CCM]; if  $S/\sigma$  is rational they are either of *Campedelli type* or *Du Val type*; if  $S/\sigma$  is an Enriques surface they are of *Enriques type*. This exhausts all cases, since (by [CCM, 4.5])

the bicanonical map is composed with  $\sigma$  and  $\sigma$  has exactly  $k = 5$  fixed points; as noted in Theorem 3.1, this implies that  $S/\sigma$  is either rational or birational to an Enriques surface.

If  $S$  is of Du Val type, there is an étale double cover  $\tilde{S}$  of  $S$  with  $p_g(\tilde{S}) = 1$  and  $K^2 = 2$ ; see [CCM, 8.1]. Since  $\pi_1(\tilde{S})$  is the kernel of  $\pi_1(S) \rightarrow \mathbb{Z}/2$ , we also have  $q(\tilde{S}) = 0$ . In addition,  $\text{Pic}(\tilde{S}) = NS(\tilde{S})$  is either torsion free or has a torsion subgroup of order 2.

**thm:DuVal**

**Theorem 4.1.** *Let  $S$  be a numerical Godeaux surface of Du Val type such that the torsion subgroup of  $\text{Pic}(S)$  has order 2. Let  $\tilde{S} \rightarrow S$  be the étale double cover of  $S$  associated to the non trivial 2-torsion element in  $\text{Pic}(S)$ . If  $\tilde{S}$  has bicanonical map of degree 4, then  $S$  satisfies Bloch's conjecture.*

*Proof.* By [CD, §3],  $\text{Pic}(\tilde{S})$  is torsion free and the image of the bicanonical map  $\Phi : \tilde{S} \rightarrow \mathbb{P}^3$  is a smooth quadric. By Lemma 4.2 below,  $\tilde{S}$  has a finite dimensional motive; this implies that  $h(S)$  is also finite dimensional. As  $p_g(S) = 0$ , Lemma 1.5 implies that  $S$  satisfies Bloch's conjecture.  $\square$

Lemma 4.2 is based upon the Catanese–Debarre paper [CD].

**lem:pg=1**

**Lemma 4.2.** *Let  $S$  be a surface of general type with  $p_g(S) = 1$ ,  $q(S) = 0$  and  $K_S^2 = 2$ . Suppose that the bicanonical map  $\Phi : S \rightarrow \mathbb{P}^3$  has degree 4 and is a morphism onto a smooth quadric  $\Sigma$ . Then the motive of  $S$  is finite dimensional.*

*Proof.* As shown in [CD, 3.2], the bicanonical map  $S \rightarrow \Sigma \subset \mathbb{P}^3$  is a bidouble cover with Galois group  $G = \{1, \sigma_1, \sigma_2, \sigma_3\}$ . Set  $S_i = S/\sigma_i$ , with  $i = 1, 2, 3$ . By [CD, 3.1], both  $S_1$  and  $S_2$  are rational surfaces, while the minimal smooth surface  $M$  over  $S_3$  is a K3 surface. Moreover  $S_3$  has 10 nodal points, and their inverse images in  $M$  are 10 disjoint smooth rational curves  $F_1, \dots, F_{10}$ .

Since  $t_2(S)^{\sigma_i} = t_2(S_i)$  and  $t_2(S_1) = t_2(S_2) = 0$ , it follows that  $\sigma_1$  and  $\sigma_2$  act as  $-1$  to  $t_2(S)$ . Hence  $\sigma_3 = \sigma_1 \circ \sigma_2$  acts as  $+1$ . It follows that  $t_2(S) = t_2(S_3) = t_2(M)$ .

On  $S_3$ , the involutions  $\bar{\sigma}_1$  and  $\bar{\sigma}_2$  agree and induce an involution  $\sigma$  on  $M$ . Since  $S_3/\bar{\sigma}_1 = \Sigma$ ,  $M^\sigma$  is the union of the ten curves  $F_i$ . Because  $\Sigma$  is a rational surface,  $H^{2,0}(M)^\sigma = H^{2,0}(\Sigma) = 0$ , so  $\sigma$  acts as  $-1$  on  $H^{2,0}(M) \cong \mathbb{C}$ . It follows from [Zh, 3.1] that the Neron-Severi group of  $M$  has rank  $\rho(M) = 20$ . By Theorems 2 and 3 of [Ped], this implies that the motive  $h(M)$  is finite dimensional in  $\mathcal{M}_{rat}$ . In particular  $t_2(M) = t_2(S)$  is finite dimensional. Since  $h_0(S)$ ,  $h_4(S)$  and  $h_2^{alg}(S)$  are finite dimensional, it follows that  $h(S)$  is finite dimensional.  $\square$

5. NUMERICAL CAMPEDELLI SURFACES ( $K^2 = 2$ )

sec:K2=2

Complex surfaces of general type with  $p_g = q = 0$  and  $K_S^2 = 2$  are called *numerical Campedelli surfaces*.

Now let  $S$  be a numerical Campedelli surface with an involution  $\sigma$ ; these have been classified in [CMP]. By [CCM, 3.32(i,iv)],  $\sigma$  has either  $k = 4$  or  $k = 6$  isolated fixed points. By Theorem 3.1, the bicanonical map  $\Phi : S \rightarrow \mathbb{P}^2$  is composed with  $\sigma$  iff  $k = 6$ ; in this case Corollary 3.3 yields  $-6 \leq D^2 \leq 2$ . (This result appeared in [CMP, 3.1(ii)]).

Campedelli surfaces with fundamental group  $(\mathbb{Z}/2)^3$  satisfy Bloch's conjecture, by the results in [IM]. In [Vois, 2.4], Voisin proves that Bloch's conjecture holds for a family of Campedelli surfaces constructed from  $5 \times 5$  symmetric matrices  $M(a)$ , with  $a \in \mathbb{P}^{11}$ , of linear forms on  $\mathbb{P}^3$  satisfying certain conditions.

Numerical Campedelli surfaces which arise as a  $(\mathbb{Z}/2)^3$ -cover of  $\mathbb{P}^2$  branched along 7 lines are called *Classical Campedelli surfaces*. They were constructed by Campedelli, and later by Kulikov in [Ku]. All 7 of the nontrivial involutions of classical Campedelli surfaces are composed with bicanonical map. The following result was first established in [IM, Thm. 3].

