

# SEVERI'S RESULTS ON CORRESPONDENCES

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## 1. INTRODUCTION

Severi developed a theory of correspondences in a series of papers which appeared in 1933, introducing the notions of *valences* and *indices*. One of the results achieved by Severi is a formula for the virtual number of fixed points of a correspondence on a smooth projective surface  $X$ . These papers are part of Severi's attempt to develop a theory of the series of equivalences on a surface. In fact Severi encountered (sometimes without being completely aware of what was going on!) the problem of not having a rigorous definition for the different equivalence relations among cycles, which are now known as rational, algebraic, homological and numerical equivalence. However as W. Fulton writes in [?, p.26]:

*It would be unfortunate if Severi's pioneering works in this area were forgotten; and if incompleteness or the presence of errors are grounds for ignoring Severi's work, few of the subsequent papers on rational equivalence would survive.*

The above considerations indicate that Severi was often wrong and certainly too bold in making conjectures. However Severi was somehow able to perceive the *motivic* content of the matter, by considering correspondences and their action both on Chow groups and cohomology groups. In fact he was the first to relate the action of a correspondence  $\Gamma \subset X \times X$  on the Chow group of 0-cycles on a smooth projective surface  $X$  to the cohomology class of  $\Gamma$  in  $H^4(X \times X, \mathbb{C})$ . In [?] (see also [?, 3.3]) he made a claim that in its original form is not correct but can be easily restated as what is now known as Bloch's conjecture.

**BC** **Conjecture 1.1.** *Let  $S$  be smooth projective surface over  $\mathbb{C}$ . If  $p_g(S) = q(S) = 0$  the Chow group  $CH_0(X)_0$  of 0-cycles of degree 0 vanishes.*

Bloch's conjecture is known to hold for all surfaces which are not of general type, see [?], and for many surfaces of general type, see [?] and [?].

In this note we will give a precise formulation of Severi's result on the virtual number of fixed points of a correspondence on a surface  $S$

(Theorem ??). We also provide a proof of what Severi claimed for the case when the diagonal  $\Delta_S$  has valence 0 (Theorem ??). Then, in Sect. 5, we apply these results to a complex surface  $S$  with an involution, in the case  $p_g(S) = q(S) = 0$  and in the case  $S$  is a K3 surface.

If  $X$  is a smooth projective variety, we will write  $CH^i(X)$  for the Chow group of codimension  $i$  cycles on  $X$ , and write  $A^i(X)$  for the Chow group with  $\mathbb{Q}$ -coefficients,  $CH^i(X) \otimes \mathbb{Q}$ . We write  $NS(S)$  for the Néron-Severi group of  $S$ , and  $\rho(S)$  for the rank of  $NS(S)$ . If  $\Gamma$  is any correspondence on  $X$ , we write  $\text{trace}_{NS}(\Gamma)$  for the trace of  $\Gamma$  acting on the vector space  $NS(X) \otimes \mathbb{Q}$ . For example,  $\text{trace}_{NS}(\Delta_X)$  is the rank  $\rho(X)$  of  $NS(X)$ .

## 2. INDICES AND VALENCE OF A CORRESPONDENCE

Let  $S$  be a smooth projective surface over  $C$  and let  $\Gamma$  be a correspondence in  $CH^2(S \times S)$ . The formula for the virtual number of fixed points of  $\Gamma$  is given in terms of the following numbers: the second Chern class  $c_2(S)$ , the trace of the action of  $\Gamma$  on the vector space  $NS(S) \otimes \mathbb{Q}$ , the *valence*  $v$  and the *indices*  $\alpha, \beta$  of  $\Gamma$  (see Definition ?? below). The following formula appeared for the first time in 1933 in [?, p. 871]:

$$\boxed{2.1} \quad (2.1) \quad \deg(\Gamma \cdot \Delta_S) = \alpha(\Gamma) + \beta(\Gamma) + \delta(\Gamma) - v(\Gamma)(I + 1),$$

where  $\delta(\Gamma)$  is the *rank* of  $\Gamma$  (see Definition ??) and  $I$  is the Zeuthen-Segre invariant,  $I = c_2(S) - 4$ . The same formula was reproduced in the first edition of Zariski's book on Algebraic Surfaces which appeared in 1935, and also in the second edition of it in 1971 (see [?, p. 246]).

The notion of the indices and valence of a correspondence  $\Gamma$  on a smooth projective variety  $X$  are about 100 years old and were well known to Severi and Lefschetz; see [?]. We give their precise definition below, following [?, §16], using the notion of degenerate correspondences in the Chow group of algebraic cycles modulo rational equivalence.

However, Severi's notion of the rank of  $\Gamma$ , as given in op.cit., is rather obscure. Also, Severi's formula is based on the assumption that the correspondence  $\Delta_S$  has rank 1, i.e., if  $\Delta_S$  does not belong to the ideal  $\mathcal{J}(S)$  of degenerate correspondences (see Definition ??). We will give the correct definition of the rank of a correspondence in Definition ??.

$\boxed{2.2}$  **Definition 2.2.** Let  $X$  be a smooth projective variety over a field  $k$ . The *indices* of a correspondence  $\Gamma \subset X \times X$  are the numbers  $\alpha(\Gamma) =$

$\deg(\Gamma \cdot [P \times X])$  and  $\beta(\Gamma) = \deg(\Gamma \cdot [X \times P])$ , where  $P$  is any rational point on  $X$ ; see [?, 16.1.4].

The indices are additive in  $\Gamma$ , and  $\beta(\Gamma) = \alpha({}^t\Gamma)$ .

