CHAPTER VI

# THE HIGHER $K$-THEORY OF FIELDS

The problem of computing the higher $K$-groups of fields has a rich history, beginning with Quillen's calculation for finite fields (IV.1.13), and Borel's calculation of $K_*(F) \otimes \mathbb{Q}$ for number fields (IV.1.18), Tate's calculations of the Milnor $K$-groups of number fields (III.7.2) and Quillen's observation that the image of the stable homotopy groups $\pi_*^s$ in $K_*(\mathbb{Z})$ contained the image of the $J$-homomorphism, whose orders are described by Bernoulli numbers. In the early 1970's, a series of conjectures were made concerning the $K$-theory of number fields, and the structure of Milnor $K$-theory. After decades of partial calculations and further conjectures, the broad picture is now in place. The goal of this chapter is to explain what we now know about the $K$-theory of fields, and especially number fields.

## §1. $K$-theory of algebraically closed fields

We begin by calculating the $K$-theory of algebraically closed fields. The results in this section are due to Suslin [Su83, Su84].

Let $C$ be a smooth curve over an algebraically closed field $k$, with function field $F$. The local ring of $C$ at any closed point $c \in C$ is a discrete valuation ring, and we have a specialization map $\lambda_c : K_*(F, \mathbb{Z}/m) \to K_*(k, \mathbb{Z}/m)$ (see V.6.7 and Ex. V.6.14). If $C = \mathbb{P}^1$, we saw in IV, Ex. 6.14 that all of the specialization maps $\lambda_c$ agree. The following result, due to Suslin [Su83], shows that this holds more generally.

THEOREM 1.1. *(Rigidity) Let $C$ be a smooth curve over an algebraically closed field $k$, with function field $F = k(C)$. If $c_0, c_1$ are two closed points of $C$, the specializations $K_*(F, \mathbb{Z}/m) \to K_*(k, \mathbb{Z}/m)$ coincide.*

PROOF. There is no loss of generality in assuming that $C$ is a projective curve. Suppose that $f : C \to \mathbb{P}^1$ is a finite map. Let $R_0$ and $R'$ be the local ring in $k(t)$ at $t = 0$ and its integral closure in $F = k(C)$, respectively. If $s_c$ is a parameter at $c$ and $e_c$ is the ramification index at $c$, so that $t = u \prod s_c^{e_c}$ in $R'$, then for $a \in K_*(F, \mathbb{Z}/m)$ we have $\partial_c(\{t, a\}) = e_c \partial_c\{s_c, a\} = e_c \lambda_c(a)$. Since $k(c) = k$ for all closed $c \in C$, we see from Chapter V, (6.6.4) that

$$\lambda_0(N_{F/k(t)}a) = \partial_0 N_{F/k(t)}(\{t, a\}) = \sum_{f(c)=0} N_c \partial_c(\{t, a\}) = \sum_{f(c)=0} e_c \lambda_c(a).$$

A similar formula holds for any other point of $\mathbb{P}^1$. In particular, since $\lambda_0(Na) = \lambda_\infty(Na)$ we have the formula

$$\sum\nolimits_{f(c)=0} e_c \lambda_c(a) = \sum\nolimits_{f(c)=\infty} e_c \lambda_c(a).$$

We can assemble this information as follows. Let $A$ denote the abelian group $\mathrm{Hom}(K_*(F;\mathbb{Z}/m), K_*(k,\mathbb{Z}/m))$, and recall that $\mathrm{Cart}(C)$ denotes the free abelian group on the closed points of $C$. There is a homomorphism $\lambda : \mathrm{Cart}(C) \to A$ sending $[c]$ to the specialization map $\lambda_c$ of IV.6.7. If we regard $f$ as an element of $F^\times$, its divisor is $\sum_{f(c)=0} e_c[c] - \sum_{f(c)=\infty} e_c[c]$. The displayed equation amounts to the formula $\lambda \circ \mathrm{div} = 0$. Now the Picard group $\mathrm{Pic}(C)$ is the cokernel of the divisor map $F^\times \to \mathrm{Cart}(C)$ (see I.5.12), so $\lambda$ factors through $\mathrm{Pic}(C)$. Since $A$ is a group of exponent $m$, $\lambda$ factors through $\mathrm{Pic}(C) \otimes \mathbb{Z}/m$. However, the kernel $J(C)$ of the degree map $\mathrm{Pic}(C) \to \mathbb{Z}$ is a divisible group (I.5.16), so $\lambda$ is zero on $J(C)$. Since $[c_0] - [c_1] \in J(C)$, this implies that $\lambda_{c_0} = \lambda_{c_1}$. $\square$

Corollary 1.2. *If $A$ is any finitely generated smooth integral $k$-algebra, and $h_0, h_1 : A \to k$ are any $k$-algebra homomorphisms, then the induced maps $h_i^* : K_*(A;\mathbb{Z}/m) \to K_*(k;\mathbb{Z}/m)$ coincide.*

Proof. The kernels of the $h_i$ are maximal ideals, and it is known that there is a prime ideal $\mathfrak{p}$ of $A$ contained in their intersection such that $A/\mathfrak{p}$ is 1-dimensional. If $R$ is the normalization of $A/\mathfrak{p}$ then the $h_i$ factor through $A \to R$. Therefore we can replace $A$ by $R$. Since $C = \mathrm{Spec}(R)$ is a smooth curve, the specializations $\lambda_i$ on $F = k(C)$ agree by Theorem 1.1. The result follows, since by Theorem V.6.7, the induced maps $h_i^*$ factor as $K_*(R;\mathbb{Z}/m) \to K_*(F;\mathbb{Z}/m) \xrightarrow{\lambda_i} K_*(k;\mathbb{Z}/m)$. $\square$

Theorem 1.3. *If $k \subset F$ is an inclusion of algebraically closed fields, the maps $K_n(k;\mathbb{Z}/m) \to K_n(F;\mathbb{Z}/m)$ are isomorphisms for all $m$.*

Proof. We saw in V.6.7.2 that both $K_n(k) \to K_n(F)$ and $K_n(k;\mathbb{Z}/m) \to K_n(F;\mathbb{Z}/m)$ are injections. To see surjectivity, we write $F$ as the union of its finitely generated subalgebras $A$. Therefore every element of $K_n(F;\mathbb{Z}/m)$ is the image of some element of $K_n(A;\mathbb{Z}/m)$ under the map induced from the inclusion $h_0 : A \hookrightarrow F$. Since the singular locus of $A$ is closed, some localization $A[1/s]$ is smooth, so we may assume that $A$ is smooth.

But for any maximal ideal $\mathfrak{m}$ of $A$ we have a second map $h_1 : A \to A/\mathfrak{m} = k \hookrightarrow F$. Both $h_0$ and $h_1$ factor through the basechange $A \to A \otimes_k F$ and the induced maps $h_i : A \otimes_k F \to F$. Since $A \otimes_k F$ is smooth over $F$, the map $h_0^* : K_n(A;\mathbb{Z}/m) \to K_n(F;\mathbb{Z}/m)$ coincides with $h_1^* : K_n(A;\mathbb{Z}/m) \to K_n(k;\mathbb{Z}/m) \to K_n(F;\mathbb{Z}/m)$ by Corollary 1.2. This finishes the proof. $\square$

*Finite characteristic.* For algebraically closed fields of characteristic $p > 0$, we may take $k = \bar{\mathbb{F}}_p$ to determine $K_*(F)$. Recall from IV.1.13 that $K_n(\bar{\mathbb{F}}_p) = 0$ for even $n > 0$, and that $K_{2i-1}(\bar{\mathbb{F}}_p) = \cup K_{2i-1}(\mathbb{F}_{p^\nu})$ is isomorphic as an abelian group to $\bar{\mathbb{F}}_p^\times \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$. In particular, $K_n(F;\mathbb{Z}/p) = K_n(\bar{\mathbb{F}}_p;\mathbb{Z}/p) = 0$; this implies that $K_n(F)$ is uniquely $p$-divisible. (We saw in IV.5.6 that this is true more generally for perfect fields of characteristic $p$.)

We also saw in IV.1.13 that if $p \nmid m$ and $\beta \in K_2(\bar{\mathbb{F}}_p;\mathbb{Z}/m) \cong \mu_m(\bar{\mathbb{F}}_p)$ is the Bott element (whose Bockstein is a primitive $m^{th}$ root of unity), then $K_*(\bar{\mathbb{F}}_p;\mathbb{Z}/m) \cong \mathbb{Z}/m[\beta]$ as a graded ring. The action of the Frobenius automorphism $\phi(x) = x^p$ on $F$ induces multiplication by $p^i$ on $K_{2i-1}(\bar{\mathbb{F}}_p)$; we say that the action is *twisted* $i$ times. The following corollary to Theorem 1.3 is immediate from these remarks.

COROLLARY 1.3.1. *Let $F$ be an algebraically closed field of characteristic $p > 0$.*
*(i) If $n$ is even and $n > 0$, $K_n(F)$ is uniquely divisible.*
*(ii) If $n = 2i-1$ is odd, $K_{2i-1}(F)$ is the direct sum of a uniquely divisible group and the torsion group $\mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$. In particular, it is divisible with no $p$-torsion, and the Frobenius automorphism acts on the torsion subgroup as multiplication by $p^i$.*
*(iii) When $p \nmid m$, the choice of a Bott element $\beta \in K_2(F;\mathbb{Z}/m)$ determines a graded ring isomorphism $K_*(F;\mathbb{Z}/m) \cong \mathbb{Z}/m[\beta]$.*

Recall that any divisible abelian group is the direct sum of a uniquely divisible group and a divisible torsion group, a divisible torsion group is the sum of its Sylow subgroups, and an $\ell$-primary divisible group is a direct sum of copies of $\mathbb{Z}/\ell^\infty$. Therefore $K_{2i-1}(F)$ is the direct sum of a uniquely divisible group and $\oplus_{\ell \neq p}\mathbb{Z}/\ell^\infty \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$.

1.3.2. If $F$ is any separably closed field of characteristic $p$, then $K_n(F)$ is non-canonically a summand of $K_n(\bar{F})$ by a transfer argument (as it is uniquely $p$-divisible for all $n > 0$, by IV.5.6). Therefore $K_n(F)$ also has the structure described in Corollary 1.3.1; see Exercise 1.1.

We now turn to the structure of $K_*(F)$ when $F$ has characteristic zero.

PROPOSITION 1.4. *If $F$ is an algebraically closed field of characteristic 0 then for every $m > 0$ the choice of a Bott element $\beta \in K_2(F;\mathbb{Z}/m)$ determines a graded ring isomorphism $K_*(F;\mathbb{Z}/m) \cong \mathbb{Z}/m[\beta]$.*

PROOF. Pick a primitive $m^{th}$ root of unity $\zeta$ in $\bar{\mathbb{Q}}$, and let $\beta$ be the corresponding Bott element in $K_2(\mathbb{Q}(\zeta);\mathbb{Z}/m)$. We use this choice to define a Bott element $\beta \in K_2(E;\mathbb{Z}/m)$ for all fields containing $\mathbb{Q}(\zeta)$, natural in $E$. By Theorem 1.3, it suffices to show that the induced ring map $\mathbb{Z}/m[\beta] \to K_*(F;\mathbb{Z}/m)$ is an isomorphism for some algebraically closed field $F$ containing $\bar{\mathbb{Q}}$.

Fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, where $p \nmid m$. For each $q = p^\nu$, let $E_q$ denote the maximal algebraic extension of $\mathbb{Q}_p$ inside $\bar{\mathbb{Q}}_p$ with residue field $\mathbb{F}_q$; $\bar{\mathbb{Q}}_p$ is the union of the $E_q$. For each $q \equiv 1 \pmod{m}$, we saw in Example V.6.10.2 (which uses Gabber rigidity) that $K_*(E_q;\mathbb{Z}/m) = \mathbb{Z}/m[\beta]$. If $q|q'$, the map $K_*(E_q;\mathbb{Z}/m) \to K_*(E_{q'};\mathbb{Z}/m)$ is an isomorphism, by naturality of $\beta$. Taking the direct limit over $q$, we have $K_*(\bar{\mathbb{Q}}_p;\mathbb{Z}/m) = \mathbb{Z}/m[\beta]$, as desired. $\square$

REMARK 1.4.1. There is a map $K_*(\mathbb{C};\mathbb{Z}/m) \to \pi_*(BU;\mathbb{Z}/m)$ arising from the change of topology; see IV.4.12.3. Suslin proved in [Su84] that this map is an isomorphism. We can formally recover this result from Proposition 1.4, since both rings are polynomial rings in one variable, and the generator $\beta \in K_2(\mathbb{C};\mathbb{Z}/m)$ maps to a generator of $\pi_2(BU;\mathbb{Z}/m)$ by IV.1.13.2.

To determine the structure of $K_*(F)$ when $F$ has characteristic 0, we need a result of Harris and Segal [HS, 3.1]. Let $m = \ell^\nu$ be a prime power and $R$ a ring containing the group $\mu_m$ of $m^{th}$ roots of unity. The group $\mu_m \wr \Sigma_n = (\mu_m)^n \rtimes \Sigma_n$ embeds into $GL_n(R)$ as the group of invertible $n \times n$ matrices with only one nonzero entry in each row and column, every nonzero entry being in $\mu_m$. Taking the union over $n$, the group $G = \mu_m \wr \Sigma_\infty$ embeds into $GL(R)$.

There is an induced map $\pi_*(BG^+) \to K_*(R)$ when $\mu_m \subset R$. Given a finite field $\mathbb{F}_q$ with $\ell \nmid q$, transfer maps for $R = \mathbb{F}_q(\zeta_{2\ell})$ give maps $\pi_*(BG^+) \to K_*(R) \to K_*(\mathbb{F}_q)$. Let $\mu_{(\ell)}(F)$ denote the $\ell$-primary subgroup of $\mu(F)$.

THEOREM 1.5. *(Harris-Segal) If $\ell \nmid q$ and $|\mu_{(\ell)}(\mathbb{F}_q(\zeta_\ell))| = m$, then each group $\pi_{2i-1}B(\mu_m \wr \Sigma_\infty)^+$ contains a cyclic summand which maps isomorphically to the $\ell$-Sylow subgroup of $\pi_{2i-1}BGL(\mathbb{F}_q)^+ = K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1)$.*

PROOF. Fix $i$ and let $N = \ell^\nu$ be the largest power of $\ell$ dividing $q^i - 1$. Theorem 3.1 of [HS] states that both $\pi_{2i-1}(BG^+)_{(\ell)} \to K_{2i-1}(\mathbb{F}_q)_{(\ell)}$ and $\pi_{2i}(BG^+; \mathbb{Z}/N) \to K_{2i}(\mathbb{F}_q; \mathbb{Z}/N) \cong \mathbb{Z}/N$ are onto. Since $\pi_{2i}(BG^+; \mathbb{Z}/N)$ is a $\mathbb{Z}/N$-module (IV.2.2), the latter map splits, *i.e.,* there is a $\mathbb{Z}/N$ summand in $\pi_{2i}(BG^+; \mathbb{Z}/N)$ and hence in $\pi_{2i-1}(BG^+)$ mapping isomorphically onto $K_{2i-1}(\mathbb{F}_q; \mathbb{Z}/N) \cong \mathbb{Z}/N$. □

COROLLARY 1.5.1. *For each prime $\ell \neq p$ and each $i > 0$, $K_{2i-1}(\bar{\mathbb{Q}}_p)$ contains a nonzero torsion $\ell$-group.*

PROOF. Fix a $q = p^\nu$ such that $q \equiv 1 \pmod{\ell}$, and $q \equiv 1 \pmod 4$ if $\ell = 2$, and let $m$ be the largest power of $\ell$ dividing $q - 1$. We consider the set of all local fields $E$ over $\mathbb{Q}_p$ (contained in a fixed common $\bar{\mathbb{Q}}_p$) whose ring of integers $R$ has residue field $\mathbb{F}_q$. For each such $E$, we have $K_{2i-1}(R) \cong K_{2i-1}(E)$ by V.6.9.2. If $R_q$ denotes the union of these $R$, and $E_q$ is the the union $E$, this implies that $K_{2i-1}(R_q) \cong K_{2i-1}(E_q)$.

Because $G$ is a torsion group, the homology groups $H_*(G; \mathbb{Q})$ vanish for $* > 0$, so the groups $\pi_*(BG^+)$ are torsion groups by the Hurewicz theorem. Since $G \to GL(\mathbb{F}_q)$ factors through $GL(R_q)$, the surjection $\pi_*(BG^+) \to \pi_*BGL(\mathbb{F}_q)^+ = K_*(\mathbb{F}_q)$ factors through $\pi_*BGL(R_q)^+ = K_*(R_q)$ and hence through $K_{2i-1}(E_q)$. It follows that $K_{2i-1}(E_q)$ contains a torsion group mapping onto the $\ell$-torsion subgroup of $K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q-1)$. Taking the direct limit over $q$, it follows that the $\ell$-torsion subgroup of $K_{2i-1}(\bar{\mathbb{Q}}_p; \mathbb{Z}/m)$ maps onto the $\ell$-torsion subgroup of $K_{2i-1}(\bar{\mathbb{F}}_p)$. □

COROLLARY 1.5.2. *If $q \equiv 1 \pmod{\ell}$ and $m$ is the order of $\mu_{(\ell)}(\mathbb{F}_q)$ then each group $K_{2i-1}(\mathbb{Z}[\zeta_m]) \cong K_{2i-1}(\mathbb{Q}(\zeta_m))$ contains a cyclic summand mapping isomorphically onto the $\ell$-primary component of $K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1)$.*

In fact, the summand is the torsion subgroup of $K_{2i-1}(\mathbb{Q}(\zeta_m))$, as we shall see in Theorem 8.2.

PROOF. We have $K_{2i-1}(\mathbb{Z}[\zeta_m]) \cong K_{2i-1}(\mathbb{Q}(\zeta_m))$ by V.6.8, and there is a canonical ring map $\mathbb{Z}[\zeta_m] \to \mathbb{F}_q$. Since $\mu_m \subset \mathbb{Z}[\zeta_m]$, the split surjection of Theorem 1.5 factors through $\pi_{2i-1}(BG^+) \to K_{2i-1}(\mathbb{Z}[\zeta_m]) \to K_{2i-1}(\mathbb{F}_q)$. □

REMARK 1.5.3. Let $S$ denote the symmetric monoidal category of finite free $\mu_m$-sets (IV.4.1.1). The space $K(S)$ is $\mathbb{Z} \times BG^+$, and the Barratt-Priddy Theorem identifies it with the zeroth space of the spectrum $\Sigma^\infty(BG_+)$. As pointed out in IV.4.10.1, the map $BG^+ \to GL(R)^+$ arises from the free $R$-module functor $S \to \mathbf{P}(R)$, and therefore $K(S) \to K(R)$ extends to a map of spectra $\mathbf{K}(S) \to \mathbf{K}(R)$.

If $k \subset F$ is an inclusion of algebraically closed fields, $K_*(k) \to K_*(F)$ is an injection by V.6.7.2. The following result implies that it is an isomorphism on torsion subgroups, and that $K_n(F)$ is divisible for $n \neq 0$.

THEOREM 1.6. *Let $F$ be an algebraically closed field of characteristic 0. Then*
*(i) If $n$ is even and $n > 0$, $K_n(F)$ is uniquely divisible.*
*(ii) If $n = 2i - 1$ is odd, $K_{2i-1}(F)$ is the direct sum of a uniquely divisible group and a torsion group isomorphic to $\mathbb{Q}/\mathbb{Z}$.*

Proof. Fix $i > 0$. For each prime $\ell$, the group $K_{2i-1}(F;\mathbb{Z}/\ell)$ is zero by Proposition 1.4. By the universal coefficient sequence IV.2.5, $K_{2i-2}(F)$ has no $\ell$-torsion and $K_{2i-1}(F)/\ell = 0$. That is, $K_{2i-1}(F)$ is $\ell$-divisible for all $\ell$ and hence divisible, while $K_{2i-2}(F)$ is torsionfree. We now consider the universal coefficient sequence

$$0 \to K_{2i}(F)/\ell \to K_{2i}(F;\mathbb{Z}/\ell) \to {}_\ell K_{2i-1}(F) \to 0.$$

The middle group is $\mathbb{Z}/\ell$ by Proposition 1.4. By Corollary 1.5.1, the exponent $\ell$ subgroup of $K_{2i-1}(F)$ is nonzero, and hence cyclic of order $\ell$. This implies that $K_{2i}(F)/\ell = 0$, *i.e.,* the torsionfree group $K_{2i}(F)$ is divisible.

Since a divisible abelian group is the direct sum of a uniquely divisible group and a divisible torsion group, a divisible torsion group is the sum of its Sylow subgroups, and an $\ell$-primary divisible group is a direct sum of copies of $\mathbb{Z}/\ell^\infty$, it follows that $K_{2i-1}(F)$ is the direct sum of a uniquely divisible group and $\oplus_\ell \mathbb{Z}/\ell^\infty \cong \mathbb{Q}/\mathbb{Z}$. $\square$

We conclude this section with a description of the torsion module of $K_{2i-1}(F)$ as a representation of the group $\mathrm{Aut}(F)$ of field automorphisms of $F$. For this we need some elementary remarks. If $F$ is algebraically closed, there is a tautological action of $\mathrm{Aut}(F)$ on the group $\mu = \mu(F)$ of roots of unity in $F$: $g \in \mathrm{Aut}(F)$ sends $\zeta$ to $g(\zeta)$. This action gives a surjective homomorphism $\mathrm{Aut}(F) \to \mathrm{Aut}(\mu)$, called the *cyclotomic* representation. To describe $\mathrm{Aut}(\mu)$, recall that the group $\mu(F)$ is isomorphic to either $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$, according to the characteristic of $F$.

Since any endomorphism of $\mathbb{Q}/\mathbb{Z}$ induces an endomorphism of its exponent $m$ subgroup $\mathbb{Z}/m$, and is equivalent to a compatible family of such, $\mathrm{End}(\mathbb{Q}/\mathbb{Z})$ is isomorphic to $\hat{\mathbb{Z}} = \lim \mathbb{Z}/m$. It is easy to see that $\hat{\mathbb{Z}}$ is the product over all primes $\ell$ of the $\ell$-adic integers $\hat{\mathbb{Z}}_\ell$, so $\mathrm{Aut}(\mu) \cong \prod \hat{\mathbb{Z}}_\ell^\times$. A similar argument, with $p \nmid m$, shows that $\mathrm{End}(\mathbb{Q}/\mathbb{Z}[\frac{1}{p}])$ is isomorphic to $\prod_{\ell\neq p} \hat{\mathbb{Z}}_\ell$, and $\mathrm{Aut}(\mu) \cong \prod_{\ell \neq p} \hat{\mathbb{Z}}_\ell^\times$.

If $\mathrm{char}(F) = 0$, the subfield of $F$ fixed by the kernel of $\mathrm{Aut}(F) \to \mathrm{Aut}(\mu)$ is the infinite cyclotomic extension $\mathbb{Q}(\mu) = \cup_m \mathbb{Q}(\zeta_m)$, by elementary Galois theory, and $\mathrm{Aut}(F)$ surjects onto $\mathrm{Aut}(\mathbb{Q}(\mu)) = \mathrm{Gal}(\mathbb{Q}(\mu)/\mathbb{Q}) \cong \mathrm{Aut}(\mu) \cong \hat{\mathbb{Z}}^\times$. If $F$ is algebraically closed of characteristic $p > 0$, the situation is similar: $\mathrm{Aut}(F)$ surjects onto $\mathrm{Aut}(\bar{\mathbb{F}}_p) = \mathrm{Gal}(\bar{\mathbb{F}}_p/F_p) \cong \mathrm{Aut}(\mu)$; the Frobenius is topologically dense in this group.

Definition 1.7. *For all $i \in \mathbb{Z}$, we shall write $\mu(i)$ for the abelian group $\mu$, made into a $\mathrm{Aut}(F)$-module by letting $g \in \mathrm{Aut}(F)$ act as $\zeta \mapsto g^i(\zeta)$. (This modified module structure is called the $i^{th}$ Tate twist of the cyclotomic module $\mu$.) If $M$ is any $\mathrm{Aut}(F)$-submodule of $\mu$, we write $M(i)$ for the abelian group $M$, considered as a submodule of $\mu(i)$. In particular, its Sylow decomposition is $\mu(i) = \oplus \mathbb{Z}/\ell^\infty(i)$.*

Proposition 1.7.1. *If $F$ is algebraically closed and $i > 0$, the torsion submodule of $K_{2i-1}(F)$ is isomorphic to $\mu(i)$ as an $\mathrm{Aut}(F)$-module.*

Proof. It suffices to show that the submodule ${}_m K_{2i-1}(F)$ is isomorphic to $\mu_m(i)$ for all $m > 0$ prime to the characteristic. Fix a primitive $m^{th}$ root of unity $\zeta$ in $F$, and let $\beta$ be the corresponding Bott element. Then $K_*(F;\mathbb{Z}/m) \cong \mathbb{Z}/m[\beta]$, by either 1.3.1 or 1.6; Since ${}_m K_{2i-1}(F) \cong K_{2i}(F;\mathbb{Z}/m)$ as $\mathrm{Aut}(F)$-modules, and the abelian group $K_{2i}(F;\mathbb{Z}/m)$ is isomorphic to $\mathbb{Z}/m$ on generator $\beta^i$.

By naturality of the product (IV, 1.10 and 2.8), the group $\mathrm{Aut}(F)$ acts on $K_*(F;\mathbb{Z}/m)$ by ring automorphisms. For each $g \in \mathrm{Aut}(F)$ there is an $a \in \mathbb{Z}/m^\times$ such that $g(\zeta) = \zeta^a$. Thus $g$ sends $\beta$ to $a\beta$, and $g$ sends $\beta^i$ to $(a\beta)^i = a^i\beta^i$. Since $\mu_m(i)$ is isomorphic to the abelian group $\mathbb{Z}/m$ with $g$ acting as multiplication by $a^i$, we have $\mu_m(i) \cong K_{2i}(F;\mathbb{Z}/m)$. $\square$

EXERCISES

**1.1** Show that the conclusion of 1.3.1 holds for any separably closed field of characteristic p: if $n > 0$ is even then $K_n(F)$ is uniquely divisible, while if $n$ is odd then $K_n(F)$ is the sum of a uniquely divisible group and $(\mathbb{Q}/\mathbb{Z})_{(p)}$. *Hint:* By IV.5.6, $K_n(F)$ is uniquely $p$-divisible for all $n > 0$.

**1.2** (Suslin) Let $k \subset F$ be an extension of algebraically closed fields, and let $X$ be an algebraic variety over $k$. Write $X_F$ for the corresponding variety $X \otimes_k F$ over $F$. In this exercise we show that the groups $G_*(X;\mathbb{Z}/m)$ are independent of $k$.

(i) If $R$ is the local ring of a smooth curve $C$ at a point $c$, show that there is a specialization map $\lambda_c : G_*(X_{k(C)};\mathbb{Z}/m) \to G_*(X;\mathbb{Z}/m)$.

(ii) (Rigidity) Show that the specialization $\lambda_c$ is independent of the choice of $c$.

(iii) If $h_i : A \to k$ is as in 1.2, show that maps $h_i^* : G_*(X_A;\mathbb{Z}/m) \to G_*(X;\mathbb{Z}/m)$ exist and coincide.

(iv) Show that the base-change $G_*(X;\mathbb{Z}/m) \to G_*(X_F;\mathbb{Z}/m)$ is an isomorphism.

**1.3** Let $E$ be a local field, finite over $\mathbb{Q}_p$ and with residue field $\mathbb{F}_q$. Use Theorem 1.6 and the proof of 1.5.1 to show that $K_{2i-1}(E)_{\mathrm{tors}}$ is the direct sum of $\mathbb{Z}/(q^i - 1)$ and a $p$-group, and that $K_{2i-1}(E)_{\mathrm{tors}} \to K_{2i-1}(\bar{\mathbb{Q}}_p)$ is an injection modulo $p$-torsion.

**1.4** The Galois group $\Gamma = \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ acts on $\mu_m$ and hence on the group $G$ of Theorem 1.5. Show that the induced action of $\Gamma$ on $\pi_*(BG^+)$ is trivial, and conclude that the summand of $K_{2i-1}(\mathbb{Q}(\zeta))$ in 1.5.2 is invariant under $\Gamma$.

## §2. The $e$-invariant of a field

The odd-indexed $K$-groups of any field $F$ have a canonical torsion summand, discovered by Harris and Segal in [HS]. It is detected by a map called the $e$-invariant, which we now define.

Let $\bar{F}$ be a separably closed field, and $\mu = \mu(\bar{F})$ the group of its roots of unity. We saw in Proposition 1.7.1 (and Ex. 1.1) that $K_{2i-1}(\bar{F})_{\mathrm{tors}}$ is isomorphic to the Tate twist $\mu(i)$ of $\mu$ as an $\mathrm{Aut}(\bar{F})$-module (see Definition 1.7). The target group $\mu(i)^G$ is always the direct sum of its $\ell$-primary Sylow subgroups $\mu_{(\ell)}(i)^G \cong \mathbb{Z}/\ell^\infty(i)^G$.