**Theorem 5.1.** *Classical Campedelli surfaces satisfy Bloch's conjecture.*

*Proof.* The automorphism group of  $S$  coincides with  $(\mathbb{Z}/2)^3$ ; see [Ku, Thm. 4.2]. By [CMP, 5.1], the bicanonical map is composed with every nontrivial involution  $\sigma$  of  $S$  and each quotient  $S/\sigma$  is either rational or else it is birationally equivalent to an Enriques surface, so  $t_2(S/\sigma) = 0$ . Fix a subgroup  $H = \langle \sigma_1, \sigma_2 \rangle$  of automorphisms, so that  $S$  is a bidouble cover of  $S/H$ . By Theorem 2.5 (applied to  $S \rightarrow S/H$ ) we get  $t_2(S) = 0$ , and hence  $S$  satisfies Bloch's conjecture.  $\square$

Here is one such family of classical Campedelli surfaces. We fix 7 distinct lines  $L_i$  in  $\mathbb{P}^2$  such that at most 3 pass through the same point, and enumerate the nontrivial elements of  $G = (\mathbb{Z}/2)^3$  as  $g_1, \dots, g_7$ , so that  $g_1, g_2, g_3$  generate  $G$  and if  $g_i + g_j + g_k = 0$  then  $L_i, L_j$  and  $L_k$  do not pass through the same point. Fix characters  $\chi_1, \chi_2, \chi_3$  generating  $G^*$ . By [Pa91], the equations  $2\mathcal{L}_i = \sum_{j=1}^7 \epsilon_{ij} L_j$  ( $i = 1, 2, 3$ ) determine a normal  $G$ -cover  $V$  of  $\mathbb{P}^2$  as long as  $\epsilon_{ij}$  is 1 if  $\chi_i(g_j) = -1$  and zero otherwise. (See Example 1 of [CMP, §5], or [Ku, 4.1].) The surface  $S$  is obtained by resolving the singular points of  $V$ , which only lie over the triple intersection points of the lines  $L_i$  in the plane.

6. THE CASE  $K_S^2 = 3, 4$ 

sec:K2=3,4

Let  $S$  be a minimal surface of general type with  $p_g(S) = 0$  and  $K_S^2 = 3$ . For each nontrivial involution  $\sigma$  of  $S$ , Corollary 3.6 and Lemma 1.5 imply that  $t_2(S/\sigma) = 0$ . Therefore if  $S$  is birational to a bidouble cover, then  $S$  satisfies Bloch's conjecture by Theorem 2.5.

**Example 6.1** ( $K_S^2 = 3$ ). Rito gives an example in [Ri1, 5.2] in which  $S$  is the minimal model of a bidouble cover of  $\mathbb{P}^2$  and  $K_S^2 = 3$ . Write  $\sigma_1, \sigma_2, \sigma_3$  for the 3 involutions of  $S$  corresponding to the bidouble cover. Rito shows that the bicanonical map is not composed with  $\sigma_1$  or  $\sigma_2$  but is composed with  $\sigma_3$ . If  $W_i$  is a minimal model of the desingularization of  $S/\sigma_i$ , he shows that  $W_1$  is an Enriques surface,  $W_2$  has Kodaira dimension 1 and  $W_3$  is a rational surface. Therefore  $t_2(W_i) = 0$  for  $i = 1, 2, 3$ . As remarked above, this implies that  $S$  satisfies Bloch's conjecture.

Similar examples with  $4 \leq K_S^2 \leq 7$  have been constructed in [Ri2].

Y. Neum and D. Naie [Na] constructed a family of surfaces of general type with  $K^2 = 4$  as double covers of an Enriques surface  $Y$  with 8 nodes. I. Bauer and F. Catanese proved in [BC3] that the connected component of the moduli space corresponding to this family is irreducible, normal, unirational of dimension 6. In Remark 3.3 of *op. cit.* they noticed that Keum's proof, given in the unpublished manuscript [Ke], of Bloch's conjecture for a subfamily of dimension 4 of the connected component of the moduli space of all Keum-Naie surfaces, can be extended to the whole family. We will now give a motivic proof of Bloch's conjecture for these surfaces.

Here is the relevant part of the construction of Keum-Naie surfaces given in [BC3]. Fix elliptic curves  $E_1, E_2$  and points  $a_i \in E_i$  of order 2. The group  $G = (\mathbb{Z}/2)^2$  acts freely on  $E_1 \times E_2$  using the involutions

$$\gamma_1(z_1, z_2) = (z_1 + a_1, -z_2) ; \gamma_2(z_1, z_2) = (-z_1, z_2 + a_2).$$

and the Enriques surface is  $Y = (E_1 \times E_2)/G$ . The automorphism  $\gamma_3(z_1, z_2) = (-z_1 + a_1, z_2)$  commutes with  $G$  and the quotient of  $Y$  by  $\gamma_3$  is a del Pezzo surface  $\Sigma$  in  $\mathbb{P}^4$ .

In [BC3, 1.4], a  $G$ -invariant double cover  $\tilde{X} \xrightarrow{\pi} E_1 \times E_2$  is constructed. If  $\tilde{S}$  is a minimal resolution of singularities of  $\tilde{X}$  then the quotient  $S = \tilde{S}/G$  is a Keum-Naie surface with  $K_S^2 = 4$  and  $p_g(S) = q(S) = 0$ . By [BC3, 4.1], the bicanonical map  $\Phi_2 : S \rightarrow \mathbb{P}^4$  of  $S$  has degree 4 and its image is  $\Sigma$ .

**Theorem 6.2.** *Let  $S$  be a Keum-Naie surface with  $K_S^2 = 4$  and  $p_g(S) = 0$ . Then  $S$  satisfies Bloch's conjecture.*

*Proof.* Because the étale degree 2 map  $\tilde{S} \rightarrow E_1 \times E_2$  is  $G$ -equivariant, it induces an étale degree 2 map  $p : S \rightarrow Y$ . Let  $H \cong (\mathbb{Z}/2)^2$  be the group of automorphisms of  $S$  generated by  $\gamma_3$  and the involution associated to  $p$ . Since the composition of  $p$  with  $Y \rightarrow \Sigma$  is the bicanonical map  $\Phi_2$ , all three nontrivial elements  $h_i$  of  $H$  are composed with  $\Phi_2$ . Hence  $t_2(S/\sigma_i) = 0$  for  $i = 1, 2, 3$  by Theorem 3.1. From Theorem 2.5 we get  $t_2(S) = 0$ .  $\square$

## 7. FIBRATIONS ON SURFACES

**sec:fibrations**

A surface  $S$  is said to be a *product-quotient surface* if it is birational to a quotient  $(C_1 \times C_2)/G$  of the product  $C_1 \times C_2$  of two curves of genera  $\geq 2$  by the action of a finite group  $G$ . Since the motives  $h(C_i)$  are finite dimensional, so are  $h(C_1 \times C_2)$  and  $h(S) = h(C_1 \times C_2)^G$ . Since  $t_2$  is a birational invariant,  $t_2(S)$  is also finite dimensional. If  $p_g(S) = 0$ , this implies that  $S$  satisfies Bloch's conjecture (by Lemma 1.5).