**2.3** **Definition 2.3.** A correspondence is said of *valence zero* if it belongs to the ideal  $\mathcal{J}(X)$  in  $A^n(X \times X)$  of degenerate correspondences, i.e., the ideal generated by correspondences of the form  $[V \times W]$ , with  $V$  or  $W$  proper subvarieties of  $X$ . We say that a correspondence  $\Gamma$  has *valence*  $v$  if  $\Gamma + v\Delta_X$  has valence 0.

For example,  $\Delta_X$  always has valence  $-1$ , but it may also have valence 0, as is the case for  $X = \mathbb{P}^1$ . If  $\Gamma_1, \Gamma_2$  in  $A^d(X \times X)$  have valences  $v_1, v_2$  then  $\Gamma = \Gamma_1 + \Gamma_2$  has valence  $v_1 + v_2$ , and  $\Gamma_1 \circ \Gamma_2$  has valence  $-v_1v_2$  by [?, 16.1.5(a)].

If  $\Delta_X$  does not have valence zero then the valence of a correspondence  $\Gamma$  is either unique or undefined. On the other hand, if  $\Delta_X$  has valence zero and the valence of a correspondence  $\Gamma$  is defined, then  $\Gamma \in \mathcal{J}(X)$ , hence it has valence  $v$  for every  $v \in \mathbb{Q}$ . This is, for example, the case if  $X$  is a rational surface.

**Example 2.4** (Chasles-Cayley-Brill-Hurwitz). Let  $C$  be a curve of genus  $g$ . If  $T \in A^1(C \times C)$  is a correspondence with valence  $v$ , then the Cayley-Brill formula is:  $\deg(T \cdot \Delta_C) = \alpha(T) + \beta(T) + 2vg$ . This is proven in [Fu 16.1.5(e)].

We thank the Referee for pointing out the following Lemma; cf. [?, 2.2.1].

**2.5** **Lemma 2.5.** *If  $\Delta_X$  has valence zero, then rational, algebraic, homological and numerical equivalence coincide in  $A^*(X)$ .*

*Proof.* If  $v(\Delta_S) = 0$  then the diagonal decomposes as  $\sum_j c_j[V_j \times W_j]$ . Hence for any cycle  $Z \in A^*(X)$  we have

$$Z = \Delta_X \cdot Z = \sum_j c_j(Z \cdot V_j)[W_j]$$

It follows that numerically equivalent cycles are rationally equivalent.  $\square$

### 3. THE CHOW MOTIVE OF A SURFACE

Let  $\mathcal{M}_{rat}(k)$  be the (covariant) category of *Chow motives* with  $\mathbb{Q}$ -coefficients over an algebraically closed field  $k$  of characteristic 0 and let  $h(X)$  be the motive associated to a smooth projective variety  $X$ .

If  $S$  is a smooth projective surface then the motive  $h(S)$  has a *reduced Chow-Künneth decomposition* as in [?, 7.2.2] of the form

$$h(S) = h_0(S) \oplus h_1(S) \oplus h_2(S) \oplus h_3(S) \oplus h_4(S),$$

where  $h_i(S) = (S, \pi_i, 0)$ ; each  $\pi_i$  is a projector whose cohomology class is the  $(i, 4 - i)$  component in the Künneth decomposition of  $\Delta_S$  in  $H^4(S \times S)$ . Here  $H^*$  means any classical Weil cohomology theory, such as Betti cohomology with  $\mathbb{Q}$  coefficients when  $k = \mathbb{C}$ .

The motive  $h_2(S)$  further decomposes as  $h_2^{\text{alg}}(S) \oplus t_2(S)$  where  $h_2^{\text{alg}}(S)$  is the algebraic part of  $h_2(S)$  and  $t_2(S)$  is the *transcendental motive*. The motive  $h_2^{\text{alg}}(S)$  may be constructed by choosing a basis  $\{E_1, \dots, E_\rho\}$  for the  $\mathbb{Q}$ -vector space  $NS(S)_\mathbb{Q}$  which is orthogonal in the sense that  $E_i \cdot E_j = 0$  for  $i \neq j$  and the self-intersections  $E_i^2$  are nonzero. The correspondences  $\varepsilon_i = \frac{[E_i \times E_i]}{(E_i)^2}$  are orthogonal and idempotent, so their sum

$$\boxed{3.1} \quad (3.1) \quad \pi_2^{\text{alg}} = \varepsilon_1 + \dots + \varepsilon_\rho = \sum_{1 \leq i \leq \rho} \frac{[E_i \times E_i]}{(E_i)^2}$$

is also an idempotent correspondence, and  $h_2^{\text{alg}}(S) = (S, \pi_2^{\text{alg}}, 0)$ . Since  $\{E_i/(E_i)^2\}$  is a dual basis to the  $\{E_i\}$ ,  $\pi_2^{\text{alg}}$  and  $h_2^{\text{alg}}(S)$  are independent of the choice of basis.

Setting  $M_i = (S, \varepsilon_i, 0)$ ,  $h_2^{\text{alg}}(S)$  is the direct sum of the  $M_i$ , and each  $M_i$  is isomorphic to the Lefschetz motive  $\mathbb{L}$  (see [?, 7.2.3]), so  $h_2^{\text{alg}}(S) \cong \mathbb{L}^{\oplus \rho}$ . Setting  $H_{\text{alg}}^2(S) = \pi_2^{\text{alg}} H^2(S)$ , we also have isomorphisms  $H_{\text{alg}}^2(S) \cong NS(S)_\mathbb{Q}$ .