Definition 2.1. Let $F$ be a field, with separable closure $\bar{F}$ and Galois group $G = \mathrm{Gal}(\bar{F}/F)$. Since $K_*(F) \to K_*(\bar{F})$ is a homomorphism of $G$-modules, with $G$ acting trivially on $K_n(F)$, it follows that there is a natural map

$$e : K_{2i-1}(F)_{\mathrm{tors}} \to K_{2i-1}(\bar{F})^G_{\mathrm{tors}} \cong \mu(i)^G.$$

We shall call $e$ the *$e$-invariant.*

If $\mu(i)^G$ is a finite group it is cyclic, and we write $w_i(F)$ for its order, so that $\mu(i)^G \cong \mathbb{Z}/w_i(F)$. If $\ell$ is a prime, we write $w_i^{(\ell)}(F)$ for the order of $\mu_{(\ell)}(i)^G$. Thus the target of the $e$-invariant is $\bigoplus_\ell \mathbb{Z}/w_i^{(\ell)}(F)$, and $w_i(F) = \prod w_i^{(\ell)}(F)$.

EXAMPLE 2.1.1 (FINITE FIELDS). It is a pleasant exercise to show that $w_i(\mathbb{F}_q) = q^i - 1$ for all $i$. Since this is the order of $K_{2i-1}(\mathbb{F}_q)$ by IV.1.13, we see that in this case, the $e$-invariant is an isomorphism. (See Exercise IV.1.26.)

EXAMPLE 2.1.2. If $i$ is odd, $w_i(\mathbb{Q}) = 2$ and $w_i(\mathbb{Q}(\sqrt{-1})) = 4$. If $i$ is even then $w_i(\mathbb{Q}) = w_i(\mathbb{Q}(\sqrt{-1}))$, and $\ell | w_i(\mathbb{Q})$ exactly when $(\ell - 1)$ divides $i$. We have: $w_2 = 24$, $w_4 = 240$, $w_6 = 504 = 2^3{\cdot}3^2{\cdot}7$, $w_8 = 480 = 2^5{\cdot}3{\cdot}5$, $w_{10} = 1320 = 2^3{\cdot}3{\cdot}5{\cdot}11$, and $w_{12} = 65,520 = 2^4{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13$. These formulas may be derived from Propositions 2.2 and 2.3 below.

In [LSz], Lee and Szczarba used a variant of the formula $K_3(R) = H_3(St(R);\mathbb{Z})$ (Ex. IV.1.10) to show that $K_3(\mathbb{Z}) \cong K_3(\mathbb{Q}) \cong \mathbb{Z}/48$. It follows that the $e$-invariant $K_3(\mathbb{Q}) \to \mathbb{Z}/24$ cannot be an injection. (We will see in Remark 2.1.3 that it vanishes on the nonzero symbol $\{-1,-1,-1\}$.)

REMARK 2.1.3. The complex Adams $e$-invariant for stable homotopy is a map from $\pi^s_{2i-1}$ to $\mathbb{Z}/w_i(\mathbb{Q})$, whence our terminology. Quillen observed in [Qlet] that the Adams $e$-invariant is the composition $\pi^s_{2i-1} \to K_{2i-1}(\mathbb{Q}) \xrightarrow{e} \mathbb{Z}/w_i(\mathbb{Q})$. (Adams defined his $e$-invariant using $\pi_{2i}(BU)/w_i(\mathbb{Q})$; Quillen's assertions have been translated using Remark 1.4.1.)

If $i$ is positive and divisible by 4, the real Adams $e$-invariant coincides with the complex $e$-invariant. If $i \equiv 2 \pmod 4$, the real (Adams) $e$-invariant is a map $\pi^s_{2i-1} \to \pi_{2i}(BO)/2w_i(\mathbb{Q}) = \mathbb{Z}/2w_i(\mathbb{Q})$. For all even $i > 0$, Adams proved in 1966 that the real $e$-invariant restricts to an injection on the image of $J : \pi_{2i-1}O \to \pi^s_{2i-1}$ and induces an isomorphism $(\mathrm{im}J)_{2i-1} \cong \mathbb{Z}/w_i(\mathbb{Q})$. His proof used the "Adams Conjecture," which was later verified by Quillen. Quillen showed in [Qlet] that the real Adams $e$-invariant factors through $K_{2i-1}(\mathbb{Z}) = K_{2i-1}(\mathbb{Q})$, so $(\mathrm{im}J)_{2i-1}$ injects into $K_{2i-1}(\mathbb{Z})$. In particular, the image $\{-1,-1,-1\}$ of $\eta^3 \in \pi^s_3$ is nonzero (see Ex. IV.1.12). Since the map from $\pi_{8k+3}(BO) = \mathbb{Z}$ to $\pi_{8k+3}(BU) = \mathbb{Z}$ has image $2\mathbb{Z}$, it follows that the $e$-invariant $K_{8k+3}(\mathbb{Q}) \to \mathbb{Z}/w_{4k+2}(\mathbb{Q})$ of Definition 2.1 is not an injection on $(\mathrm{im}J)_{8k+3}$.

Not all of the image of $J$ injects into $K_*(\mathbb{Z})$. If $n \equiv 0, 1 \pmod 8$ then $J(\pi_n O) \cong \mathbb{Z}/2$, but Waldhausen showed (in 1982) that these elements map to zero in $K_n(\mathbb{Z})$.

*Formulas for $w_i(F)$*

We now turn to formulas for the numbers $w_i^{(\ell)}(F)$. Let $\zeta_m$ denote a primitive $m^{th}$ root of unity. For odd $\ell$, we have the following simple formula.

PROPOSITION 2.2. *Fix a prime $\ell \neq 2$, and let $F$ be a field of characteristic $\neq \ell$. Let $a \le \infty$ be maximal such that $F(\zeta_\ell)$ contains a primitive $\ell^a$th root of unity and set $r = [F(\zeta_\ell) : F]$. If $i = c\ell^b$, where $\ell \nmid c$, then the numbers $w_i^{(\ell)} = w_i^{(\ell)}(F)$ are $\ell^{a+b}$ if $r \mid i$, and 1 otherwise. That is:*
*(a) If $\zeta_\ell \in F$ then $w_i^{(\ell)} = \ell^{a+b}$;*
*(b) If $\zeta_\ell \notin F$ and $i \equiv 0 \pmod r$ then $w_i^{(\ell)} = \ell^{a+b}$;*
*(c) If $\zeta_\ell \notin F$ and $i \not\equiv 0 \pmod r$ then $w_i^{(\ell)} = 1$.*

PROOF. Since $\ell$ is odd, $G = \mathrm{Gal}(F(\zeta_{\ell^\nu})/F)$ is a cyclic group of order $r\ell^{\nu-a}$ for all $\nu \ge a$. If a generator of $G$ acts on $\mu_{\ell^{a+\nu}}$ by $\zeta \mapsto \zeta^g$ for some $g \in (\mathbb{Z}/\ell^{a+\nu})^\times$ then it acts on $\mu^{\otimes i}$ by $\zeta \mapsto \zeta^{g^i}$. Now use the criterion of Lemma 2.2.1; if $r \mid i$ then

$\mathrm{Gal}(F(\zeta_{\ell^{a+b}})/F)$ is cyclic of order $r\ell^b$, while if $r \nmid i$ the exponent $r$ of $\mathrm{Gal}(F(\zeta_\ell)/F)$ does not divide $i$. $\square$

LEMMA 2.2.1. $w_i^{(\ell)}(F) = \max\{\ell^\nu \mid \mathrm{Gal}(F(\zeta_{\ell^\nu})/F)$ *has exponent dividing* $i\}$

PROOF. Set $\zeta = \zeta_{\ell^\nu}$. Then $\zeta^{\otimes i}$ is invariant under $g \in \mathrm{Gal}(\bar{F}/F)$ precisely when $g^i(\zeta) = \zeta$, and $\zeta^{\otimes i}$ is invariant under all of $G$ precisely when the group $\mathrm{Gal}(F(\zeta_{\ell^\nu})/F)$ has exponent $i$. $\square$

EXAMPLE 2.2.2. Consider $F = \mathbb{Q}(\zeta_{p^a})$. If $i = cp^b$ then $w_i^{(p)}(F) = p^{a+b}$ $(p \neq 2)$. If $\ell \neq 2, p$ then $w_i^{(\ell)}(F) = w_i^{(\ell)}(\mathbb{Q})$ for all $i$. This number is 1 unless $(\ell - 1) \mid i$; if $(\ell - 1)|i$ but $\ell \nmid i$ then $w_i^{(\ell)}(F) = \ell$. In particular, if $\ell = 3$ and $p \neq 3$ then $w_i^{(3)}(F) = 1$ for odd $i$, and $w_i^{(3)}(F) = 3$ exactly when $i \equiv 2, 4 \pmod 6$.

The situation is more complicated when $\ell = 2$, because $\mathrm{Aut}(\mu_{2^\nu}) = (\mathbb{Z}/2^\nu)^\times$ contains two involutions if $\nu \geq 3$. We say that a field $F$ is *exceptional* if $\mathrm{char}(F) = 0$ and the Galois groups $\mathrm{Gal}(F(\zeta_{2^\nu})/F)$ are not cyclic for large $\nu$. If $F$ is not exceptional, we say that it is *non-exceptional.*

PROPOSITION 2.3. *($\ell = 2$) Let $F$ be a field of characteristic $\neq 2$. Let $a$ be maximal such that $F(\sqrt{-1})$ contains a primitive $2^a$th root of unity. If $i = c2^b$, where $2 \nmid c$, then the 2-primary numbers $w_i^{(2)} = w_i^{(2)}(F)$ are:*
*(a) If $\sqrt{-1} \in F$ then $w_i^{(2)} = 2^{a+b}$ for all $i$.*
*(b) If $\sqrt{-1} \notin F$ and $i$ is odd then $w_i^{(2)} = 2$.*
*(c) If $\sqrt{-1} \notin F$, $F$ is exceptional and $i$ is even then $w_i^{(2)} = 2^{a+b}$.*
*(d) If $\sqrt{-1} \notin F$, $F$ is non–exceptional and $i$ is even then $w_i^{(2)} = 2^{a+b-1}$.*

The proof of Proposition 2.3(a,b,d) is almost identical to that of 2.2 with $r = 1$. The proof in the exceptional case (c) is relegated to Exercise 2.2.

Both $\mathbb{R}$ and $\mathbb{Q}_2$ are exceptional, and so are each of their subfields. In particular, real number fields (like $\mathbb{Q}$) are exceptional, and so are some totally imaginary number fields, like $\mathbb{Q}(\sqrt{-7})$.

EXAMPLE 2.3.1 (LOCAL FIELDS). Let $E$ be a local field, finite over $\mathbb{Q}_p$ and with residue field $\mathbb{F}_q$. Then $w_i(E)$ is $w_i(\mathbb{F}_q) = q^i - 1$ times a power of $p$. (The precise power of $p$ is given in Exercise 2.3 when $E = \mathbb{Q}_p$.) This follows from Propositions 2.2 and 2.3, using the observation that (for $\ell \neq p$) the number of $\ell$-primary roots of unity in $E(\zeta_\ell)$ is the same as in $\mathbb{F}_q(\zeta_\ell)$.

By Exercise 1.3, the map $K_{2i-1}(E)_{\mathrm{tors}} \xrightarrow{e} \mathbb{Z}/w_i(E)$ is a surjection up to $p$-torsion, and induces an isomorphism on $\ell$-primary torsion subgroups $K_{2i-1}(E)\{\ell\} \cong \mathbb{Z}/w_i^{(\ell)}$ for $\ell \neq p$. We will see in Proposition 7.3 that the torsion subgroup of $K_{2i-1}(E)$ is exactly $\mathbb{Z}/w_i(E)$.

*Bernoulli numbers*

The numbers $w_i(\mathbb{Q})$ are related to the *Bernoulli numbers* $B_k$. These were defined by Jacob Bernoulli in 1713 as coefficients in the power series

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{t^{2k}}{(2k)!}.$$

(We use the topologists' $B_k$ from [MSt], all of which are positive. Number theorists would write it as $(-1)^{k+1}B_{2k}$.) The first few Bernoulli numbers are:

$$B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30}, B_5 = \frac{5}{66}, B_6 = \frac{691}{2730}, B_7 = \frac{7}{6}, B_8 = \frac{3617}{510}.$$

The denominator of $B_k$ is always squarefree, divisible by 6, and equal to the product of all primes with $(p-1)|2k$. Moreover, if $(p-1) \nmid 2k$ then $p$ is not in the denominator of $B_k/k$ even if $p|k$; see [MSt]. From this information, it is easy to verify the following identity. Recall from 2.1.2 that $w_i(\mathbb{Q}) = 2$ when $i$ is odd.

LEMMA 2.4. *If $i = 2k$ is even then $w_i(\mathbb{Q})$ is the denominator of $B_k/4k$. The prime $\ell$ divides $w_i(\mathbb{Q})$ exactly when $(\ell - 1)$ divides $i$.*

Although the numerator of $B_k$ is difficult to describe, it is related to the notion of irregular primes, which we now define.

IREGULAR PRIMES 2.4.1. A prime $p$ is called *irregular* if $p$ divides the order $h_p$ of $\mathrm{Pic}(\mathbb{Z}[\mu_p])$; if $p$ is not irregular it is called *regular*. Iwasawa proved that a prime $p$ is regular if and only if $\mathrm{Pic}(\mathbb{Z}[\mu_{p^\nu}])$ has no $p$-torsion for all $\nu$. The smallest irregular primes are $37, 59, 67, 101, 103, 131$ and $149$. Siegel conjectured that asymptotically about 39% of all primes are irregular, and about 39% of the primes less than 4 million are irregular.

Kummer proved that $p$ is irregular if and only if $p$ divides the numerator of one of the Bernoulli numbers $B_k$, $k \le (p-3)/2$ (see [Wash, 5.34]). By Kummer's congruences ([Wash, 5.14]), a regular prime $p$ does not divide the numerator of *any* $B_k/k$ (but $5|B_5$). Thus only irregular primes can divide the numerator of $B_k/k$.

The historical interest in regular primes is Kummer's 1847 proof of Fermat's Last Theorem (case I) for regular primes: $x^p + y^p = z^p$ has no solution in which $p \nmid xyz$. For us, certain calculations of $K$-groups become easier at regular primes. (See Example 8.3.2 and Proposition 10.5.)

REMARK 2.4.2. Bernoulli numbers also arise as values of the Riemann zeta function. Euler proved (in 1735) that $\zeta_{\mathbb{Q}}(2k) = B_k(2\pi)^{2k}/2(2k)!$. By the functional equation, we have $\zeta_{\mathbb{Q}}(1-2k) = (-1)^k B_k/2k$. By Lemma 2.4, the denominator of $\zeta_{\mathbb{Q}}(1-2k)$ is $\frac{1}{2}w_{2k}(\mathbb{Q})$.

REMARK 2.4.3. The Bernoulli numbers are of interest to topologists because if $n = 4k-1$ the image of $J : \pi_n SO \to \pi_n^s$ is cyclic of order equal to the denominator of $B_k/4k$, and the numerator determines the number of exotic $(4k-1)$-spheres which bound parallelizable manifolds; see [MSt, App.B].

HARRIS-SEGAL THEOREM 2.5. *Let $F$ be a field with $1/\ell \in F$; if $\ell = 2$, we also suppose that $F$ is non-exceptional. Set $w_i = w_i^{(\ell)}(F)$. Then each $K_{2i-1}(F)$ has a direct summand isomorphic to $\mathbb{Z}/w_i(F)$, detected by the e-invariant.*

*If $F$ is the field of fractions of an integrally closed domain $R$ then $K_{2i-1}(R)$ also has a direct summand isomorphic to $\mathbb{Z}/w_i(F)$, detected by the e-invariant.*

*The splitting $\mathbb{Z}/w_i \to K_{2i-1}(R)$ is called the Harris-Segal map, and its image is called the ($\ell$-primary) Harris-Segal summand of $K_{2i-1}(R)$.*

We will see in Theorem 8.2 below that $\mathbb{Z}/w_i$ is the torsion subgroup of $K_{2i-1}(\mathbb{Z}[\zeta_{\ell^a}])$. It follows that the Harris-Segal map is unique, and hence so is the Harris-Segal summand of $K_{2i-1}(R)$. This uniqueness was originally established by Kahn and others.

Proof. Suppose first that either $\ell \neq 2$ and $\zeta_\ell \in R$, or that $\ell = 2$ and $\zeta_4 \in R$. If $R$ has $m = \ell^a$ $\ell$-primary roots of unity, then $w_i^{(\ell)}(\mathbb{Q}(\zeta_m))$ equals $w_i = w_i^{(\ell)}(F)$ by 2.2 and 2.3. Thus there is no loss in generality in assuming that $R = \mathbb{Z}[\zeta_m]$. Pick a prime $p$ with $p \not\equiv 1 \pmod{\ell^{a+1}}$. Then $\zeta_{\ell^{a+1}} \notin \mathbb{F}_p$, and if $\mathfrak{p}$ is any prime ideal of $R = \mathbb{Z}[\zeta_m]$ lying over $p$ then the residue field $R/\mathfrak{p}$ is $\mathbb{F}_q = \mathbb{F}_p(\zeta_m)$. We have $w_i = w_i^{(\ell)}(\mathbb{F}_q)$ by Example 2.2.2.

The quotient map $R \to R/\mathfrak{p}$ factors through the $\mathfrak{p}$-adic completion $\hat{R}_\mathfrak{p}$, whose field of fractions is the local field $E = \mathbb{Q}_p(\zeta_m)$. By Example 2.3.2, $w_i^{(\ell)}(E) = w_i$ and $K_{2i-1}(E)\{\ell\} \xrightarrow{e} \mathbb{Z}/w_i$ is an isomorphism. Now the $e$-invariant for the finite group $K_{2i-1}(R)$ is the composite

$$K_{2i-1}(R)_{(\ell)} \cong K_{2i-1}(\mathbb{Q}(\zeta_m))_{(\ell)} \to K_{2i-1}(E)\{\ell\} \xrightarrow{e} \mathbb{Z}/w_i.$$

By Corollary 1.5.2, $K_{2i-1}(R)$ contains a cyclic summand $A$ of order $w_i$, mapping to the summand $\mathbb{Z}/w_i$ of $K_{2i-1}(\mathbb{F}_q)$ under $K_{2i-1}(R) \to K_{2i-1}(\hat{R}_\mathfrak{p}) \to K_{2i-1}(\mathbb{F}_q)$. Therefore $A$ injects into (and is isomorphic to) $K_{2i-1}(\hat{R}_\mathfrak{p})\{\ell\} \cong K_{2i-1}(E)\{\ell\} \cong \mathbb{Z}/w_i$. The theorem now follows in this case.

Suppose now that $\zeta_\ell \notin R$. By Exercise 2.5, we may assume that $F$ is a subfield of $\mathbb{Q}(\zeta_m)$, that $\mathbb{Q}(\zeta_m) = F(\zeta_\ell)$, and that $R$ is the integral closure of $\mathbb{Z}$ in $F$. We may suppose that $r = [\mathbb{Q}(\zeta_m) : F]$ divides $i$ since otherwise $w_i = 1$, and set $\Gamma = \mathrm{Gal}(\mathbb{Q}(\zeta_m)/F)$. By Proposition 2.2, $w_i = w_i^{(\ell)}(\mathbb{Q}[\zeta_m])$. We have just seen that there is a summand $A$ of $K_{2i-1}(\mathbb{Q}[\zeta_m])$ mapping isomorphically to $\mathbb{Z}/w_i$ by the $e$-invariant. By Ex. 1.4, $\Gamma$ acts trivially on $A$.

Since $f : R \to \mathbb{Z}[\zeta_m]$ is Galois, the map $f^*f_*$ is multiplication by $\sum_{g \in G} g$ on $K_{2i-1}(\mathbb{Z}[\zeta_m])$, and hence multiplication by $r$ on $A$ (see Ex. IV.6.13). Since $f_*f^*$ is multiplication by $r$ on $f_*(A)$, we see that $f^* : f_*(A) \to A$ is an isomorphism with inverse $f_*/r$. Hence $f_*(A)$ is a summand of $K_{2i-1}(R)$, and the $e$-invariant $K_{2i-1}(R) \xrightarrow{f^*} K_{2i-1}(\mathbb{Z}[\zeta_m]) \xrightarrow{e} \mathbb{Z}/w_i$ maps $f_*(A)$ isomorphically to $\mathbb{Z}/w_i$.

When $\ell = 2$ and $F$ is non-exceptional but $\sqrt{-1} \notin F$, we may again assume by Ex. 2.5 that $F$ is a subfield of index 2 in $\mathbb{Q}(\zeta_m) = F(\sqrt{-1})$. By Proposition 2.3, $w_i^{(2)}(\mathbb{Q}(\zeta_m)) = 2w_i$ and there is a summand $A$ of $K_{2i-1}(\mathbb{Q}(\zeta_m))$ mapping isomorphically to $\mathbb{Z}/2w_i$ by the $e$-invariant; by Ex. 1.4, $\Gamma$ acts trivially on $A$ and we set $\bar{A} = f_*(A)$. Since $f^*f_*$ is multiplication by 2 on $A$, the image of $\bar{A}$ is $2A$. From the diagram

$$\begin{array}{ccc}
\bar{A} \subset K_{2i-1}(F) & \xrightarrow{e} & \mathbb{Z}/w_i \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \text{include}} \\
A \subset K_{2i-1}(\mathbb{Q}(\zeta_m)) & \xrightarrow{e} & \mathbb{Z}/2w_i
\end{array}$$

we see that $\bar{A} \cong 2A \cong \mathbb{Z}/w_i$, as desired. $\square$

Remark 2.5.1. If $F$ is an exceptional field, a transfer argument using $F(\sqrt{-1})$ shows that there is a cyclic summand in $K_{2i-1}(F)$ whose order is either $w_i(F)$, $2w_i(F)$ or $w_i(F)/2$. (Exercise 2.4); we will also call these *Harris-Segal summands*.

When $F$ is a totally imaginary number field, we will see in Theorem 8.4 below that the Harris-Segal summand always has order $w_i(F)$. The following theorem, extracted from Theorem 9.5 below, shows that all possibilities occur for real number fields, *i.e.,* number fields embeddable in $\mathbb{R}$.

THEOREM 2.6. *Let $F$ be a real number field. Then the Harris-Segal summands in $K_{2i-1}(F)$ and $K_{2i-1}(\mathcal{O}_F)$ are isomorphic to:*

(1) $\mathbb{Z}/w_i(F)$, *if* $i \equiv 0 \pmod 4$ *or* $i \equiv 1 \pmod 4$, *i.e.,* $2i-1 \equiv \pm 1 \pmod 8$;
(2) $\mathbb{Z}/2w_i(F)$, *if* $i \equiv 2 \pmod 4$, *i.e.,* $2i-1 \equiv 3 \pmod 8$;
(3) $\mathbb{Z}/\frac{1}{2}w_i(F)$, *if* $i \equiv 3 \pmod 4$, *i.e.,* $2i-1 \equiv 5 \pmod 8$.

EXAMPLE 2.7. Let $F = \mathbb{Q}(\zeta + \zeta^{-1})$ be the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta)$, $\zeta^p = 1$ with $p$ odd. Then $w_i(F) = 2$ for odd $i$, and $w_i(F) = w_i(\mathbb{Q}(\zeta))$ for even $i > 0$ by 2.2 and 2.3 (see Ex. 2.6). Note that $p \,|\, w_i(F(\zeta))$ for all $i$, $p \,|\, w_i(F)$ if and only if $i$ is even, and $p \,|\, w_i(\mathbb{Q})$ only when $(p-1)|i$; see 2.2.2.

If $n \equiv 3 \pmod 4$, the groups $K_n(\mathbb{Z}[\zeta + \zeta^{-1}]) = K_n(F)$ are classically finite (see 8.1); the order of their Harris-Segal summands are given by Theorem 2.6. When $n \not\equiv -1 \pmod{2p-2}$, the group $K_n(F)$ has an extra $p$-primary factor not coming from the image of $J$ (see 2.1.3).

EXERCISES

**2.1** For every prime $\ell$ with $1/\ell \in F$, show that the following are equivalent:
(i) $F(\zeta_\ell)$ has only finitely many $\ell$-primary roots of 1;
(ii) $w_i^{(\ell)}(F)$ is finite for some $i \equiv 0 \pmod{2(\ell-1)}$;
(iii) $w_i^{(\ell)}(F)$ is finite for all $i > 0$.

**2.2** Prove Proposition 2.3(c), giving the formula $w_i^{(2)}(F) = 2^{a+b}$ when $i$ is even and $F$ is exceptional. *Hint:* Consider $\mu(i)^H$, $H = \mathrm{Gal}(\bar{F}/F(\sqrt{-1}))$.

**2.3** If $p$ is odd, show that $w_i(\mathbb{Q}_p) = p^i - 1$ unless $(p-1)|i$, and if $i = m(p-1)p^b$ $(p \nmid m)$ then $w_i(\mathbb{Q}_p) = (p^i - 1)p^{1+b}$.

For $p = 2$, show that $w_i(\mathbb{Q}_2) = 2(2^i - 1)$ for $i$ odd; if $i$ is even, say $i = 2^b m$ with $m$ odd, show that $w_i(\mathbb{Q}_2) = (2^i - 1)2^{2+b}$.

**2.4** Let $f : F \to E$ be a field extension of degree 2, and suppose $x \in K_*(E)$ is fixed by $\mathrm{Gal}(E/F)$. If $x$ generates a direct summand of order $2m$, show that $f_*(x)$ is contained in a cyclic summand of $K_*(F)$ of order either $m$, $2m$ or $4m$.

**2.5** Let $F$ be a field of characteristic 0. If $\ell \neq 2$ and $a < \infty$ is as in 2.2, show that there is a subfield $F_0$ of $\mathbb{Q}(\zeta_{\ell^a})$ such that $w_i^{(\ell)}(F_0) = w_i^{(\ell)}(F)$. If $\ell = 2$ and $a < \infty$ is as in 2.3, show that there is a subfield $F_0$ of $\mathbb{Q}(\zeta_{2^a})$ such that $w_i^{(2)}(F_0) = w_i^{(2)}(F)$, and that $F_0$ is exceptional (resp., non-exceptional) if $F$ is.

**2.6** Let $\ell$ be an odd prime, and $F = \mathbb{Q}(\zeta_\ell + \zeta_\ell^{-1})$ the maximal real subfield of $\mathbb{Q}(\zeta_\ell)$. Show that $w_i(F) = 2$ for odd $i$, and that $w_i(F) = w_i(\mathbb{Q}(\zeta_\ell))$ for even $i > 0$. In particular, $\ell | w_i(\mathbb{Q}(\zeta_\ell))$ for all $i$, but $\ell | w_i(F)$ if and only if $i$ is even.

## §3. The $K$-theory of $\mathbb{R}$

In this section, we describe the algebraic $K$-theory of the real numbers $\mathbb{R}$, or rather the torsion subgroup $K_n(\mathbb{R})_{\text{tors}}$ of $K_n(\mathbb{R})$. Here is the punchline:

THEOREM 3.1. *(Suslin) For all $n \geq 1$,*
*(a) $K_n(\mathbb{R})$ is the direct sum of a uniquely divisible group and $K_n(\mathbb{R})_{\text{tors}}$.*
*(b) The torsion groups and $K_n(\mathbb{R})_{\text{tors}} \to K_n(\mathbb{C})_{\text{tors}}$ are given by Table 3.1.1.*
*(c) The map $K_n(\mathbb{R};\mathbb{Z}/m) \to \widetilde{KO}(S^n;\mathbb{Z}/m) = \pi_n(BO;\mathbb{Z}/m)$ is an isomorphism.*

| $i$ (mod 8) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $K_i(\mathbb{R})$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Q}/\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Q}/\mathbb{Z}$ | 0 |
| $\downarrow$ | injects | 0 | $a \mapsto 2a$ | 0 | 0 | 0 | $\cong$ | 0 |
| $K_i(\mathbb{C})$ | $\mathbb{Q}/\mathbb{Z}$ | 0 | $\mathbb{Q}/\mathbb{Z}$ | 0 | $\mathbb{Q}/\mathbb{Z}$ | 0 | $\mathbb{Q}/\mathbb{Z}$ | 0 |
| $\downarrow$ | 0 | 0 | $\cong$ | 0 | 0 | 0 | $a \mapsto 2a$ | 0 |
| $K_i(\mathbb{H})$ | 0 | 0 | $\mathbb{Q}/\mathbb{Z}$ | 0 | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Q}/\mathbb{Z}$ | 0 |

**Table 3.1.1**. The torsion subgroups of $K_n(\mathbb{R})$, $K_n(\mathbb{C})$ and $K_n(\mathbb{H})$, $n > 0$.

REMARK 3.1.2. Table 3.1.1 shows that $K_n(\mathbb{R})_{\text{tors}} \cong \pi_{n+1}(BO;\mathbb{Q}/\mathbb{Z})$ is the Harris-Segal summand of $K_n(\mathbb{R})$ (in the sense of 2.5.1) for all $n$ odd; the $e$-invariant (Definition 2.1) is the map from $K_n(\mathbb{R})_{\text{tors}}$ to $K_n(\mathbb{C})_{\text{tors}}$. When $n$ is $8k+1$, the $\mathbb{Z}/2$-summand in $K_n(\mathbb{R})$ is generated by the image of Adams' element $\mu_n \in \pi_n^s$; Adams showed that $\mu_n$ is detected by the complex Adams $e$-invariant. Adams also showed that the elements $\mu_{8k+2} = \eta \cdot \mu_{8k+1}$ and $\mu_{8k+3} = \eta^2 \cdot \mu_{8k+1}$ are nonzero and detected by the real Adams $e$-invariant which, as Quillen showed in [Qlet], maps $\pi_n^s$ to $K_n(\mathbb{R})_{\text{tors}} \cong \pi_{n+1}(BO;\mathbb{Q}/\mathbb{Z})$. (See Remark 2.1.3.) It follows that when $n = 8k+3$, the kernel of the $e$-invariant $K_n(\mathbb{R})_{\text{tors}} \to \mathbb{Q}/\mathbb{Z}$ is isomorphic to $\mathbb{Z}/2$, and is generated by the nonzero element $\{-1,-1,\mu_{8k+1}\}$. When $n = 8k+5$, the Harris-Segal summand is zero even though the target of the $e$-invariant is $\mathbb{Z}/2$.