A complete classification is given in [BCGP] of surfaces  $S$  with  $p_g(S) = q(S) = 0$ , whose canonical models arise as product-quotient surfaces. If  $G$  acts freely then the quotient surface is minimal of general type and it is said to be *isogenous to a product*. If  $S$  is isogenous to a product then  $K_S^2 = 8$ , see [BCG, 0.1] or [BCP, Thm. 4.3].

In the case when  $G$  acts freely on both  $C_1$  and  $C_2$ , then the projection  $C_1 \times_G C_2 \rightarrow C_2/G$  has fibers  $C_1$ . More generally, a fibration  $S \rightarrow B$  from a smooth projective surface onto a smooth curve is said to be *isotrivial* if the smooth fibers are mutually isomorphic.

**thm:isotriv=fd**

**Theorem 7.1.** *Let  $S$  be a complex surface of general type with  $p_g = 0$  which has a fibration  $S \xrightarrow{\pi} B$  with  $B$  a smooth curve of genus  $b$ , and general fibre a curve  $F$  of genus  $g \geq 1$ . If  $\pi$  is isotrivial, then  $t_2(S) = 0$ , and  $S$  satisfies Bloch's conjecture.*

*Proof.* By [Se, 2.0.1] there is a finite group  $G$  acting on the fiber  $F$  and a Galois cover  $C \rightarrow B$  so that  $B = C/G$  and  $S$  is birational to  $S' = F \times_G C$ .  $\square$

One source of isotrivial fibrations comes from the following observation of Beauville. Let  $S$  be a smooth projective complex surface and  $S \rightarrow B$  a fibration with general fibre a smooth curve  $F$  of genus  $g \geq 1$ . Let  $b$  denote the genus of the curve  $B$ . Beauville proved in [Be] that

**eq:Beauville**

$$(7.2) \quad K_S^2 \geq 8(b-1)(g-1),$$

and if equality holds then the fibration is isotrivial.

Let  $S$  be a smooth complex surface with a fibration  $f : S \rightarrow \mathbb{P}^1$  with general fiber  $F$  a smooth curve of genus  $g(F)$ . In many cases, we

can construct a finite map  $h : C \rightarrow \mathbb{P}^1$  with  $C$  smooth such that the normalization  $X$  of  $C \times_{\mathbb{P}^1} S$  is nonsingular and the map  $\tilde{h} : X \rightarrow S$  is étale. Because  $\tilde{h}$  is étale,  $X$  is smooth and  $K_X = \tilde{h}^* K_S$ , we have  $K_X^2 = \deg(\tilde{h}) K_S^2$ . This information is summarized in the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\tilde{h}} & S \\ \downarrow f_C & & \downarrow f \\ C & \xrightarrow{h} & \mathbb{P}^1. \end{array} \quad (7.3)$$

**square:XSC**

**4.3** **Lemma 7.4.** *Suppose that the singular fibers of a fibration  $S \rightarrow \mathbb{P}^1$  consist only in double fibers, over points  $P_1, \dots, P_r$  of  $\mathbb{P}^1$ , and that there is a smooth cover  $h : C \rightarrow \mathbb{P}^1$  branched along the  $P_k$ , with  $h^{-1}(P_k)$  consisting only in double points. Then, in the diagram (7.3),  $X$  is nonsingular and fibered over  $C$ , and  $\tilde{h}$  is an étale map.*

*Proof.* The base change,  $C \times_{\mathbb{P}^1} S \rightarrow S$  is étale except over the double fibers, where it is a simple normal crossings divisor. Therefore the normalization  $X$  of  $C \times_{\mathbb{P}^1} S$  is étale over  $S$ .  $\square$

**doublecover**

**Example 7.5.** Suppose that  $f : S \rightarrow \mathbb{P}^1$  has an even number of singular fibers, all double fibers. Let  $C$  be the double cover of  $\mathbb{P}^1$  branched at the points of  $\mathbb{P}^1$  supporting the double fibers of  $f$ . Then the hypotheses of Lemma 7.4 are satisfied, so we have a diagram (7.3) with  $\tilde{h}$  étale.

**Pardini's method.** We recall Pardini's method (from [Pa91, (2.21)]) for producing a smooth cover  $C \rightarrow \mathbb{P}^1$ , branched along a given set of  $r$  points  $P_k$ , together with a faithful action of the group  $G = (\mathbb{Z}/2)^s$  on  $C$  so that  $C/G = \mathbb{P}^1$ . Fix linearly independent 1-dimensional characters  $\chi_1, \dots, \chi_s$  of  $G = (\mathbb{Z}/2)^s$  and enumerate the  $2^s - 1$  cyclic subgroups  $H_j$  of  $G$  in some order. Define integers  $\epsilon_{ij}$  to be 1 if the character  $\chi_i$  is nontrivial on  $H_j$  and 0 otherwise.

Suppose that we can partition the  $P_k$  into  $2^s - 1$  subsets  $D_j$  of cardinality  $n_j$ , and find integers  $L_1, \dots, L_s$  such that  $2L_i = \sum \epsilon_{ij} n_j$  for all  $i$ . Regarding each  $D_j$  as an effective divisor of degree  $n_j$  and the  $L_i$  as the degrees of line bundles, this yields a family of "reduced building data" in the sense of [Pa91, Prop. 2.1]. Pardini constructs a  $G$ -cover of  $\mathbb{P}^1$  from this data in *loc. cit.* Since the fiber over each  $P_k$  has  $2^{s-1}$  double points, the ramification divisor on  $C$  has degree  $2^{s-1}r$  and  $C$  has genus  $2^{s-2}r + 1 - 2^s$ , by Hurwitz' theorem. The case  $s = 1$  recovers the classical result that any even number of points in  $\mathbb{P}^1$  forms the branch locus of a double cover. Here is the case  $s = 2$ :

Pardini's cover

**Lemma 7.6.** *Let  $P_1, \dots, P_r$  be distinct points on  $\mathbb{P}^1$ ,  $r \geq 3$ . Then there exists a smooth curve  $C$  and a  $(\mathbb{Z}/2)^2$ -cover  $h : C \rightarrow \mathbb{P}^1$  such that the ramification divisor on  $C$  has degree  $2r$  and  $C$  has genus  $r - 3$ .*

*Proof.* We use Pardini's method with  $s = 2$ . We need to partition the points into three subsets of cardinalities  $n_i$ , corresponding to the three subgroups  $H_1, H_2, H_3$  of order 2 in  $(\mathbb{Z}/2)^2$ . As above, we need to solve the equations  $n_1 + n_2 + n_3 = r$ ,  $n_2 + n_3 = 2L_1$  and  $n_1 + n_3 = 2L_2$  for positive integers  $n_i$  and  $L_j$ . If  $r$  is even and at least 4, we can take  $n_1 = n_2 = 2$  and  $n_3 = r - 4$ ; if  $r$  is odd and at least 3, we can take  $n_1 = n_3 = 1$  and  $n_2 = r - 2$ .  $\square$

