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Algèbre homologique/*Homological Algebra*

# The 2-torsion in the $K$-theory of the Integers

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**Abstract.** Using recent results of Voevodsky, Suslin-Voevodsky and Bloch-Lichtenbaum, we completely determine the 2-torsion subgroups of the $K$-theory of the integers $\mathbb{Z}$. The result is periodic with period 8, and there are no 2-torsion elements except those known for over 20 years. There is no 2-torsion except for the $\mathbb{Z}/2$ summands in degrees $8n+1$ and $8n+2$, the $\mathbb{Z}/16$ in degrees $8n+3$ and the image of the $J$-homomorphism in degrees $8n+7$. In particular, the 2-part of $\zeta(1-2n)$ is twice the 2-part of the ratio $|K_{4n-2}(\mathbb{Z})|/|K_{4n-1}(\mathbb{Z})|$ for all $n > 0$. This corrects a conjecture of Lichtenbaum.

### *La torsion 2-primaire de la $K$-théorie de $\mathbb{Z}$*

**Résumé.** *À partir de résultats récents de Voevodsky, Suslin-Voevodsky et Bloch-Lichtenbaum, nous déterminons la torsion 2-primaire de la $K$-théorie de $\mathbb{Z}$, qui est périodique de période 8. Il n'existe pas d'autres éléments 2-primaires que ceux connus depuis 20 ans. La torsion 2-primaire se réduit à $\mathbb{Z}/2$ en degrés $8n+1$ et $8n+2$, à $\mathbb{Z}/16$ en degré $8n+3$, et à l'image de l'homomorphisme $J$ en degré $8n+7$. La partie 2-primaire de $\zeta(1-2n)$ est donc le double de la partie 2-primaire de $|K_{4n-2}(\mathbb{Z})|/|K_{4n-1}(\mathbb{Z})|$ pour tout $n > 0$. Ce résultat corrige un conjecture de Lichtenbaum.*

## *Version française abrégée*

Les résultats récents de Voevodsky (*voir* [16]) nous permettent de déterminer la torsion 2-primaire de la $K$-théorie de $\mathbb{Z}$, qui est périodique de période 8. Il n'existe pas d'autres éléments 2-primaires que ceux connus depuis 20 ans (*voir* [9] et [4]). Comme les groupes $K_j(\mathbb{Z})$ sont de type fini (*voir* [8]), et que leur rang est connu par [3], nous pouvons énoncer le résultat suivant.

THÉORÈME. – *Les groupes $K_*(\mathbb{Z})$ sont donnés dans la Table 1 ci-dessus, où $w_i$ dénote la plus grande puissance de 2 qui divise $4i$, et l'indication* « odd » *dans la table représente un groupe fini d'ordre impair.*

Les premières assertions non déjà connues sont que $K_4(\mathbb{Z}) = 0$ et que la seule torsion dans $K_5(\mathbb{Z})$ est 3-primaire. Cela découle des résultats de Lee-Szczarba et Soulé [11], de Rognes [10], et de notre calcul qui montre qu'il n'existe pas de torsion 2-primaire.

**Note présentée par Alain CONNES.**

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Comme la partie 2-primaire de $\zeta(1-2n) = (-1)^n B_n/2n$ est $\frac{1}{2}w_i$, cela implique :

Corollaire 2. – *La partie 2-primaire de la formule conjecturée par Lichtenbaum* [7] *est correcte à un facteur 2 près :*

$$\frac{|K_{4n-2}(\mathbb{Z})|}{|K_{4n-1}(\mathbb{Z})|} = \frac{1}{w_{2n}} = \frac{1}{2}\zeta(1-2n) \qquad \text{à un facteur impair près.}$$

Pour obtenir ces résultats, nous utilisons la suite spectrale

$$(\dagger) \qquad E_2^{pq} = H_{et}^{p-q}(F;\mathbb{Z}/2) \Rightarrow K_{-p-q}(F;\mathbb{Z}/2)(F;\mathbb{Z}/2), \qquad (p \le 0,\ p \ge q).$$

C'est la suite spectrale de Bloch et Lichtenbaum [2], le terme $E_2^{pq}$ étant déduit de l'isomorphisme

$$CH^i(F, 2i-j;\mathbb{Z}/2) \cong \begin{cases} H_{et}^j(F;\mathbb{Z}/2), & j \le i \\ 0, & j > i \end{cases}$$

(*voir* les travaux récents de Voevodsky [16] et Suslin [13] [14]).