A similar calculation for the quaternions $\mathbb{H}$ is also due to Suslin [Su86, 3.5]. The proof uses the algebraic group $SL_n(\mathbb{H})$ in place of $SL_n(\mathbb{R})$.

THEOREM 3.2. *(Suslin) For all $n \geq 1$,*
*(a) $K_n(\mathbb{H})$ is the direct sum of a uniquely divisible group and $K_n(\mathbb{H})_{\text{tors}}$.*
*(b) The torsion groups and $K_n(\mathbb{C})_{\text{tors}} \to K_n(\mathbb{H})_{\text{tors}}$ are given by Table 3.1.1.*
*(c) The map $K_n(\mathbb{H};\mathbb{Z}/m) \to \widetilde{KSp}(S^n;\mathbb{Z}/m) = \pi_n(BSp;\mathbb{Z}/m)$ is an isomorphism.*

The method of proof uses a universal homotopy construction which is of independent interest, and also gives an alternative calculation of $K_*(\mathbb{C})$ to the one we gave in Proposition 1.4. As observed in 1.4.1, the punchline of that calculation is that, for all $n \geq 1$, $K_n(\mathbb{C};\mathbb{Z}/m) \to \widetilde{KU}(S^n;\mathbb{Z}/m) = \pi_n(BU;\mathbb{Z}/m)$ is an isomorphism.

We begin with some general remarks. If $G$ is a topological group, then it is important to distinguish between the classifying space $BG^\delta$ of the discrete group $G$, which is the simplicial set of IV.3.4.1, and the classifying space $BG^{\text{top}}$ of the topological group $G^{\text{top}}$, which is discussed in IV.3.9. For example, the homotopy groups of $BGL(\mathbb{C})^\delta$ are zero, except for the fundamental group, while the homotopy groups of $BGL(\mathbb{C})^{\text{top}} \simeq BU$ are given by Bott Periodicity II.3.1.1.

Next, suppose that $G$ is a Lie group having finitely many components, equipped with a left invariant Riemannian metric. Given $\varepsilon > 0$, let $G_\varepsilon$ denote the $\varepsilon$–ball about 1. If $\varepsilon$ is small, then $G_\varepsilon$ is geodesically convex; the geodesic between any two points lies in $G_\varepsilon$.

Definition 3.3. Let $BG_\varepsilon$ denote the simplicial subset of $BG^\delta$ whose $p$-simplices are the $p$-tuples $[g_1, \dots, g_p]$ such that there is a point in the intersection of $G_\varepsilon$ with all the translates $g_1 \cdots g_i G_\varepsilon$, $i \le p$. (This condition is preserved by the face and degeneracy maps of $BG^\delta$.)

Suslin proved the following result in [Su84, 4.1].

Theorem 3.4. *(Suslin) Let $G$ be a Lie group. If $\varepsilon$ is small enough so that $G_\varepsilon$ is geodesically convex, then $BG_\varepsilon \to BG^\delta \to BG^{\mathrm{top}}$ is a homotopy fibration.*

The next step is the construction of a universal chain homotopy. Given a commutative ring $R$, the algebraic group $GL_n$ is $\mathrm{Spec}(H)$, where $H$ is the Hopf algebra $R[\{x_{ij}\}_{i,j=1}^n][\det(X)^{-1}]$, where $\det(X)$ is the determinant of the universal matrix $X = (x_{ij})$ in $GL_n(H)$. For every commutative $R$-algebra $B$ there is a bijection $\mathrm{Hom}_R(H, B) \to GL_n(B)$ sending $f$ to the matrix $f(X)$; the counit structure map $H \to R$ corresponds to the identity matrix of $GL_n(R)$.

For each positive integer $p$, let $A^p$ denote the henselization of the $p$-fold tensor product $H^{\otimes p}$ along the kernel of the evident structure map $H^{\otimes p} \to R$, so that $(A^p, I^p)$ is a hensel pair, where $I^p$ is the kernel of $A^p \to R$. For $i = 1, \dots, p$, the coordinates $pr_i : H \to H^{\otimes p} \to A^p$ determine matrices $X^i = pr_i(X)$ in $GL_n(A^p)$, and since $X^i$ is congruent to the identity modulo $I^p$ we even have $X^i \in GL_n(I^p)$.

Recall that for any discrete group $G$ and integer $m$, the homology $H_*(G, \mathbb{Z}/m)$ of $G$ is the homology of a standard complex, which we will write as $C_*(G)$, whose degree $p$ piece is $\mathbb{Z}/m[G^p]$; see [WHomo, 6.5]. We write $u^p$ for the $p$-chain $[X^1, ..., X^p]$ in $C_p(GL_n(I^p))$. The differential $d$ sends $u^p$ to

$$[X^2, ..., X^p] + \sum_{i=1}^{p-1} (-1)^i [\dots, X^i X^{i+1}, \dots] + (-1)^{p-1}[X^1, ..., X^{p-1}].$$

Now the $A^p$ fit together to form a cosimplicial $R$-algebra $A^\bullet$, whose cofaces $\partial^j : A^p \to A^{p+1}$ are induced by the comultiplication $\Delta : H \to H \otimes H$. Applying $GL_n$ yields cosimplicial groups $GL_n(A^\bullet)$ and $GL_n(I^\bullet)$. We are interested in the cosimplicial chain complex $C_*(GL_n(I^\bullet))$, which we may regard as a third quadrant double chain complex, with $C_p(GL_n(I^{-q}))$ in the $(p, q)$ spot. Thus $(u^p) = (0, u^1, u^2, \dots)$ is an element of total degree 0 in the associated product total complex, i.e., in $\prod_{p=0}^\infty C_p(GL_n(I^p))$ (see [WHomo, 1.2.6]). By construction, $d(u^p) = \sum (-1)^j \partial^j (u^{p-1})$, so $u^p$ is a cycle in this total complex.

Proposition 3.5. *For each $n$, the image of the cycle $(u^p)$ in $\prod_{p=0}^\infty C_p(GL(I^p))$ is a boundary in the product total complex of $C_*(GL(I^\bullet))$. That is, there are chains $c^p \in C_{p+1}(GL(I^p))$ so that $d(c^p) + \sum (-1)^j \partial^j (c^{p-1}) = u^p$ for all $p \ge 1$.*

Proof. Since the reduced complex $\widetilde{C}_*(G)$ is the subcomplex of $C_*(G)$ obtained by setting $C_0 = 0$, and $u^p \in \widetilde{C}_p$, it suffices to show that $(u^p)$ is a boundary in the total complex $T_*$ of $\widetilde{C}_*(GL(I^\bullet))$. By Gabber Rigidity IV.2.10, the reduced homology $\widetilde{H}_*(GL(I^t),\mathbb{Z}/m)$ is zero for each $t$. Thus the rows $\widetilde{C}_*(GL(I^{-q}))$ of the double complex are exact. By the Acyclic Assembly Lemma [WHomo, 2.7.3], the product total complex $T_*$ is exact, so every cycle is a boundary. □

Lemma 3.6. *For $G = GL_n(\mathbb{R})$, if $\varepsilon$ is small enough then the embedding $BG_\varepsilon \to BG^\delta \to BGL(\mathbb{R})$ induces the zero map on $\widetilde{H}_*(-,\mathbb{Z}/m)$.*

Proof. Let $B^p$ denote the ring of germs of continuous $\mathbb{R}$-valued functions on the topological space $G^p = G\times\cdots\times G$ defined in some neighborhood of $(1,\dots,1)$; $B^p$ is a hensel ring. The coordinate functions give a canonical map $H^{\otimes p} \to B^p$, whose henselization is a map $A^p \to B^p$, and we write $c^p_{\text{ctn}}$ for the image of $c^p$ in $C_{p+1}(GL(B^p))$. Since $GL(B^p)$ is the group of germs of continuous $GL(\mathbb{R})$-valued functions on $G^p$, $u^p$ is the germ of the function $G^p \to GL(\mathbb{R})^p$ sending $(g_1,\dots,g_p)$ to $(g_1,\dots,g_p)$, and $c^p_{\text{ctn}}$ is a $\mathbb{Z}/m$-linear combination of germs of continuous functions $\gamma : G^p \to GL(\mathbb{R})^{p+1}$. That is, we may regard each $c^p_{\text{ctn}}$ as a continuous map of some neighborhood of $(1,\dots,1)$ to $C_{p+1}(GL(\mathbb{R})^{\text{top}})$.

If $N$ is a fixed integer, there is an $\varepsilon>0$ so that the $c^p_{\text{ctn}}$ are defined on $(G_\varepsilon)^p$ for all $p\le N$. Extending $c^p_{\text{ctn}}$ by linearity, we get homomorphisms $s^p : C_p(BG_\varepsilon) \to C_{p+1}(BSL(\mathbb{R})^{\text{top}})$. It is clear from Proposition 3.5 that $s$ is a chain contraction for the canonical embedding $\widetilde{C}_*(BG_\varepsilon) \to \widetilde{C}_*(BGL(\mathbb{R})^{\text{top}})$, defined in degrees at most $N$. This proves the Lemma. □

Proposition 3.7. *For $G = SL_n(\mathbb{R})$, if $\varepsilon$ is small enough then*
*(a) the embedding $BG_\varepsilon \to BG^\delta \to BSL(\mathbb{R})$ induces the zero map on $\widetilde{H}_*(-,\mathbb{Z}/m)$.*
*(b) $\widetilde{H}_i(BG_\varepsilon,\mathbb{Z}/m) = 0$ for all $i \le (n-1)/2$.*
*(c) $H_i(BG^\delta,\mathbb{Z}/m) \to H_i(BG^{\text{top}},\mathbb{Z}/m)$ is an isomorphism for all $i\le (n-1)/2$.*

Proof. Since $H_*(BSL(\mathbb{R})) \to H_*(BGL(\mathbb{R}))$ is a split injection (see Ex. 3.1), part (a) follows from Lemma 3.6. For (b) and (c), let $q$ be the smallest integer such that $\widetilde{H}_q(BG_\varepsilon,\mathbb{Z}/m)$ is nonzero; we will show that $q > (n-1)/2$. Since $BG_\varepsilon$ has only one 0-simplex, $q\ge 1$. Consider the Serre spectral sequence associated to 3.4:

$$E^2_{p,q} = H_p(BG^{\text{top}}, H_q(BG_\varepsilon,\mathbb{Z}/m)) \Rightarrow H_*(BG^\delta,\mathbb{Z}/m).$$

Then $H_i(BG^\delta,\mathbb{Z}/m) \to H_i(BG^{\text{top}},\mathbb{Z}/m)$ is an isomorphism for $i<q$ and the exact sequence of low degree terms for the spectral sequence is

$$H_{q+1}(BG^\delta,\mathbb{Z}/m) \xrightarrow{\text{onto}} H_{q+1}(BG^{\text{top}},\mathbb{Z}/m) \xrightarrow{d^q} H_q(BG_\varepsilon,\mathbb{Z}/m) \to H_q(BG^\delta,\mathbb{Z}/m).$$

Milnor proved in [M-Lie, Thm. 1] that the left-hand map is a split surjection; it follows that the right-hand map is an injection. By Homological Stability IV.1.15, $H_i(BSL_n(\mathbb{R})^\delta) \to H_i(BSL(\mathbb{R})^\delta)$ is an isomorphism for $i\le (n-1)/2$; part (a) implies that $q>(n-1)/2$. This proves parts (b) and (c). □

Corollary 3.8. *$BSL(\mathbb{R})^\delta \to BSL(\mathbb{R})^{\text{top}}$ and $BGL(\mathbb{R})^\delta \to BGL(\mathbb{R})^{\text{top}}$ induce isomorphisms on $\widetilde{H}_*(-,\mathbb{Z}/m)$.*

Proof. Set $G = SL_n(\mathbb{R})$. Passing to the limit as $n\to\infty$ in 3.7 proves the assertion for $SL$. □

Proof of Theorem 3.1. Since $BSL(\mathbb{R})^+$ and $BSL(\mathbb{R})^{\text{top}} \simeq BSO$ are simply connected, Corollary 3.8 implies that $\pi_n(BSL(\mathbb{R})^+;\mathbb{Z}/m) \to \pi_n(BSO;\mathbb{Z}/m)$ is an isomorphism for all $n$. We saw in Chapter IV, Ex. 1.8 that $\pi_n(BSL(\mathbb{R})^+;\mathbb{Z}/m) \to K_n(\mathbb{R};\mathbb{Z}/m)$ is an isomorphism for $n \geq 3$, and an injection for $n = 2$ with cokernel $\mathbb{Z}/2$; the same is true for $\pi_n(BSO;\mathbb{Z}/m) \to \pi_n(BO;\mathbb{Z}/m)$. This proves part (c) for $n \geq 3$; the result for $n = 1, 2$ is classical (see III.1.5.4 and IV.2.5.1, or [Milnor, p. 61]).

Now consider the action of complex conjugation $c$ on $K_*(\mathbb{C})$. The image of $i^* : K_n(\mathbb{R}) \to K_n(\mathbb{C})$ lands in the invariant subgroup $K_n(\mathbb{C})^c$, which by Theorem 1.6 is the direct sum of a uniquely divisible group and a torsion group (which is one of 0, $\mathbb{Z}/2$, 0 or $\mathbb{Q}/\mathbb{Z}$ depending on $n$ modulo 4). The transfer $i_*$ satisfies $i_*i^* = 2$ and $i^*i_* = 1 + c$; see Ex. IV.6.13. Hence (for $n \geq 1$) $K_n(\mathbb{R})$ is the direct sum of its torsion submodule $K_n(\mathbb{R})_{\text{tors}}$ and the uniquely divisible abelian group $K_n(\mathbb{C})^c \otimes \mathbb{Q}$, and the kernel and cokernel of $K_n(\mathbb{R})_{\text{tors}} \to K_n(\mathbb{C})^c_{\text{tors}}$ are elementary abelian 2-groups. By Ex. IV.2.6, we have $K_n(\mathbb{R})_{\text{tors}} \cong K_{n+1}(\mathbb{R};\mathbb{Q}/\mathbb{Z}) \cong \pi_{n+1}(BO,\mathbb{Q}/Z)$. These torsion groups may be read off from Bott Periodicity II.3.1.1. □

EXERCISES

**3.1** For any commutative ring $R$ and ideal $I$, show that $R^\times$ acts trivially on the homology of $SL(R)$ and $GL(R)$, while $(1 + I)^\times$ acts trivially on the homology of $SL(I)$ and $GL(I)$. Conclude that $SL(R) \to GL(R)$ and $SL(I) \to GL(I)$ are split injections on homology.

**3.2** Using Exercise 3.1, show that there is a universal homotopy construction for $SL_n$ parallel to the one in Proposition 3.5 for $GL_n$. Use this to prove Proposition 3.7 directly, modifying the proof of Lemma 3.6.

**3.3** Check that 3.6, 3.7 and 3.8 go through with $\mathbb{R}$ replaced by $\mathbb{C}$. Using these, prove the analogue of Theorem 3.1 for $\mathbb{C}$ and compare it to Theorem 1.5.

**3.4** If $F$ is any formally real field, such as $\mathbb{R} \cap \bar{\mathbb{Q}}$, show that $K_*(F;\mathbb{Z}/m) \cong K_*(\mathbb{R};\mathbb{Z}/m)$ for all $m$.

## §4. Relation to motivic cohomology

Motivic cohomology theory, developed by Voevodsky, is intimately related to algebraic $K$-theory. For every abelian group $A$ and every $(n, i)$, the motivic cohomology of a smooth scheme $X$ over a field consists of groups $H^n(X, A(i))$, defined as the hypercohomology of a certain chain complex $A(i)$ of Nisnevich sheaves. An introduction to motivic cohomology is beyond the scope of this book, and we refer the reader to [MVW] for the definitions and properties of motivic cohomology.

When $A = \mathbb{Q}$, we have isomorphisms $H^n(X, \mathbb{Q}(i)) \cong K^{(i)}_{2i-n}(X)$, where the right side refers to the eigenspace of $K_{2i-n}(X) \otimes \mathbb{Q}$ on which the Adams operations $\psi^k$ act as multiplication by $k^i$, described in IV, Theorem 5.11. This fact is due to Bloch [Bl86], and follows from 4.1 and 4.9 below; see Ex. 4.5.

Here is the fundamental structure theorem for motivic cohomology with finite coefficients, due to Rost and Voevodsky. Since the proof is scattered over 15-20 research papers, we refer the reader to the book [HW] for the proof.

For any smooth $X$, there is a natural map $H^n(X, \mathbb{Z}/m(i)) \to H^n_{\text{et}}(X, \mu_m^{\otimes i})$ from motivic to étale cohomology. It arises from the forgetful functor $a_*$ from étale sheaves to Nisnevich sheaves on $X$, via the isomorphism $H^n_{\mathrm{et}}(X,\mu_m^{\otimes i})\cong H^n_{nis}(X,Ra_*\mu_m^{\otimes i})$ and a natural map $\mathbb{Z}/m(i)\to Ra_*\mu_m^{\otimes i}$.

Norm Residue Theorem 4.1 (Rost-Voevodsky). *If $k$ is a field containing $1/m$, the natural map induces isomorphisms*

$$H^n(k,\mathbb{Z}/m(i))\cong\begin{cases}H^n_{\mathrm{et}}(k,\mu_m^{\otimes i}) & n\le i\\ 0 & n>i.\end{cases}$$

*If $X$ is a smooth scheme over $k$, the natural map $H^n(X,\mathbb{Z}/m(i))\to H^n_{\mathrm{et}}(X,\mu_m^{\otimes i})$ is an isomorphism for $n\le i$. For $n>i$, the map identifies $H^n(X,\mathbb{Z}/m(i))$ with the Zariski hypercohomology on $X$ of the truncated direct image complex $\tau^{\le i}Ra_*(\mu_m^{\otimes i})$.*

A result of Totaro, and Nesterenko-Suslin [NS, Tot] states that the $i^{th}$ Galois symbol (III.7.11) factors through an isomorphism $K_i^M(k)\xrightarrow{\simeq}H^i(k,\mathbb{Z}(i))$, compatibly with multiplication. This means that the Milnor $K$-theory ring $K_*^M(k)$ (III.7.1) is isomorphic to the ring $\oplus H^i(k,\mathbb{Z}(i))$. Since $H^i(k,\mathbb{Z}(i))/m\cong H^i(k,\mathbb{Z}/m(i))$, we deduce the following special case of Theorem 4.1. This special case was once called the Bloch-Kato conjecture, and is in fact equivalent to Theorem 4.1; see [SV00], [GL01] or [HW] for a proof of this equivalence.

Corollary 4.1.1. *If $k$ is a field containing $1/m$, the Galois symbols are isomorphisms for all $n$: $K_i^M(k)/m\xrightarrow{\simeq}H^i_{\mathrm{et}}(k,\mu_m^{\otimes i})$. They induce a ring isomorphism:*

$$\oplus K_i^M(k)/m\xrightarrow{\simeq}\oplus H^i(k,\mathbb{Z}(i))/m\cong\oplus H^i(k,\mathbb{Z}/m(i))\cong\oplus H^i_{\mathrm{et}}(k,\mu_m^{\otimes i}).$$

The Merkurjev-Suslin isomorphism $K_2(k)/m\cong H^2_{\mathrm{et}}(k,\mu_m^{\otimes 2})$ of [MS], mentioned in III.6.10.4, is the case $i=2$ of 4.1.1. The isomorphism $K_3^M(k)/m\cong H^3_{\mathrm{et}}(k,\mu_m^{\otimes 3})$ for $m=2^\nu$ was established by Rost and Merkurjev-Suslin; see [MS2].

The key technical tool which allows us to use Theorem 4.1 in order to make calculations is the *motivic-to-$K$-theory spectral sequence*, so-named because it goes from motivic cohomology to algebraic $K$-theory. The construction of this spectral sequence is given in the references cited in the Historical Remark 4.4; the final assertion in 4.2 is immediate from the first assertion and Theorem 4.1.

Theorem 4.2. *For any coefficient group $A$, and any smooth scheme $X$ over a field $k$, there is a spectral sequence, natural in $X$ and $A$:*

$$E_2^{p,q}=H^{p-q}(X,A(-q))\Rightarrow K_{-p-q}(X;A).$$

*If $X=\mathrm{Spec}(k)$ and $A=\mathbb{Z}/m$, where $1/m\in k$, then the $E_2$ terms are just the étale cohomology groups of $k$, truncated to lie in the octant $q\le p\le 0$.*

Addendum 4.2.1. If $A=\mathbb{Z}/m$ and $m\not\equiv 2 \pmod 4$, the spectral sequence has a multiplicative structure which is the product in motivic cohomology on the $E_2$ page and the $K$-theory product (IV.8) on the abutment.

For any coefficients $\mathbb{Z}/m$, there is a pairing between the spectral sequence 4.2 for $\mathbb{Z}$ coefficients and the spectral sequence 4.2 with coefficients $\mathbb{Z}/m$. On the $E_2$ page, it is the product in motivic cohomology and on the abutment it is the pairing $K_*(X)\otimes K_*(X;\mathbb{Z}/m)\to K_*(X;\mathbb{Z}/m)$ of IV, Ex. 2.5.

Remark 4.2.2. The spectral sequence (4.2) has an analogue for non-smooth schemes over $k$, in which the motivic cohomology groups are replaced by higher Chow groups $CH^i(X,n)$. It is established in [FS, 13.12] and [L01, 8.9]. For any equidimensional quasi-projective scheme $X$, there is a convergent spectral sequence

$$E_2^{p,q} = CH^{-q}(X,-p-q) \Rightarrow G_{-p-q}(X).$$

If $X$ is smooth and $k$ is perfect, then $H^n(X,\mathbb{Z}(i)) \cong CH^i(X,2i-n)$; see [MVW, 19.1]. This identifies the present spectral sequence with (4.2). Since $CH^i(X,n)$ is the same as the Borel-Moore homology group $H^{BM}_{2i+n}(X,\mathbb{Z}(i))$, this spectral sequence is sometimes cited as a homology spectral sequence with $E^2_{p,q} = H^{BM}_{p-q}(X,\mathbb{Z}(-q))$.

Edge map 4.3. Let $k$ be a field. The edge map $K_{2i}(k;\mathbb{Z}/m) \to H^0_{\mathrm{et}}(k,\mu_m^{\otimes i})$ in (4.2) is the $e$-invariant of 2.1, and is an isomorphism for the algebraic closure $\bar{k}$ of $k$; the details are given in Example 4.5(ii) below.

We now consider the other edge map, from $E_2^{p,-n} = H^n(k,\mathbb{Z}(n)) \cong K^M_n(k)$ to $K_n(k)$. Since the ring $K^M_*(k)$ is generated by its degree 1 terms, and the low degree terms of (4.2) yield isomorphisms $H^1(k,\mathbb{Z}(1)) \cong K_1(k)$, and $H^1(k,\mathbb{Z}/m(1)) \cong K_1(k)/m$, the multiplicative structure described in 4.2.1 implies that the edge maps in the spectral sequence are canonically identified with the maps $K^M_*(k) \to K_*(k)$ and $K^M_*(k)/m \to K_*(k;\mathbb{Z}/m)$ described in IV.1.10.1. (This was first observed in [GeilL, 3.3] and later in [FS, 15.5].)

By V.11.13, the kernel of the edge map $K^M_n(k) \to K_n(k)$ is a torsion group of exponent $(n-1)!$. This is not best possible; we will see in 4.3.2 that the edge map $K^M_3(k) \to K_3(k)$ is an injection.

Since $\{-1,-1,-1,-1\}$ is nonzero in $K^M_4(\mathbb{Q})$ and $K^M_4(\mathbb{R})$ but zero in $K_4(\mathbb{Q})$ (by Ex. IV.1.12), the edge map $K^M_4(k) \to K_4(k)$ is not an injection for subfields of $\mathbb{R}$. This means that the differential $d_2 : H^1(\mathbb{Q},\mathbb{Z}(3)) \to K^M_4(\mathbb{Q})$ is nonzero.

Similarly, using the étale Chern class $c_{3,3}\colon K_3(k,\mathbb{Z}/m) \to H^n_{\mathrm{et}}(k,\mu_m^{\otimes n})$ of V.11.10, we see that the kernel of the edge map $K^M_n(k)/m \to K_n(k;\mathbb{Z}/m)$ has exponent $(n-1)!$. (The composition of $c_{3,3}$ with the isomorphism $H^n_{\mathrm{et}}(k,\mu_m^{\otimes n}) \cong K^M_n(k)/m$ of Corollary 4.1.1 satisfies $c_{3,3}(x) = -2x$ for all $x \in K^M_3(k)/m$.) Since $K^M_3(\mathbb{Q}) \cong \mathbb{Z}/2$ on $\{-1,-1,-1\}$ (Remark 2.1.3), and this element dies in $K_3(\mathbb{Q})/8 \cong \mathbb{Z}/8$ and hence $K_3(\mathbb{Q};\mathbb{Z}/8)$, the edge map $K^M_3(\mathbb{Q})/8 \to K_3(\mathbb{Q};\mathbb{Z}/8)$ is not an injection.

Low degree terms 4.3.1. When $k$ is a field, the edge map $K^M_2(k) \to K_2(k)$ is an isomorphism by Matsumoto's Theorem III.6.1, so the low degree sequence $0 \to K^M_2(k)/m \to K_2(k;\mathbb{Z}/m) \to \mu_m(k) \to 0$ of 4.2 may be identified with the Universal Coefficient sequence IV.2.5. This yields the Merkurjev-Suslin formula $K_2(k)/m \cong H^2_{\mathrm{et}}(k,\mu_m^{\otimes 2})$ of III(6.10.4). Since $H^n(X,\mathbb{Z}(0)) = 0$ for $n < 0$ and $H^n(X,\mathbb{Z}(1)) = 0$ for $n \le 0$, by [MVW, 4.2], we also obtain the exact sequences

$$K_4(k) \to H^0(k,\mathbb{Z}(2)) \xrightarrow{d_2} K^M_3(k) \to K_3(k) \to H^1(k,\mathbb{Z}(2)) \to 0,$$

$$K_4(k;\mathbb{Z}/m) \to H^0_{\mathrm{et}}(k,\mu_m^{\otimes 2}) \xrightarrow{d_2} K^M_3(k)/m \to K_3(k;\mathbb{Z}/m) \to H^1_{\mathrm{et}}(k,\mu_m^{\otimes 2}) \to 0.$$

Since the kernel of $K^M_3(k)/m \to K_3(k;\mathbb{Z}/m)$ is nonzero for $k = \mathbb{Q}$, the differential $d_2$ can be nontrivial with finite coefficients. The integral $d_2$ is always zero:

PROPOSITION 4.3.2. *The map $K_3^M(k) \to K_3(k)$ is an injection for every field $k$.*

PROOF. We have seen that the kernel of $K_3^M(k) \to K_3(k)$ has exponent 2. If $\mathrm{char}(k) = 2$, then $K_3^M(k)$ has no 2-torsion (Izhboldin's Theorem III.7.8) and the result holds, so we may suppose that $\mathrm{char}(k) \neq 2$. Consider the motivic group $H(k) = H^0(k, \mathbb{Z}(2))/2$. Since the differential $d_2 : H^0(k, \mathbb{Z}(2)) \to K_3^M(k)$ in 4.3.1 factors through $H(k)$, it suffices to show that $H(k) = 0$. By universal coefficients, $H(k)$ is a subgroup of $H^0(k, \mathbb{Z}/2(2)) \cong H^0_{\mathrm{et}}(k, \mathbb{Z}/2) = \mathbb{Z}/2$; by naturality this implies that $H(k) \subseteq H(k') \subset \mathbb{Z}/2$ for any field extension $k'$ of $k$. Thus we may suppose that $k$ is algebraically closed. In this case, $K_4(k)$ is divisible (by 1.6) and $K_3^M(k)$ is uniquely divisible (by III.7.2), so it follows from 4.3.1 that $H^0(k, \mathbb{Z}(2))$ is divisible and hence $H(k) = 0$. $\square$

HISTORICAL REMARK 4.4. This spectral sequence 4.2 has an awkward history. In 1972, Lichtenbaum [Li2] made several conjectures relating the $K$-theory of integers in number fields to étale cohomology and (via this) to values of Zeta functions at negative integers (see 6.10 below). Expanding on these conjectures, Quillen speculated that there should be a spectral sequence like (4.2) (with finite coefficients) at the 1974 Vancouver ICM, and Beilinson suggested in 1982 that one might exist with coefficients $\mathbb{Z}$.