$K^2$  bound

**Theorem 7.7.** *Let  $S$  be a minimal surface of general type with  $p_g(S) = 0$ . Suppose that there exists a fibration  $f : S \rightarrow \mathbb{P}^1$  with general fiber  $F$  and  $r$  double fibers as singular fibers. Then*

$$K_S^2 \geq 2(r - 4)(g(F) - 1).$$

*If equality holds then  $S$  satisfies Bloch's conjecture.*

*Proof.* By Lemma 7.6 we can find a  $(\mathbb{Z}/2)^2$ -cover  $h : C \rightarrow \mathbb{P}^1$  branched over the  $r$  points of  $\mathbb{P}^1$  corresponding to the double fibres of  $f$ . We have seen that the degree of the ramification divisor of  $h$  on  $C$  is  $2r$ , and that  $C$  has genus  $r - 3$ . By Lemma 7.4 we can find a square (7.3) with  $\tilde{h} : X \rightarrow S$  étale. By Beauville's bound (7.2),

$$K_X^2 \geq 8(r - 4)(g(F) - 1),$$

and if equality holds then the fibration  $X \rightarrow C$  is isotrivial. Since  $K_X^2 = 4K_S^2$ , we have

$$K_S^2 = 2^{-2}K_X^2 \geq 2(r - 4)(g(F) - 1).$$

and if equality holds then (by Theorem 7.1)  $t_2(S) = 0$  and  $S$  satisfies Bloch's conjecture.  $\square$

$K^2=8$  rational

**Corollary 7.8.** *Let  $S$  be a minimal complex surface of general type with  $K_S^2 = 8$ . If  $S$  has an involution  $\sigma$  such that  $S/\sigma$  is rational, then  $S$  satisfies Bloch's conjecture.*

*Proof.* By [Pa03, Thm. 2.2],  $S$  has a fibration  $S \rightarrow \mathbb{P}^1$  with general fiber  $F$  and  $r$  double fibers, such that either  $r = 6$  and  $g(F) = 3$  or  $r = 5$  and  $g(F) = 5$ . In both cases there is an equality  $K_S^2 = 2(r - 4)(g(F) - 1)$ . By Theorem 7.7,  $S$  satisfies Bloch's conjecture.  $\square$

8. INOUE'S SURFACE WITH  $K^2 = 7$ sec:K2=7

Until recently, the only known family of examples of surfaces  $S$  of general type with  $K_S^2 = 7$  and  $p_g(S) = 0$  was constructed by M. Inoue in [I]. It is a quotient of a complete intersection in the product of four elliptic curves, by a free action of  $\mathbb{Z}/5$ . (Another family was found recently by Y. Chen in [Ch]; see Remark 8.4). In this section, we use Theorem 8.1 to show that Bloch's conjecture holds for Inoue's surfaces; Bauer's recent preprint [Bau] also gives a proof of Theorem 8.1, and her proof is similar to ours.

An alternative description of Inoue's surface as a bidouble cover of a rational surface was given in [MP01a]. Let  $\Pi$  be the blow up of  $\mathbb{P}^2$  in 6 points  $P_1, \dots, P_6$  as in [MP01a, Ex. 4.1] and let  $\pi : X \rightarrow \Pi$  be the bidouble cover with branch locus  $D = D_1 + D_2 + D_3$ , where

$$\begin{aligned} D_1 &= \Delta_1 + F_2 + L_1 + L_2 ; D_2 = \Delta_2 + F_3 ; \\ D_3 &= \Delta_3 + F_1 + F'_1 + L_3 + L_4. \end{aligned}$$

Here  $\Delta_i$  ( $i = 1, 2, 3$ ) is the strict transform in  $\Pi$  of the diagonal lines  $P_1P_3, P_2P_4$  and  $P_5P_6$ , respectively;  $L_i$  is the strict transform of the line between  $P_i$  and  $P_{i+1}$  of the quadrilateral  $P_1, P_2, P_3, P_4$ ;  $F_1$  is the strict transform of a general conic through  $P_2, P_4, P_5, P_6$  and  $F'_1 \in |F_1|$ ;  $F_2$  is the strict transform of a general conic through  $P_1, P_3, P_5, P_6$ ; and  $F_3$  is that of a general conic through  $P_1, P_2, P_3, P_4$ .

The image of the morphism  $f : \Pi \rightarrow \mathbb{P}^3$  given by  $|-K_\Pi|$  is a cubic surface  $V \subset \mathbb{P}^3$ ;  $f$  contracts each  $L_i$  to a point  $A_i$ , and is an isomorphism on  $\Pi \setminus \cup_i L_i$ . The (set-theoretic) inverse image in  $X$  of the 4 lines  $L_i$  is the disjoint union of two  $(-1)$ -curves  $E_{i,1}$  and  $E_{i,2}$ . The surface  $S$  is obtained by contracting these eight exceptional curves on  $X$ ; the results in [MP01a] show that  $S$  is a surface of general type with  $p_g(S) = 0$ ,  $K_S^2 = 7$ , that the bicanonical map  $\Phi_2 : S \rightarrow \mathbb{P}^7$  has degree 2, and the bicanonical map is one of the maps  $S \rightarrow S/\sigma$  associated to the bidouble cover.

K2=7

**Theorem 8.1.** *Inoue's surface  $S$  with  $K_S^2 = 7$  satisfies Bloch's conjecture.*

*Proof.* Let  $\sigma_1, \sigma_2, \sigma_3$  be the nontrivial involutions of  $X$  over  $\Pi$ , or equivalently, of  $S$  over  $V$ . We will determine the number  $k_i$  of isolated fixed points of  $\sigma_i$  on  $S$  in Lemma 8.2 below. Let  $Y_i$  be the desingularization of  $S/\sigma_i$  given by (2.1). For  $\sigma_1$  we have  $k_1 = 11 = K_S^2 + 4$  so, by Theorem 3.1,  $\Phi_2$  is composed with the involution  $\sigma_1$  and  $Y_1$  is either rational or birational to an Enriques surface. In particular,  $t_2(Y_1) = 0$ .