Nous commençons par calculer la suite spectrale (†) pour $\mathbb{R}$ et pour les corps locaux $\mathbb{Q}_p$. Comme $H_{et}^i(\mathbb{Q};\mathbb{Z}/2) \to H_{et}^i(\mathbb{R};\mathbb{Z}/2)$ est un isomorphisme pour $i \ne 1, 2$ (*voir* [15]), nous pouvons utiliser ces calculs pour déterminer les différentielles dans la suite spectrale pour $\mathbb{Q}$.

Théorème 5. – *Pour $n \ge 0$, la $K$-théorie modulo 2 de $\mathbb{Q}$ est donnée par la Table 2 ci-dessous.*

La comparaison des suites exactes de localisation en $K$-théorie et en cohomologie étale, pour $\mathbb{Q}$ et pour les $\mathbb{Q}_p$, donne le résultat suivant.

Théorème 7. – *Pour $n \ge 0$, la $K$-théorie modulo 2 de $\mathbb{Z}[\frac{1}{2}]$ est donnée par la Table 3 ci-dessous.*

Ce résultat implique le Théorème 1, car $K_j(\mathbb{Z};\mathbb{Z}/2) \cong K_j(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2)$ pour $j \ge 2$. En effet, tous les termes de la Table 3 proviennent des facteurs connus de $K_*(\mathbb{Z})$.

Le Théorème 1 implique aussi que $K_j(\mathbb{Z}[\frac{1}{2}]) \otimes \hat{\mathbb{Z}}_2$ est la $K$-théorie étale dyadique $K_j^{et}(\mathbb{Z}[\frac{1}{2}])$. Bökstedt, et plus tard Dwyer et Friedlander [6, Proposition 4.2], ont remarqué que cela implique le résultat suivant :

Corollaire 8. – *Soit $K(R)\hat{_2}$ le complété dyadique de l'espace de la $K$-théorie $K(R)$ d'un anneau $R$. On fixe un plongement de $\hat{\mathbb{Z}}_3$ dans le corps $\mathbb{C}$ des nombres complexes. Alors le carré suivant est homotopiquement cartésien :*

$$\begin{array}{ccc} K(\mathbb{Z}[\frac{1}{2}])\hat{_2} & \longrightarrow & K(\mathbb{R})\hat{_2} \\ \downarrow & & \downarrow \\ K(\hat{\mathbb{Z}}_3)\hat{_2} & \longrightarrow & K(\mathbb{C})\hat{_2}. \end{array}$$

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The recent spectacular results of Voevodsky (*see* [16]) allow us to completely determine the 2-torsion in the $K$-theory groups $K_j(\mathbb{Z})$ associated with the integers $\mathbb{Z}$. The result is that there are no new elements. Hence 2-torsion has been fully known for over 20 years. Since each group $K_j(\mathbb{Z})$ is finitely generated (*see* [8]), and since their ranks were computed by Borel (*see* [3]), our 2-primary calculation yields the following assertion.

Theorem 1. – *The groups $K_*(\mathbb{Z})$ are given in Table 1. Here $w_i = w_i(\mathbb{Q})$ is defined (for even $i$) to be the largest power of 2 dividing $4i$. The annotation "(odd)" in the table denotes a finite group of odd order.*

Of course, there is nothing new up to $K_3(\mathbb{Z})$. The first new entry is the assertion that $K_4(\mathbb{Z}) = 0$ and that $K_5(\mathbb{Z})$ has at most 3-torsion. This follows from our calculation that there is no 2-torsion in $K_4(\mathbb{Z})$ or $K_5(\mathbb{Z})$, because Lee-Szczarba and Soulé (*see* [11]) showed that there is no $p$-torsion in $K_4(\mathbb{Z})$ or $K_5(\mathbb{Z})$ for $p > 3$, and Rognes (*see* [10]) has shown that there is no 3-torsion in $K_4(\mathbb{Z})$ either.