The transcendental motive  $t_2(S)$  is defined as  $t_2(S) = (S, \pi_2^{\text{tr}}, 0)$ , where

$$\pi_2^{\text{tr}} = \pi_2 - \pi_2^{\text{alg}}.$$

Setting  $H_{\text{tr}}^2(S) = \pi_2^{\text{tr}} H^2(S)$ , we have  $H^2(S) = H_{\text{alg}}^2(S) \oplus H_{\text{tr}}^2(S)$ , so  $H^2(t_2(S))$  is the “transcendental part” of  $H^2(S)$ . In addition,  $A_0(t_2(S))$  is the Albanese Kernel  $T(S)$ ; see [?, 7.2.3]. The Chow motive  $t_2(S)$  does not depend on the choices made to define the refined Chow-Künneth decomposition, it is functorial on  $S$  for the action of correspondences, and it is a birational invariant of  $S$  (see [KMP]).

If  $p_g(S) = 0$  then  $H_{\text{tr}}^2(S) = 0$  and  $t_2(S) = 0$  iff  $T(S) = 0$ , i.e.,  $S$  satisfies Bloch’s conjecture [?]. The condition  $t_2(S) = 0$  is also equivalent to the finite dimensionality of  $h(X)$ , see [?]. If  $p_g(S) > 0$  then  $T(S) \neq 0$ , hence  $t_2(S) \neq 0$ .

Recall that  $A^2(S \times S)$  is the endomorphism ring of  $h(S)$  in  $\mathcal{M}_{\text{rat}}$ , with the diagonal correspondence  $\Delta_S$  acting as the identity. Consider

the ring projection

$$(3.2) \quad \Psi_S : A^2(S \times S) = \text{End}_{\mathcal{M}_{rat}}(h(S)) \rightarrow \text{End}_{\mathcal{M}_{rat}}(t_2(S))$$

sending  $\Gamma$  to  $\pi_2^{tr} \circ \Gamma \circ \pi_2^{tr}$ . By construction,  $\Psi_S$  sends the class  $[\Delta_S]$  of the diagonal to  $\pi_2^{tr}$ , which is the identity map of the motive  $t_2(S)$ . In fact,  $\Psi_S$  induces a ring isomorphism

$$\Psi_S : A^2(S \times S)/J_{nd}(S) \xrightarrow{\cong} \text{End}_{\mathcal{M}_{rat}}(t_2(S)),$$

where  $J_{nd}(S)$  is the ideal of  $A^2(S \times S)$  generated by the classes of correspondences which are not dominant over  $S$  by at least one of the two projections  $S \times S \rightarrow S$ , see [?, 7.4.3].

**3.3 Lemma 3.3.** *Let  $S$  be a smooth projective surface with  $q(S) = 0$ . Then  $\mathcal{J}(S) = J_{nd}(S)$  in  $A^2(S \times S)$ .*

*Proof.* From the definition of the ideals  $\mathcal{J}(S)$  and  $J_{nd}(S)$  we get  $\mathcal{J}(S) \subseteq J_{nd}(S)$ . Let  $\Gamma \in J_{nd}(S)$  such that  $\Gamma$  is not dominant over  $S$  under the first projection. We claim that  $\Gamma$  belongs to the ideal of degenerate correspondences.  $\Gamma$  vanishes on some  $V \times S$ , with  $V$  open in  $S$ , hence it has support on  $W \times S$ , with  $\dim W \leq 1$ . If  $\dim W = 0$  then  $\Gamma = \sum_i n_i [S \times P_i]$  in  $A^2(S \times S)$ , where  $P_i$  are closed points in  $S$ . Hence  $\Gamma \in \mathcal{J}(S)$ . If  $\dim W = 1$  then  $\Gamma \in A^1(W \times S)$ , where  $A^1(W \times S) = p_1^*(A^1(W)) \times p_2^*(A^1(S))$ , with  $p_i$  the projections, because  $H^1(S, \mathcal{O}_S) = 0$  (see [?, p. 292]). Therefore  $\Gamma \in \mathcal{J}(S)$ .  $\square$

#### 4. SEVERI'S FORMULA

In [?, p. 761], Severi claims that if on a surface  $S$  there exists a correspondence  $\Gamma \in A^2(S \times S)$  with two distinct valences, i.e.,  $v(\Delta_S) = 0$ , then  $S$  is "regular of genus 0", i.e.  $q(S) = p_g(S) = 0$ . The following theorem verifies Severi's claim.

**4.1 Theorem 4.1.** *Let  $S$  be a smooth projective surface. Then the following conditions are equivalent*

- (1) *There exists a correspondence  $\Gamma \in A^2(S \times S)$  with two distinct valences  $v$  and  $v'$ ;*
- (2)  *$v(\Delta_S) = 0$ ;*
- (3)  *$p_g(S) = q(S) = 0$  and  $S$  satisfies Bloch's conjecture;*
- (4)  *$A_0(S) \simeq \mathbb{Q}$ .*

*Proof.* Since both  $\Gamma + v\Delta_S$  and  $\Gamma + v'\Delta_S$  are in  $\mathcal{J}(S)$ , so is  $(v - v')\Delta_S$ . Therefore  $\Delta_S \in \mathcal{J}(S)$ , i.e.,  $\Delta_S$  has valence 0, which shows that (1)  $\implies$  (2). Clearly (2)  $\implies$  (1), by taking  $\Gamma = \Delta_S$ . If  $v(\Delta_S) = 0$  then  $\Delta_S \in \mathcal{J}(S)$ . Since  $\mathcal{J}(S) \subseteq J_{nd}(S)$  we get  $\Psi_S(\Delta_S) = 0$ , i.e., the identity map on  $t_2(S)$  is 0 in  $\mathcal{M}_{rat}$ . This is equivalent

to  $t_2(S) = 0$  and hence  $T(S) = A_0(t_2(S))$  equals 0. The condition  $T(S) = 0$  forces  $p_g(S) = 0$  and is a form of Bloch's conjecture. By Lemma ??,  $A^1(S)$  injects into  $H^2(S, \mathbb{Q})$ , so  $q(S) = 0$ . This shows that (2) implies (3). The equivalence (3)  $\iff$  (4) is well known.