Since  $k_2$  and  $k_3$  are less than  $K_S^2 + 4$ ,  $\Phi_2$  is *not* composed with either  $\sigma_2$  or  $\sigma_3$ , by Theorem 3.1. By Theorem 3.5, the minimal models of  $Y_2$  and  $Y_3$  cannot have Kodaira dimension 2, because  $K_S^2 = 7$  is odd and  $k_2 = k_3 = 9 = K_S^2 + 2$ . It follows from [BKL] that  $Y_2$  and  $Y_3$  satisfy Bloch's conjecture, and so  $t_2(Y_2) = t_2(Y_3) = 0$ . By Theorem 2.5, this shows that  $t_2(S) = 0$  and finishes the proof.  $\square$

**k-Inoue**

**Lemma 8.2.** *The involutions  $\sigma_i$  on  $S$  have  $k_1 = 11$ ,  $k_2 = 9$  and  $k_3 = 9$  fixed points, respectively.*

*Proof.* There is a smooth bidouble cover  $p : S \rightarrow V$ , where  $V$  is obtained from  $\Pi$  by contracting the curves  $L_i$  to 4 singular points  $A_1, \dots, A_4$ . Hence we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \Pi \\ g \downarrow & & \downarrow f \\ S & \xrightarrow{p} & V \end{array}$$

where  $g(E_{i,j}) = Q_{i,j}$  and  $p(Q_{i,j}) = A_i$  for  $i = 1, \dots, 4$  and  $j = 1, 2$ . The divisors on  $\Pi$  satisfy the following relations, see [MP01a, Ex.4.1]:

$$\begin{aligned} -K_\Pi &\equiv \Delta_1 + \Delta_2 + \Delta_3; & F_i &\equiv \Delta_{i+1} + \Delta_{i+2}, i \in \mathbb{Z}/3; \\ \Delta_i \cdot L_j &= 0; & \Delta_i \cdot F_j &= 2\delta_{ij}. \end{aligned}$$

From these relations it is easy to calculate that the number  $k'_i$  of isolated fixed points of  $\sigma_i$  on  $X$  is:  $k'_1 = (D_2 \cdot D_3) = 7$ ;  $k'_2 = (D_1 \cdot D_3) = 9$ ; and  $k'_3 = (D_1 \cdot D_2) = 5$ . The surface  $S$  may have additional isolated fixed points  $Q_{i,j}$ , because some of the curves  $E_{i,j}$  may be fixed on  $X$ .

By construction the divisorial part of the fixed locus of  $\sigma_i$  on  $X$  lies over  $D_i$ . Since  $L_1 + L_2$  is contained in  $D_1$ , the fixed locus of  $\sigma_1$  on  $X$  contains the four  $(-1)$ -curves  $E_{i,1}$  ( $i = 1, 2$ ) lying over  $L_1$  and  $L_2$ . Their images  $Q_{i,1}$  and  $Q_{i,2}$  are fixed points of  $\sigma_1$  on  $S$ , so  $\sigma_1$  has  $k_1 = k'_1 + 4 = 11$  isolated fixed points on  $S$ . Similarly, since  $L_3$  and  $L_4$  are contained in  $D_3$  the fixed locus of  $\sigma_3$  on  $S$  contains the four  $(-1)$ -curves  $E_{3,j}$  and  $E_{4,j}$ , so  $\sigma_3$  has  $k_3 = k'_3 + 4 = 9$  isolated fixed points on  $S$ . Finally, we have  $k_2 = k'_2 = 9$  because none of the  $L_i$  are contained in the divisorial part  $D_2$  of  $\sigma_2$ .  $\square$

**Remark 8.3.** In the recent preprint [LS12], Lee and Shin consider the case of the Inoue's surface  $S$  with  $K_S^2 = 7$  and  $p_g(S) = 0$  and compute  $K_{W_i}^2$ , where  $W_i$  is a minimal model of  $S/\sigma_i$  ( $i = 1, 2, 3$ ). In particular they show that  $W_1$  and  $W_3$  (hence  $S/\sigma_1$  and  $S/\sigma_3$ ) are rational, while  $W_2$  (and hence  $S/\sigma_2$ ) is birational to an Enriques surface. Theorem 8.1 also follows from this via Theorem 2.5.

**rem:Chen**

**Remark 8.4.** Recently Y. Chen in [Ch] has produced a family of surfaces of general type with  $K^2 = 7$  and  $p_g = 0$ , whose bicanonical map is not composed with any involution on  $S$ . Chen has verified that Bloch's conjecture holds for these surfaces.

## 9. BURNIAT SURFACES AND SURFACES WITH $K^2 = 6$ , $K^2 = 7$

In this section we show that Burniat surfaces satisfy Bloch's conjecture and consider other surfaces with  $K^2 = 6$ .

### *Burniat surfaces*

Burniat surfaces are certain surfaces of general type with  $2 \leq K_S^2 \leq 6$ . It is proven in [BC1, Thm. 2.3] that Burniat surfaces are exactly the surfaces constructed by Inoue in [I] as the  $(\mathbb{Z}/2)^3$ -quotient of an invariant hypersurface in the product of 3 elliptic curves.

To construct them, one first forms a del Pezzo surface  $\Pi$  as the blowup of the plane at 3 non-collinear points  $P_1, P_2, P_3$ , see [Pet]. Then one forms a bidouble cover  $\bar{S}$  of  $\Pi$  whose branch locus is the union of the exceptional curves and the proper transform of 9 other lines. Then  $S$  is the minimal resolution of  $\bar{S}$ , and  $\bar{S}$  has  $k = 6 - K_S^2$  singular points.

If  $K_S^2 = 6$  (so  $S = \bar{S}$ ), these are called *primary* Burniat surfaces. If  $K_S^2 = 4, 5$  (so  $k = 1, 2$ )  $S$  is called *secondary*; if  $K_S^2 = 2, 3$  (so  $k = 3, 4$ )  $S$  is called *tertiary*. see [BC1, 2.2].

An important feature of the Burniat surfaces  $S$  is that their bicanonical map is the composition of a bidouble cover  $S \rightarrow \Pi'$  onto a normal del Pezzo surface  $\Pi'$  (a blowup of  $\Pi$ ) followed by the anticanonical embedding of  $\Pi'$  into  $\mathbb{P}^6$ . (See [MP01, 3.1] and [BC1, 4.1]). Thus  $\Phi_2$  is composed with each of the three nontrivial involutions  $\sigma_i$  on  $S$ .

The following result was first proven by Inoue-Mizukmai in [IM, Thm. 2]; cf. [BC2, 4.4].