Table 1. - The $K$-theory of $\mathbb{Z}$.
*Tableau 1. - La $K$-théorie de $\mathbb{Z}$.*

| | |
|---|---|
| $K_0(\mathbb{Z}) = \mathbb{Z}$ | $K_{8n}(\mathbb{Z}) = (\text{odd})$ for $n \geq 1$ |
| $K_1(\mathbb{Z}) = \mathbb{Z}/2$ | $K_{8n+1}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2 \oplus (\text{odd})$ *for* $n \geq 1$ |
| $K_2(\mathbb{Z}) = \mathbb{Z}/2$ | $K_{8n+2}(\mathbb{Z}) = \mathbb{Z}/2 \oplus (\text{odd})$ |
| $K_3(\mathbb{Z}) = \mathbb{Z}/48$ | $K_{8n+3}(\mathbb{Z}) = \mathbb{Z}/16 \oplus (\text{odd})$ |
| $K_4(\mathbb{Z}) = 0$ | $K_{8n+4}(\mathbb{Z}) = (\text{odd})$ |
| $K_5(\mathbb{Z}) = \mathbb{Z} \oplus (3\text{-torsion})$ | $K_{8n+5}(\mathbb{Z}) = \mathbb{Z} \oplus (\text{odd})$ |
| $K_6(\mathbb{Z}) = (\text{odd})$ | $K_{8n+6}(\mathbb{Z}) = (\text{odd})$ |
| $K_7(\mathbb{Z}) = \mathbb{Z}/240 \oplus (\text{odd})$ | $K_{8n+7}(\mathbb{Z}) = (\mathbb{Z}/w_i) \oplus (\text{odd})$. $i = 4(n+1)$. |

Essentially, the 2-torsion summands of $K_j(\mathbb{Z})$ all come from the stable homotopy groups $\pi_j^s$ of spheres via the natural map $\pi_j^s \to K_j(\mathbb{Z})$, and were found by Quillen in [9]. When $j$ is $8n+1$ or $8n+2$, the $\mathbb{Z}/2$-summand is generated by the image of Adams' element $\mu_j$. When $j = 8n+3$, the 2-Sylow subgroup of $J(\pi_j O)$ is cyclic of order 8, and is contained in a $\mathbb{Z}/16$ summand of $K_j(\mathbb{Z})$; this result is due to Browder (*see* [4]), although Quillen proved in [9] that $J(\pi_j O)$ injects into $K_j(\mathbb{Z})$. When $j = 8n+7$, there is a cyclic summand of $K_j(\mathbb{Z})$ isomorphic to the subgroup $J(\pi_j O)$ of $\pi_j^s$. The 2-Sylow subgroup of $J(\pi_{8n+7} O)$ is isomorphic to $\mathbb{Z}/w_i$, where $i = 4(n+1)$ and $w_i$ is defined in Theorem 1 (*see* [17], p. 284).

The number $w_i$ also arises in the following ways (*see* [17], Theorem 6.7). If $i = 2n$ then: (1) for $\nu \gg 0$, $H^0_{et}(\mathbb{Q}, \mu_{2^\nu}^{\otimes i}) \cong \mathbb{Z}/w_i$; (2) the 2-part of the denominator of $\zeta(1-2n) = (-1)^n B_n/2n$ is $\frac{1}{2} w_{2n}$. The first of these is used to detect the image of $J$, while the second lets us relate the orders of $K$-groups to the zeta-function.

COROLLARY 2. – *The 2-part of the formula conjectured by Lichtenbaum in* [7] *holds up to a factor of 2:*

$$\frac{|K_{4n-2}(\mathbb{Z})|}{|K_{4n-1}(\mathbb{Z})|} \equiv \frac{1}{w_{2n}} \equiv \frac{1}{2}\zeta(1-2n) \qquad \text{(up to odd torsion).}$$

We now turn to the derivation of these results. Voevodsky proved (*see* [16]) that the Galois symbol $K_j^M(F)/2K_j^M(F) \to H^j_{et}(F; \mathbb{Z}/2)$ $H^j_{et}(F; \mathbb{Z}/2)$ is an isomorphism for every field $F$ of characteristic $\neq 2$ and every $j$. Suslin and Voevodsky have shown in [13] and [14] that this implies that Bloch's higher Chow groups (with coefficients mod 2) (*see* [1]) for a field $F$ are:

$$CH^i(F, 2i-j; \mathbb{Z}/2) = \begin{cases} H^j_{et}(F; \mathbb{Z}/2), & j \leq i \\ 0, & j > i. \end{cases}$$

Bloch and Lichtenbaum have defined a third quadrant spectral sequence in [2], converging to groups $K_*(F)$. Using $K$-theory with coefficients modulo 2 in their construction yields a similar third quadrant spectral sequence; using the Suslin-Voevodsky formula above, we may rewrite it as

$$(\dagger) \qquad E_2^{pq} = H^{p-q}_{et}(F; \mathbb{Z}/2) \Rightarrow K_{-p-q}(F; \mathbb{Z}/2), \qquad (p \leq 0,\ p \geq q).$$

Note that every column $E_2^{p*}$ of (†) equals the mod 2 étale cohomology of $F$.