The existence of such a spectral sequence was claimed by Bloch and Lichtenbaum in their 1994 preprint [BL], which was heavily cited for a decade, but there is a gap in their proof. Friedlander and Suslin showed in [FS] that one could start with the construction of [BL] to get a spectral sequence for all smooth schemes, together with the multiplicative structure of 4.2.1. The spectral sequence in [BL] was also used to construct the Borel-Moore spectral sequence in 4.2.3 for quasi-projective $X$ in [FS, 13.12] and [L01, 8.9]

Also in the early 1990s, Grayson constructed a spectral sequence in [Gr95], following suggestions of Goodwillie and Lichtenbaum. Although it converged to the $K$-theory of regular rings, it was not clear what the $E_2$ terms were until 2001, when Suslin showed (in [Su03]) that the $E_2$ terms in Grayson's spectral sequence agreed with motivic cohomology for fields. Using the machinery of [FS], Suslin then constructed the spectral sequence of Theorem 4.2 for all smooth varieties over a field, and also established the multiplicative structure of 4.2.1.

In 2000–1, Voevodsky observed (in [VV02a, VV02b]) that the slice filtration for the motivic spectrum representing $K$-theory (of smooth varieties) gave rise to a spectral sequence, and showed that it had the form given in Theorem 4.2 modulo two conjectures about motivic homotopy theory (since verified). Yet a third construction was given by Levine in [Le01] [Le08]; a proof that these three spectral sequences agree is also given in [Le08].

REMARK 4.4.1. A similar motivic spectral sequence was established by Levine in [Le01, (8.8)] over a Dedekind domain, in which the group $H^n_M(X, A(i))$ is defined to be the $(2i - n)$th hypercohomology on $X$ of the complex of higher Chow group sheaves $z^i \otimes A$.

We now give several examples in which the motivic-to-$K$-theory spectral sequence degenerates at the $E_2$ page, quickly yielding the $K$-groups.

EXAMPLES 4.5. (i) When $k$ is a separably closed field, $H^n_{\text{et}}(k,-)=0$ for $n>0$ and the spectral sequence degenerates along the line $p=q$ to yield $K_{2i}(k;\mathbb{Z}/m)\cong\mathbb{Z}/m$, $K_{2i-1}(k;\mathbb{Z}/m)=0$. This recovers the calculations of 1.3.1 and 1.4 above. In particular, the Bott element $\beta\in K_2(k,\mathbb{Z}/m)$ (for a fixed choice of $\zeta$) corresponds to the canonical element $\zeta$ in $E_2^{-1,-1}=H^0_{\text{et}}(k,\mu_m)$.

(ii) If $k$ is any field containing $1/m$, and $G=\text{Gal}(\bar{k}/k)$, then $H^0_{\text{et}}(k,\mu_m^{\otimes i})$ is the subgroup of $\mu_m^{\otimes i}$ invariant under $G$; by Definition 2.1 it is isomorphic to $\mathbb{Z}/(m,w_i(k))$. By naturality in $k$ and (i), the edge map of 4.2 (followed by the inclusion) is the composition $K_{2i}(k;\mathbb{Z}/m)\to K_{2i}(\bar{k};\mathbb{Z}/m)\to\mu_m^{\otimes i}$. Therefore the edge map vanishes on $K_{2i}(k)/m$ and (by the Universal Coefficient Sequence of IV.2.5) induces the $e$-invariant ${}_mK_{2i-1}(k)\to\text{Hom}(\mathbb{Z}/m,\mathbb{Z}/w_i(k))=\mathbb{Z}/(m,w_i(k))$ of 2.1.

(iii) For a finite field $\mathbb{F}_q$ with $m$ prime to $q$, we have $H^n_{\text{et}}(\mathbb{F}_q,-)=0$ for $n>1$ [Shatz, p. 69]. There is also a duality isomorphism $H^1_{\text{et}}(\mathbb{F}_q,\mu_m^{\otimes i})\cong H^0_{\text{et}}(\mathbb{F}_q,\mu_m^{\otimes i})$. Thus each diagonal $p+q=-n$ in the spectral sequence 4.2 has only one nonzero entry, so $K_{2i}(\mathbb{F}_q;\mathbb{Z}/m)$ and $K_{2i-1}(\mathbb{F}_q;\mathbb{Z}/m)$ are both isomorphic to $\mathbb{Z}/(m,w_i(k))$. This recovers the computation for finite fields given in 2.1.1 and IV.1.13.1.

(iv) Let $F$ be the function field of a curve over a separably closed field containing $1/m$. Then $H^n_{\text{et}}(F,-)=0$ for $n>1$ (see [Shatz, p. 119]) and $H^0_{\text{et}}(F,\mu_m^{\otimes i})\cong\mathbb{Z}/m$ as in (i). By Kummer theory,

$$H^1_{\text{et}}(F,\mu_m^{\otimes i})\cong H^1_{\text{et}}(F,\mu_m)\otimes\mu_m^{\otimes i-1}\cong F^\times/F^{\times m}\otimes\mu_m^{\otimes i-1}.$$

(The twist by $\mu_m^{\otimes i-1}$ is to keep track of the action of the Galois group.) As in (iii), the spectral sequence degenerates to yield $K_{2i}(F;\mathbb{Z}/m)\cong\mathbb{Z}/m$, $K_{2i-1}(F;\mathbb{Z}/m)\cong F^\times/F^{\times m}$. Since the spectral sequence is multiplicative, it follows that the map $F^\times/F^{\times m}\to K_{2i+1}(F;\mathbb{Z}/m)$ sending $u$ to $\{\beta^i,u\}$ is an isomorphism because it corresponds to the isomorphism $E_2^{0,-1}\to E_2^{-i,-i-1}$ obtained by multiplication by the element $\zeta^{\otimes i}$ of $E_2^{-i,-i}$. Thus

$$K_n(F;\mathbb{Z}/m)\cong\begin{cases}\mathbb{Z}/m \text{ on } \beta^i, & n=2i,\\ F^\times/F^{\times m} \text{ on } \{\beta^i,u\} & n=2i+1.\end{cases}$$

When $X$ has dimension $d>0$, the spectral sequence (4.2) extends to the fourth quadrant, with terms only in columns $\le d$. This is because $H^n(X,A(i))=0$ for $n>i+d$; see [MVW, 3.6]. To illustrate this, we consider the case $d=1$, *i.e.,* when $X$ is a curve.

EXAMPLE 4.6. Let $X$ be a smooth projective curve over a field $k$ containing $1/m$, with function field $F$. By Theorem 4.1, $E_2^{p,q}=H^{p-q}(X,\mathbb{Z}/m(-q))$ is $H^{p-q}_{\text{et}}(X,\mu_m^{\otimes q})$ for $p\le 0$, and $E_2^{p,q}=0$ for $p\ge 2$ by the above remarks. That is, the $E_2$-terms in the third quadrant of (4.2) are étale cohomology groups, but there are also modified terms in the column $p=+1$. To determine these, we note that a comparison of the localization sequences for $\text{Spec}(F)\to X$ in motivic cohomology [MVW, 14.5] and étale cohomology yields an exact sequence

$$0\to H^{i+1}(X,\mathbb{Z}/m(i))\to H^{i+1}_{\text{et}}(X,\mu_m^{\otimes i})\to H^{i+1}_{\text{et}}(F,\mu_m^{\otimes i}).$$

In particular, $E_2^{1,0} = 0$ and $E_2^{1,-1} = E_\infty^{1,-1} = \mathrm{Pic}(X)/m$. In this case, we can identify the group $K_0(X;\mathbb{Z}/m) = \mathbb{Z}/m \oplus \mathrm{Pic}(X)/m$ (see II.8.2.1) with the abutment of (4.2) in total degree 0.

Now suppose that $k$ is separably closed and $m = \ell^\nu$. Then $X$ has ($\ell$-primary) étale cohomological dimension 2, and it is well known that $H^1_{\mathrm{et}}(X, \mu_m) \cong {}_m\mathrm{Pic}(X)$ and $H^2_{\mathrm{et}}(X, \mu_m) \cong \mathbb{Z}/m$; see [Milne, pp. 126, 175]. Thus the spectral sequence has only three diagonals ($p - q = 0, 1, 2$) with terms $\mathbb{Z}/m$, ${}_m\mathrm{Pic}(X) \cong (\mathbb{Z}/m)^{2g}$ and $\mathrm{Pic}(X)/m \cong \mathbb{Z}/m$ (see I.5.15); the only nonzero term in the column $p = +1$ is $\mathrm{Pic}(X)/m \cong \mathbb{Z}/m$. By 4.5(ii), there is simply no room for any differentials, so the spectral sequence degenerates at $E_2$. Since the $e$-invariant maps $K_{2i}(k;\mathbb{Z}/m)$ isomorphically onto $\mu_m^{\otimes i} \cong E_\infty^{-i,-i}$, the extensions split and we obtain

PROPOSITION 4.6.1. *Let $X$ be a smooth projective curve over a separably closed field containing $1/m$. Then*

$$K_n(X;\mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m \oplus \mathbb{Z}/m, & n = 2i, \quad n \geq 0, \\ {}_m\mathrm{Pic}(X) \cong (\mathbb{Z}/m)^{2g}, & n = 2i-1, n > 0. \end{cases}$$

The multiplicative structure of $K_*(X;\mathbb{Z}/m)$ is given in Exercise 4.3. When $k = \bar{\mathbb{F}}_p$, the structure of $K_*(X)$ is given in Theorem 6.4 below.

Geisser and Levine proved in [GeiL] that if $k$ is a field of characteristic $p > 0$ then the motivic cohomology groups $H^{n,i}(X, \mathbb{Z}/p^\nu)$ vanish for all $i \neq n$. This allows us to clarify the relationship between $K_*^M(k)$ and $K_*(k)$ at the prime $p$. Part (b) should be compared with Izhboldin's Theorem III.7.8 that $K_n^M(k)$ has no $p$-torsion.

THEOREM 4.7. *Let $k$ be a field of characteristic $p$. Then for all $n \geq 0$,*
*(a) for all $\nu > 0$, the map $K_n^M(k)/p^\nu \to K_n(k;\mathbb{Z}/p^\nu)$ is an isomorphism;*
*(b) $K_n(k)$ has no $p$-torsion;*
*(c) the kernel and cokernel of $K_n^M(k) \to K_n(k)$ are uniquely $p$-divisible groups.*

PROOF. The Geisser-Levine result implies that the spectral sequence (4.2) with coefficients $\mathbb{Z}/p^\nu$ collapses at $E_2$, with all terms zero except for $E_2^{0,q} = K_{-q}^M(k)/p^\nu$. Hence the edge maps of 4.2.2 are isomorphisms. This yields (a). Since the surjection $K_n^M(k) \to K_n^M(k)/p \cong K_n(k;\mathbb{Z}/p)$ factors through $K_n(k)/p$, the Universal Coefficient sequence of IV.2.5 implies (b), that $K_{n-1}(k)$ has no $p$-torsion. Finally (c) follows from the 5-lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n^M(k) & \xrightarrow{p} & K_n^M(k) & \longrightarrow & K_n^M(k)/p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\cong} & & \\ 0 & \longrightarrow & K_n(k) & \xrightarrow{p} & K_n(k) & \longrightarrow & K_n(k;\mathbb{Z}/p) & \longrightarrow & 0. \end{array} \qquad \square$$

4.8 PERIODICITY FOR $\ell > 2$. Let $\beta$ denote the Bott element in $K_2(\mathbb{Z}[\zeta_\ell];\mathbb{Z}/\ell)$ corresponding to the primitive $\ell^{th}$ root of unity $\zeta_\ell$, and let $b \in K_{2(\ell-1)}(\mathbb{Z};\mathbb{Z}/\ell)$ denote the image of $-\beta^{\ell-1}$ under the transfer map $i_*$. Since $i^*(b) = -(\ell-1)\beta^{\ell-1} = \beta^{\ell-1}$, the $e$-invariant of $b$ is the canonical generator $\zeta^{\otimes \ell-1}$ of $H^0(\mathbb{Z}[1/\ell], \mu_\ell^{\ell-1})$ by naturality. If $X$ is any smooth variety over a field containing $1/\ell$, multiplication by $b$ gives a map $K_n(X;\mathbb{Z}/\ell) \to K_{n+2(\ell-1)}(X;\mathbb{Z}/\ell)$; we refer to this as a *periodicity map.* Indeed, the multiplicative pairing in Addendum 4.2.1, of $b$ with the spectral sequence converging to $K_*(X;\mathbb{Z}/\ell)$, gives a morphism of spectral sequences $E_r^{p,q} \to E_r^{p+1-\ell,q+1-\ell}$ from (4.2) to a shift of itself. On the $E_2$ page, these maps are isomorphisms for $p \le 0$, induced by $\mu_\ell^{\otimes i} \cong \mu_\ell^{\otimes i+\ell-1}$.

The term 'periodicity map' comes from the fact that the periodicity map is an isomorphism $K_n(X;\mathbb{Z}/\ell) \xrightarrow{\simeq} K_{n+2(\ell-1)}(X;\mathbb{Z}/\ell)$ for all $n > \dim(X) + \mathrm{cd}_\ell(X)$, $\mathrm{cd}_\ell(X)$ being the étale cohomological dimension of $X$ for $\ell$-primary sheaves. This follows from the comparison theorem for the morphism $\cup b$ of the spectral sequence (4.2) to itself.

4.8.1 Periodicity for $\ell = 2$. Pick a generator $v_1^4$ of $\pi^s(S^8;\mathbb{Z}/16) \cong \mathbb{Z}/16$; it defines a generator of $K_8(\mathbb{Z}[1/2];\mathbb{Z}/16)$ and, by the edge map in (4.2), a canonical element of $H^0_{\mathrm{et}}(\mathbb{Z}[1/2];\mu_{16}^{\otimes 4})$ which we shall also call $v_1^4$. If $X$ is any scheme, smooth over $\mathbb{Z}[1/2]$, the multiplicative pairing of $v_1^4$ (see Addendum 4.2.1) with the spectral sequence converging to $K_*(X;\mathbb{Z}/2)$ gives a morphism of spectral sequences $E_r^{p,q} \to E_r^{p-4,q-4}$ from (4.2) to itself. For $p \le 0$ these maps are isomorphisms, induced by $E_2^{p,q} \cong H^{p-q}_{\mathrm{et}}(X,\mathbb{Z}/2)$; we shall refer to these isomorphisms as *periodicity isomorphisms.*

Adams Operations 4.9. The Adams operations $\psi^k$ act on the spectral sequence (4.2), commuting with the differentials and converging to the action of $\psi^k$ on $K_*(k)$ and $K_*(k;\mathbb{Z}/m)$ (IV.5), with $\psi^k = k^i$ on the row $q = -i$. This was proven by Soule; see [GiS, 7.1]. Since $(k^i - k^{i+r-1})\, d_r(x) = d_r(\psi^k x) - \psi^k(d_r x) = 0$ for all $x$ in row $-i$, we see that the image of the differentials $d_r$ are groups of bounded exponent. That is, the spectral sequence (4.2) degenerates modulo bounded torsion.

EXERCISES

**4.1** If $cd_\ell(k) = d$ and $\mu_{\ell^\nu} \subset k$, show that $\cup\beta : K_n(k;\mathbb{Z}/\ell^\nu) \to K_{n+2}(k;\mathbb{Z}/\ell^\nu)$ is an isomorphism for all $n \ge d$. This is a strong form of periodicity.

**4.2** (Browder) Show that the periodicity maps $K_n(\mathbb{F}_q;\mathbb{Z}/\ell) \to K_{n+2(\ell-1)}(\mathbb{F}_q;\mathbb{Z}/\ell)$ of 4.8 are isomorphisms for finite fields $\mathbb{F}_q$ (with $\ell \nmid q$) for all $n \ge 0$.

**4.3** Let $X$ be a smooth projective curve over an algebraically closed field $k$, and $[x] \in \mathrm{Pic}(X)$ the class of a closed point $x$. By I.5.15, $\mathrm{Pic}(X)/m \cong \mathbb{Z}/m$ on $[x]$, and by II.8.2.1 we have $K_0(X)/m \cong \mathbb{Z}/m \oplus \mathbb{Z}/m$ with basis $\{1,[x]\}$. In this exercise, we clarify (4.2.1), assuming $1/m \in k$.

(i) Show that multiplication by the Bott element $\beta^i$ induces an isomorphism $K_0(X)/m \xrightarrow{\simeq} K_{2i}(X;\mathbb{Z}/m)$.

(ii) Show that $K_1(X)$ is divisible, so that the map $K_1(X;\mathbb{Z}/m) \to {}_m\mathrm{Pic}(X)$ in the Universal Coefficient sequence is an isomorphism.

(iii) Show that multiplication by the Bott element $\beta^i$ induces an isomorphism $K_1(X;\mathbb{Z}/m) \xrightarrow{\simeq} K_{2i+1}(X;\mathbb{Z}/m)$.

(iv) Conclude that the ring $K_*(X;\mathbb{Z}/m)$ is $\mathbb{Z}/m[\beta] \otimes \mathbb{Z}/m[M]$, where $M = \mathrm{Pic}(X)/m \oplus {}_m\mathrm{Pic}(X)$ is a graded ideal of square zero.

**4.4** Use the formula $H^n(\mathbb{P}^1_k, A(i)) \cong H^n(k, A(i)) \oplus H^{n-2}(k, A(i-1))$ (see [MVW, 15.12]) to show that the spectral sequence (4.2) for $\mathbb{P}^1_k$ is the direct sum of two copies of the spectral sequence for $k$, on generators $1 \in E_2^{0,0}$ and $[L] \in E_2^{1,-1}$. Using this, re-derive the calculation of V.6.14 that $K_n(\mathbb{P}^1_k) \cong K_n(k) \otimes K_0(\mathbb{P}^1)$.

**4.5** Use (4.2) and 4.9 to recover Bloch's isomorphism $H^n(X, \mathbb{Q}(i)) \cong K_{2i-n}^{(i)}(X)$.

**4.6** The *Vanishing Conjecture* in $K$-theory states that $K_n^{(i)}(X)$ vanishes whenever $i \leq n/2$, $n > 0$. (See [Sou85, p. 501].) Using the Universal Coefficient sequence

$$0 \to H^j(X, \mathbb{Z}(i))/m \to H^j(X, \mathbb{Z}/m(i)) \to {}_mH^{j+1}(X, \mathbb{Z}(i)) \to 0,$$

(a) show that $H^j(X, \mathbb{Z}(i))$ is uniquely divisible for $j \leq 0$, and (b) conclude that the Vanishing Conjecture is equivalent to the assertion that $H^j(X, \mathbb{Z}(i))$ vanishes for all $j \leq 0$ ($i \neq 0$). This and Exercise 4.5 show that the Vanishing Conjecture holds for any field $k$ whose groups $K_n(k)$ are finitely generated, such as number fields.

## §5. $K_3$ of a field

In this section, we study the group $K_3(F)$ of a field $F$. By Proposition 4.3.2, $K_3^M(F)$ injects into $K_3(F)$. By IV.1.20, the map $K_3(F) \to H_3(SL(F))$ is onto, and its kernel is the subgroup of $K_3^M(F)$ generated by the symbols $\{-1, a, b\}$. Assuming that $K_3^M(F)$ is known, we may use homological techniques. The focus of this section will be to relate the group $K_3^{\text{ind}}(F) := K_3(F)/K_3^M(F)$ to Bloch's group $B(F)$ of a field $F$, which we now define.

For any abelian group $A$, let $\tilde{\wedge}^2 A$ denote the quotient of the group $A \otimes A$ by the subgroup generated by all $a \otimes b + b \otimes a$. The exterior product $\wedge^2 A$ is the quotient of $\tilde{\wedge}^2 A$ by the subgroup (isomorphic to $A/2A$) of all symbols $x \wedge x$.

Definition 5.1. For any field $F$, let $\mathcal{P}(F)$ denote the abelian group presented with generators symbols $[x]$ for $x \in F - \{0\}$, with relations $[1] = 0$ and

$$[x] - [y] + [y/x] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] = 0, \quad x \neq y \text{ in } F - \{0, 1\}.$$

There is a canonical map $\mathcal{P}(F) \to \tilde{\wedge}^2 F^\times$ sending $[1]$ to 0 and $[x]$ to $x \wedge (1-x)$ for $x \neq 1$, and *Bloch's group* $B(F)$ is defined to be its kernel. Thus we have an exact sequence

$$0 \to B(F) \to \mathcal{P}(F) \to \tilde{\wedge}^2 F^\times \to K_2(F) \to 0.$$

Remark 5.1.1. Since the cases $B(\mathbb{F}_2) = 0$ and $B(\mathbb{F}_3) = \mathbb{Z}$ are pathological, we will tacitly assume that $|F| \geq 4$ in this section. Theorem 5.2 below implies that if $q > 3$ is odd then $B(\mathbb{F}_q)$ is cyclic of order $(q+1)/2$, while if $q > 3$ is even then $B(\mathbb{F}_q)$ is cyclic of order $q+1$. This is easy to check for small values of $q$.

Remark 5.1.2. The group $\mathcal{P}(F)$ is closely related to the *scissors congruence* group for polyhedra in hyperbolic 3-space $\mathcal{H}^3$ with vertices in $\mathcal{H}^3$ or $\partial\mathcal{H}^3$, and has its origins in Hilbert's Third Problem. It was first studied for $\mathbb{C}$ by Wigner, Bloch and Thurston and later by Dupont and Sah; see [DSah, 4.10].

For any finite cyclic abelian group $A$ of even order $m$, there is a unique nontrivial extension $\tilde{A}$ of $A$ by $\mathbb{Z}/2$. If $A$ is cyclic of odd order, we set $\tilde{A} = A$. Since the group $\mu(F)$ of roots of unity is a union of finite cyclic groups, we may define $\tilde{\mu}(F)$ as the union of the $\tilde{\mu}_n(F)$. Here is the main result of this section.

Theorem 5.2. *(Suslin) For any infinite field $F$, there is an exact sequence*

$$0 \to \tilde{\mu}(F) \to K_3^{\text{ind}}(F) \to B(F) \to 0.$$

The proof is taken from [Su91, 5.2], and will be given at the end of this section. To prepare for the proof, we introduce the element $c$ in Lemma 5.4 and construct a map $\psi : H_3(GL(F), \mathbb{Z}) \to B(F)$ in Theorem 5.7. In Theorem 5.16 we connect $\psi$ to the group $M$ of monomials matrices, and the group $\tilde{\mu}(F)$ appears in 5.20 as part of the calculation of $\pi_3(BM^+)$. The proof of Theorem 5.2 is obtained by collating all this information.

Remark 5.2.1. In fact, $B(\mathbb{Q}) \cong \mathbb{Z}/6$. This follows from Theorem 5.2 and the calculations that $K_3(\mathbb{Q}) \cong \mathbb{Z}/48$ (2.1.2), $K_3^M(\mathbb{Q}) \cong \mathbb{Z}/2$ (III.7.2.d) and $\tilde{\mu}(\mathbb{Q}) \cong \mathbb{Z}/4$. In fact, the element $c = [2]+[-1]$ has order exactly 6 in both $B(\mathbb{Q})$ and $B(\mathbb{R})$. This may be proven using the Rogers $L$-function, which is built from the dilogarithm function. See [Su91, pp. 219–220].

As an application, we compute $K_3$ of a number field $F$. Let $r_1$ and $r_2$ denote the number of real and complex embeddings, *i.e.,* the number of factors of $\mathbb{R}$ and $\mathbb{C}$ in the $\mathbb{R}$-algebra $F \otimes_{\mathbb{Q}} \mathbb{R}$. Then $K_3^M(F) \cong (\mathbb{Z}/2)^{r_1}$ by III.7.2(d), and $K_3(F)$ is finitely generated by IV.6.9 and V.6.8. By Borel's Theorem IV.1.18, $K_3(F)$ is the sum of $\mathbb{Z}^{r_2}$ and a finite group. We can make this precise.

Corollary 5.3. *Let $F$ be a number field, with $r_1$ real embeddings and $r_2$ complex embeddings, and set $w = w_2(F)$. Then $K_3^{\text{ind}}(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w$, and:*

*(a) If $F$ is totally imaginary then $K_3(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w$;*

*(b) If $F$ has $r_1 > 0$ embeddings into $\mathbb{R}$ then $K_3(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/(2w) \oplus (\mathbb{Z}/2)^{r_1-1}$.*

Proof. By 4.3.1 and Proposition 4.3.2, there is an exact sequence

$$0 \to K_3^M(F) \to K_3(F) \to H^1(F, \mathbb{Z}(2)) \to 0.$$

Therefore $H^1(F, \mathbb{Z}(2)) \cong K_3^{\text{ind}}(F)$ is the direct sum of $\mathbb{Z}^{r_2}$ and a finite group, say of order $m'$. Choose $m$ divisible by $m'$ and $w$. Because $H^0(F, \mathbb{Z}(2))$ is divisible by Ex. 4.6(a), the map $H^0(F, \mu_m^{\otimes 2}) \to H^0(F, \mathbb{Z}(2))_{\text{tors}}$ is an isomorphism. But $H^0(F, \mu_m^{\otimes 2}) \cong \mathbb{Z}/w$, so $K_3^{\text{ind}}(F) \cong \mathbb{Z}/w$. This establishes the result when $F$ is totally imaginary, since in that case $K_3^M(F) = 0$.

When $F = \mathbb{Q}$ then $w = 24$ and $K_3(\mathbb{Q}) \cong \mathbb{Z}/48$ is a nontrivial extension of $\mathbb{Z}/w$ by $\mathbb{Z}/2$; $K_3(\mathbb{Q})$ embeds in $K_3(\mathbb{R})$ by Theorem 3.1. When $F$ has a real embedding, it follows that $K_3(\mathbb{Q}) \subseteq K_3(F)$ so $\{-1,-1,-1\}$ is a nonzero element of $2K_3(F)$. Hence the extension is nontrivial, as claimed. $\square$

Rigidity Conjecture 5.3.1. (Suslin [Su86, 5.4]) Let $F_0$ denote the algebraic closure of the prime field in $F$. The Rigidity Conjecture states that $K_3^{\text{ind}}(F_0) \to K_3^{\text{ind}}(F)$ is an isomorphism. If $\text{char}(F) > 0$ then $K_3^{\text{ind}}(F_0)$ is $\mathbb{Z}/w_2(F)$; if $\text{char}(F) = 0$, $K_3^{\text{ind}}(F_0)$ is given by Corollary 5.3.

*The element $c$ of $B(F)$*

The elements $c = [x]+[1-x]$ and $\langle x\rangle = [x]+[x^{-1}]$ of $B(F)$ play an important role, as illustrated by the following calculations.

Lemma 5.4. *Assuming that $|F| > 3$,*

*(a) $c = [x] + [1-x]$ is independent of the choice of $x \in F - \{0,1\}$.*

*(b) For each $x$ in $F - \{0,1\}$, $2\langle x\rangle = 0$.*

*(c) There is a homomorphism $F^\times \to B(F)$ sending $x$ to $\langle x\rangle$.*

*(d) $3c = \langle -1\rangle$ and hence $6c = 0$ in $B(F)$.*

Proof. Given $x \neq y$, we have the relations in $\mathcal{P}(F)$:

$$[1-y] - [1-x] + \left[\frac{1-x}{1-y}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + [y/x] = 0;$$

$$[x^{-1}] - [y^{-1}] + [x/y] - \left[\frac{1-x}{1-y}\right] + \left[\frac{1-x^{-1}}{1-y^{-1}}\right] = 0.$$

Subtracting the first from the relation in 5.1 yields $[x]+[1-x]-[y]-[1-y]=0$, whence (a) holds. Adding the second to the relation in 5.1 yields $\langle y\rangle - \langle x\rangle = \langle y/x\rangle$. Interchanging $x$ and $y$, and using $\langle y/x\rangle = \langle x/y\rangle$, we obtain $2\langle x\rangle = 0$. Because $|F|\geq 4$, any $z\in F-\{0,1\}$ has the form $z=zx/x$ for $x\neq 1$ and hence $2\langle z\rangle = 0$. For (d), we compute using (b) and (c):

$$\begin{aligned}3c =& [x]+[1-x]+[x^{-1}]+[1-x^{-1}]+[(1-x)^{-1}]+[1-(1-x)^{-1}]\\ =&\langle x\rangle + \langle 1-x\rangle + \langle 1-x^{-1}\rangle = \langle -(1-x)^2\rangle = \langle -1\rangle. \qquad \square\end{aligned}$$

Corollary 5.4.1. *If* $\mathrm{char}(F)=2$ *or* $\sqrt{-1}\in F$ *then* $3c=0$ *in* $B(F)$*; if* $\mathrm{char}(F)=3$ *or* $\sqrt[3]{-1}\in F$ *then* $2c=0$ *in* $B(F)$*.*

*The map* $\psi : H_3(GL_2)\to B(F)$

We will now construct a canonical map $H_3(GL_2(F),\mathbb{Z})\to B(F)$; see Theorem 5.7. To do this, we use the group hyperhomology of $GL_2(F)$ with coefficients in the chain complex arising from the following construction (for a suitable $X$).

Definition 5.5. If $X$ is any set, let $C_*(X)$ denote the "configuration" chain complex in which $C_n$ is the free abelian group on the set of $(n{+}1)$-tuples $(x_0,\dots,x_n)$ of distinct points in $X$, with differential

$$d(x_0,\dots,x_n)=\sum_{i=0}^{n}(-1)^i(x_0,\dots,\hat{x}_i,\dots,x_n).$$

There is a natural augmentation $C_0(X)\to\mathbb{Z}$ sending each $(x)$ to 1.