**thm:Burniat**

**Theorem 9.1.** *Every Burniat surface  $S$  satisfies Bloch's conjecture*

*Proof.* As noted above, the bicanonical map  $\Phi_2$  is the composition of a bidouble cover  $S \rightarrow \Pi'$  with the anticanonical embedding of  $\Pi'$ . Thus  $\Phi_2$  is composed with each of the three nontrivial involutions  $\sigma_i$  on  $S$  associated to this cover. By Theorem 3.1 each  $S/\sigma_i$  is either rational or birational to an Enriques surface, so in particular a desingularization  $W_i$  of  $S/\sigma_i$  has  $t_2(W_i) = 0$ . From Theorem 2.5 we have  $t_2(S) = 0$ , viz.,  $S$  satisfies Bloch's conjecture.  $\square$

**Remark 9.1.1.** As noted in [BC1, §5], there is only one Burniat surface with  $K_S^2 = 2$ , and it is a surface in the 6-dimensional family of classical Campedelli surfaces with  $\pi_1(S) = (\mathbb{Z}/2)^3$ . Thus this surface fits into the discussion in Section 5, where we also noted that such surfaces satisfy Bloch’s conjecture.

*Surfaces with  $K^2 = 6$*

We now consider a minimal surface  $S$  of general type with  $p_g(S) = 0$  and  $K_S^2 = 6$ . Either the bicanonical map  $\Phi_2 : S \rightarrow \mathbb{P}^6$  is birational or the degree of  $\Phi_2$  is either 2 or 4, by [MP04, Thm.1.1].

When the degree of  $\Phi_2$  is 4 and  $K_S^2 = 6$ , a complete classification has been given in [MP01]. The classification shows that all these surfaces are Burniat surfaces. It follows that they satisfy Bloch’s conjecture, by Theorem 9.1.

Now suppose that the degree of  $\Phi_2$  is 2. Then  $\Phi_2$  determines an involution  $\tau$  on  $S$ , and  $\Phi_2$  is composed with  $\tau$ . By [MP04, 1.2] there is a fibration  $f : S \rightarrow \mathbb{P}^1$ , whose general fibre  $F$  is hyperelliptic of genus 3, such that the bicanonical involution  $\tau$  on  $S$  induces the hyperelliptic involution on  $F$  and  $f$  has either  $r = 4$  or  $r = 5$  double fibres. (Both possibilities occur; see [MP04, §4]).

In particular  $S/\tau$  is rational. Forming the square (2.1), the desingularization  $Y$  of  $S/\tau$  is rational by [MP04, 2.1]. Hence  $t_2(Y) = 0$ . By Theorem 3.1,  $-4 \leq K_Y^2 \leq 0$  and the fixed locus of  $\tau$  has 10 isolated points. The cases of  $r = 4$  or 5 double fibers corresponds to the cases  $K_Y^2 = -4$  and  $K_Y^2 > -4$ .

Although we have been unable to show that Bloch’s conjecture holds for every surface with  $K_S^2 = 6$  whose bicanonical map has degree 2, we can show this for the examples constructed in [MP04, §4]. We shall use the notation of Section 8, since the construction of these examples are variations of the construction of Inoue’s surface described there.

**construct.1**

**Construction 9.2.** In the example in [MP04, 4.1] we assume that the conics  $F_1, F_2, F_3$  all pass through a general point  $P$  and that pairwise they intersect transversally at  $P$ . In this case the branch locus  $D = D_1 + D_2 + D_3$  acquires a singular point of type  $(1, 1, 1)$ , and the resulting bidouble cover  $X \rightarrow \Pi$  has a singularity over the image  $P'$  of  $P$ . Blowing up  $P'$  gives  $\Pi'$ ; normalizing  $X \times_{\Pi} \Pi'$  yields a bidouble cover  $S \rightarrow \Pi'$ , and the surface  $S$  has  $p_g = 0$  and  $K_S^2 = 6$ . The surface  $S$  has a fibration  $f : S \rightarrow \mathbb{P}^1$ , whose general fiber is hyperelliptic of genus 3 and  $f$  has 4 double fibers. Moreover the exceptional divisor of

$S \rightarrow X$  is a smooth rational curve  $C$  with  $C^2 = -4$ ,

$$\begin{array}{ccc} S & \xrightarrow{p} & \Pi' \\ \downarrow & & \downarrow \\ X & \xrightarrow{p_0} & \Pi \end{array}$$

**K2=6.1** **Proposition 9.3.** *The surface  $S$  of 9.2 satisfies Bloch's conjecture.*

*Proof.* We shall write  $\sigma_i$  for the three nontrivial involutions on  $X$  and on  $S$  associated to the bidouble covers. We first compute the number  $k'_i$  of isolated fixed points on  $X$  of  $\sigma_i$ , as we did in Lemma 8.2. Since  $P \in D_1 \cap D_2 \cap D_3$  and  $F_1 \cdot F_3$  contains only 1 point outside of  $P_1, \dots, P_6$  we get  $k'_1 = (D_2 \cdot D_3) = 6$ . Similarly we get  $k'_2 = 8$  and  $k'_3 = 4$ . Arguing again as in Lemma 8.2, this implies that the number  $k_i$  of isolated fixed points of  $\sigma_i$  on  $S$  is:  $k_1 = 10$ ,  $k_2 = k_3 = 8$ .

By Theorem 3.1, the bicanonical map  $\Phi_2$  is composed with  $\sigma_1$  but not with  $\sigma_2$  or  $\sigma_3$ . In particular,  $S/\sigma_1$  is rational. We will show that the desingularizations  $Y_2$  and  $Y_3$  of  $S/\sigma_2$  and  $S/\sigma_3$  satisfy Bloch's conjecture, so  $t_2(Y_2) = t_2(Y_3) = 0$ ; Theorem 2.5 will imply that  $t_2(S) = 0$ , i.e., that  $S$  satisfies Bloch's conjecture.

By symmetry it suffices to consider  $Y_2$ . Let  $X_2$  denote the blowup of  $S$  along the  $k_2 = 8$  isolated fixed points of  $\sigma_2$ , and form the square

$$\begin{array}{ccc} X_2 & \xrightarrow{h} & S \\ \bar{\sigma}_2 \downarrow & & \sigma_2 \downarrow \\ Y_2 & \xrightarrow{g} & S/\sigma_2 \end{array}$$

as in (2.1). The images in  $Y_2$  of the 8 exceptional curves on  $X_2$  are 8 disjoint nodal curves  $C_1, \dots, C_8$ . As pointed after (2.1) the branch locus of  $X_2 \rightarrow Y_2$  is  $B = g^*(D_2) + C + \sum C_j$ . We get a minimal resolution  $W$  of  $S/\sigma_2$  with branch locus  $\bar{B}$  by blowing down curves on  $Y_2$ . By Theorem 3.5,  $W$  cannot have Kodaira dimension 2:  $\bar{B}$  is not the disjoint union of 4 nodal curves, and  $k = 8 > K_S^2 = 6$ . Hence  $W$  and  $Y_2$  are not of general type, so  $t_2(Y_2) = 0$  as desired.  $\square$