If  $p_g(S) = q(S) = 0$  and Bloch's conjecture holds for  $S$ , then, by [?, Prop. 1], there exist a closed  $V \subset S$  of dimension 0 and a divisor  $D$  on  $S$  such that  $\Delta_S = \Gamma_1 + \Gamma_2$  in  $A^2(S \times S)$ , with  $\Gamma_1$  supported on  $V \times S$  and  $\Gamma_2$  supported on  $S \times D$ . Hence  $\Delta_S \in J_{\text{nd}}(S)$ . By Lemma ?? we get  $\Delta_S \in \mathcal{J}(S)$ , because  $q(S) = 0$ , so that  $v(\Delta_S) = 0$ , i.e., (3)  $\implies$  (2).

The equivalence (3)  $\iff$  (4) is well known.  $\square$

**Remark 4.2.** For a surface  $S$ , if  $v(\Delta_S) = 0$ , then  $p_g(S) = q(S) = 0$  and also  $t_2(S) = 0$ . Therefore  $h(S) = \mathbf{1} \oplus \mathbb{L}^{\oplus \rho(S)} \oplus \mathbb{L}^2$ , so that  $h(S)$  coincides with the motive of  $S$  in the category of numerical motives  $\mathcal{M}_{\text{num}}$ .

**Example 4.3.** Let  $S$  be a hyperelliptic surface over  $\mathbb{C}$ , i.e., a smooth projective surface with  $p_g(S) = 0$  and  $q(S) = 1$ , which is isomorphic to a quotient  $E \times F/G$ , with  $E, F$  elliptic curves and  $G$  a finite group. By [BKL] the Albanese kernel of  $S$  vanishes, hence, by [?, Prop.1],  $\Delta_S = \Gamma_1 + \Gamma_2$  with  $\Gamma_1 \subset V \times S$ ,  $\Gamma_2 \subset S \times D$  and  $V \neq S, D \neq S$ . Therefore  $\Delta_S \in J_{\text{nd}}(S)$ . Because  $q(S) \neq 0$ , Theorem ?? implies that  $v(\Delta_S) \neq 0$ , so that  $\Delta_S \notin \mathcal{J}(S)$ .

**4.2 Proposition 4.4.** *Let  $S$  be a smooth projective surface. Then for every correspondence  $T \in A^2(S \times S)$  of valence zero:*

$$\deg(T \cdot \Delta_S) = \alpha(T) + \beta(T) + \text{trace}_{NS(S)}(T).$$

*Proof.* Since the correspondence  $T$  has valence 0, it belongs to the ideal  $\mathcal{J}(S)$  of degenerate correspondences. Therefore we may write  $T = T_0 + T_1$ , where

$$T_0 = \sum p_j [P_j \times S] + \sum q_k [S \times Q_k], \quad T_1 = \sum m_i [D_i \times D'_i];$$

here  $D_i, D'_i \in A^1(S)$  and  $P_j, Q_j$  are points. We may move  $[D_i \times D'_i]$  within its class in  $A^2(S \times S)$  in such a way that it does not meet any of the  $P_j \times S$  or  $S \times Q_k$ , so that

$$[D_i \times D'_i] \cdot [P_j \times S] = [D_i \times D'_i] \cdot [S \times Q_k] = 0$$

for all  $i, j, k$ . With this reduction, we have  $\alpha(T_1) = \beta(T_1) = 0$ ,  $\alpha(T_0) = \deg(T_0 \cdot [P \times S]) = \sum q_k$ , and  $\beta(T_0) = \deg(T_0 \cdot [S \times P]) = \sum p_j$ . We also have

$$\deg(T_0 \cdot \Delta_S) = \sum p_j + \sum q_k = \alpha(T) + \beta(T).$$

Now for any divisor  $C$  on  $S$ , we have  $[P_j \times S]_*(C) = [S \times Q_k]_*(C) = 0$ . Thus  $T_0$  acts as zero on  $NS(S)$ , so  $\text{trace}_{NS}(T_0) = 0$ . Therefore we may assume that  $T = T_1$ , and need to evaluate

$$\deg(T_1 \cdot \Delta_S) = \deg\left(\sum m_i [D_i \times D'_i] \cdot \Delta_S\right) = \sum m_i (D_i \cdot D'_i).$$

Choose an orthogonal basis  $\{E_\ell, 1 \leq \ell \leq \rho(S)\}$  for the  $\mathbb{Q}$ -vector space  $NS(S) \otimes \mathbb{Q}$ . In terms of this basis,

$$D_i = \sum_k a_{ik} E_k, \quad D'_i = \sum_\ell b_{i\ell} E_\ell.$$

Since  $E_k \cdot E_\ell = 0$  when  $k \neq \ell$ , we may expand  $D_i \cdot D'_i$  to get

$$\deg(T \cdot \Delta_S) = \sum_i m_i (D_i \cdot D'_i) = \sum_{i,k} m_i a_{ik} b_{ik} (E_k)^2.$$