If a group $G$ acts on $X$, then $C_*(X)$ is a complex of $G$-modules, and we may form its hyperhomology $\mathbb{H}_n(G,C_*(X))$; see [WHomo, 6.1.15]. There is a canonical map from $\mathbb{H}_n(G,C_*(X))$ to $H_n(C_G)$, where $C_G$ denotes $C_*(X)\otimes_G\mathbb{Z}$.

Lemma 5.5.1. *If* $X$ *is infinite then* $C_*(X)\to\mathbb{Z}$ *is a quasi-isomorphism.*

Proof. If $X_0$ is a proper subset of $X$ and $z\in X-X_0$ then $s_n(x_0,\dots)=(z,x_0,\dots)$ defines a chain homotopy $s_n : C_n(X_0)\to C_{n+1}(X)$ from the inclusion $C_*(X_0)\to C_*(X)$ to the projection $C_*(X_0)\to\mathbb{Z}\to C_*(X)$, where the last map sends 1 to $(z)$. $\square$

If $X$ is finite and $|X|>n+1$, we still have $H_nC_*(X)=0$, by Exercise 5.1.

Corollary 5.5.2. *If a group* $G$ *acts on* $X$*, and* $X$ *is infinite (or* $|X|>n+1$*), then* $\mathbb{H}_n(G,C_*(X))\cong H_n(G,\mathbb{Z})$*.*

The case of most interest to us is the action of the group $G=GL_2(F)$ on $X=\mathbb{P}^1(F)$. If $n\leq 2$ then $G$ acts transitively on the basis of $C_n(X)$, and $C_n(X)\otimes_G\mathbb{Z}$ is an induced module from the stablizer subgroup $G_x$ of an element $x$. By Shapiro's Lemma [WH, 6.3.2] we have $H_q(C_n(X)\otimes_G\mathbb{Z})=H_q(G_x,\mathbb{Z})$.

Lemma 5.6. $H_0(C_G)=\mathbb{Z}$*,* $H_n(C_G)=0$ *for* $n=1,2$ *and* $H_3(C_G)\cong\mathcal{P}(F)$*.*

Proof. By right exactness of $\otimes_G$, we have $H_0(C_*(X)\otimes_G\mathbb{Z})=\mathbb{Z}$. The differential from $C_2\otimes_G\mathbb{Z}\cong\mathbb{Z}$ to $C_1\otimes_G\mathbb{Z}\cong\mathbb{Z}$ is an isomorphism, since $d(0,1,\infty)\equiv(0,1)$. For $n=3$ we write $[x]$ for $(0,\infty,1,x)$; $C_3$ is a free $\mathbb{Z}[G]$-module on the set $\{[x], x\in F-\{0,1\}\}$. Similarly, $C_4$ is a free $\mathbb{Z}[G]$-module of the set of all 5-tuples $(0,\infty,1,x,y)$ and we have

$$\begin{aligned} d(0,\infty,1,x,y) =&(\infty,1,x,y)-(0,1,x,y)+(0,\infty,x,y)-(0,\infty,1,y)+(0,\infty,1,x)\\ =&\left[\frac{1-x}{1-y}\right]-\left[\frac{1-x^{-1}}{1-y^{-1}}\right]+[y/x]-[y]+[x]. \end{aligned}$$

Thus the cokernel $H_3(C_G)$ of $d: C_4\otimes_G\mathbb{Z}\to C_3\otimes_G\mathbb{Z}$ is $\mathcal{P}(F)$. $\square$

Remark 5.6.1. The proof of Lemma 5.6 goes through for all finite fields, since $|\,\mathbb{P}^1(F)|\ge|\,\mathbb{P}^1(\mathbb{F}_2)|=7$. Hence $H_n(C_*(X))=0$ for $n\le 3$ by Exercise 5.1.

Let $T_2$ denote the diagonal subgroup (isomorphic to $F^\times\times F^\times$) of $GL_2(F)$; the semidirect product $T_2\rtimes\Sigma_2$ is the subgroup of $M_2$ of monomial matrices in $GL_2(F)$ (matrices with only one nonzero term in every row and column).

Theorem 5.7. *For all $F$, $H_2(GL_2(F),\mathbb{Z})=F^\times$, and there is an exact sequence*

$$H_3(M_2,\mathbb{Z})\to H_3(GL_2(F),\mathbb{Z})\xrightarrow{\psi} B(F)\to 0.$$

To prove Theorem 5.7, we consider the hyperhomology spectral sequence

$$E^1_{p,q}=H_q(G,C_p(X))\Rightarrow\mathbb{H}_{p+q}(G,C_*(X))\cong H_{p+q}(G,\mathbb{Z}) \tag{5.8}$$

with $G=GL_2(F)$; see [WHomo, 6.1.15]. By Lemma 5.6, the edge map $H_3(G,\mathbb{Z})\to E^\infty_{3,0}$ lands in a subset of $E^1_{3,0}=H_3(C_G)\cong\mathcal{P}(F)$, which we must show is $B(F)$.

It is not hard to determine all 10 nonzero terms of total degree at most 4 in (5.8). Indeed, the stabilizer of $0\in X$ is the group $B$ of upper triangular matrices, so $E^1_{0,q}=H_q(G,C_0)=H_q(B,\mathbb{Z})$; the stabilizer of $(0,\infty)\in X^2$ is the diagonal subgroup $T_2=F^\times\times F^\times$, so $E^1_{1,q}=H_q(G,C_1)=H_q(T_2,\mathbb{Z})$, and the stabilizer of $(0,\infty,1)\in X^3$ is the subgroup $\Delta=\{(a,a^{-1})\}$ of $T_2$, isomorphic to $F^\times$, so $E^1_{2,q}=H_q(\Delta,\mathbb{Z})$. By [Su84, §3], the inclusion $T_2\subset B$ induces an isomorphism on homology.

It is not hard to see that the differential $d^1: H_q(\Delta)\to H_q(T_2)$ is induced by the inclusion $\Delta\subset T_2$. Since the inclusion is split (by projection onto the first component of $T_2$), the map $d_1: H_q(\Delta)\to H_q(T_2)$ is a split injection, and hence $E^2_{2,q}=0$ for all $q$. The following lemma is proven in Exercise 5.2.

Lemma 5.8.1. *Let $\sigma: T_2\to T_2$ be the involution $\sigma(a,b)=(b,a)$. Then the differential $d^1: H_q(T_2)\to H_q(B)\cong H_q(T_2)$ is induced by $1-\sigma$.*
*Thus $E^2_{0,q}=H_q(T_2)_\sigma$ and $E^2_{1,q}=H_q(T_2)^\sigma/H_q(F^\times)$.*

Proof of Theorem 5.7. ([Su91, Thm. 2.1]) By Lemma 5.8.1, the row $q=1$ in (5.8) has $E^2_{0,1}=(T_2)_\sigma=F^\times$ and $E^2_{1,1}=0$ (because $T_2^\sigma=F^\times$). Writing $(H_n)_\sigma$ for $H_n(T_2,\mathbb{Z})_\sigma$, the low degree terms of the $E^2$ page are depicted in Figure 5.8.2.

$$\begin{array}{|cccc}
(H_3)_\sigma & & & \\
(H_2)_\sigma & (H_2)^\sigma & 0 & \\
F^\times & 0 & 0 & E^2_{3,1} \\
\mathbb{Z} & 0 & 0 & \mathcal{P}(F) \\
\hline
\end{array}$$

Figure 5.8.2. The $E^2$ page of (5.8)

By the Künneth formula [WH, 6.1.13], $H_*(T_2) \cong H_*(F^\times) \otimes H_*(F^\times)$, with $\sigma$ interchanging the factors; if $x, y \in H_i(F^\times)$ then $\sigma(x \otimes y) = y \otimes x = (-1)^i x \otimes y$. Since $H_2(F^\times) = \wedge^2 F^\times$, the group $H_2(T_2)_\sigma$ is the direct sum of $\wedge^2 F^\times$ and $\tilde{\wedge}^2(F^\times)$. A routine but tedious calculation shows that the differential $d^3 : \mathcal{P}(F) \to H_2(T_2)_\sigma$ is the canonical map $\mathcal{P}(F) \to \tilde{\wedge}^2(F^\times)$ of 5.1 followed by the split inclusion of $\tilde{\wedge}^2(F^\times)$ into $H_2(T_2)_\sigma$; see [Su91, 2.4]. Thus the cokernel of $d^3$ is $E^3_{0,2} \cong \wedge^2 F^\times \oplus K_2(F)$. In particular, we have $E^3_{3,0} = B(F)$ and $H_2(GL_2(F), \mathbb{Z}) \cong E^3_{0,2} \cong \wedge^2 F^\times \oplus K_2(F)$.

Let $K$ denote the kernel of the edge map $H_3(GL_2(F), \mathbb{Z}) \to B(F)$. From (5.8.2) we see that $K$ is an extension of a quotient $Q_2$ of $H_2(T_2)^\sigma$ by a quotient $Q_3$ of $H_3(T_2)_\sigma$. It follows that $H_3(GL_2(F), \mathbb{Z})$ is an extension of $B(F)$ by $K$.

Recall that $M_2 \cong T_2 \rtimes \Sigma_2$. Since $H_p(\Sigma_2, T_2) = 0$ for $p \neq 0$, the Hochschild-Serre spectral sequence $H_p(\Sigma_2, H_q T_2) \Rightarrow H_{p+q}(M_2)$ degenerates enough to show that the cokernel of $H_3(T_2) \oplus H_3(\Sigma_2) \to H_3(M_2)$ is a quotient of $H_2(T_2)_\sigma$. Analyzing the subquotient $Q_2$ in (5.8), Suslin showed in [Su91, p. 223] that $K$ is the image of $H_3(M_2, \mathbb{Z}) \to H_3(GL_2(F), \mathbb{Z})$. The result follows. $\square$

In order to extend the map $H_3(GL_2, \mathbb{Z}) \xrightarrow{\psi} B(F)$ of Theorem 5.7 to a map $\psi : H_3(GL_3, \mathbb{Z}) \to \mathcal{P}(F)$, we need a small digression.

*A cyclic homology construction*

Recall that under the Dold-Kan correspondence [WHomo, 8.4], a nonnegative chain complex $C_*$ (*i.e.,* one with $C_n = 0$ if $n < 0$) corresponds to to a simplicial abelian group $\{\widetilde{C}_n\}$. Conversely, given a simplicial abelian group $\{\widetilde{C}_n\}$, $C_*$ is the associated reduced chain complex.

For example, the chain complex $C_*(X)$ of Definition 5.5 corresponds to a simplicial abelian group; $\widetilde{C}_n(X)$ is the free abelian group on the set $X^{n+1}$ of all $(n+1)$-tuples $(x_0, \dots, x_n)$ in $X$, including duplication. In fact, $\widetilde{C}_n(X)$ is a cyclic abelian group in the following sense. (These assertions are relegated to Ex. 5.6.)

Definition 5.9. ([WHomo, 9.6]) A *cyclic abelian group* is a simplicial abelian group $\{\widetilde{C}_n\}$ together with an automorphism $t_n$ of each $\widetilde{C}_n$ satisfying: $t_n^{n+1} = 1$;

$$\partial_i t_n = t_n \partial_{i-1} \text{ and } \sigma_i t_n = t_n \sigma_i \text{ for } i \neq 0; \quad \partial_0 t_n = \partial_n \text{ and } \sigma_0 t_n = t^2_{n+1} \sigma_n.$$

The associated *acyclic complex* $(C^a_*, d^a)$ is the complex obtained from the reduced complex $C_*$ by omitting the last face operator; $C^a_*$ is acyclic, and there is a chain map $N : C_* \to C^a_*$ defined by $N = \sum_{i=0}^n (-1)^i t^i_n$ on $C_n$. The mapping cone of $N$ has $C_{n-1} \oplus C^a_n$ in degree $n$, and $(b, c) \mapsto b$ defines a natural quasi-isomorphism $\operatorname{cone}(N)[1] \xrightarrow{\simeq} C_*$ (see [WHomo, 1.5]). In fact, $\operatorname{cone}(N)[1]$ is a reduced form of two columns of Tsygan's double complex [WHomo, 9.6.6].

Since $N_0 : C_0 \to C_0^a$ is the natural identification isomorphism, we may truncate the zero terms to get a morphism of chain complexes

$$\begin{tikzcd} 0 & C_1 \arrow[l] \arrow[d, "N"] & C_2 \arrow[l, "d"'] \arrow[d, "N"] & \cdots \arrow[l, "d"'] \arrow[d, "N"] & C_n \arrow[l, "d"'] \arrow[d, "N"] & \arrow[l, "d"'] \\ 0 & C_1^a \arrow[l] & C_2^a \arrow[l, "d^a"'] & \cdots \arrow[l, "d^a"'] & C_n^a \arrow[l, "d^a"'] & \arrow[l, "d^a"'] \end{tikzcd} \qquad (5.9.1)$$

We write $D_*$ for the the associated mapping cone of this morphism. Thus, $D_0 = C_1^a$, and $D_n = C_{n+1}^a \oplus C_n$ for $n > 0$ with differential $(x, y) \mapsto (d^a x - Ny, -dy)$. Then $\text{cone}(N)[1] \to D_*$ is a quasi-isomorphism.

Example 5.9.2. When $X = \mathbb{P}^2(F)$, let $C_n$ denote the subgroup of $C_n(X)$ generated by the $(n+1)$-tuples of points $(x_0, \dots, x_n)$ for which no three $x_i$ are collinear. Since $C_n$ is closed under the operator $t_n$, the associated simplicial abelian subgroup $\{\widetilde{C}_n\}$ of $\{\widetilde{C}_n(X)\}$ has the structure of a cyclic abelian subgroup. The proof of Lemma 5.5.1 goes through to show that if $X$ is infinite then $C_* \to \mathbb{Z}$ and hence $C_* \to C_*(X)$ are quasi-isomorphisms. It follows that the map $\varepsilon : D_0 \to \mathbb{Z}$ sending $(x, y)$ to 1 induces a quasi-isomorphism $D_* \to \mathbb{Z}$.

Under the canonical action of the group $GL_3 = GL_3(F)$ on $X = \mathbb{P}^2(F)$, $GL_3$ sends the subcomplex $C_*$ of $C_*(X)$ to itself, so $GL_3$ acts on $C_*$ and $D_*$.

*The map $\psi : H_3(GL_3) \to B(F)$*

We shall now construct a map $H_3(GL_3, \mathbb{Z}) \xrightarrow{\psi} \mathcal{P}(F)$ whose image is $B(F)$. We will relate it to the map of Theorem 5.7 in Lemma 5.10.

Let $C_*$ be the subcomplex of Example 5.9.2 for $X = \mathbb{P}^2(F)$. By 5.5 and Lemma 5.5.1, the hyperhomology $\mathbb{H}_n(GL_3, C_*)$ of $GL_3 = GL_3(F)$ is just $H_n(GL_3, \mathbb{Z})$ when $F$ is infinite (or $|F| > n + 1$), and there is a canonical map from $H_n(GL_3, \mathbb{Z}) = \mathbb{H}_n(GL_3, D_*)$ to $H_n(D_G)$, where $D_G$ denotes $D_* \otimes_{GL_3} \mathbb{Z}$.

The four points $p_1 = (1 : 0 : 0)$, $p_2 = (0 : 1 : 0)$, $p_3 = (0 : 0 : 1)$ and $q = (1 : 1 : 1)$ play a useful role in any analysis of the action of $GL_3$ on $\mathbb{P}^2(F)$. For example, the $\mathbb{Z}[GL_3]$-module $C_3$ is generated by $P = (p_1, p_2, q, p_3)$. Just as in Lemma 5.6, $C_n \otimes_G \mathbb{Z} = \mathbb{Z}$ for $n \le 3$, while $C_4$ and $C_5$ are free $\mathbb{Z}[G]$-modules on the set of all 5-tuples and 6-tuples

$$\begin{bmatrix} a \\ x \end{bmatrix} := \quad (p_1, p_2, q, (1 : a : x), p_3)$$

$$\begin{bmatrix} a\ b \\ x\ y \end{bmatrix} := \quad (p_1, p_2, q, (1 : a : x), (1 : b, y), p_3)\,,$$

where $a \ne x$, $b \ne y$, $a \ne b$, $x \ne y$ and $ay \ne bx$. By inspection of $D_G$, $d(\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right]) = P$, $d^a(\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right]) = N(\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right]) = 0$, and $d : (D_G)_3 \to (D_G)_2$ is zero. Thus the terms $n \le 4$ of the complex $D_G$ have the form:

$$\oplus \mathbb{Z} \begin{bmatrix} a \\ x \end{bmatrix} \oplus \mathbb{Z} \begin{bmatrix} a\ b \\ x\ y \end{bmatrix} \xrightarrow{\begin{pmatrix} -1 & 0 \\ -N & d^a \end{pmatrix}} \mathbb{Z}\, P \oplus \bigoplus \mathbb{Z} \begin{bmatrix} a \\ x \end{bmatrix} \xrightarrow{0} \mathbb{Z}^2 \to \mathbb{Z}^2 \to \mathbb{Z} \to 0.$$

The map $\psi' : (D_G)_3 \to \mathcal{P}(F)$ defined by $\psi'(P) = 2c$, $\psi'(\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right]) = a$ vanishes on the image of $(D_G)_4$, by a straightforward calculation done in [Su91, 3.3], so it induces a homomorphism $\psi : H_3(GL_3, \mathbb{Z}) \to H_3(D_G) \to \mathcal{P}(F)$.

LEMMA 5.10. *The composition $H_3(GL_2, \mathbb{Z}) \to H_3(GL_3, \mathbb{Z}) \xrightarrow{\psi} \mathcal{P}(F)$ is the homomorphism $H_3(GL_2, \mathbb{Z}) \to B(F) \subset \mathcal{P}(F)$ of Theorem 5.7.*

PROOF. ([Su91, 3.4]) The subgroup $GL_2 = GL_2(F)$ also acts on $X = \mathbb{P}^2(F)$ and fixes the origin $p_3 = (0 : 0 : 1)$, so it acts on $X_0 = X - \{p_3\}$. The maps $f_n : C_n(X_0) \to C_{n+1}(X) = C^a_{n+1}(X) \subset D_n$ sending $(x_0, \dots, x_n)$ to $(x_0, \dots, x_n, p_3)$ satisfy $fd = d^a f$, so they form a $GL_2$-equivariant chain map $f : C_*(X_0) \to C^a_*(X)/C_0(X)[1]$.

If $C'_n$ denotes the subgroup $f_n^{-1}(C_{n+1})$ of $C_n(X_0)$, then the restriction of $f$ defines a $GL_2$-equivariant chain map $C'_* \to C^a_*/C^a_0[1] \subset D_*$. Therefore the composition of Lemma 5.10 factors as

$$H_3(GL_2, \mathbb{Z}) \to H_3(C'_{*GL_2}) \xrightarrow{f} H_4(C^a_G) \to H_3(D_G) \xrightarrow{\psi'} \mathcal{P}(F).$$

The projection from $\mathbb{P}^2(F) - \{p_3\}$ to $\mathbb{P}^1(F)$ with center $p_3$ is $GL_2$-equivariant and determines a homomorphism from $C'_*$ to $C^1_* = C_*(\mathbb{P}^1(F))$ over $\mathbb{Z}$. By inspection, the composition $H_3(GL_2) \to H_3(C'_{*GL_2}) \to H_3(C^1_{*GL_2}) \cong \mathcal{P}(F)$ is the inclusion of Theorem 5.7. $\square$

LEMMA 5.11. *Let $T_3$ denote the diagonal subgroup $F^\times \times F^\times \times F^\times$ of $GL_3(F)$. If $F$ is infinite, $H_n(T_3, \mathbb{Z}) \to H_n(D_{*T_3})$ is zero for $n > 0$.*

PROOF. Since $p_1 = (1 : 0 : 0)$ and $p_2 = (0 : 0 : 1)$ are fixed by $T_3$, the augmentation $D_0 \to \mathbb{Z}$ has a $T_3$-equivariant section sending 1 to $(p_1, p_2)$. Therefore $D_* \cong \mathbb{Z} \oplus D'_*$ as $T_3$-modules, and $D'_*$ is acyclic. Therefore if $n > 0$ we have $\mathbb{H}_n(T_3, D'_*) = 0$. Since $H_n(D_{*T_3}) = H_n(D'_{*T_3})$ for $n > 0$, the map $H_n(T_3, \mathbb{Z}) \to H_n(D_{*T_3})$ factors through $\mathbb{H}_n(T_3, D'_*) = 0$. $\square$

PROPOSITION 5.12. *The image of $\psi$ is $B(F)$, and there is an exact sequence*

$$H_3(M_2, \mathbb{Z}) \oplus H_3(T_3, \mathbb{Z}) \to H_3(GL_3(F), \mathbb{Z}) \xrightarrow{\psi} B(F) \to 0.$$

PROOF. By [Su-KM, 3.4], $H_3(T_3, \mathbb{Z})$ and $H_3(GL_2, \mathbb{Z})$ generate $H_3(GL_3, \mathbb{Z})$. The restriction of $\psi$ to $H_3(T_3, \mathbb{Z})$ is zero by Lemma 5.11, since it factors as

$$H_3(T_3, \mathbb{Z}) \to H_3(D_{T_3}) \to H_3(D_G) \xrightarrow{\psi'} \mathcal{P}(F).$$

The proposition now follows from Theorem 5.7 and Lemma 5.10. $\square$

REMARK 5.12.1. The map $\psi$ extends to a map defined on $H_3(GL(F), \mathbb{Z})$, because of the stability result $H_3(GL_3(F), \mathbb{Z}) \cong H_3(GL(F), \mathbb{Z})$; see IV.1.15.

We now consider the image of $H_3(\Sigma_\infty, \mathbb{Z})$ in $B(F)$, where we regard $\Sigma_\infty$ as the subgroup of permutation matrices in $GL(F)$, i.e., as the direct limit of the permutation embeddings $\iota_n : \Sigma_n \subset GL_n(F)$. We will use the following trick.

Let $S$ be the $p$-Sylow subgroup of a finite group $G$. Then the transfer-corestriction composition $H_n(G) \to H_n(S) \to H_n(G)$ is multiplication by $[G : S]$, which is prime to $p$. Therefore the $p$-primary torsion in $H_n(G, \mathbb{Z})$ is the image of $H_n(S, \mathbb{Z})$ when $n > 0$.

PROPOSITION 5.13. *The image of* $H_3(\Sigma_\infty, \mathbb{Z}) \xrightarrow{\iota_*} H_3(GL(F), \mathbb{Z}) \xrightarrow{\psi} B(F)$ *is the subgroup generated by* $2c$*, which is trivial or cyclic of order 3.*

PROOF. We saw in Ex. IV.1.13 that $H_3(\Sigma_\infty, \mathbb{Z}) \cong \mathbb{Z}/12 \oplus (\mathbb{Z}/2)^2$, so the image of $H_3(\Sigma_\infty, \mathbb{Z}) \to H_3(GL(F), \mathbb{Z})$ has at most 2- and 3-primary torsion. Nakaoka proved in [Nak] that $H_3(\Sigma_6, \mathbb{Z}) \cong H_3(\Sigma_\infty, \mathbb{Z})$.

Using the Sylow 2-subgroup of $\Sigma_6$, Suslin proves [Su-KM, 4.4.1] that the 2-primary subgroup of $H_3(\Sigma_6, \mathbb{Z})$ maps to zero in $B(F)$. We omit the details.

Now the 3-primary component of $H_3(\Sigma_6, \mathbb{Z})$ is the image of $H_3(S)$, where $S$ is a Sylow 3-subgroup of $\Sigma_6$. We may take $S = A_3 \times \sigma A_3 \sigma^{-1}$, generated by the 3-cycles (123) and (456), where $\sigma = (14)(25)(36)$. Since $H_2(A_3, \mathbb{Z}) = 0$, the Künneth formula yields $H_3(S, \mathbb{Z}) = H_3(A_3, \mathbb{Z}) \oplus H_3(\sigma A_3 \sigma^{-1}, \mathbb{Z})$. By [WHomo, 6.7.8], these two summands have the same image in $H_3(\Sigma_6, \mathbb{Z})$. By Lemma 5.14, the 3-primary component of the image of $H_3(S, \mathbb{Z})$ in $B(F)$ is generated by $2c$, as desired. □

We are reduced to the alternating group $A_3$, which is cyclic of order 3, embedded in $GL_3(F)$ as the subgroup of even permutation matrices. We need to analyze the homomorphism $H_3(A_3, \mathbb{Z}) \to H_3(GL_3(F), \mathbb{Z}) \xrightarrow{\psi} B(F)$.

LEMMA 5.14. *If* $|F| \geq 4$*, the image of* $\mathbb{Z}/3 \cong H_3(A_3)$ *in* $B(F)$ *is generated by* $2c$*.*

PROOF. If $\mathrm{char}(F) = 3$ then the permutation representation of $A_3$ is conjugate to an upper triangular representation, so $H_*(A_3, \mathbb{Z}) \to H_*(GL_3(F), \mathbb{Z})$ is trivial (see [S82]). Since $2c = 0$ by Corollary 5.4.1, the result follows. Thus we may assume that $\mathrm{char}(F) \neq 3$.

When $\mathrm{char}(F) \neq 3$, the permutation representation is conjugate to the representation $A_3 \to GL_2(F)$ with generator $\lambda = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. For example, if $\mathrm{char}(F) = 2$, this representation identifies $A_3$ with a subgroup of $\Sigma_3 \cong GL_2(\mathbb{F}_2)$. By the Comparison Theorem [WHomo, 2.2.6], there is a morphism between $\mathbb{Z}[A_3]$-module resolutions of $\mathbb{Z}$, from the standard periodic free resolution (see [WHomo, 6.2.1]) to $C_*$:

$$\begin{tikzcd}
\mathbb{Z}[A_3] \arrow[d, "f_0"'] & \mathbb{Z}[A_3] \arrow[l, "1-\lambda"'] \arrow[d, "f_1"'] & \mathbb{Z}[A_3] \arrow[l, "1+\lambda+\lambda^2"'] \arrow[d, "f_2"'] & \mathbb{Z}[A_3] \arrow[l, "1-\lambda"'] \arrow[d, "f_3"'] & \cdots \arrow[l, "1+\lambda+\lambda^2"'] \\
C_0 & C_1 \arrow[l, "d"'] & C_2 \arrow[l, "d"'] & C_3 \arrow[l, "d"'] & \cdots . \arrow[l, "d"']
\end{tikzcd} \tag{5.14.1}$$

We can build the morphisms $f_n$ by induction on $n$, starting with $f_0(1) = (0)$, $f_1(1) = (\infty, 0)$ and $f_2(1) = (\infty, 0, x) + (1, \infty, x) = (0, 1, x)$ for any $x \in F$ $(x \neq 0, 1)$. If we set $w = 1 - 1/x$ and choose $y \neq \infty, 0, 1, x, w$ (which is possible when $|F| \geq 4$) then we may also take

$$\begin{aligned} f_3(1) = &- (\infty, 0, x, y) - (1, \infty, x, y) - (0, 1, x, y) \\ &+ (1, \infty, w, y) + (0, 1, w, y) + (\infty, 0, w, y). \end{aligned}$$

Here we have regarded $F$ as embedded in $\mathbb{P}^1(F)$ via $x \mapsto \binom{1}{x}$. Taking coinvariants in (5.14.1), the generator 1 of $H_3(A_3)$ in the periodic complex maps to $f_3(1)$, representing an element of $H_3(C_{*G})$. Applying $\psi$ sends this element to

$$-\left[\frac{x}{y}\right] - \left[\frac{y-1}{x-1}\right] - \left[\frac{y(x-1)}{x(y-1)}\right] + [x(1-y)] + \left[\frac{1-x}{xy}\right] + \left[\frac{y}{(1-x)(1-y)}\right]$$

in $B(F)$. As $|F| \geq 4$, we can take $x \neq -1$ and $y = x^{-1}$, so this expression becomes $-[x^2] - 2[-x^2] + 2[x-1] + [1/x^2]$. This equals $-2c$, by Exercise 5.5. □

*Monomial matrices*

By definition, a *monomial matrix* in $GL_n(F)$ is one which has only one nonzero entry in each row and column. We write $M_n$ for the group of all monomial matrices in $GL_n(F)$ and $M$ for the union of the $M_n$ in $GL(F)$; $M_n$ is isomorphic to the wreath products $F^\times \wr \Sigma_n = (F^\times)^n \rtimes \Sigma_n$ and $M \cong F^\times \wr \Sigma_\infty$. We encountered $M_2$ in Theorem 5.7, and the subgroup $\mu(F) \wr \Sigma_n$ of $M_n$ in Theorem 1.5.

PROPOSITION 5.15. *Let $\iota : \Sigma_\infty \to GL(F)$ be the inclusion. Then there is an exact sequence:*

$$H_3(M,\mathbb{Z}) \to H_3(GL(F),\mathbb{Z}) \oplus H_3(\Sigma_\infty,\mathbb{Z}) \xrightarrow{(\psi,-\psi\iota_*)} B(F) \to 0.$$

PROOF. (Cf. [Su91, 4.3]) Let $H_3^0 M_n$ denote the kernel of $H_3(M_n,\mathbb{Z}) \to H_3(\Sigma_n,\mathbb{Z})$; it suffices to show that the image of $H_3^0 M_n \to H_3(GL(F),\mathbb{Z})$ is the kernel of $\psi$ for large $n$. The image contains the kernel of $\psi$ because, by Proposition 5.12, the kernel of $\psi$ comes from the image of $H_3(T_3,\mathbb{Z})$, which is in $H_3^0 M_n$, and $H_3(M_2,\mathbb{Z})$, which is in $H_3^0 M_n$ by Ex. 5.10(d) since the image of $H_3(\Sigma_2,\mathbb{Z})$ is in the image of $H_3([M,M],\mathbb{Z})$ by Ex. 5.12.