*Example 3 (Local fields).* – Let $F_v$ be a local field with residue field $k_v$ of characteristic $\neq 2$. By [15] we know that $H^i_{et}(F_v;\mathbb{Z}/2)$ is $\mathbb{Z}/2$ for $i = 0,2$, $(\mathbb{Z}/2)^2$ for $i=1$, and 0 for $i>2$. Thus the spectral sequence (†) degenerates to yield the well-known calculation that for $j>0$ the group $K_j(F_v;\mathbb{Z}/2)$ is $\mathbb{Z}/2\oplus\mathbb{Z}/2$ for all $j>0$. Comparing with an unramified extension of $F_v$, we see that the tame symbol $K_{2i}(F_v;\mathbb{Z}/2)\to K_{2i-1}(k_v;\mathbb{Z}/2)=\mathbb{Z}/2$ is the projection onto the summand $H^2_{et}(F_v;\mathbb{Z}/2)$, while the tame symbol $K_{2i+1}(F_v;\mathbb{Z}/2)\to K_{2i}(k_v;\mathbb{Z}/2)=\mathbb{Z}/2$ may be identified with the canonical map $H^1_{et}(F_v;\mathbb{Z}/2)\to H^0_{et}(k_v;\mathbb{Z}/2)$.

When $F=\mathbb{R}$, every column of the spectral sequence (†) is the polynomial ring $H^*_{et}(\mathbb{R};\mathbb{Z}/2)=\mathbb{Z}/2[\eta]$, where $\eta$ is the nonzero element in $H^1_{et}(\mathbb{R};\mathbb{Z}/2)$. From [12], we know that $K_j(\mathbb{R};\mathbb{Z}/2)$ is $\mathbb{Z}/2$ for $j\equiv 0,1,3,4 \pmod 8$; $\mathbb{Z}/4$ for $j\equiv 2 \pmod 8$; 0 for $j\equiv 5,6,7 \pmod 8$. Thus the $d_2$ differential out of $E_2^{-2,-2}$ must be nonzero. Now multiplication by $\eta$ commutes with the differentials, and induction yields the following result.

Proposition 4. – *When $F=\mathbb{R}$, the spectral sequence (†) degenerates at $E_3=E_\infty$ to the mod 2 $K$-theory of $\mathbb{R}$. Indeed, the differentials $E_2^{-p,q}\to E_2^{2-p,q-1}$ are isomorphisms for all $p\equiv 2,3 \pmod 4$ and all $q\le -p$.*

We are now ready to consider the rational numbers $\mathbb{Q}$. Let us write $\widetilde{H}^1(\mathbb{Q};\mathbb{Z}/2)$ for the kernel of the surjection $H^1_{et}(\mathbb{Q};\mathbb{Z}/2)\to H^1_{et}(\mathbb{R};\mathbb{Z}/2)$ and $\widetilde{H}^2(\mathbb{Q};\mathbb{Z}/2)$ for the kernel of the surjection $H^2_{et}(\mathbb{Q};\mathbb{Z}/2)\to H^2_{et}(\mathbb{R};\mathbb{Z}/2)$.

Theorem 5. – *For $n\ge 0$, the mod 2 $K$-theory of $\mathbb{Q}$ is given by Table 2. Here the notation $H\rtimes\mathbb{Z}/2$ denotes the nontrivial extension of $\mathbb{Z}/2$ by the group $H$.*

Table 2. - The mod 2 $K$-theory of $\mathbb{Q}$.
*Tableau 2.- La $K$-théorie modulo 2 de $\mathbb{Q}$.*