Because  $(D_i \times D'_i)_*(E_k) = (D_i \cdot E_k)D'_i = a_{ik}(E_k)^2 D'_i$ ,

$$T_*(E_k) = \left(\sum m_i [D_i \times D'_i]\right)_*(E_k) = \sum m_i a_{ik} b_{ik} (E_k)^2 E_\ell.$$

Thus  $\text{trace}_{NS(S)}(T_*) = \sum m_i a_{ik} b_{ik} (E_k)^2$ , and the result follows.  $\square$

**LFP**

**Remark 4.5.** If  $S$  is a smooth projective surface over  $\mathbb{C}$  and  $\Gamma$  is a correspondence in  $A^2(S \times S)$  then, by the Lefschetz fixed point formula (see [?, 16.1.15])

$$\deg(\Gamma \cdot \Delta_S) = \sum_{0 \leq i \leq 4} (-1)^i \text{trace}_{H^i(S)}(\Gamma).$$

Note that  $\alpha(\Gamma)$  and  $\beta(\Gamma)$  are the traces of  $\Gamma$  acting on  $H^4(S)$  and  $H^0(S)$ , respectively. This is immediate from Definition ??, since  $\pi_0 = [S \times P]$  and  $\pi_4 = [P \times S]$  in our covariant setting. If  $v(\Gamma) = 0$  then  $\Psi_S(\Gamma) = 0$ , so  $\Gamma$  acts as 0 on  $h_2^{tr}(S)$  and on  $H_{tr}^2(S, \mathbb{Q})$ . Therefore Proposition ?? says that a correspondence of valence 0 has trace 0 on the odd cohomology of  $S$ .

In [?, p.871], Severi gave a definition of the *rank*  $\delta(T)$  of a correspondence  $T$  of valence 0 and gave an argument asserting that if  $T$  is a correspondence of valence 0 on a surface  $S$ , then

$$\deg(T \cdot \Delta_S) = \alpha(T) + \beta(T) + \delta(T).$$

Severi pointed out that  $\delta(T)$  may be computed by taking any decomposition of  $T$  into the sum of degenerate correspondences. Proposition ?? shows that  $\delta(T) = \text{trace}_{NS}(T)$ .

Severi then defines the *rank* of a correspondence  $\Gamma$  of valence  $v$  to be the number  $\delta(\Gamma)$  such that  $\delta(\Gamma) + v$  is the rank of the correspondence  $T = \Gamma + v\Delta_S$  of valence 0. He also sets the rank of the the diagonal

$\Delta_S$  to be 1, which is consistent when  $\Delta_S$  has valence -1. Thus we may reinterpret Severi's definition as follows.

**4.4** **Definition 4.6.** If  $T \in A^2(X \times X)$  is a correspondence on a smooth projective variety  $X$  of valence 0, we define its *rank*  $\delta(T)$  to be the trace  $\text{trace}_{NS}(T)$  of  $T$  acting on  $NS(X)$ . If  $\Delta_X$  does not have valence 0, and  $\Gamma$  is a correspondence of valence  $v$ , we define the rank of  $\Gamma$  to be

$$\delta(\Gamma) = \text{trace}_{NS}(\Gamma) + v(\rho(X) - 1).$$

With this definition of  $\delta(\Gamma)$  we recover Severi's formula (2.1) for a surface  $S$ . If  $v = v(\Gamma)$  then  $T = \Gamma + v\Delta_S$  has valence 0 and by ?? we have

$$\begin{aligned} \delta(T) &= \delta(\Gamma + v\Delta_S) = \text{trace}_{NS}(\Gamma + v\Delta_S) = \text{trace}_{NS}(\Gamma) + v \text{trace}_{NS}(\Delta_S) \\ &= \text{trace}_{NS}(\Gamma) + v \cdot \rho(S) = \delta(\Gamma) + v. \end{aligned}$$

Recall that the second Chern class of  $S$  satisfies  $c_2(S) = \deg(\Delta_S \cdot \Delta_S)$  (see [?, 8.1.12]).

**4.5** **Theorem 4.7.** *Let  $S$  be a smooth projective surface. If  $\Delta_S$  does not have valence 0 and  $\Gamma \in A^2(S \times S)$  is a correspondence with valence  $v$ , then*

$$\deg(\Gamma \cdot \Delta_S) = \alpha(\Gamma) + \beta(\Gamma) + \text{trace}_{NS}(\Gamma) + v \cdot (2 + \rho(S) - c_2(S)).$$

Therefore

$$\deg(\Gamma \cdot \Delta_S) = \alpha(\Gamma) + \beta(\Gamma) + \delta(\Gamma) - v \cdot (c_2(S) - 3)$$

as in (2.1).