The group $M_n = T_n \rtimes \Sigma_n$ contains $M_2 \times \Sigma_{n-2}$ as a subgroup. Let $S$ denote the group $\Sigma_2 \times \Sigma_{n-2}$, and write $A$ for the kernel of the split surjection $H_3(M_2 \times \Sigma_{n-2},\mathbb{Z}) \to H_3(S,\mathbb{Z})$. By Ex. 5.10(e), $H_3(T_3,\mathbb{Z}) + A$ maps onto $H_3^0 M_n$.

By the Künneth formula for $M_2 \times \Sigma_{n-2}$, $A$ is the direct sum of $H_3^0(M_2,\mathbb{Z})$, $H_2^0(M_2,\mathbb{Z}) \otimes H_1(\Sigma_{n-2},\mathbb{Z})$ and $F^\times \otimes H_2(\Sigma_{n-2},\mathbb{Z})$. Suslin proved in [Su-KM, 4.2] that the images of the latter two summands in $H_3(M_n)$ are contained in the image of $H_3(T_n,\mathbb{Z})$ for large $n$. It follows that the image of $H_3^0 M_n$ is the kernel of $\psi$. □

In Example IV.4.10.1 and Ex. IV.1.27 we saw that the homotopy groups of $K(F^\times\text{-}\mathbf{Sets}_{\text{fin}}) \simeq \mathbb{Z} \times BM^+$ form a graded-commutative ring with $\pi_1(BM^+) \cong F^\times \times \pi_1^s$. By Ex. IV.4.12 , $K_*(F^\times\text{-}\mathbf{Sets}_{\text{fin}}) \to K_*(F)$ is a ring homomorphism. By Matsumoto's Theorem III.6.1, it follows that $\pi_2(BM^+) \to K_2(F)$ is onto; in fact, $\pi_2(BM^+) = \pi_2^s \oplus \tilde{\wedge}^2 F^\times$ by Exercise 5.11. Multiplying by $\pi_1(BM^+)$, this implies that $K_3^M(F)$ lies in the image of $\pi_3(BM^+) \to K_3(F)$.

We saw in Theorem 5.7 that the kernel of the map $H_3(GL_2) \to B(F)$ is the image of $H_3(M_2)$. The monomial matrices $M_3$ in $GL_3(F)$ contain $T_3$, so Proposition 5.12 shows that the kernel of $\psi$ is contained in the image of $H_3(M_3)$.

THEOREM 5.16. *(*[Su91, Lemma 5.4]*) The cokernel of $\pi_3 BM^+ \to K_3(F)$ is $B(F)/(2c)$, and there is an exact sequence:*

$$\pi_3(BM^+) \to K_3(F) \oplus \pi_3^s \xrightarrow{(\psi,-\psi\iota_*)} B(F) \to 0.$$

PROOF. Write $H_n G$ for $H_n(G,\mathbb{Z})$, and let $P$ denote the commutator subgroup $[M,M]$; we saw in IV, Ex. 1.27 that $P$ is perfect. By IV.1.19–20, $\pi_3^s$ maps onto

$H_3A_\infty$ and there is a commutative diagram

$$\begin{array}{ccccc}
\pi_2(BP^+) & \xrightarrow{\circ\eta} & \pi_3(BP^+) & \xrightarrow{h} & H_3P \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
K_2(F) & \xrightarrow{[-1]} & K_3(F) & \xrightarrow{h} & H_3SL(F) \to 0.
\end{array}$$

Since $\pi_n BP^+ \cong \pi_n BM^+$ for $n \geq 2$ (Ex. IV.1.8), and $\pi_2(BM^+) \to K_2(F)$ is onto (as we noted above), a diagram chase shows that $K_3(F)/\pi_3(BM^+) \cong H_3SL(F)/H_3P$.

Let $SM$ denote the kernel of det : $M \to F^\times$. We saw in Chapter IV, Ex. 1.27 that $BM^+ \simeq BP^+ \times B(F^\times) \times B\Sigma_2$; it follows that $BSM^+ \simeq BP^+ \times B\Sigma_2$. By the Künneth formula, $H_3SM \cong H_3P \oplus (H_2P \otimes H_1\Sigma_2) \oplus H_3\Sigma_2$. Under the map $H_3SM \to H_3SL(F)$, the final term lands in the image of $H_3P$ by Ex. 5.12, and the middle term factors through $H_2SL(F) \otimes H_1SL(F)$, which is zero as $SL(F)$ is perfect. Hence $H_3SL(F)/H_3P = H_3SL(F)/H_3SM$.

This explains the top half of the following diagram.

$$\begin{array}{ccccc}
H_3(P) & \longrightarrow & H_3(SL) & \longrightarrow & K_3(F)/\pi_3(BM^+) \to 0 \\
\downarrow & & \downarrow & & \downarrow \cong \\
H_3(SM) & \longrightarrow & H_3(SL) & \longrightarrow & H_3(SL)/H_3(SM) \to 0 \\
\downarrow & & \downarrow & & \downarrow \cong \\
H_3(M) & \longrightarrow & H_3(GL) & \longrightarrow & B(F)/(2c) \to 0.
\end{array}$$

Similarly, Ex. IV.1.8 implies that $H_nGL(F) \cong \oplus_{i+j=n} H_iSL(F) \otimes H_jF^\times$. There is a compatible splitting $H_nM \cong \oplus_{i+j=n} H_iSM \otimes H_jF^\times$ by Ex. IV.1.27. This implies that $H_nSL(F)/H_nSM \to H_nGL(F)/H_nM$ is injective for all $n$. For $n=3$, the summands $H_3F^\times$ and $H_2SL(F) \otimes F^\times$ of $H_3GL(F)$ are in the image of $H_3SM$ (because $\pi_2BSM^+ \to K_2(F) \cong H_2SL(F)$ is onto). Since $H_1SL(F) = 0$, we conclude that $H_3SL(F)/H_3SM \to H_3GL(F)/H_3M$ is onto and hence an isomorphism.

Finally, combining Propositions 5.12 and 5.13 with Exercise 5.10 and Lemma 5.14, we see that $H_3SL(F) \to B(F)$ is onto, and the cokernel of $H_3M \to H_3GL(F)$ is $B(F)/(2c)$. Concatenating the isomorphisms yields the first assertion. The second assertion follows from this by the argument in the proof of Proposition 5.15. □

We define $\pi_3^{\text{ind}}(BM^+)$ to be the quotient of $\pi_3(BM^+)$ by all products from $\pi_1(BM^+) \otimes \pi_2(BM^+)$. There is a natural map $\pi_3^{\text{ind}}(BM^+) \to \pi_3^s/(\eta^3) \cong \mathbb{Z}/12$. Since the products map to $K_3^M(F)$ in $K_3(F)$, we have the following reformulation.

Corollary 5.16.1. *The sequence of Theorem 5.16 induces an exact sequence*

$$\pi_3^{\text{ind}}(BM^+) \to K_3^{\text{ind}}(F) \oplus \mathbb{Z}/12 \to B(F) \to 0.$$

Thus to prove Theorem 5.2, we need to study $\pi_3^{\text{ind}}(BM^+)$.

Remark 5.16.2. The diagram in the proof of Theorem 5.16 shows that the map $\pi_3^{\text{ind}}(BM^+) \to K_3^{\text{ind}}(F)$ is a quotient of the map $H_3(P, \mathbb{Z}) \to H_3(SL(R), \mathbb{Z})$.

If $E$ is any homology theory and $X$ any pointed topological space, the Atiyah-Hirzebruch spectral sequence converging to $E_*(X)$ has $E^2_{p,q} = H_p(X, E_q(*))$. For stable homotopy we have $E_*(X) = \pi^s_*(X)$. When $X = BG_+$, the Barratt-Priddy Theorem (IV.4.10.1) states that $\pi^s_*(BG_+) = \pi_*(\mathbb{Z} \times B(G \wr \Sigma_\infty)^+)$.

Proposition 5.17. *There is an exact sequence*

$$0 \to \mu_2(F) \to \pi_3^{\text{ind}}(BM^+) \xrightarrow{\gamma} \mu(F) \oplus \mathbb{Z}/12 \to 0.$$

Proof. ([Su91, §5]) We analyze the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(F^\times, \pi^s_q) \Rightarrow \pi_{p+q}(\mathbb{Z} \times BM^+)$$

which has a module structure over the stable homotopy ring $\pi^s_*$. Note that the $y$-axis $E^2_{0,*} = \pi^s_*$ is a canonical summand of $\pi_*(\mathbb{Z} \times BM^+)$, so it survives to $E^\infty$. By Ex. 5.11, $\pi_2 BM^+/\pi^s_2 \cong \tilde{\wedge}^2 F^\times$. It is easy to see that the map from $E^2_{1,1} = F^\times/F^{\times 2}$ to $E^\infty_{1,1} = \tilde{\wedge}^2 F^\times$ is injective, as it sends $x$ to $x \otimes x$. It follows that the differential from $E^2_{3,0} = H_3(F^\times, \mathbb{Z})$ to $E^2_{1,1}$ is zero, so $E^\infty_{3,0} = H_3(F^\times, \mathbb{Z})$; see Figure 5.17.1.

$$\begin{array}{|lllll}
\pi^s_3 & & & & \\
\pi^s_2 & F^\times/F^{\times 2} & & & \\
\pi^s_1 & F^\times/F^{\times 2} & E^2_{2,1} & & \\
\mathbb{Z} & F^\times & \wedge^2 F^\times & H_3(F^\times) & H_4(F^\times) \\
\hline
\end{array}$$

Figure 5.17.1. The $E^2$ page converging to $\pi_*(\mathbb{Z} \times BM^+)$

The universal coefficient sequence expresses $E^2_{2,1} = H_2(F^\times, \mathbb{Z}/2)$ as an extension:

$$0 \to (\wedge_2 F^\times)/2 \to H_2(F^\times, \mathbb{Z}/2) \to \mu_2(F) \to 0.$$

The differential $E^2_{4,0} \to E^2_{2,1}$ lands in $(\wedge_2 F^\times)/2$ because the composite to $\mu_2(F)$ is zero: by naturality in $F$, it factors through the divisible group $H_4(\bar{F}^\times, \mathbb{Z})$.

The cokernel of the product maps from $F^\times \otimes \pi^s_2$ and $\wedge^2 F^\times \otimes \pi^s_1 \to H_3(F^\times, \mathbb{Z})$ is therefore the direct sum of $\pi^s_3/(\eta^3)$ and an extension of $H_3(F^\times, \mathbb{Z})$ by $\mu_2(F)$. Modding out by the products from $\wedge^2 F^\times \otimes F^\times$ replaces $H_3(F^\times, \mathbb{Z})$ by $H_3(F^\times, \mathbb{Z})/\wedge^3 F^\times$, which by Ex. 5.9 is isomorphic to $\mu(F)$. $\square$

Since $H_3(P, \mathbb{Z}) \cong \pi_3(BM^+)/\eta \circ \pi_2(BM^+)$ (see IV.1.19), there is a natural surjection $H_3(P, \mathbb{Z}) \to \pi_3^{\text{ind}}(BM^+)$. Consider the homomorphism $\delta : \mu(F) \to P$ sending $x$ to $\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$. The composition with the map $\gamma$ of Proposition 5.17 is:

$$\mu(F) \cong H_3(\mu(F), \mathbb{Z}) \xrightarrow{\delta} H_3(P, \mathbb{Z}) \xrightarrow{\text{onto}} \pi_3^{\text{ind}}(BM^+) \xrightarrow{\gamma} \mu(F).$$

LEMMA 5.18. *The composition $\mu(F) \xrightarrow{\delta} \pi_3^{\mathrm{ind}}(BM^+) \xrightarrow{\gamma} \mu(F)$ sends $x$ to $x^2$.*

PROOF. Let us write $\mu$ for $\mu(F)$ and $H_nG$ for $H_n(G,\mathbb{Z})$. The homomorphism $\mu \xrightarrow{\delta} P$ factors through $D = \mu \rtimes \Sigma_2$ (where $\Sigma_2 \to A_\infty \subset P$ is given by (12)(34)), and the map $\mu \cong H_3(\mu) \to H_3(D)$ lands in the subgroup $H_3(\mu)_{\Sigma_2} \cong \mu/\{\pm 1\}$. By Exercise 5.13, the composition of $\mu \to \mu/\{\pm 1\}$ with $H_3(D) \to H_3(P) \xrightarrow{\gamma} \mu$ sends $x \in \mu$ to $x^2$. $\square$

Recall that the $e$-invariant maps $K_3^{\mathrm{ind}}(F)$ to $H^0(F,\mu^{\otimes 2}) \subset \mu(\bar{F})$ (see Definition 2.1). We use it to detect the image of $\mu(F)$ in $\pi_3^{\mathrm{ind}}(BM^+)$.

LEMMA 5.19. *The following composition is an injection:*

$$\mu(F) \xrightarrow{\delta} \pi_3^{\mathrm{ind}}(BM^+) \to K_3^{\mathrm{ind}}(F) \xrightarrow{e} H^0(F,\mu^{\otimes 2}).$$

*If $F$ is algebraically closed it is an isomorphism.*

PROOF. If $F \subset F'$, $\mu(F) \subseteq \mu(F')$. Therefore we may enlarge $F$ to assume that it is algebraically closed. In this case $H^0(F,\mu^{\otimes 2}) = \mu(F)$ and $K_3^M(F)$ is uniquely divisible (III.7.2), so it is a summand of $K_3(F)$ (by V.11.13) and therefore ${}_mK_3(F) \cong {}_mK_3^{\mathrm{ind}}(F)$.

Now consider the étale Chern class $c_{2,4} : K_4(F;\mathbb{Z}/m) \to H^0(F,\mu_m^{\otimes 2})$, where $1/m \in F$ and $m \not\equiv 2 \pmod 4$. Since $K_4(F;\mathbb{Z}/m) \cong \mathbb{Z}/m$ on generator $\beta^2$ (1.4) and $c_1(\beta) = \zeta$ by V.11.10.1, the product rule yields $c_2(\beta^2) = \zeta^{-1} \otimes \zeta$. Since the Bockstein is an isomorphism: $K_4(F;\mathbb{Z}/m) \cong {}_mK_3(F)$, this implies that the $e$-invariant is $-c_{2,4}$ on ${}_mK_3(F)$.

The isomorphism $K_4(F;\mathbb{Z}/m) \cong {}_mK_3^{\mathrm{ind}}(F)$ factors through the Hurewicz map, the map $H_4(SL(F),\mathbb{Z}/m) \to {}_mH_3(SL(F),\mathbb{Z})$ and the quotient ${}_mH_3(SL(F),\mathbb{Z}) \to {}_mK_3^{\mathrm{ind}}(F)$ of IV.1.20. Therefore we have a commutative diagram:

$$\begin{array}{ccccccc}
H_4(\mu,\mathbb{Z}/m) & \xrightarrow{\delta} & H_4(P,\mathbb{Z}/m) & \longrightarrow & H_4(SL(F),\mathbb{Z}/m) & \xrightarrow{c_{2,4}} & H^0_{\mathrm{et}}(F,\mu_m^{\otimes 2}) \\
\downarrow\cong & & \downarrow & & \downarrow & & \| \\
\mu_m & \xrightarrow{\delta} & {}_m\pi_3^{\mathrm{ind}}(BM^+) & \longrightarrow & {}_mK_3^{\mathrm{ind}}(F) & \xrightarrow{-e} & \mu_m \otimes \mu_m.
\end{array}$$

Since the top composition is an isomorphism by Ex. V.11.5, the result follows. $\square$

COROLLARY 5.20. $\pi_3^{\mathrm{ind}}(BM^+) \cong \tilde{\mu}(F) \oplus \mathbb{Z}/12.$

PROOF. If $\mathrm{char}(F) = 2$, the result is immediate from Proposition 5.17, so we may suppose that $\mathrm{char}(F) \neq 2$. If $F \subset E$ then $\mu(F) \subseteq \mu(E)$ and therefore, by naturality of Proposition 5.17 in $F$, the map on $\pi_3(BM^+)$ groups is an injection. We may therefore assume that $F$ is algebraically closed. In this case, it follows from 5.18 and 5.19 that $(\delta, i) : \mu(F) \oplus \mathbb{Z}/12 \to \pi_3(BM^+)$ is an isomorphism, because $\gamma\delta$ is onto and $\delta(-1)$ is a nonzero element of the kernel $\mu_2(F)$ of $\gamma$. $\square$

PROOF OF THEOREM 5.2. (Suslin) By Corollaries 5.16.1 and 5.20, the kernel of $K_3^{\mathrm{ind}}(F) \to B(F)$ is the image of $\tilde{\mu}(F)$. Thus it suffices to show that the summand $\tilde{\mu}(F)$ of $\pi_3(BM^+)$ (given by Corollary 5.20) injects into $K_3^{\mathrm{ind}}(F)$. When $F$ is algebraically closed, this is given by Lemma 5.19. Since $\tilde{\mu}(F) \to \tilde{\mu}(E)$ is an injection for all field extensions $F \subset E$, the general case follows. $\square$

EXERCISES

**5.1** (Hutchinson) If $|X| = d$, $C_n(X) = 0$ for $n \geq d$ by 5.5. Show that $H_n(C_*(X)) = 0$ for all $n \neq 0, d-1$, and that $H_n(C'_*(X)) = 0$ for all $n \neq d-1$. ($C'_*$ is defined in 5.10.) Conclude that if $|X| \geq 5$ then $H_3(G, \mathbb{Z}) \cong \mathbb{H}_3(G, C_*(X))$ and $H_4(G, C'_*(X)) = 0$.

**5.2** In this exercise we prove Lemma 5.8.1. In the hypercohomology spectral sequence (5.8), the differential $d^1$ from $H_q(G, C_1)$ to $H_q(G, C_0) \cong H_q(B, \mathbb{Z}) \cong H_q(T_2, \mathbb{Z})$ is induced by the map $d^1_0 - d^1_1 : C_1 \to C_0$, where $d^1_0(x_0, x_1) = x_1$ and $d^1_1(x_0, x_1) = x_0$. Let $T_2$ be the diagonal subgroup, and $B$ the upper triangular subgroup of $GL_2(F)$. We saw after (5.8) that the inclusion of $T_2$ and $B$ induces isomorphisms $\iota : H_q(T_2, \mathbb{Z}) \xrightarrow{\cong} H_q(G, C_1)$ and $H_q(B, \mathbb{Z}) \xrightarrow{\cong} H_q(G, C_0)$. Show that:
(i) $d^1_0\, \iota = \sigma(d^1_1\, \iota)$, where $\sigma$ is the involution $\sigma(a, b) = (b, a)$ of $F^2$;
(ii) The composition $d^1_0\, \iota$ is the natural map $H_q(T_2, \mathbb{Z}) \to H_q(B, \mathbb{Z}) \cong H_q(G, C_0)$;
(iii) The fact that $H_q(T_2, \mathbb{Z}) \to H_q(B, \mathbb{Z})$ is an isomorphism ([Su84, §3]) implies that $d^1 = 1 - \sigma$.

**5.3** Show that $B(\mathbb{F}_5) \cong \mathbb{Z}/3$ on generator $c = [2] = 2[3]$, with $[-1] = 0$. Then show that $B(\mathbb{F}_7) \cong \mathbb{Z}/4$ on generator $[-1] = 2[3]$ with $c = [4] = 2[-1]$. (In both cases, $[3] \notin B(\mathbb{F})$.)

**5.4** Show that $c = 0$ in $B(\mathbb{F}_q)$ if either: (a) $\mathrm{char}(\mathbb{F}_q) = 2$ and $q \equiv 1 \pmod 3$; (b) $\mathrm{char}(\mathbb{F}_q) = 3$ and $q \equiv 1 \pmod 4$; or (c) $\mathrm{char}(\mathbb{F}_q) > 3$ and $q \equiv 1 \pmod 6$;

**5.5** (Dupont-Sah) Show that $[x^2] = 2\,([x] + [-x] + [-1])$ in $\mathcal{P}(F)$ for all $x \neq \pm 1$. Using this, show that $[x^2] + 2[-x^2] - 2[x-1] - [1/x^2] = 2c$.

**5.6** Given an arbitrary set $X$, let $\widetilde{C}_n(X)$ be the free abelian group on the set $X^{n+1}$ of all $(n+1)$-tuples $(x_0, \dots, x_n)$ in $X$, including duplication.

a) Show that $\widetilde{C}_n(X)$ is a simplicial abelian group, whose degeneracy operators $\sigma_i$ are duplication of the $i^{th}$ entry. Under the Dold-Kan correspondence, the simplicial abelian group $\widetilde{C}_n(X)$ corresponds to the chain complex $C_*(X)$ of Definition 5.5.

b) Show that the rotations $t_n(x_0, \dots, x_n) = (x_n, x_0, \dots, x_{n-1})$ satisfy $t_n^{n+1} = 1$, $\partial_i t_n = t_n \partial_{i-1}$ and $\sigma_i t_n = t_n \sigma_i$ for $i \neq 0$, $\partial_0 t_n = \partial_n$ and $\sigma_0 t_n = t^2_{n+1} \sigma_n$. This shows that $\widetilde{C}_n(X)$ is a cyclic abelian group [WHomo, 9.6.1].

**5.7** *Transfer maps.* Let $F \subset E$ be a finite field extension. If the transfer map $K_3(E) \to K_3(F)$ induces a map $N_{E/F} : B(E) \to B(F)$ (via Theorem 5.2), we call $N_{E/F}$ a transfer map. If $N_{E/F}$ exists, the composition $B(F) \to B(E) \to B(F)$ must be multiplication by $[E : F]$.

(a) Conclude that there is no transfer map $N_{E/F} : B(E) \to B(F)$ defined for all $F \subset E$. *Hint:* Consider $\mathbb{F}_5 \subset \mathbb{F}_{25}$ or $\mathbb{R} \subset \mathbb{C}$.

(b) Show that a transfer map $B(E) \to B(F)$ exists if $\mu(F) = \mu(E)$, or more generally if $E$ has an $F$-basis such that $\mu(E)$ is represented by monomial matrices over $F$.

**5.8** If $F_0$ is the field of constants in $F$, show that $B(F)/B(F_0)$ is the cokernel of $K^M_3(F) \oplus K_3(F_0) \to K_3(F)$. (The Rigidity Conjecture 5.3.1 implies that it is zero.) Conclude that $B(F(t)) \cong B(F)$, using V.6.7.1.

**5.9** If $A$ is an abelian group, $H_*(A,\mathbb{Z})$ is a graded-commutative ring [WHomo, 6.5.14]. Since $H_1(A,\mathbb{Z}) \cong A$, there is a ring map $\wedge^* A \to H_*(A,\mathbb{Z})$. The Künneth formula [WHomo, 6.1.13] for $H_*(A\times A)$ and the diagonal provide a natural map $H_3(A,\mathbb{Z}) \to H_3(A\times A,\mathbb{Z}) \to \mathrm{Tor}(A,A)$, whose image is invariant under the transposition involution $\tau$ on $A\times A$.

(a) Show that $\wedge^2 A \to H_2(A,\mathbb{Z})$ is an isomorphism, and that $\wedge^3 A \to H_3(A,\mathbb{Z})$ is an injection whose cokernel $H_3^{\mathrm{ind}}(A)$ is canonically isomorphic to $\mathrm{Tor}(A,A)^\tau$. *Hint:* It is true for cyclic groups; use the Künneth formula to check it for finitely generated $A$.

(b) If $A$ is a finite cyclic group and $\sigma$ is an automorphism of $A$, show that the composite

$$A \cong H_3^{\mathrm{ind}}(A) \xrightarrow{\sigma} H_3^{\mathrm{ind}}(A) \cong A$$

sends $a\in A$ to $\sigma^2(a)$. In particular, $\sigma$ acts trivially when $\sigma^2=1$.

**5.10** In this exercise we analyze $H_3$ of the group $M_n = T_n \rtimes \Sigma_n$ of monomial matrices in $GL_n(F)$, using the Hochschild-Serre spectral sequence. Here $F$ is a field and $T_n=(F^\times)^n$ denotes the subgroup of diagonal matrices. In this exercise we write $H_i(G)$ for $H_i(G,\mathbb{Z})$.

(a) Show that the images of $H_3(T_3)$ and $H_3(T_n)$ in $H_3(M_n(F))$ are the same.

(b) Show that $H_*(\Sigma_n, T_n) \cong H_*(\Sigma_{n-1}, F^\times)$. ($\Sigma_{n-1}$ is the stabilizer of $T_1 = F^\times$.)

(c) Show that $H_*(\Sigma_n, \wedge^2 T_n) \cong H_*(\Sigma_{n-1}, \wedge^2 F^\times) \oplus H_*(\Sigma_2\times\Sigma_{n-2}, F^\times\otimes F^\times)$.

(d) When $n=2$, conclude that $H_3(M_2) \cong H_3(\Sigma_2) \oplus H_3(T_2)_{\Sigma_2} \oplus \tilde{\wedge}^2 F^\times$.

(e) Let $A$ denote the kernel of $H_3(M_2\times\Sigma_{n-2},\mathbb{Z}) \to H_3(\Sigma_2\times\Sigma_{n-2},\mathbb{Z})$. Conclude that $H_3(T_3)+A \to H_3(M_n) \to H_3(\Sigma_n,\mathbb{Z})$ is exact for $n\ge 5$.

**5.11** Show that $H_2(P,\mathbb{Z}) \cong \pi_2 BM^+$ equals $\pi_2^s \oplus \tilde{\wedge}^2 F^\times$, where $P=[M,M]$. *Hint:* Recall from IV, Ex. 1.13 that $BM^+ \simeq BP^+ \times B(F^\times)\times B\Sigma_2$. Thus it suffices to compute $H_2(M,\mathbb{Z})$. Now use Exercise 5.10(b,c).

**5.12** Show that the linear transformation $\alpha(x_1,x_2,x_3)=(x_2,x_1,-x_3)$ is in $SM$, inducing a decomposition $SM\cong P\rtimes\Sigma_2$. If $\mathrm{char}(F)\ne 2$, show that $\alpha$ is conjugate in $GL_3(F)$ to the matrix $\mathrm{diag}(-1,-1,+1)$ in $P$, and hence that the image of $H_*(\Sigma_2,\mathbb{Z}) \to H_*(SM,\mathbb{Z}) \to H_*(GL(F),\mathbb{Z})$ lies in the image of $H_*(P,\mathbb{Z})$. If $\mathrm{char}(F)=2$, show that $\alpha$ is conjugate to an upper triangular matrix in $GL_3(F)$ and hence that the map $H_*(\Sigma,\mathbb{Z}) \to H_*(GL(F),\mathbb{Z})$ is trivial.

**5.13** Let $T'$ denote the group of diagonal matrices in $SL(F)$.

(a) Show that $P\cong T'\rtimes A_\infty$, and that $H_3(T',\mathbb{Z})_{A_\infty} \cong H_3(F^\times,\mathbb{Z})$.

(b) Use the proof of Proposition 5.17 to show that $\mu(F)\cong H_3(F^\times,\mathbb{Z})/\wedge^3 F^\times$ is the image of $H_3(T',\mathbb{Z})$ in $H_3(P,\mathbb{Z})/\wedge^3 F^\times \cong \pi_3(BM^+)$.

(c) If $D=\mu(F)\rtimes\Sigma_2$, show that the map from $H_3(D,\mathbb{Z})$ to $\pi_3(BM^+)$ sends $\mu(F)/\{\pm 1\}$ to $\mu(F)$.

## §6. Global fields of finite characteristic

A *global field* of finite characteristic $p$ is a field $F$ which is finitely generated of transcendence degree one over $\mathbb{F}_p$; the algebraic closure of $\mathbb{F}_p$ in $F$ is a finite field $\mathbb{F}_q$ of characteristic $p$. It is classical (see [Hart, I.6]) that there is a unique smooth projective curve $X$ over $\mathbb{F}_q$ whose function field is $F$. If $S$ is any nonempty set of closed points of $X$, then $X\backslash S$ is affine; we call the coordinate ring $R$ of $X\backslash S$ the ring of *$S$-integers in $F$*. In this section, we discuss the $K$-theory of $F$, $X$ and the rings of $S$-integers of $F$.

Any discussion of the $K$-theory of $F$ must involve the $K$-theory of $X$. For example, $K_1(F)$ is related to the Picard group $\mathrm{Pic}(X)$ by the Units-Pic sequence I.5.12:

$$1 \to \mathbb{F}_q^\times \to F^\times \to \oplus_{x\in X}\mathbb{Z} \to \mathrm{Pic}(X) \to 0.$$

Recall that $K_0(X) = \mathbb{Z}\oplus \mathrm{Pic}(X)$, and that $\mathrm{Pic}(X)\cong \mathbb{Z}\oplus J(X)$, where $J(X)$ is a finite group (see I.5.16).