**construct. 2**

**Construction 9.4.** The second example of a surface of general type with  $p_g = 0$ ,  $K_S^2 = 6$  is constructed in [MP04, 4.2]; the fibration  $S \rightarrow \mathbb{P}^1$  has a hyperelliptic curve of genus 3 as its general fiber, and 5 double fibers. We start with the same configuration of 6 points  $P_1, \dots, P_6$  as in Section 8, and consider the point  $P_7 = \Delta_2 \cap \Delta_3$ . Form the blowup  $\Pi'$  of  $\pi$  at  $P_7$ , and write  $e_7$  for the corresponding exceptional divisor. Let  $\Delta'_2$  and  $\Delta'_3$  denote the strict transform of  $\Delta_2$  and  $\Delta_3$  on  $\Pi'$  and set

$D = D_1 + D_2 + D_3$ , where

$$D_1 = C + L_1 + L_2 ; D_2 = F_3 ; D_3 = F_1 + F'_1 + \Delta'_2 + \bar{\Delta}'_3 + L_3 + L_4.$$

Here  $F_1, F'_1 \in |F_1|$ ,  $F_3 \in |F_3|$  and  $C \in |F_2 + F_3 - 2e_7|$  are general curves. As noted in [MP04],  $C$  is a smooth irreducible curve of genus 0, and we get a smooth bidouble cover  $\pi : X \rightarrow \Pi'$  with  $p_g(X) = 0$ .

For  $i = 1, \dots, 4$  the inverse image of  $L_i$  in  $X$  is the disjoint union of two  $(-1)$ -curves  $E_{i1}, E_{i2}$ . Also the inverse image of  $\Delta'_2$  is the disjoint union of two  $(-1)$ -curves  $E_1$  and  $E_2$ . The system  $|-K_{\Pi'}|$  gives a degree 2 morphism  $\Pi' \rightarrow \mathbb{P}^2$ . The surface  $S$  obtained from  $X$  by contracting  $E_1, E_2$  and  $E_{ij}$ , with  $1 \leq i \leq 4$  and  $1 \leq j \leq 2$  is minimal of general type with  $K_S^2 = 6$  and  $p_g(S) = 0$ .

**K2=6.2**

**Proposition 9.5.** *The surface  $S$  of 9.4 satisfies Bloch's conjecture.*

*Proof.* We shall write  $\sigma_i$  for the three nontrivial involutions on  $X$  and on  $S$  associated to the bidouble covers. As noted in the proof of Proposition 9.5, the bicanonical map  $\Phi_2$  is composed with  $\sigma_1$  and hence  $S/\sigma_1$  is rational and  $t_2(S/\sigma_1) = 0$ .

Next, we compute the number  $k'_2$  and  $k'_3$  of isolated fixed points on  $X$  of  $\sigma_2$  and  $\sigma_3$  and the corresponding number  $k_2$  and  $k_3$  of isolated fixed points on  $S$ , as we did in Lemma 8.2.

We have  $k'_2 = (D_1 \cap D_3) = (C \cdot F_1) + (C \cdot F'_1) + (C \cdot \Delta'_2) + (C \cdot \Delta'_3)$ . Now  $(C \cdot F_1) = (C \cdot F'_1) = 4$  and  $(C \cdot \Delta'_2) = (C \cdot \Delta'_3) = 0$ , because  $e_7 \cdot \Delta'_2 = e_7 \cdot \Delta'_3 = 1$ . Therefore  $k'_2 = 8$  and  $k_2 = k'_2 = 8$  because  $D_2 = F_3$  does not contain any of the  $L_i$ .

Similarly  $k'_3 = (D_1 \cdot D_2) = (F_2 + F_3) \cdot F_3 = 2$ , because  $(F_3)^2 = 0$ . This last equality follows from adjunction because  $F_3$  has genus 0 and  $K_{\Pi} \cdot F_3 = (-\Delta_1 + \Delta_2 + \Delta_3) \cdot F_3 = -2$ . The fixed divisorial part  $D_3$  of  $\sigma_3$  contains  $L_3 + L_4$  and  $\Delta'_2$  which contract on  $S$  to the 6 points corresponding to  $E_{ij}$  and to  $E_1, E_2$ . Therefore  $k_3 = k'_3 + 6 = 8$ .

Since  $k_2 = k_3 = 8$  are less than  $K_S^2 + 4 = 10$ , Theorem 3.1 implies that  $\Phi_2$  is not composed with either  $\sigma_2$  or  $\sigma_3$ . As in the previous example let  $X_i$  be the blowup of  $S$  along the fixed points of  $\sigma_i$ , and let  $\pi_i : X_i \rightarrow Y_i$  be the map to the desingularization  $Y_i$  of  $S/\sigma_i$  ( $i = 2, 3$ ). Then  $\pi_i$  is branched on  $B_i = g_i^*(D_i) + \sum_{1 \leq h \leq 8} C_i^h$ , with  $(C_i^h)^2 = -2$ . Because the image  $\bar{B}_i$  of the branch locus  $B_i$  in a minimal model  $W_i$  of  $Y_i$  is not the disjoint union of 4 nodal curves, it follows from Theorem 3.5 that the Kodaira dimension of  $W_i$  cannot be 2. Therefore both  $Y_2$  and  $Y_3$  satisfy Bloch's conjecture, so  $t_2(Y_2) = t_2(Y_3) = 0$ . By Theorem 2.5, this shows that  $t_2(S) = 0$  and finishes the proof.  $\square$

**Remark 9.5.1.** The same argument applies to the surface in Example 4.3 in [MP04], showing that it also satisfies Bloch's conjecture.