*Proof.* By definition,  $T = \Gamma + v\Delta_S$  has valence 0. From Proposition ?? we have

$$\deg(T \cdot \Delta_S) = \alpha(T) + \beta(T) + \text{trace}_{NS}(T),$$

with  $\alpha(T) = \alpha(\Gamma) + v$ ,  $\beta(T) = \beta(\Gamma) + v$  and  $\text{trace}_{NS}(T) = \text{trace}_{NS}(\Gamma) + v \rho(S)$  by additivity of the trace. We also have

$$\deg(T \cdot \Delta_S) = \deg(\Gamma \cdot \Delta_S) + v \cdot \deg(\Delta_S \cdot \Delta_S) = \deg(\Gamma \cdot \Delta_S) + v \cdot c_2(S).$$

Equating the formulas yields the desired formula for  $\deg(\Gamma \cdot \Delta_S)$ .  $\square$

## 5. SURFACES WITH AN INVOLUTION

We now consider the case of the correspondence on a smooth projective surface  $S$  over  $\mathbb{C}$  which is the graph of an involution  $\sigma$ , i.e.  $\Gamma_\sigma = \{(x, \sigma(x)) \in S \times S\}$  and show that, if  $p_g(S) = q(S) = 0$ , then  $\deg(\Gamma_\sigma \cdot \Delta_S)$  is given by the same formula as in Proposition ??. We also apply Theorem 4.5 to the case of a K3 surface with an involution, see Example 5.5.

The fixed locus of  $\sigma$  consists of a 1-dimensional part  $D$  (possibly empty) and  $k \geq 0$  isolated fixed points  $\{P_1, \dots, P_k\}$ . The images  $Q_i$  in  $S/\sigma$  of the  $P_i$  are nodes, and  $S/\sigma$  is smooth elsewhere. The blow-up  $X$  of  $S$  at the  $k$  set of isolated fixed points resolves these singularities,  $\sigma$  lifts to an involution on  $X$  (which we will still call  $\sigma$ ), and the quotient  $Y = X/\sigma$  is a desingularization of  $S/\sigma$ . The images  $C_1, \dots, C_k$  in  $Y$  of the exceptional divisors of  $X$  are disjoint nodal curves, i.e., smooth rational curves with self-intersection  $-2$ . In summary, we have a commutative diagram

$$(5.1) \quad \begin{array}{ccc} X & \xrightarrow{h} & S \\ \downarrow \pi & & \downarrow f \\ Y & \xrightarrow{g} & S/\sigma. \end{array}$$

If  $p_g(S) = q(S) = 0$  then  $k = K_S \cdot D + 4$ , see [?, 3.2].

For brevity, we write  $t$  for  $\text{trace}_{NS}(\Gamma_\sigma)$ , the trace of the action of  $\sigma$  on  $NS(S)_\mathbb{Q}$ .

The following result is immediate from the Lefschetz fixed point formula and Remark ??, but we prefer to give an elementary proof.

**5.2** **Theorem 5.2.** *Let  $S$  be smooth projective surface over  $\mathbb{C}$ , with  $p_g(S) = q(S) = 0$ . Let  $\sigma$  be an involution on  $S$  and let  $\Gamma_\sigma = (1 \times \sigma)\Delta_S$ . Then*

$$\deg(\Gamma_\sigma \cdot \Delta_S) = \alpha(\Gamma_\sigma) + \beta(\Gamma_\sigma) + t = 2 + t = 4 - D^2.$$

Moreover  $\sigma$  acts as the identity on  $H^2(S, \mathbb{Q})$  (i.e.,  $t = \rho(S)$ ) iff

$$K_S^2 = D^2 + 8.$$

*Proof.* Since  $S$  has no odd cohomology, the motive  $h(S)$  has a Chow-Künneth decomposition

$$h(S) = h_0(S) \oplus h_2^{\text{alg}}(S) \oplus t_2(S) \oplus h_4(S),$$

where  $\pi_0 = [S \times P]$ ,  $\pi_4 = [P \times S]$ , with  $P$  a rational point on  $S$ , so that

$$\Delta_S = [S \times P] + \pi_2^{\text{alg}} + \pi_2^{\text{tr}} + [P \times S].$$

Since  $h_2^{\text{alg}} = (S, \pi_2^{\text{alg}}, 0)$ , where  $\pi_2^{\text{alg}} = \sum \varepsilon_i$  is as in (??), the action of  $\sigma$  on  $h_2^{\text{alg}}$  is determined by  $\Gamma_\sigma(\varepsilon_i) = \frac{[E_i \times \sigma(E_i)]}{(E_i)^2}$ . Let  $a_{ij}$  be such that  $\sigma(E_i) = \sum_j a_{ij} E_j$ . Then

$$(1 \times \sigma)\pi_2^{\text{alg}} \cdot \Delta_S = \sum_{1 \leq i \leq \rho} \frac{[E_i \times \sigma(E_i)]}{(E_i)^2} \cdot \Delta_S = \sum_{1 \leq i \leq \rho} a_{ii}.$$

Therefore  $\deg(\Gamma_\sigma \cdot \pi_2^{\text{alg}}) = \text{trace}_{NS}(\sigma)$ . We have  $\pi_2^{\text{tr}} H^2(S) = H_{\text{tr}}^2(S) = 0$ , because  $p_g(S) = 0$ , hence  $((1 \times \sigma)\pi_2^{\text{tr}})H^2(S) = 0$ . By [?, 19.2]

$$cl((1 \times \sigma)\pi_2^{\text{tr}} \cdot \Delta_S) = cl((1 \times \sigma)\pi_2^{\text{tr}}) \cdot cl(\Delta_S) = 0$$

in  $H_0(S \times S)$ , hence the 0-cycle  $(\Gamma_\sigma \cdot \pi_2^{\text{tr}})$  has degree 0 in  $A_0(S \times S)$ . We also have

$$\beta(\Gamma_\sigma) = \deg(\Gamma_\sigma \cdot \pi_0) = 1 ; \alpha(\Gamma_\sigma) = \deg(\Gamma_\sigma \cdot \pi_4) = 1.$$

Summing up we get

$$\deg(\Gamma_\sigma \cdot \Delta_S) = \deg(\Gamma_\sigma \cdot \pi_0) + \deg(\Gamma_\sigma \cdot \pi_2^{\text{alg}}) + \deg(\Gamma_\sigma \cdot \pi_4) = 2 + t.$$

From [?, 4.2], we get  $t = 2 - D^2$ , hence  $\deg(\Gamma_\sigma \cdot \Delta_S) = 4 - D^2$ .