Since $K_2$ vanishes on finite fields (III.6.1.1), the localization sequence V.6.12 for $X$ ends in the exact sequence

$$0 \to K_2(X) \to K_2(F) \xrightarrow{\partial} \oplus_{x\in X} k(x)^\times \to K_1(X) \to \mathbb{F}_q^\times \to 0.$$

By classical Weil reciprocity (V.6.12.1), the cokernel of $\partial$ is $\mathbb{F}_q^\times$, so $K_1(X)\cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. By III.2.5.1, if $R$ is the coordinate ring of any affine open in $X$ then $SK_1(R) = 0$. A diagram chase shows that the image of $K_1(X)$ in $K_1(F) = F^\times$ is $\mathbb{F}_q^\times$.

By III.7.2(a) (due to Bass and Tate), the kernel $K_2(X)$ of $\partial$ is finite of order prime to $p$. This establishes the low dimensional cases of the following theorem, first proven by Harder [Har], using the method pioneered by Borel [Bo].

Harder's Theorem 6.1. *Let $X$ be a smooth projective curve over a finite field of characteristic $p$. For $n\geq 1$, each $K_n(X)$ is a finite group of order prime to $p$.*

Parshin has conjectured that if $X$ is any smooth projective variety over a finite field then $K_n(X)$ is a torsion group for $n \geq 1$. Harder's Theorem (6.1) shows that Parshin's conjecture holds for curves.

Proof. By III.7.2(a), $K_n^M(F) = 0$ for all $n \geq 3$. By Geisser and Levine's Theorem 4.7, the Quillen groups $K_n(F)$ are uniquely $p$-divisible for $n \geq 3$. For every closed point $x \in X$, the groups $K_n(x)$ are finite of order prime to $p$ ($n > 0$) because $k(x)$ is a finite field extension of $\mathbb{F}_q$. From the localization sequence

$$\oplus_{x\in X} K_n(x) \to K_n(X) \to K_n(F) \to \oplus_{x\in X} K_{n-1}(x),$$

a diagram chase shows that $K_n(X)$ is uniquely $p$-divisible. By IV.6.9 (due to Quillen), the abelian groups $K_n(X)$ are finitely generated. As any finitely generated $p$-divisible abelian group $A$ is finite with $p\nmid |A|$, this is true for each $K_n(X)$. $\square$

Corollary 6.2. *If $R$ is the ring of $S$-integers in $F = \mathbb{F}_q(X)$ (and $S\neq\emptyset$) then:*
*a) $K_1(R)\cong R^\times \cong \mathbb{F}_q^\times \times \mathbb{Z}^s$, $|S| = s-1$;*
*b) For $n\geq 2$, $K_n(R)$ is a finite group of order prime to $p$.*

Proof. Recall (III.1.1.1) that $K_1(R) = R^\times \oplus SK_1(R)$. We saw in III.2.5.1 that $SK_1(R) = 0$, and the units of $R$ were determined in I.5.16, whence (a). The rest follows from the localization sequence $K_n(X) \to K_n(R) \to \oplus_{x\in S} K_{n-1}(x)$. $\square$

THE $e$-INVARIANT 6.3. The targets of the $e$-invariant of $X$ and $F$ are the same groups as for $\mathbb{F}_q$, because every root of unity is algebraic over $\mathbb{F}_q$. Hence the inclusions of $K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1)$ in $K_{2i-1}(X)$ and $K_{2i-1}(F)$ are split by the $e$-invariant, and this group is the Harris-Segal summand.

The inverse limit of the finite curves $X_\nu = X \times \mathrm{Spec}(\mathbb{F}_{q^\nu})$ is the curve $\bar{X} = X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ over the algebraic closure $\bar{\mathbb{F}}_q$. To understand $K_n(X)$ for $n > 1$, it is useful to know not only what the groups $K_n(\bar{X})$ are, but how the (geometric) Frobenius $\varphi : x \mapsto x^q$ acts on them.

By II.8.2.1 and I.5.15, $K_0(\bar{X}) = \mathbb{Z} \oplus \mathbb{Z} \oplus J(\bar{X})$, where $J(\bar{X})$ is the group of points on the Jacobian variety over $\bar{\mathbb{F}}_q$; it is a divisible torsion group. If $\ell \neq p$, the $\ell$-primary torsion subgroup $J(\bar{X})_\ell$ of $J(\bar{X})$ is isomorphic to the abelian group $(\mathbb{Z}/\ell^\infty)^{2g}$. The group $J(\bar{X})$ may or may not have $p$-torsion. For example, if $X$ is an elliptic curve then the $p$-torsion in $J(\bar{X})$ is either 0 or $\mathbb{Z}/p^\infty$, depending on whether $X$ is supersingular (see [Hart, Ex. IV.4.15]). Note that the localization $J(\bar{X})[1/p]$ is the direct sum over all $\ell \neq p$ of the $\ell$-primary groups $J(\bar{X})_\ell$.

Next, recall that the group of units $\bar{\mathbb{F}}_q^\times$ may be identified with the group $\mu$ of all roots of unity in $\bar{\mathbb{F}}_q$; its underlying abelian group is isomorphic to $\mathbb{Q}/\mathbb{Z}[1/p]$. Passing to the direct limit of the $K_1(X_\nu)$ yields $K_1(\bar{X}) \cong \mu \oplus \mu$.

For $n \geq 1$, the groups $K_n(\bar{X})$ are all torsion groups, of order prime to $p$, because this is true of each $K_n(X_\nu)$ by 6.1. We can now determine the abelian group structure of the $K_n(\bar{X})$ as well as the action of the Galois group on them. Recall from Definition 1.7 that $M(i)$ denotes the $i^{th}$ Tate twist of a Galois module $M$.

THEOREM 6.4. *Let $X$ be a smooth projective curve over $\mathbb{F}_q$, and set $\bar{X} = X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. Then for all $n \geq 0$ we have isomorphisms of* $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$*-modules:*

$$K_n(\bar{X}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus J(\bar{X}), & n = 0 \\ \mu(i) \oplus \mu(i), & n = 2i - 1 > 0 \\ J(\bar{X})[1/p](i), & n = 2i > 0. \end{cases}$$

*For $\ell \neq p$, the $\ell$-primary subgroup of $K_{n-1}(X)$ is isomorphic to $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$, $n > 0$, whose Galois module structure is given by:*

$$K_n(\bar{X}; \mathbb{Z}/\ell^\infty) \cong \begin{cases} \mathbb{Z}/\ell^\infty(i) \oplus \mathbb{Z}/\ell^\infty(i), & n = 2i \geq 0 \\ J(\bar{X})_\ell(i-1), & n = 2i - 1 > 0. \end{cases}$$

PROOF. Since the groups $K_n(\bar{X})$ are torsion for all $n > 0$, the universal coefficient theorem (Ex. IV.2.6) shows that $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$ is isomorphic to the $\ell$-primary subgroup of $K_{n-1}(\bar{X})$. Thus we only need to determine the Galois modules $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$. For $n = 0, 1, 2$ they may be read off from the above discussion. For $n > 2$ we consider the motivic spectral sequence (4.2); by Theorem 4.1, the terms $E_2^{p,q}$ vanish unless $p = q, q+1, q+2$. There is no room for differentials, so the spectral sequence degenerates at $E_2$ to yield the groups $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$. There are no extension issues because the edge maps are the $e$-invariants $K_{2i}(X; \mathbb{Z}/\ell^\infty) \to H^0_{\mathrm{et}}(\bar{X}, \mathbb{Z}/\ell^\infty(i)) = \mathbb{Z}/\ell^\infty(i)$ of 6.3, and are therefore split surjections. Finally, we note that as Galois modules we have $H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}/\ell^\infty(i)) \cong J(\bar{X})_\ell(i-1)$, and (by Poincaré Duality [Milne2, V.2]) $H^2_{\mathrm{et}}(\bar{X}, \mathbb{Z}/\ell^\infty(i+1)) \cong \mathbb{Z}/\ell^\infty(i)$. $\square$

Passing to invariants under the group $G = \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, there is a natural map from $K_n(X)$ to $K_n(\bar{X})^G$. For odd $n$, we see from 6.4 and 2.3 that $K_{2i-1}(\bar{X})^G \cong \mathbb{Z}/(q^i-1)\oplus\mathbb{Z}/(q^i-1)$; for even $n$, we have the less concrete description $K_{2i}(\bar{X})^G \cong J(\bar{X})[1/p](i)^G$. One way of studying this group is to consider the action of the Frobenius on $H^*_{\mathrm{et}}(\bar{X},\mathbb{Q}_\ell(i))$ and use the identity $H^*_{\mathrm{et}}(X,\mathbb{Q}_\ell(i)) = H^*_{\mathrm{et}}(\bar{X},\mathbb{Q}_\ell(i))^G$, which follows from the spectral sequence $H^p(G,H^*_{\mathrm{et}}(\bar{X},\mathbb{Q}_\ell(i))) \Rightarrow H^*_{\mathrm{et}}(X,\mathbb{Q}_\ell(i))$ since $H^p(G,-)$ is torsion for $p>0$; see [WHomo, 6.11.14].

EXAMPLE 6.5. $\varphi^*$ acts trivially on $H^0_{\mathrm{et}}(\bar{X},\mathbb{Q}_\ell) = \mathbb{Q}_\ell$ and $H^2_{\mathrm{et}}(\bar{X},\mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$. It acts as $q^{-i}$ on the twisted groups $H^0_{\mathrm{et}}(\bar{X},\mathbb{Q}_\ell(i))$ and $H^2_{\mathrm{et}}(\bar{X},\mathbb{Q}_\ell(i+1)$. Taking $G$-invariants yields $H^0_{\mathrm{et}}(X,\mathbb{Q}_\ell(i)) = 0$ for $i\neq 0$ and $H^2_{\mathrm{et}}(X,\mathbb{Q}_\ell(i)) = 0$ for $i \neq 1$.

Weil's 1948 proof of the Riemann Hypothesis for Curves implies that the eigenvalues of $\varphi^*$ acting on $H^1_{\mathrm{et}}(\bar{X},\mathbb{Q}_\ell(i))$ have absolute value $q^{1/2-i}$. Taking $G$-invariants yields $H^1_{\mathrm{et}}(X,\mathbb{Q}_\ell(i)) = 0$ for all $i$.

For any $G$-module $M$, we have an exact sequence [WH, 6.1.4]

$$0 \to M^G \to M \xrightarrow{\varphi^*-1} M \to H^1(G,M) \to 0. \tag{6.5.1}$$

The case $i=1$ of the following result reproduces Weil's theorem that the $\ell$-primary torsion part of the Picard group of $X$ is $J(\bar{X})^G_\ell$.

LEMMA 6.6. *For a smooth projective curve $X$ over $\mathbb{F}_q$, $\ell \nmid q$ and $i \geq 2$ we have:*

1. $H^{n+1}_{\mathrm{et}}(X,\mathbb{Z}_\ell(i)) \cong H^n_{\mathrm{et}}(X,\mathbb{Z}/\ell^\infty(i)) \cong H^n_{\mathrm{et}}(\bar{X},\mathbb{Z}/\ell^\infty(i))^G$ *for all $n$;*
2. $H^0_{\mathrm{et}}(X,\mathbb{Z}/\ell^\infty(i)) \cong \mathbb{Z}/w_i^{(\ell)}(F)$*;*
3. $H^1_{\mathrm{et}}(X,\mathbb{Z}/\ell^\infty(i)) \cong J(\bar{X})_\ell(i-1)^G$*;*
4. $H^2_{\mathrm{et}}(X,\mathbb{Z}/\ell^\infty(i)) \cong \mathbb{Z}/w_{i-1}^{(\ell)}(F)$*; and*
5. $H^n_{\mathrm{et}}(X,\mathbb{Z}/\ell^\infty(i)) = 0$ *for all $n\geq 3$.*

PROOF. Since $i \geq 2$, we see from 6.5 that $H^n_{\mathrm{et}}(X,\mathbb{Q}_\ell(i)) = 0$ for all $n$. Since $\mathbb{Q}_\ell/\mathbb{Z}_\ell = \mathbb{Z}/\ell^\infty$, this yields $H^n_{\mathrm{et}}(X,\mathbb{Z}/\ell^\infty(i)) \cong H^{n+1}_{\mathrm{et}}(X,\mathbb{Z}_\ell(i))$ for all $n$.

Since each $H^n = H^n_{\mathrm{et}}(\bar{X},\mathbb{Z}/\ell^\infty(i))$ is a quotient of $H^n_{\mathrm{et}}(\bar{X},\mathbb{Q}_\ell(i))$, $\varphi^*-1$ is a surjection, *i.e.*, $H^1(G,H^n)=0$. Since $H^n(G,-)=0$ for $n>1$, the Leray spectral sequence $E_2^{p,q} = H^p(G,H^q_{\mathrm{et}}(\bar{X},\mathbb{Z}/\ell^\infty(i))) \Rightarrow H^{p+q}_{\mathrm{et}}(X,\mathbb{Z}/\ell^\infty(i))$ for $X \to \mathrm{Spec}(\mathbb{F}_q)$ [Milne, III.1.18], collapses for $i>1$ to yield exact sequences

$$0 \to H^n_{\mathrm{et}}(X,\mathbb{Z}/\ell^\infty(i)) \to H^n_{\mathrm{et}}(\bar{X},\mathbb{Z}/\ell^\infty(i)) \xrightarrow{\varphi^*-1} H^n_{\mathrm{et}}(\bar{X},\mathbb{Z}/\ell^\infty(i)) \to 0.$$

In particular, $H^n_{\mathrm{et}}(X,\mathbb{Z}/\ell^\infty(i)) = 0$ for $n>2$. As in the proof of 6.4, $H^1_{\mathrm{et}}(X,\mathbb{Z}/\ell^\infty(i))$ is $J(\bar{X})_\ell(i-1)$, so $H^1_{\mathrm{et}}(X,\mathbb{Z}/\ell^\infty(i)) \cong J(\bar{X})_\ell(i-1)^G$, and $H^2_{\mathrm{et}}(\bar{X},\mathbb{Z}/\ell^\infty(i))$ is $\mathbb{Z}/\ell^\infty(i-1)$ by duality, so $H^2_{\mathrm{et}}(X,\mathbb{Z}/\ell^\infty(i))$ is $\mathbb{Z}/\ell^\infty(i-1)^G \cong \mathbb{Z}/w_{i-1}^{(\ell)}$. $\square$

Given the calculation of $K_n(\bar{X})^G$ in 6.4 and the calculation of $H^n_{\mathrm{et}}(X,\mathbb{Z}/\ell^\infty(i))$ in 6.6, we see that the natural map $K_n(X) \to K_n(\bar{X})^G$ is a surjection. Thus the real content of the following theorem is that $K_n(X) \to K_n(\bar{X})^G$ is an isomorphism.

THEOREM 6.7. *Let $X$ be the smooth projective curve corresponding to a global field $F$ over $\mathbb{F}_q$. Then $K_0(X) = \mathbb{Z}\oplus\mathrm{Pic}(X)$, and the finite groups $K_n(X)$ for $n>0$ are given by:*

$$K_n(X) \cong K_n(\bar{X})^G \cong \begin{cases} K_n(\mathbb{F}_q)\oplus K_n(\mathbb{F}_q), & n \text{ odd}, \\ \bigoplus_{\ell\neq p} J(\bar{X})_\ell(i)^G, & n=2i \text{ even}. \end{cases}$$

PROOF. We may assume that $n \neq 0$, so that the groups $K_n(X)$ are finite by 6.1. It suffices to calculate the $\ell$-primary part $K_{n+1}(X;\mathbb{Z}/\ell^\infty)$ of $K_n(X)$. But this follows from the motivic spectral sequence (4.2), which degenerates by 6.6. $\square$

THEOREM 6.8. *If $F$ is the function field of a smooth projective curve $X$ over $\mathbb{F}_q$, then for all $i \geq 1$: $\mathbb{F}_q \subset F$ induces an isomorphism $K_{2i+1}(\mathbb{F}_q) \cong K_{2i+1}(F)$, and there is an exact reciprocity sequence (generalizing V.6.12.1):*

$$0 \to K_{2i}(X) \to K_{2i}(F) \xrightarrow{\oplus \partial_x} \oplus_{x\in X} K_{2i-1}(\mathbb{F}_q(x)) \xrightarrow{N} K_{2i-1}(\mathbb{F}_q) \to 0.$$

PROOF. The calculation of $K_2(F)$ was carried out at the beginning of this section, so we restrict attention to $K_n(F)$ for $n \geq 3$. Because $K_{2i}(\mathbb{F}_q) = 0$, the localization sequence V.6.12 breaks up into exact sequences for $i \geq 2$:

$$0 \to K_{2i}(X) \to K_{2i}(F) \xrightarrow{\oplus \partial_x} \oplus_{x\in X} K_{2i-1}(\mathbb{F}_q(x)) \to K_{2i-1}(X) \to K_{2i-1}(F) \to 0.$$

Let $SK_{2i-1}(X)$ denote the kernel of $K_{2i-1}(X) \to K_{2i-1}(F)$. Since $\mathbb{F}_q \subset F \subset \bar{F}$ induces an injection from the subgroup $K_{2i-1}(\mathbb{F}_q)$ of $K_{2i-1}(\bar{\mathbb{F}}_q)$ into $K_{2i-1}(\bar{F})$ by V.6.7.2, we see from Theorem 6.7 that $|SK_{2i-1}(X)|$ is at most $|K_{2i-1}(\mathbb{F}_q)|$. As in V.6.12, for each closed point $x$ the composition of the transfer $K_{2i-1}(\mathbb{F}_q(x)) \to SK_{2i-1}(X)$ with the proper transfer $\pi_* : K_{2i-1}(X) \to K_{2i-1}(\mathbb{F}_q)$ is the transfer associated to $\mathbb{F}_q \subset \mathbb{F}_q(x)$, *i.e.*, the transfer map $K_{2i-1}(\mathbb{F}_q(x)) \to K_{2i-1}(\mathbb{F}_q)$; each of these transfer maps are onto by IV.1.13. It follows that $\pi_* : SK_{2i-1}(X) \to K_{2i-1}(\mathbb{F}_q)$ is an isomorphism. The theorem now follows. $\square$

THE ZETA FUNCTION 6.9. We can relate the orders of the $K$-groups of the curve $X$ to values of the zeta function $\zeta_X(s)$. By definition, $\zeta_X(s) = Z(X, q^{-s})$, where

$$Z(X,t) = \exp\left(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \, \frac{t^n}{n}\right).$$

Weil proved that $Z(X,t) = P(t)/(1-t)(1-qt)$ for every smooth projective curve $X$, where $P(t) \in \mathbb{Z}[t]$ is a polynomial of degree $2 \cdot \mathrm{genus}(X)$ with all roots of absolute value $1/\sqrt{q}$. This formula is a restatement of Weil's proof of the Riemann Hypothesis for $X$ (6.5 above), given Grothendieck's formula $P(t) = \det(1-\varphi^* t)$, where $\varphi^*$ is regarded as an endomorphism of $H^1_{\mathrm{et}}(\bar{X};\mathbb{Q}_\ell)$. Note that by 6.5 the action of $\varphi^*$ on $H^0_{\mathrm{et}}(\bar{X};\mathbb{Q}_\ell)$ has $\det(1-\varphi^* t) = (1-t)$, and the action on $H^2_{\mathrm{et}}(\bar{X};\mathbb{Q}_\ell)$ has $\det(1-\varphi^* t) = (1-qt)$.

Here is application of Theorem 6.7, which was conjectured by Lichtenbaum in [Li3] and proven by Thomason in [Th, (4.7)]. For legibility, let $\#A$ denote the order of a finite abelian group $A$.

PROPOSITION 6.10. *If $X$ is a smooth projective curve over $\mathbb{F}_q$ then for all $i \geq 2$,*

$$\frac{\#K_{2i-2}(X) \cdot \#K_{2i-3}(\mathbb{F}_q)}{\#K_{2i-1}(\mathbb{F}_q) \cdot \#K_{2i-3}(X)} = \prod_\ell \frac{\#H^2_{\mathrm{et}}(X;\mathbb{Z}_\ell(i))}{\#H^1_{\mathrm{et}}(X;\mathbb{Z}_\ell(i)) \cdot \#H^3_{\mathrm{et}}(X;\mathbb{Z}_\ell(i))} = \big|\zeta_X(1-i)\big|.$$

PROOF. We have seen that all the groups appearing in this formula are finite. The first equality follows from 6.6 and 6.7. The second equality follows from the formula for $\zeta_X(1-i)$ in 6.9. $\square$

EXERCISES

**6.1** Let $X$ be the projective line $\mathbb{P}^1_{\mathbb{F}_q}$ over $\mathbb{F}_q$. Use Theorem 6.7 to recover the calculation of $K_*(X)$ in IV.1.5. Show directly that $Z(X,t) = 1/(1-t)(1-qt)$ and use this to verify the formula in 6.9 for $\zeta_X(i-1)$.

**6.2** Let $R$ be the ring of $S$-integers in a global field $F = \mathbb{F}_q(X)$ of finite characteristic. Show that $K_n(R) \to K_n(F)$ is an injection for all $n \geq 1$, and that $K_{2i-1}(R) \to K_{2i-1}(F)$ is an isomorphism for all $i > 1$. (This generalizes the Bass-Milnor-Serre Theorem III.2.5, and provides another proof of 6.2(a).)

**6.3** Let $F = \mathbb{F}_q(X)$ be a global field, of degree $d$ over a function field $\mathbb{F}_q(t)$. For $i > 0$, show that the transfer $K_{2i}(F) \to K_{2i}(\mathbb{F}_q(t))$ is onto, and that the transfer $K_{2i-1}(F) \to K_{2i-1}(\mathbb{F}_q(t))$ is multiplication by $d$ (under the identification of both groups with $K_{2i-1}(\mathbb{F}_q)$ in 6.8).

## §7. Local Fields

A *local field* is a field $E$ which is complete under a discrete valuation $v$, and whose residue field $k_v$ is finite. The subring $V$ of elements of positive valuation is a complete valuation domain. It is classical that every local field is either a finite extension of the $p$-adic rationals $\hat{\mathbb{Q}}_p$ or of $\mathbb{F}_q((t))$. (See [S-LF].)

We saw in II.2 and III.1.4 that $K_0(V) = K_0(E) = \mathbb{Z}$ and $K_1(V) = V^\times$, $K_1(E) = E^\times \cong (V^\times)\times\mathbb{Z}$, where the factor $\mathbb{Z}$ is identified with the powers $\{\pi^m\}$ of a parameter $\pi$ of $V$. It is well known that $V^\times \cong \mu(E) \times U_1$, where $\mu(E)$ is the group of roots of unity in $E$ (or $V$), and that $U_1$ is a torsionfree $\mathbb{Z}_p$-module.

We also saw in Moore's Theorem (Chapter III, Theorem 6.2.4 and Ex. 6.11) that $K_2(E) \cong U_2 \oplus \mathbb{F}_q^\times$, where $U_2$ is an uncountable, uniquely divisible abelian group. Since $K_2(E) \cong K_2(V) \oplus \mathbb{F}_q^\times$ by V.6.9.2, this implies that $K_2(V) \cong U_2$.

Proposition 7.1. *Let $E$ be a local field. For $n \geq 3$, $K_n^M(E)$ is an uncountable, uniquely divisible group. The group $K_2(E)$ is the sum of $\mathbb{F}_q^\times$ and an uncountable, uniquely divisible group.*

Proof. The group is uncountable by Ex. III.7.14, and divisibility follows easily from Moore's Theorem (see III, Ex. 7.4). We give a proof that it is uniquely divisible using the isomorphism $K_n^M(E) \cong H^n(E, \mathbb{Z}(n))$. If $\mathrm{char}(E) = p$, $K_n^M(E)$ has no $p$-torsion by Izhboldin's Theorem III.7.8, so we consider $m$-torsion when $1/m \in E$. The long exact sequence in motivic cohomology associated to the coefficient sequence $0 \to \mathbb{Z}(n) \xrightarrow{m} \mathbb{Z}(n) \to \mathbb{Z}/m(n) \to 0$ yields the exact sequence for $m$:

$$H_{\mathrm{et}}^{n-1}(E, \mu_m^{\otimes n}) \to K_n^M(E) \xrightarrow{m} K_n^M(E) \to H_{\mathrm{et}}^n(E, \mu_m^{\otimes n}). \tag{7.1.1}$$

Since $H_{\mathrm{et}}^n(E, -)$=0 for $n \geq 3$, this immediately implies that $K_n^M(E)$ is uniquely $m$-divisible for $n > 3$ (and $m$-divisible for $n = 3$). Moreover, the $m$-torsion subgroup of $K_3^M(E)$ is a quotient of the group $H_{\mathrm{et}}^2(E, \mu_m^{\otimes 3})$, which by duality is $\mathbb{Z}/(w_2, m)$, where $w_2 = w_2(E)$ is $q^2 - 1$ by 2.3.1. Thus the torsion subgroup of $K_3^M(E)$ is a quotient of $w_2$. We may therefore assume that $w_2$ divides $m$. Now map the sequence (7.1.1) for $m^2$ to the sequence (7.1.1) for $m$; the map from $\mathbb{Z}/w_i = H_{\mathrm{et}}^2(E, \mu_{m^2}^{\otimes 3})$ to $\mathbb{Z}/w_i = H_{\mathrm{et}}^2(E, \mu_m^{\otimes 3})$ is the identity but the map from the image ${}_{m^2}K_3^M(E)$ to ${}_mK_3^M(E)$ is multiplication by $m$ and thus zero, as required. □

*Equicharacteristic local fields*

We first dispose of the equi-characteristic case, where $E = \mathbb{F}_q((t))$, $V \cong \mathbb{F}_q[[\pi]]$ and $\mathrm{char}(E) = p$. In this case, $\mu(E) = \mathbb{F}_q^\times$, and $U_1 = 1 + \pi\mathbb{F}_q[[\pi]]$ is isomorphic to the big Witt vectors of $\mathbb{F}_q$ (II.4.3), which is the product of a countably infinite number of copies of $\mathbb{Z}_p$ (see Ex. 7.2). (In fact, it is a countably infinite product of copies of the ring $\mathbb{Z}_p[\zeta_{q-1}]$ of Witt vectors over $\mathbb{F}_q$.

Here is a description of the abelian group structure on $K_n(V)$ for $n \geq 2$.

Theorem 7.2. *Let $V = \mathbb{F}_q[[\pi]]$ be the ring of integers in the local field $E = \mathbb{F}_q((\pi))$. For $n \geq 2$ there are uncountable, uniquely divisible abelian groups $U_n$ and canonical isomorphisms:*

$$K_n(E) \cong K_n(V) \oplus K_{n-1}(\mathbb{F}_q), \quad K_n(V) \cong K_n(\mathbb{F}_q) \oplus U_n.$$

PROOF. The splitting $K_n(E) \cong K_n(V) \oplus K_{n-1}(\mathbb{F}_q)$ was established in V.6.9.2. Now let $U_n$ denote the kernel of the canonical map $K_n(V) \to K_n(\mathbb{F}_q)$. Since $V \to \mathbb{F}_q$ splits, naturality yields $K_n(V) = U_n \oplus K_n(\mathbb{F}_q)$. By Gabber rigidity (IV.2.10), $U_n$ is uniquely $\ell$-divisible for all $\ell \neq p$ and $n > 0$. It suffices to show that $U_n$ is uncountable and uniquely $p$-divisible when $n \geq 2$; this holds for $n = 2$ by 7.1.

Now the Milnor groups $K_n^M(E)$ are uncountable, uniquely divisible abelian groups for $n \geq 3$, by Proposition 7.1. The group $K_n^M(E)$ is a summand of the Quillen $K$-group $K_n(E)$ by 4.3 or 4.9. On the other hand, the Geisser-Levine Theorem 4.7 shows that the complementary summand is uniquely $p$-divisible. $\square$

*p-adic local fields*

In the mixed characteristic case, when $\mathrm{char}(E) = 0$, even the structure of $V^\times$ is quite interesting. The torsionfree part $U_1$ is a free $\mathbb{Z}_p$-module of rank $[E : \mathbb{Q}_p]$; it is contained in $(1 + \pi V)^\times$ and injects into $E$ as a lattice by the convergent power series for $x \mapsto \ln(x)$.

The quotient $V^\times \to \mathbb{F}_q^\times$ splits, and the subgroup of $V^\times$ isomorphic to $\mathbb{F}_q^\times$ is called the group of *Teichmüller units*. Thus $K_1(V) = V^\times$ is a product $U_1 \times \mathbb{F}_1^\times \times \mu_{p^\infty}(E)$, where $\mu_{p^\infty}(E)$ is the finite cyclic group of $p$-primary roots of unity in $V$. There seems to be no simple formula for the order of the cyclic $p$-group $\mu_{p^\infty}(E)$.

To understand the groups $K_n(E)$ for $n \geq 3$, recall from Proposition 7.1 that $K_n^M(E)$ is an uncountable, uniquely divisible abelian group. From Example 4.3, this is a direct summand of $K_n(E)$; since $K_n(E) \cong K_n(V) \oplus K_{n-1}(\mathbb{F}_q)$ by V.6.9.2, it is also a summand of $K_n(V)$. Thus, as in the equicharacteristic case, $K_n(E)$ will contain an uncountable uniquely divisible summand about which we can say very little.