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## REFERENCES

- [Bar] R. Barlow, *Rational equivalence of zero cycles for some more surfaces with  $p_g = 0$* , Invent. math. **79** (1985), 303-308.
- [Bau] I. Bauer, *Bloch's conjecture for Inoue surfaces with  $p_g = 0, K^2 = 7$* , preprint arXiv:1210.4287v 2 [math.AG], 29 Nov. 2012.
- [BC1] I. Bauer and F. Catanese, *Burniat surfaces I: fundamental groups and moduli of primary Burniat surfaces*, pp. 49–76 in *Classification of algebraic varieties*, EMS Ser. Congr. Rep., Eur. Math. Soc., 2011.
- [BC2] I. Bauer and F. Catanese, *Burniat surfaces II: secondary Burniat surfaces form three connected components of the moduli space*, Invent. Math. **180** (2010), 559-588.
- [BC-KN] [BC3] I. Bauer and F. Catanese, *The moduli space of Keum-Naie surfaces*, Groups, Geometry, and Dynamics **5** (2011), 231-250.
- [BCG] [BCG] I. Bauer, F. Catanese and F. Grunewald *The classification of surfaces with  $p_g = 0$  isogenous to a product of curves*, Pure Appl. Math Q. **4** (2008), 547-586.
- [BCGP] [BCGP] I. Bauer, F. Catanese, F. Grunewald and R. Pignatelli, *Quotients of products of curves, new surfaces with  $p_g = 0$* , Amer. J. Math. **134** (2012), 993-1049.
- [BCP] [BCP] I. Bauer, F. Catanese, and R. Pignatelli, *Surfaces of general type with geometric genus 0: a survey*, Complex and Differential Geometry, Springer Proc. in Math. **8** (2011) 1-48.
- [Be] [Be] A. Beauville, *L'inegalité  $p_g \geq 2q-4$  pour les surfaces de type general*, Appendix to [Deb], Bull. Soc Math. France **110** (1982), 343–346.
- [BKL] [BKL] S. Bloch, A. Kas and D. Lieberman, *Zero cycles on surfaces with  $p_g = 0$* , Compositio Math. **33** (1976), 135–145.
- [BHPV] [BHPV] W. Barth, K. Hulek, C. Peters and A. Van de Ven, *Compact complex surfaces*, Second edition, Springer, 2004.
- [CCM] [CCM] A. Calabri, C. Ciliberto and M. Mendes Lopes, *Numerical Godeaux surfaces with an involution*, Trans. AMS **359** (2007), 1605–1632.
- [CD] [CD] F. Catanese and O. Debarre, *Surfaces with  $K^2 = 2, p_g = 1, q = 0$* , J. Reine angew. Math. **395** (1989), 1-55.
- [CMP] [CMP] A. Calabri, M. Mendes Lopes and R. Pardini, *Involutions on numerical Campedelli surfaces*, Tohoku Math. J. **60** (2008), 1–22.
- [Ch] [Ch] Y. Chen, *A new Family of surfaces of general type with  $K^2 = 7$  and  $p_g = 0$* , preprint arXiv:1210.4633v 3 [math.AG], 1 Nov. 2012.
- [CG] [CG] H. Clemens and P. Griffiths, *The intermediate Jacobian of a cubic three-fold*, Annals of Math., **95**,(1972), 281-356.
- [DMP] [DMP] I. Dolgachev, M. Mendes Lopes and R. Pardini, *Rational surfaces with many nodes*, Compositio Math. **132** (2002), 349–363.
- [Fu] [Fu] W. Fulton, *Intersection Theory*, Springer-Verlag, Heidelberg-New-York, 1984.
- [GP2] [GP2] V. Guletskii and C. Pedrini, *Finite-dimensional motives and the conjectures of Beilinson and Murre*, K-Theory **30** (2003), 243–263.

- [GP1] [GP1] V. Guletskii and C. Pedrini, *The Chow motive of the Godeaux surface*, Algebraic Geometry, a Volume in Memory of Paolo Francia, Walter de Gruyter (2002), 180–195.
- [IM] [IM] H. Inose and M. Mizukami, *Rational equivalence of 0-cycles on some surfaces of general type with  $p_g = 0$* , Math. Ann. **244** (1979), 205–217.
- [I] [I] M. Inoue, *Some new surfaces of general type*, Tokyo J. Math. **17** (1994), 295–319.
- [KMP] [KMP] B. Kahn, J. Murre and C. Pedrini, *On the transcendental part of the motive of a surface*, pp. 143–202 in "Algebraic cycles and Motives Vol II", London Math. Soc. LNS **344**, Cambridge University Press, 2008.
- [Keum] [Ke] J. H. Keum, *Some new surfaces of general type with  $p_g = 0$* , Unpublished preprint, 1988.
- [Ki] [Ki] S. Kimura, *Chow groups are finite dimensional, in some sense*, Math. Ann. **331** (2005), 173–201. Preprint, 1998.
- [Ku] [Ku] V.S. Kulikov, *Old examples and a new example of surfaces of general type with  $p_g = 0$* , Izv. Math **68** (2004), 965–1008.
- [LS12] [LS12] Y. Lee and Y. Shin *Involutions on a surface of general type with  $p_g = q = 0$  and  $K^2 = 7$* , preprint arXiv:1003.935 v 4 [math.AG], Oct 2012.
- [MP01] [MP01] M. Mendes Lopes and R. Pardini, *A connected component of the moduli space of surfaces with  $p_g = 0$* , Topology **40**(2001), 977–991.
- [MP01a] [MP01a] M. Mendes Lopes and R. Pardini, *The bicanonical map of surfaces with  $p_g = 0$  and  $K^2 \geq 7$* , Bull. London Math. Soc. **33** (2001), 265–274.
- [MP02] [MP02] M. Mendes Lopes and R. Pardini, *A survey on the bicanonical map of surfaces with  $p_g=0$  and  $K^2 \geq 2$* , pp. 277–287 in Algebraic Geometry, a Volume in Memory of Paolo Francia, Walter de Gruyter (2002).
- [MP04] [MP04] M. Mendez Lopez and R. Pardini, *Surfaces of general type with  $p_g = 0$ ,  $K^2 = 6$  and non birational bicanonical map*, Math. Ann. **329** (2004), 535–552.
- [Na] [Na] D. Naie, *Surfaces d'Enriques et un construction de surfaces de type general avec  $p_g = 0$* , Math. Zeit. **215** (1994), 269–280.
- [Pa91] [Pa91] R. Pardini, *Abelian covers of algebraic varieties*, J. reine angew. Math. **417** (1991), 191–213.
- [Pa03] [Pa03] R. Pardini, *The classification of double planes of general type with  $K^2 = 8$  and  $p_g = 0$* , J. Algebra **259** (2003), 95–118.
- [Ped] [Ped] C. Pedrini *On the finite dimensionality of a  $K3$  surface*, Manuscripta Math. **138** (2012), 59–72.
- [Pet] [Pet] C. Peters, *On certain examples of surfaces with  $p_g = 0$  due to Burniat*, Nagoya Math J. **66** (1977), 109–119.
- [Ri1] [Ri1] C. Rito, *Involutions on surfaces with  $p_g = q = 0$  and  $K_S^2 = 3$* , Geom. Dedicata **157** (2012), 319–330.
- [Ri2] [Ri2] C. Rito, *Some bidouble planes with  $p_g = q = 0$  and  $4 \leq K^2 \leq 7$* , preprint arXiv:1103.2940 v 2 [math.AG], 2012.
- [Se] [Se] F. Serrano, *Isotrivial fibred surfaces*, Ann. Mat. Pura Appl. (IV) **CLXXI** (1996), 63–81.
- [Vois] [Vois] C. Voisin, *Bloch's conjecture for Catanese and Barlow surfaces*, preprint arXiv:1203.935 v2 [math.AG], Jan 2013.
- [Zh] [Zh] D.Q. Zhang, *Quotients of  $K3$  surfaces modulo involutions*, Japan J. of Math **24** n.2 (1998), 335–366.