Finally the trace  $t = 2 - D^2$  of  $\sigma$  on  $H^2(S, \mathbb{Q})$  is at most the rank  $\rho(S)$  of  $NS(S)$ . By Noether's formula  $c_2(S) = 12 - K_S^2$  because  $q(S) = p_g(S) = 0$ , hence  $\rho(S) = 10 - K_S^2$ . Therefore  $D^2 \geq K_S^2 - 8$ , with equality iff  $t = \rho(S)$ , i.e., iff  $\sigma$  acts as the identity on  $H^2(S, \mathbb{Q})$ .  $\square$

**Remark 5.3.** Theorem 5.2 gives a simplified version of a formula that appears in [?, p. 874] and also in [?, 16.2.4], showing that in this case  $\deg(\Gamma_\sigma \cdot \Delta_S)$  only depends on  $D^2$ .

**Example 5.4.** (1) Let  $S$  be a minimal surface of general type with  $p_g(S) = 0$  and  $K_S^2 = 8$  with an involution. By [?, 4.4]  $\sigma$  acts as the identity on  $H^2(S, \mathbb{Q})$ ,  $\rho(S) = t = 2$  and  $D^2 = 0$ . Therefore  $\deg(\Gamma_\sigma \cdot \Delta_S) = 4$ . If  $D = 0$  the number  $k$  of isolated fixed points of  $\sigma$  is 4, otherwise  $k$  is even and  $6 \leq k \leq 12$ .

(2) Let  $S$  be a numerical Godeaux surface with an involution  $\sigma$ .  $S$  is a minimal surface of general type with  $p_g(S) = q(S) = 0$  and  $K_S^2 = 1$ . By [?, 4.5]  $\sigma$  has  $k = 5$  isolated fixed points and  $-7 \leq D^2 \leq 1$ . If  $D^2 = -7$  then  $\sigma$  acts as the identity on  $H^2(S, \mathbb{Q})$ ,  $t = \rho(S) = 9$  and  $\deg(\Gamma_\sigma \cdot \Delta_S) = 11 = c_2(S)$ . If  $D^2 = 1$  then  $t = 1$  and  $\deg(\Gamma_\sigma \cdot \Delta_S) = 3$ .

**Example 5.5.** (*K3 Surfaces with an involution*) Let  $S$  be a complex K3 surface with an involution  $\sigma$ . We have:  $p_g(S) = 1$ ,  $q(S) = 0$ , hence  $H^{1,0}(S) = 0$  and  $H^{2,0}(S) \simeq \mathbb{C}$ . Also  $c_2(S) = 24$ ,  $H^2(S, \mathbb{Q}) = NS(X)_\mathbb{Q} \oplus H_{\text{tr}}^2(S, \mathbb{Q})$  with  $\dim H^2(S, \mathbb{Q}) = 22$ ,  $\dim H_{\text{tr}}^2(S, \mathbb{Q}) = 22 - \rho(S)$  and  $\rho(S) = \dim(NS(S)_\mathbb{Q}) \leq 20$ . If  $\sigma$  is an involution on  $S$  then  $\sigma(\omega) = \pm\omega$ , where  $\omega$  is a generator of the vector space  $H^{2,0}(S)$ . Then the same argument as in [V2,3.10] shows that  $\sigma$  either acts as +1 or as -1 on  $H_{\text{tr}}^2(S, \mathbb{Q})$ . The correspondence  $\Gamma_\sigma$  induces an involution  $\pi_2^{\text{tr}} \circ \Gamma_\sigma \circ \pi_2^{\text{tr}}$  on  $t_2(S)$ , which we will still denote by  $\sigma$ . Let  $\pi = 1/2(\pi_2^{\text{tr}} - \sigma)$ . Then  $\pi \in A^2(S \times S)$  is a projector of  $t_2(S)$ .  $\pi_*$  acts either as 0 or as the identity on  $H^{2,0}(S) \simeq \mathbb{C}$ . In the first case  $\ker(\pi_*)|_{H_{\text{tr}}^2(S)}$  is a sub-Hodge structure with (2,0)-component equal to  $H^{2,0}(S)$ . Its orthogonal complement is

then contained in  $NS(S)_{\mathbb{Q}}$ , which implies that  $\pi_* = 0$  on  $H_{tr}^2(S, \mathbb{Q})$ . Therefore  $\sigma$  acts as the identity on  $H_{tr}^2(S, \mathbb{Q})$ . In the second case  $\pi_*$  equals the identity on  $H_{tr}^2(S, \mathbb{Q})$ , so that  $\sigma$  acts as  $-1$  on  $H_{tr}^2(S, \mathbb{Q})$ .

