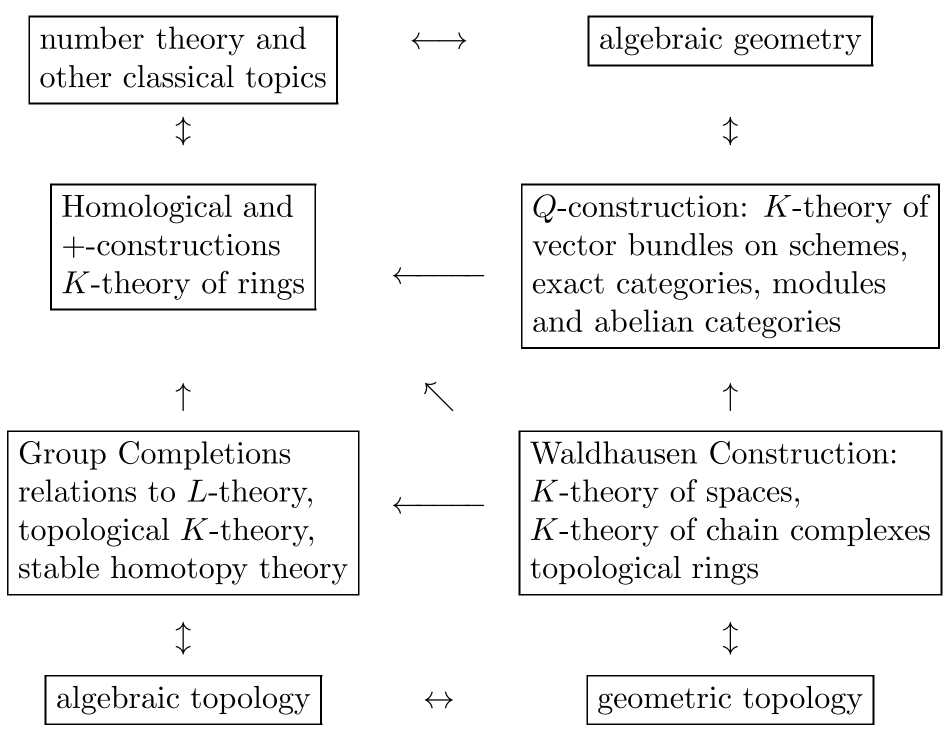
INTRODUCTION

Algebraic K-theory has two components: the classical theory which centers around the Grothendieck group K_0 of a category and uses explicit algebraic presentations, and higher algebraic K-theory which requires topological or homological machinery to define.

There are three basic versions of the Grothendieck group K_0. One involves the group completion construction, and is used for projective modules over rings, vector bundles over compact spaces and other symmetric monoidal categories. Another adds relations for exact sequences, and is used for abelian categories as well as exact categories; this is the version first used in algebraic geometry. A third adds relations for weak equivalences, and is used for categories of chain complexes and other categories with cofibrations and weak equivalences ("Waldhausen categories").

Similarly, there are four basic constructions for higher algebraic K-theory: the $+$–construction (for rings), the group completion constructions (for symmetric monoidal categories), Quillen's Q-construction (for exact categories), and Waldhausen's wS. construction (for categories with cofibrations and weak equivalences). All these constructions give the same K-theory of a ring, but are useful in various distinct settings. These settings fit together like this:



All the constructions have one feature in common: Some category C is concocted from the given setup, and one defines a K-theory space associated to the geometric realization BC of this category. The K-theory groups are then the homotopy groups of the K-theory space. In the first chapter, we introduce the basic cast of characters: projective modules and vector bundles (over a topological space, and over a scheme). Large segments of this chapter will be familiar to many readers, but which segments are familiar will depend upon the background and interests of the reader. The unfamiliar parts of this material may be skipped at first, and referred back to when relevant. We would like to warn the complacent reader that the material on the Picard group and Chern classes for topological vector bundles is in this first chapter.

In the second chapter, we define K_0 for all the settings in the above figure, and give the basic definitions appropriate to these settings: group completions for symmetric monoidal categories, K_0 for rings and topological spaces, λ-operations, abelian and exact categories, Waldhausen categories. All definitions and manipulations are in terms of generators and relations. Our philosophy is that this algebraic beginning is the most gentle way to become acquainted with the basic ideas of higher K-theory. The material on K-theory of schemes is isolated in a separate section, so it may be skipped by those not interested in algebraic geometry.

In the third chapter we give a brief overview of the classical K-theory for K_1 and K_2 of a ring. Via the Fundamental Theorem, this leads to Bass' "negative K-theory," meaning groups K_{-1}, K_{-2}, etc. We cite Matsumoto's presentation for K_2 of a field from [Milnor], and "Hilbert's Theorem 90 for K_2" (from chapter VI) in order to get to the main structure results. This chapter ends with a section on Milnor K-theory, including the transfer map, Izhboldin's theorem on the lack of p-torsion, the norm residue symbol and the relation to the Witt ring of a field.

In the fourth chapter we shall describe the four constructions for higher K-theory, starting with the original BGL^+ construction. In the case of $\mathbf{P}(R)$, finitely generated projective R-modules, we show that all the constructions give the same K-groups: the groups $K_n(R)$. The λ-operations are developed in terms of the $S^{-1}S$ construction. Non-connective spectra and homotopy K-theory are also presented. Very few theorems are present here, in order to keep this chapter short. We do not want to get involved in the technicalities lying just under the surface of each construction, so the key topological results we need are cited from the literature when needed.

The fundamental structural theorems for higher K-theory are presented in chapter V. This includes Additivity, Approximation, Cofinality, Resolution, Devissage and Localization (including the Thomason-Trobaugh localization theorem for schemes). As applications, we compute the K-theory and G-theory of projective spaces and Severi-Brauer varieties (§2), construct transfer maps satisfying a projection formula (§3), prove the Fundamental Theorem for G-theory (§6) and K-theory (§9). Several cases of Gersten's DVR conjecture are established in §6 and the Gersten-Quillen conjecture in §7. This is used to interpret the coniveau spectral sequence in terms of K-cohomology, and establish Bloch's Formula that $CH^p(X) \cong H^p(X, \mathcal{K}_p)$ for regular varieties.

In chapter 6 we describe the structure of the K-theory of fields. First we handle algebraically closed fields (§1), and the real numbers $\mathbb{R}$ (§3), following Suslin and

Harris-Segal. The group $K_3(F)$ can also be handled by comparison to Bloch's group $B(F)$ using these methods (§5). In order to say more, using classical invariants such as étale cohomology, we introduce the spectral sequence from Motivic Cohomology to K-theory in §4 and use it in §6–10 to describe the K-theory of local and global fields.