| | |
|---|---|
| $K_{8n+1}(\mathbb{Q};\mathbb{Z}/2)=H^1_{et}(\mathbb{Q};\mathbb{Z}/2)$ | $K_{8n+2}(\mathbb{Q};\mathbb{Z}/2)=H^2_{et}(\mathbb{Q};\mathbb{Z}/2)\rtimes\mathbb{Z}/2$ |
| $K_{8n+3}(\mathbb{Q};\mathbb{Z}/2)=H^1_{et}(\mathbb{Q};\mathbb{Z}/2)$ | $K_{8n+4}(\mathbb{Q};\mathbb{Z}/2)=H^2_{et}(\mathbb{Q};\mathbb{Z}/2)$ |
| $K_{8n+5}(\mathbb{Q};\mathbb{Z}/2)=\widetilde{H}^1(\mathbb{Q};\mathbb{Z}/2)$ | $K_{8n+6}(\mathbb{Q};\mathbb{Z}/2)=\widetilde{H}^2(\mathbb{Q};\mathbb{Z}/2)$ |
| $K_{8n+7}(\mathbb{Q};\mathbb{Z}/2)=\widetilde{H}^1(\mathbb{Q};\mathbb{Z}/2)$ | $K_{8n+8}(\mathbb{Q};\mathbb{Z}/2)=\widetilde{H}^2(\mathbb{Q};\mathbb{Z}/2)\oplus\mathbb{Z}/2$ |

*Proof.* – We know from [15] that the natural map $H^i_{et}(\mathbb{Q};\mathbb{Z}/2)\to H^i_{et}(\mathbb{R};\mathbb{Z}/2)$ is an isomorphism for $i\neq 1,2$. The differentials in (†) for $F=\mathbb{Q}$ are thus determined by the differentials when $F=\mathbb{R}$, and we can read off the result from $E_3=E_\infty$. The two extension problems are resolved by comparison with $F=\mathbb{R}$ and $F=\mathbb{Q}_p$ (*see* [4]).

In order to go from $\mathbb{Q}$ to $\mathbb{Z}[\frac{1}{2}]$, we need to break up the localization sequence:

Lemma 6. – *For each $j>0$, there is a short exact sequence*

$$0\to K_j(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2)\to K_j(\mathbb{Q};\mathbb{Z}/2)\xrightarrow{\partial}\oplus_{p\neq 2}K_{j-1}(\mathbb{Z}/p;\mathbb{Z}/2)\to 0.$$

*Proof.* – Fix an even number $j$. Then the Universal Coefficient Theorem implies that $K_j(\mathbb{Q};\mathbb{Z}/2)$ maps onto $\oplus K_{j-1}(\mathbb{Z}/p;\mathbb{Z}/2)$. In addition, since $K_j(\mathbb{Z}/p;\mathbb{Z}/2)=\mathbb{Z}/2$ for all $p\neq 2$, the natural map from $\oplus_{p\neq 2}K_j(\mathbb{Z}/p;\mathbb{Z}/2)=\oplus_{p\neq 2}\mathbb{Z}/2$ to $K_j(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2)$ factors through $\mathrm{Pic}(\mathbb{Z}[\frac{1}{2}])/2=0$.

Theorem 7. – *For $n\ge 0$, the mod 2 $K$-theory of $\mathbb{Z}$ is given by Table 3.*

Table 3. - The mod 2 $K$-theory of $\mathbb{Z}[\frac{1}{2}]$
*Tableau 3. - La $K$-théorie modulo 2 de $\mathbb{Z}[\frac{1}{2}]$.*

$$\begin{array}{llll} K_{8n+1}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2) = (\mathbb{Z}/2)^2 & K_{8n+2}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2) = \mathbb{Z}/4 \\ K_{8n+3}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2) = (\mathbb{Z}/2)^2 & K_{8n+4}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2) = \mathbb{Z}/2 \\ K_{8n+5}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2) = \mathbb{Z}/2 & K_{8n+6}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2) = 0 \\ K_{8n+7}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2) = \mathbb{Z}/2 & K_{8n+8}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2) = \mathbb{Z}/2 \end{array}$$

*Proof.* – Since $H^1_{et}(\mathbb{Q};\mathbb{Z}/2) = \mathbb{Q}^*/\mathbb{Q}^{*2}$ and $H^1_{et}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2) = \mathbb{Z}[\frac{1}{2}]^*/\mathbb{Z}[\frac{1}{2}]^{*2} \cong (\mathbb{Z}/2)^2$, we have a short exact sequence

$$0 \to H^1_{et}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2) \to H^1_{et}(\mathbb{Q};\mathbb{Z}/2) \to \oplus_{p\neq 2}\mathbb{Z}/2 \to 0.$$

Comparing with the direct sum of the corresponding sequences for the local fields $\mathbb{Q}_p$ allows us to deduce the calculation of $K_j(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2)$ for odd $j$. The calculation for $j$ even follows similarly from the commutative diagram (in which the left vertical isomorphism follows from class field theory).