Now suppose that  $\sigma(\omega) = -\omega$ , so that  $\sigma$  acts as  $-1$  on  $H_{tr}^2(S)$ . By [?, 1.2], the quotient surface  $S/\sigma$  is an Enriques surface iff  $S^\sigma = \emptyset$ , while  $S/\sigma$  is a rational surface if  $S^\sigma \neq \emptyset$ . In either case  $t_2(S/\sigma) = 0$ , by [BKL], hence  $t_2(S) \neq t_2(S/\sigma)$  because  $p_g(S) \neq 0$ . Let  $\Psi_S$  be the map defined in (3.2). Then, by [?, Prop. 1 (iv)]  $\Psi_S(\Gamma_\sigma) = -id_{t_2(S)} = \Psi_S(-\Delta_S)$ . Since  $q(S) = 0$  the kernel of  $\Psi_S$  is  $\mathcal{J}(S)$  by ???. Hence  $\Gamma_\sigma + \Delta_S \in \mathcal{J}(S)$ , i.e.,  $\Gamma_\sigma$  has valence 1. Since  $p_g(S) \neq 0$ ,  $\Delta_S$  cannot have valence 0 by Theorem ??. The correspondence  $\Gamma_\sigma$  has indices  $\alpha(\Gamma_\sigma) = \beta(\Gamma_\sigma) = 1$ . By Theorem 4.7, if  $t = \text{trace}_{NS(S)}(\Gamma_\sigma)$  we get

$$\deg(\Gamma_\sigma \cdot \Delta_S) = 2 + t + \rho(S) - 22 = 2 + t - \dim H_{tr}^2(S, \mathbb{Q}).$$

By [?, Th.3],  $\sigma$  has no isolated fixed points and a 1-dimensional (possibly empty) fixed locus  $D$ . Let  $\tau$  be the trace of  $\sigma$  on  $H^2(S, \mathbb{C})$ . By the topological fixed point formula (see [?, 4])  $\tau + 2 = e(D)$ , where  $e(D) = -D^2 - D \cdot K_S$  is the topological Euler characteristic of  $D$ . By the holomorphic fixed point formula, see [?, 4]

$$-D \cdot K_S = 4(\text{trace}_{H^0(S, \mathcal{O}_S)}(\sigma) + \text{trace}_{H^2(S, \mathcal{O}_S)}(\sigma)) = 0,$$

because  $\sigma$  acts as  $-1$  on  $H^2(S, \mathcal{O}_S)$ . Therefore  $e(D) = -D^2$  and

$$t = \tau - \text{trace}_{H_{tr}^2(S)}(\Gamma_\sigma) = \tau + \dim H_{tr}^2(S) = -D^2 - 2 + \dim H_{tr}^2(S),$$

because  $\sigma$  acts as  $-1$  on  $H_{tr}^2(S)$ , so that

$$\deg(\Gamma_\sigma \cdot \Delta_S) = -D^2.$$

If  $S^\sigma = \emptyset$ , in which case  $S \rightarrow S/\sigma$  is the canonical unramified cover of an Enriques surface, then  $D = 0$  and  $(\Gamma_\sigma \cdot \Delta_S) = 0$ .

If  $S^\sigma \neq \emptyset$  then, by [?, Th 3], the fixed locus  $D$  of  $\sigma$  is the disjoint union of  $m$  smooth curves  $C_i$ , with  $1 \leq m \leq 10$ , so that  $D^2 = \sum_{1 \leq i \leq m} (2g(C_i) - 2)$ , by the adjunction formula. If  $m = 10$  and the curves  $C_i$  are rational then  $D^2 = -20$  and  $\rho(S) = 20$ , see [?, Th. 3' (1)]. Therefore  $\deg(\Gamma_\sigma \cdot \Delta_S) = 20 = \rho(S)$ ,  $\dim H_{tr}^2(S) = 2$  and  $t = 20 = \rho(S)$ . Hence  $\sigma$  acts as the identity on  $NS(S)_{\mathbb{Q}}$ .  $S$  is the unique (up to isomorphism) complex K3 surface described in [?, Th.1].

If  $\sigma^*(\omega) = \omega$ , with  $\omega \in H^{2,0}(S)$ , then  $\sigma$  is a *Nikulin involution*. A Nikulin involution has 8 isolated fixed points, no 1-dimensional fixed locus and the desingularization  $Y$  of the quotient surface  $S/\sigma$  is a K3 surface, see [?, 5.2]. Therefore  $c_2(S) = c_2(Y) = 24$ , and  $\rho(S) = \rho(Y)$  because  $\sigma$  acts as the identity on  $H_{tr}^2(S)$ . From the Lefschetz fixed

point formula (see Remark 4.5) we get

$$\deg(\Gamma_\sigma \cdot \Delta_S) = \tau + 2 = 8$$

where  $\tau$  is the trace of  $\sigma$  on  $H^2(S, \mathbb{C})$ . In order to compute  $\deg(\Gamma_\sigma \cdot \Delta_S)$  using Theorem 4.7 we should know that the correspondence  $\Gamma_\sigma$  has a valence  $v$ . By [?, Th.4] this is equivalent to  $t_2(S) \simeq t_2(Y)$  in which case  $v(\Gamma_\sigma) = -1$ . Then we get

$$\deg(\Gamma_\sigma \cdot \Delta_S) = 2 + t - (\rho(S) - 22)$$

where

$$t = \text{trace}_{NS}(\Gamma_\sigma) = \tau - \dim H_{tr}^2(S) = \tau - (22 - \rho(S)) = \rho(S) - 16,$$

so that  $\deg(\Gamma_\sigma \cdot \Delta_S) = 8$ . The condition  $t_2(S) \simeq t_2(Y)$  is satisfied if the K3 surface  $S$  has a finite dimensional motive, which is in particular the case if  $\rho(S) = 19$  or  $\rho(S) = 20$ , see [?, Th 2 and 3].

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