$$\begin{array}{ccccc} \tilde{H}^2_{et}(\mathbb{Q};\mathbb{Z}/2) & \to & K_{2i}(\mathbb{Q};\mathbb{Z}/2) & \xrightarrow{\partial} & \otimes_{p\neq 2} K_{2i-1}(\mathbb{Z}/p;\mathbb{Z}/2) \\ \downarrow\cong & & \downarrow & & \downarrow = \\ \otimes_{p\neq 2} H^2_{et}(\mathbb{Q}_p;\mathbb{Z}/2) & \to & \otimes_{p\neq 2} K_{2i}(\mathbb{Q}_p;\mathbb{Z}/2) & \xrightarrow{\partial} & \otimes_{p\neq 2} K_{2i-1}(\mathbb{Z}/p;\mathbb{Z}/2) \end{array}$$

Theorem 7 implies Theorem 1. Indeed, $K_j(\mathbb{Z};\mathbb{Z}/2) \cong K_j(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2)$ for $j \geq 2$, and the known summands of $K_j(\mathbb{Z})$ account for all of $K_j(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2)$ in this range.

Theorem 1 also shows that $K_j(\mathbb{Z}[\frac{1}{2}]) \otimes \hat{\mathbb{Z}}_2$ is the 2-adic étale $K$-theory $K_j^{et}(\mathbb{Z}[\frac{1}{2}])\hat{_2}$. As observed by Bökstedt, and again by Dwyer and Friedlander in (*see* [6], Proposition 4.2), it also implies the following topological result. Let $K(R)\hat{_2}$ denote the 2-completion of the algebraic $K$-theory space $K(R)$ of a ring $R$.

Corollary 8. – *Choose an embedding of $\hat{\mathbb{Z}}_3$ in the complex numbers $\mathbb{C}$ representing the Brauer lifting of $K(\mathbb{Z}/3) \to BU$. Then the square*

$$\begin{array}{ccc} \mathbb{Z}[\frac{1}{2}] & \to & \mathbb{R} \\ \downarrow & & \downarrow \\ \hat{\mathbb{Z}}_3 & \to & \mathbb{C} \end{array}$$

*induces a homotopy cartesian square upon 2-adic completion.*

$$\begin{array}{ccc} K(\mathbb{Z}[\frac{1}{2}])\hat{_2} & \to & K(\mathbb{R})\hat{_2} \\ \downarrow & & \downarrow \\ K(\hat{\mathbb{Z}}_3)\hat{_2} & \to & K(\mathbb{C})\hat{_2} \end{array}$$

*Example* 9. – Another case when the spectral sequence (†) degenerates is when $F$ is a totally imaginary number field. In this case $K_j(F;\mathbb{Z}/2)$ is $\mathbb{Z}/2 \oplus H^2_{et}(F;\mathbb{Z}/2)$ for $j > 0$ even, and $H^1_{et}(F;\mathbb{Z}/2)$ for $j$ odd. That is, $K_j(F;\mathbb{Z}/2)$ equals the étale $K$-theory $K_j^{et}(F;\mathbb{Z}/2)$ of Dwyer and Friedlander (*see* [5]). However, the results of [5] do not apply here unless $F$ contains $\sqrt{-1}$.

When the primes over 2 generate the class group of $F$, the methods used for $\mathbb{Z}$ allow us to calculate the $K$-theory of the ring of integers in $F$. For example, when $F = \mathbb{Q}(i)$, $i = \sqrt{-1}$, we obtain the following result. Recall that the groups $K_0(\mathbb{Z}[i]) = \mathbb{Z}$, $K_1(\mathbb{Z}[i]) = \mathbb{Z}/4$, $K_2(\mathbb{Z}[i]) = 0$ and $K_3(\mathbb{Z}[i]) = \mathbb{Z} \oplus \mathbb{Z}/24$ are known.

THEOREM 10. – *For all $n \geq 2$, $K_{2n}(\mathbb{Z}[i])$ is a finite group of odd order, while*

$$K_{2n-1}(\mathbb{Z}[i]) = \mathbb{Z} \oplus \mathbb{Z}/w_n \oplus (\text{odd}).$$

As in Theorem 1, $w_n = w_n(\mathbb{Q}(i))$ denotes the largest power of 2 dividing $4n$. (For example, if $n$ is odd then $w_n = 4$.)

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