Before stating our next result, we need an étale calculation. Since $E$ has étale cohomological dimension 2, we may ignore $H^n_{\mathrm{et}}(E, -)$ for $n > 2$. By Tate–Poitou duality [Milne2, I.2.3], $H^2_{\mathrm{et}}(E, \mu_m^{\otimes i+1})$ is isomorphic to $H^0_{\mathrm{et}}(E, \mu_m^{\otimes i})$. We shall assume that $i > 0$ and $m$ is divisible by $w_i(E)$, so that these groups are isomorphic to $\mathbb{Z}/w_i(E)$. Now consider the change of coefficients $\mu_m^{\otimes i} \subset \mu_{m^2}^{\otimes i}$. The induced endomorphisms of $\mathbb{Z}/w_i(E) = H^0_{\mathrm{et}}(E, \mu_m^{\otimes i})$ and $\mathbb{Z}/w_i(E) = H^2_{\mathrm{et}}(E, \mu_m^{\otimes i+1})$ are the identity map and the zero map, respectively. Since $\mu = \cup \mu_m$, passing to the limit over $m$ yields:

$$H^0_{\mathrm{et}}(E, \mu^{\otimes i}) \cong \mathbb{Z}/w_i(E) \quad \text{and} \quad H^2_{\mathrm{et}}(E, \mu^{\otimes i+1}) = 0, \quad i > 0.$$

Recall that an abelian group which is uniquely $\ell$-divisible for all $\ell \neq p$ is the same thing as a $\mathbb{Z}_{(p)}$-module.

PROPOSITION 7.3. *For $n > 0$ we have $K_n(E) \cong K_n(V) \oplus K_{n-1}(\mathbb{F}_q)$, and the groups $K_n(V)$ are $\mathbb{Z}_{(p)}$-modules.*

*When $n = 2i-1$, $K_n(V) \cong K_n(E)$ is the direct sum of a torsionfree $\mathbb{Z}_{(p)}$-module and the Harris-Segal summand (see 2.5), which is isomorphic to $\mathbb{Z}/w_i(E)$.*

*When $n = 2i$, $K_n(V)$ is the direct sum of $\mathbb{Z}/w_i^{(p)}(E)$ and a divisible $\mathbb{Z}_{(p)}$-module.*

PROOF. The decomposition $K_n(E) \cong K_n(V) \oplus K_{n-1}(\mathbb{F}_q)$ was established in V.6.9.2. In particular, $K_{2i-1}(V) \cong K_{2i-1}(E)$.

To see that $K_{2i-1}(E)$ has a cyclic summand of order $w_i(E)$, consider the spectral sequence (4.2) with coefficients $\mathbb{Q}/\mathbb{Z}$. By the above remarks, it degenerates completely to yield $K_{2i}(E;\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/w_i(E)$. Since this injects into $K_{2i}(\bar{E};\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ (by Theorem 1.6, $\bar{E}$ being the algebraic closure of $E$), this implies that the Adams $e$-invariant is an isomorphism: $K_{2i}(E;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\simeq} \mathbb{Z}/w_i(E)$. By Ex. IV.2.5, this implies that $K_{2i-1}(E) \cong T_i \oplus \mathbb{Z}/w_i(E)$ where $T_i$ is torsionfree.

To see that $K_{2i}(E)$ is the sum of a divisible group and $\mathbb{Z}/w_i(E)$, fix $i$ and suppose that $w_i(E)$ divides $m$. Since $H^n_{\text{et}}(E,-) = 0$ for $n > 2$, the spectral sequence (4.2) with coefficients $\mathbb{Z}/m$ degenerates completely and describes $K_{2i}(E;\mathbb{Z}/m)$ as an extension of $H^0_{\text{et}}(E,\mu_m^{\otimes i}) \cong \mathbb{Z}/w_i(E)$ by $H^2_{\text{et}}(E,\mu_m^{\otimes i+1}) \cong \mathbb{Z}/w_i(E)$. By the previous paragraph, the quotient $\mathbb{Z}/w_i(E)$ is identified with the $m$-torsion in $K_{2i-1}(E)$, so the kernel $\mathbb{Z}/w_i(E)$ is identified with $K_{2i}(E)/m$. Setting $m = m'w_i(E)$, it follows that the subgroup $D_i = w_i(E)K_{2i}(E)$ of $K_{2i}(E)$ is a divisible group. Thus $K_{2i}(E) \cong D_i \oplus \mathbb{Z}/w_i(E)$, and $K_{2i}(V) \cong D_i \oplus \mathbb{Z}/w_i^{(p)}(E)$.

It remains to show that $T_i$ and $D_i$ are uniquely $\ell$-divisible for $\ell \neq p$ and $i > 0$, *i.e.,* that $T_i/\ell T_i = {}_{\ell}(D_i) = 0$. By Universal Coefficients IV.2.5 and the calculations above, $K_{2i-1}(V;\mathbb{Z}/\ell^\nu)$ is isomorphic to $\mathbb{Z}/w_i^{(\ell)}(E) \oplus T_i/\ell^\nu T_i \oplus {}_{\ell^\nu}(D_i)$ for large $\nu$. By Gabber Rigidity IV.2.10, $K_{2i-1}(V;\mathbb{Z}/\ell^\nu)$ and $K_{2i-1}(\mathbb{F}_q;\mathbb{Z}/\ell^\nu) \cong w_i^{(\ell)}(\mathbb{F}_q)$ are isomorphic. Since $w_i^{(\ell)}(E) = w_i^{(\ell)}(\mathbb{F}_q)$ by 2.3.1, we have $T_i/\ell^\nu T_i = {}_{\ell^\nu}(D_i) = 0$, as required. $\square$

Remark 7.3.1. Either $T_i$ fails to be $p$-divisible, or else $D_{i-1}$ has $p$-torsion. This follows from Corollary 7.4.1 below: $\dim(T_i/pT_i) + \dim({}_pD_{i-1}) = [E:\mathbb{Q}_p]$.

We now consider the $p$-adic $K$-groups $K_*(E;\mathbb{Z}_p)$ of $E$, as in IV.2.9. This result was first proved in [RW, 3.7] for $p = 2$, and in [HM, thm. A] for $p > 2$.

Theorem 7.4. *Let $E$ be a local field, of degree $d$ over $\mathbb{Q}_p$, with ring of integers $V$. Then for $n \geq 2$ we have:*

$$K_n(V;\mathbb{Z}_p) \cong K_n(E;\mathbb{Z}_p) \cong \left\{ \begin{array}{ll} \mathbb{Z}/w_i^{(p)}(E), & n = 2i, \\ (\mathbb{Z}_p)^d \oplus \mathbb{Z}/w_i^{(p)}(E), & n = 2i-1. \end{array} \right\}$$

Proof. It is classical that the groups $H^*_{\text{et}}(E;\mathbb{Z}/p^\nu)$ are finitely generated groups, and that the $H^*_{\text{et}}(E;\mathbb{Z}/p)$ are finitely generated $\mathbb{Z}_p$-modules. Since this implies that the groups $K_n(E;\mathbb{Z}/p)$ are finitely generated, this implies (by IV.2.9) that $K_n(E;\mathbb{Z}_p)$ is an extension of the Tate module of $K_{n-1}(E)$ by the $p$-adic completion of $K_n(E)$. Since the Tate module of $K_{2i-1}(E)$ is trivial by 7.3, this implies that $K_{2i}(E;\mathbb{Z}_p) \cong \varprojlim K_{2i}(E)/p^\nu \cong \mathbb{Z}/w_i^{(p)}(E)$.

By 7.3 and IV.2.9, $K_n(V;\mathbb{Z}_p) \cong K_n(E;\mathbb{Z}_p)$ for all $n > 0$. Hence it suffices to consider the $p$-adic group $K_{2i-1}(V;\mathbb{Z}_p)$. By 7.3 and IV.2.9 again, $K_{2i-1}(V;\mathbb{Z}_p)$ is the direct sum of the finite $p$-group $\mathbb{Z}/w_i^{(p)}(E)$ and two finitely generated torsionfree $\mathbb{Z}_p$-modules: the Tate module of $D_{i-1}$ and $T_i \otimes_{\mathbb{Z}} \mathbb{Z}_p$. All that is left is to calculate the rank of $K_{2i-1}(V;\mathbb{Z}_p)$.

Wagoner proved in [Wag] that the $\mathbb{Q}_p$-vector space $K_n(V;\mathbb{Z}_p)\otimes\mathbb{Q}$ has dimension $[E:\mathbb{Q}_p]$ when $n$ is odd and $n \geq 3$. (Wagoner's continuous $K$-groups were identified with $K_*(E;\mathbb{Z}_p)$ in [Pa].) Hence $K_{2i-1}(V;\mathbb{Z}_p)$ has rank $d = [E:\mathbb{Q}_p]$. $\square$

COROLLARY 7.4.1. *For $i > 1$ and all large $\nu$ we have*

$$K_{2i-1}(E;\mathbb{Z}/p^\nu) \cong H^1_{\text{et}}(E,\mu_{p^\nu}^{\otimes i}) \cong (\mathbb{Z}/p^\nu)^{[E:\mathbb{Q}_p]} \oplus \mathbb{Z}/w_i^{(p)}(E) \oplus \mathbb{Z}/w_{i-1}^{(p)}(E).$$

PROOF. This follows from Universal Coefficients and Theorem 7.4. □

COROLLARY 7.4.2. *$K_3(V)$ contains a torsionfree subgroup isomorphic to $\mathbb{Z}^d_{(p)}$, whose $p$-adic completion is isomorphic to $K_3(V;\mathbb{Z}_p) \cong (\mathbb{Z}_p)^d$.*

PROOF. Combine 7.4 with Moore's theorem III.6.2.4 that $D_1$ is torsionfree. □

REMARK 7.4.3. Surprisingly, the cohomology groups $H^1_{\text{et}}(E;\mu_m^{\otimes i})$ (for $m = p^\nu$) were not known before the $K$-group $K_{2i-1}(E;\mathbb{Z}/m)$ was calculated, circa 2000.

WARNING 7.5. Unfortunately, I do not know how to reconstruct the homotopy groups $K_n(V)$ from the information in 7.4. Any of the $\mathbb{Z}_p$'s in $K_{2i-1}(V;\mathbb{Z}_p)$ could come from either a $\mathbb{Z}_{(p)}$ in $K_{2i-1}(V)$ or a $\mathbb{Z}/p^\infty$ in $K_{2i-2}(V)$. Another problem is illustrated by the case $n = 1$, $V = \mathbb{Z}_p$ and $p \neq 2$. The information that $\varprojlim K_1(V)/p^\nu \cong (1+\pi V)^\times \cong V$ is not enough to deduce that $K_1(V) \otimes \mathbb{Z}_{(p)} \cong V$.

Even if we knew that $\varprojlim K_n(V)/p^\nu = \mathbb{Z}_p$, we would still not be able to determine the underlying abelian group $K_n(V)$ exactly. To see why, note that the extension $0 \to \mathbb{Z}_{(p)} \to \mathbb{Z}_p \to \mathbb{Z}_p/\mathbb{Z}_{(p)} \to 0$ doesn't split, because there are no $p$-divisible elements in $\mathbb{Z}_p$, yet $\mathbb{Z}_p/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Q}$ is a uniquely divisible abelian group. For example, I doubt that the extension $0 \to \mathbb{Z}^d_{(p)} \to K_3(V) \to U_3 \to 0$ splits in Corollary 7.4.1.

Here are some more cases when I can show that the $\mathbb{Z}_p$'s in $K_{2i-1}(V;\mathbb{Z}_p)$ come from torsionfree elements in $K_{2i-1}(E)$; I do not know any example where a $\mathbb{Z}/p^\infty$ appears in $K_{2i}(E)$.

EXAMPLE 7.6. If $k > 0$, then $K_{4k+1}(\mathbb{Z}_2)$ contains a subgroup $T$ isomorphic to $\mathbb{Z}_{(2)}$, and the quotient $K_{4k+1}(\mathbb{Z}_2)/(T \oplus \mathbb{Z}/w_i(\mathbb{Q}_2)$ is uniquely divisible. (By Exercise 2.3, $w_i(\mathbb{Q}_2) = 2(2^{2k+1}-1)$.) This follows from Rognes' theorem [R1, 4.13] that the map from $K_{4k+1}(\mathbb{Z}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus (\mathbb{Z}/2)$ to $K_{4k+1}(\mathbb{Z}_2;\mathbb{Z}_2)$ is an isomorphism for all $k > 1$. (The information about the torsion subgroups, missing in [R1], follows from 7.4 and 7.4.1.) Since this map factors through $K_{4k+1}(\mathbb{Z}_2)$, the assertion follows.

EXAMPLE 7.7. Let $F$ be a totally imaginary number field of degree $d = 2r_2$ over $\mathbb{Q}$, with $s$ prime ideals over $p$, and let $E_1, ..., E_s$ be the completions of $F$ at these primes. By Borel's Theorem IV.1.18, there is a subgroup of $K_{2i-1}(F)$ isomorphic to $\mathbb{Z}^{r_2}$; its image in $\oplus K_{2i-1}(E_j)$ is a subgroup of rank at most $r_2$, while $\oplus K_{2i-1}(E_j;\mathbb{Z}_p)$ has rank $2r_2 = \sum [E_j : \mathbb{Q}_p]$. So these subgroups of $K_{2i-1}(E_j)$ can account for at most half of the torsionfree part of $\oplus K_{2i-1}(E_j;\mathbb{Z}_p)$.

EXAMPLE 7.8. Suppose that $F$ is a totally real number field, of degree $d = r_1$ over $\mathbb{Q}$, and let $E_1, ..., E_s$ be the completions of $F$ at the prime ideals over $p$. By Borel's Theorem IV.1.18, there is a subgroup of $K_{4k+1}(F)$ isomorphic to $\mathbb{Z}^d$ for all $k > 0$; its image in $\oplus K_{4k+1}(E_j)$ is a subgroup of rank $d$. Although $\oplus K_{4k+1}(E_j;\mathbb{Z}_p)$ also has rank $d = \sum [E_j : \mathbb{Q}_p]$, this does not imply that the $p$-adic completion $\mathbb{Z}^d_p$ of the subgroup injects into $\oplus K_{4k+1}(E_j;\mathbb{Z}_p)$.

Implications like this are related to Leopoldt's conjecture, which states that the torsionfree part $\mathbb{Z}_p^{d-1}$ of $(\mathcal{O}_F)^\times \otimes \mathbb{Z}_p$ injects into the torsionfree part $\mathbb{Z}_p^d$ of $\prod_{j=1}^s \mathcal{O}_{E_j}^\times$; see [Wash, 5.31]. This conjecture has been proven when $F$ is an abelian extension of $\mathbb{Q}$; see [Wash, 5.32].

When $F$ is a totally real abelian extension of $\mathbb{Q}$, and $p$ is a regular prime, Soulé shows in [Sou81, 3.1, 3.7] that the torsion free part $\mathbb{Z}_p^d$ of $K_{4k+1}(F) \otimes \mathbb{Z}_p$ injects into $\oplus K_{4k+1}(E_j;\mathbb{Z}_p) \cong (\mathbb{Z}_p)^d$, because the cokernel is determined by the Leopoldt $p$-adic $L$-function $L_p(F, \omega^{2k}, 2k+1)$, which is a $p$-adic unit in this favorable scenario. Therefore in this case we also have a summand $\mathbb{Z}_{(p)}^d$ in each of the groups $K_{4k+1}(E_j)$.

We conclude this section with a description of the topological type of $\hat{K}(V)$ and $\hat{K}(E)$, when $p$ is odd, due to Hesselholt and Madsen [HM]. Recall that $F\Psi^k$ denotes the homotopy fiber of $\Psi^k - 1 : \mathbb{Z} \times BU \to BU$. Since $\Psi^k = k^i$ on $\pi_{2i}(BU) = \mathbb{Z}$ when $i > 0$, and $\pi_{2i-1}(BU) = 0$, we see that $\pi_{2i-1} F\Psi^k \cong \mathbb{Z}/(k^i - 1)$, and that all even homotopy groups of $F\Psi^k$ vanish, except for $\pi_0(F\Psi^k) = \mathbb{Z}$.

Theorem 7.9. ([HM, thm.D]) *Let $E$ be a local field, of degree $d$ over $\mathbb{Q}_p$, with $p$ odd. Then after $p$-completion, there is a number $k$ (given below) so that*

$$\hat{K}(V) \simeq SU \times U^{d-1} \times F\Psi^k \times BF\Psi^k, \qquad \hat{K}(E) \simeq U^d \times F\Psi^k \times BF\Psi^k.$$

The number $k$ is defined as follows. As in Proposition 2.2, let $p^a$ be the number of $p$-primary roots of unity in $E(\mu_p)$ and set $r = [E(\mu_p) : E]$. If $\gamma$ is a topological generator of $\mathbb{Z}_p^\times$, then $k = \gamma^n$, where $n = p^{a-1}(p-1)/r$. (See Exercise 7.4.)

EXERCISES

**7.1** Show that $M = \prod_{i=1}^\infty \mathbb{Z}_p$ is not a free $\mathbb{Z}_p$-module. *Hint:* Consider the submodule $S$ of all $(a_1, ...)$ in $M$ where all but finitely many $a_i$ are divisible by $p^\nu$ for all $\nu$. If $M$ were free, $S$ would also be free. Show that $S/p$ is countable, and that $\prod p^i \mathbb{Z}_p$ is an uncountably generated submodule of $S$. Hence $S$ cannot be free.

**7.2** When $E = \mathbb{F}_q[[\pi]]$, show that the subgroup $W(\mathbb{F}_q) = 1 + \pi\mathbb{F}_q[[\pi]]$ of units is a module over $\mathbb{Z}_p$, by defining $(1 + \pi f)^a$ for all power series $f$ and all $a \in \mathbb{Z}_p$. If $\{u_i\}$ is a basis of $\mathbb{F}_q$ over $\mathbb{F}_p$, show that every element of $W(\mathbb{F}_q)$ is uniquely the product of terms $(1 + u_i t^n)^{a_{ni}}$, where $a_{ni} \in \mathbb{Z}_p$. This shows that $W(\mathbb{F}_q)$ is a countably infinite product of copies of $\mathbb{Z}_p$. Using Ex. 7.1, conclude that $W(\mathbb{F}_q)$ is not a free $\mathbb{Z}_p$-module.

**7.3** Show that the first étale Chern classes $K_{2i-1}(E;\mathbb{Z}/p^\nu) \cong H^1(E, \mu_{p^\nu}^{\otimes i})$ are natural isomorphisms for all $i$ and $\nu$.

**7.4** In Theorem 7.9, check that $\pi_{2i-1} F\Psi^k \cong \mathbb{Z}_p/(k^i - 1)$ is $\mathbb{Z}/w_i(E)$ for all $i$.

## §8. Number fields at primes where $cd = 2$

In this section we quickly obtain a cohomological description of the odd torsion in the $K$-groups of a number field, and also the 2-primary torsion in the $K$-groups of a totally imaginary number field. These are the cases where $cd_\ell(\mathcal{O}_S) = 2$; see [Milne2, 4.10]. This bound forces the motivic spectral sequence 4.2 to degenerate completely, leaving an easily-solved extension problem.

Classical Data 8.1. Let $\mathcal{O}_S$ be a ring of integers in a number field $F$. By Chapter IV, 1.18 and 6.9, the groups $K_n(F)$ are finite when $n$ is even and nonzero; if $n$ is odd and $n \geq 3$ the groups $K_n(F)$ are the direct sum of a finite group and $\mathbb{Z}^r$, where $r$ is $r_2$ when $n \equiv 3 \pmod 4$ and $r_1 + r_2$ when $n \equiv 1 \pmod 4$. Here $r_1$ is the number of real embeddings of $F$, and $r_2$ is the number of complex embeddings (up to conjugacy), so that $[F : \mathbb{Q}] = r_1 + 2r_2$. The formulas for $K_0(\mathcal{O}_S) = \mathbb{Z} \oplus \mathrm{Pic}(\mathcal{O}_S)$ and $K_1(\mathcal{O}_S) = \mathcal{O}_S^\times \cong \mathbb{Z}^{r_2+|S|-1} \oplus \mu(F)$ are different; see II.2.6.3 and III.1.3.6.

The Brauer group of $\mathcal{O}_S$ is determined by the sequence

$$0 \to \mathrm{Br}(\mathcal{O}_S) \to (\mathbb{Z}/2)^{r_1} \oplus \coprod_{\substack{v \in S \\ \text{finite}}} (\mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{add}} \mathbb{Q}/\mathbb{Z} \to 0. \tag{8.1.1}$$

The notation $A_{(\ell)}$ will denote the localization of an abelian group $A$ at the prime $\ell$, and the notation ${}_\ell A$ will denote the subgroup $\{a \in A \mid \ell a = 0\}$.

Theorem 8.2. *Let $F$ be a number field, and let $\mathcal{O}_S$ be a ring of integers in $F$. Fix a prime $\ell$; if $\ell = 2$ we suppose $F$ totally imaginary. Then for all $n \geq 2$:*

$$K_n(\mathcal{O}_S)_{(\ell)} \cong \begin{cases} H^2_{\text{et}}(\mathcal{O}_S[1/\ell]; \mathbb{Z}_\ell(i+1)) & \text{for } n = 2i > 0; \\ \mathbb{Z}_{(\ell)}^{r_2} \oplus \mathbb{Z}/w_i^{(\ell)}(F) & \text{for } n = 2i-1,\ i \text{ even}; \\ \mathbb{Z}_{(\ell)}^{r_2+r_1} \oplus \mathbb{Z}/w_i^{(\ell)}(F) & \text{for } n = 2i-1,\ i \text{ odd}. \end{cases}$$

Proof. Set $R = \mathcal{O}_S[1/\ell]$. Then by the localization sequence (V, 6.6 or 6.8), $K_n(\mathcal{O}_S)_{(\ell)} = K_n(R)_{(\ell)}$. Thus we may replace $\mathcal{O}_S$ by $R = \mathcal{O}_S[1/\ell]$. Since the rank of $K_n(R)$ is classically known (see 8.1), it suffices by IV.2.9 and Ex. IV.2.6 to determine $K_{2i-1}(R)\{\ell\} = K_{2i}(R; \mathbb{Z}/\ell^\infty)$ and $K_{2i}(R)\{\ell\} = K_{2i}(R; \mathbb{Z}_\ell)$.

If $F$ is a number field, the étale $\ell$-cohomological dimension of $R$ (and of $F$) is 2, unless $\ell = 2$ and $r_1 > 0$ ($F$ has a real embedding). Since $H^2_{\text{et}}(R; \mathbb{Z}/\ell^\infty(i)) = 0$ by Ex. 8.1, the motivic spectral sequence 4.2 with coefficients $\mathbb{Z}/\ell^\infty$ has at most one nonzero entry in each total degree at the $E_2$ page. Thus we may read off:

$$K_n(R; \mathbb{Z}/\ell^\infty) \cong \begin{cases} H^0(R; \mathbb{Z}/\ell^\infty(i)) = \mathbb{Z}/w_i^{(\ell)}(F) & \text{for } n = 2i \geq 2, \\ H^1(R; \mathbb{Z}/\ell^\infty(i)) & \text{for } n = 2i-1 \geq 1. \end{cases}$$

The description of $K_{2i-1}(R)\{\ell\} = K_{2i}(R; \mathbb{Z}/\ell^\infty)$ follows.

The same argument works for coefficients $\mathbb{Z}_\ell$; for $i > 0$ we have $H^n_{\text{et}}(R, \mathbb{Z}_\ell(i)) = 0$ for $n \neq 1, 2$, so the spectral sequence (4.2) degenerates to yield $K_{2i-1}(R; \mathbb{Z}_\ell) \cong H^1_{\text{et}}(R, \mathbb{Z}_\ell(i))$ and $K_{2i}(R; \mathbb{Z}_\ell) \cong H^2_{\text{et}}(R, \mathbb{Z}_\ell(i+1))$ (which is a finite group by Ex. 8.1). The description of $K_{2i}(R)\{\ell\} = K_{2i}(R; \mathbb{Z}_\ell)$ follows. $\square$

COROLLARY 8.3. *For all odd $\ell$ and $i > 0$, $K_{2i}(\mathcal{O}_S)/\ell \cong H^2_{\text{et}}(\mathcal{O}_S[1/\ell], \mu_\ell^{\otimes i+1})$. The same formula holds for $\ell = 2$ if $F$ is totally imaginary.*

PROOF. Immediate from 8.2 since $H^2_{\text{et}}(R, \mathbb{Z}_\ell(i+1))/\ell \cong H^2_{\text{et}}(R, \mu_\ell^{\otimes i+1})$. $\square$

EXAMPLE 8.3.1. Let $F$ be a number field containing a primitive $\ell^{th}$ root of unity, $\ell \neq 2$, and let $S$ be the set of primes over $\ell$ in $\mathcal{O}_F$. If $t$ is the rank of $\text{Pic}(\mathcal{O}_S)/\ell$, then $H^2_{\text{et}}(\mathcal{O}_S, \mu_\ell)$ has rank $t+|S|-1$ by (8.1.1). Since $H^2_{\text{et}}(\mathcal{O}_S, \mu_\ell^{\otimes i+1}) \cong H^2_{\text{et}}(\mathcal{O}_S, \mu_\ell) \otimes \mu_\ell^{\otimes i}$, it follows from Corollary 8.3 that $K_{2i}(\mathcal{O}_S)/\ell$ has rank $t+|S|-1$. Hence the $\ell$-primary subgroup of the finite group $K_{2i}(\mathcal{O}_F)$ has $t+|S|-1$ nonzero summands for each $i \geq 1$.

EXAMPLE 8.3.2. If $\ell \neq 2$ is a regular prime (see 2.4.1), we claim that $K_{2i}(\mathbb{Z}[\zeta_\ell])$ has no $\ell$-torsion. The case $K_0$ is tautological since $\text{Pic}(\mathcal{O}_F)/\ell = 0$ by definition. Setting $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$, we have $|S| = 1$ and $\text{Br}(R) = 0$ by (8.1.1). The case $K_2$ is known classically; see III.6.9.3. By Theorem 8.2, $K_{2i}(\mathbb{Z}[\zeta_\ell])_{(\ell)} \cong H^2_{\text{et}}(R, \mathbb{Z}_\ell(i+1))$. By Example 8.3.1, $H^2_{\text{et}}(R, \mathbb{Z}_\ell(i+1)) = 0$ and the claim now follows.

Note that every odd-indexed group $K_{2i-1}(\mathbb{Z}[\zeta_\ell]) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F)$ has nontrivial $\ell$-torsion, because $w_i^{(\ell)}(F) \geq \ell$ for all $i$ by 2.2.

Combining Theorems 8.1 and 8.2, we obtain a description of $K_*(\mathcal{O}_S)$ when $F$ is totally imaginary. This includes exceptional number fields such as $\mathbb{Q}(\sqrt{-7})$.

THEOREM 8.4. *Let $F$ be a totally imaginary number field, and let $\mathcal{O}_S$ be the ring of $S$-integers in $F$ for some set $S$ of finite places. Then:*

$$K_n(\mathcal{O}_S) \cong \begin{cases} \mathbb{Z} \oplus \text{Pic}(\mathcal{O}_S), & \textit{for } n = 0; \\ \mathbb{Z}^{r_2+|S|-1} \oplus \mathbb{Z}/w_1(F), & \textit{for } n = 1; \\ \oplus_\ell \ H^2_{\text{et}}(\mathcal{O}_S[1/\ell]; \mathbb{Z}_\ell(i+1)) & \textit{for } n = 2i \geq 2; \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F) & \textit{for } n = 2i-1 \geq 3. \end{cases}$$

PROOF. The cases $n = 0, 1$ and the ranks of $K_n$ are part of the Classical Data 8.1. Since $F$ is totally imaginary, the torsion comes from Theorem 8.2. $\square$

Similarly, the mod-$\ell$ spectral sequence (4.2) collapses when $\ell$ is odd to yield the $K$-theory of $\mathcal{O}_S$ with coefficients $\mathbb{Z}/\ell$, as our next example illustrates.

EXAMPLE 8.5. If $\mathcal{O}_S$ contains a primitive $\ell^{th}$ root of unity and $1/\ell \in \mathcal{O}_S$ then $H^1(\mathcal{O}_S; \mu_\ell^{\otimes i}) \cong \mathcal{O}_S^\times / \mathcal{O}_S^{\times \ell} \oplus {}_\ell \text{Pic}(\mathcal{O}_S)$ and $H^2(\mathcal{O}_S; \mu_\ell^{\otimes i}) \cong \text{Pic}(\mathcal{O}_S)/\ell \oplus {}_\ell\text{Br}(\mathcal{O}_S)$ for all $i$, so $K_0(\mathcal{O}_S; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell \oplus \text{Pic}(\mathcal{O}_S)/\ell$ and

$$K_n(\mathcal{O}_S; \mathbb{Z}/\ell) \cong \begin{cases} \mathcal{O}_S^\times / \mathcal{O}_S^{\times \ell} \oplus {}_\ell \text{Pic}(\mathcal{O}_S) & \text{for } n = 2i-1 \geq 1, \\ \mathbb{Z}/\ell \oplus \text{Pic}(\mathcal{O}_S)/\ell \oplus {}_\ell\text{Br}(\mathcal{O}_S) & \text{for } n = 2i \geq 2. \end{cases}$$

The $\mathbb{Z}/\ell$ summands in degrees $2i$ are generated by the powers $\beta^i$ of the Bott element $\beta \in K_2(\mathcal{O}_S; \mathbb{Z}/\ell)$ (see IV.2.5.2). In fact, $K_*(\mathcal{O}_S; \mathbb{Z}/\ell)$ is free as a graded $\mathbb{Z}[\beta]$-module on $\text{Pic}(\mathcal{O}_S)/\ell$, $K_1(\mathcal{O}_S; \mathbb{Z}/\ell)$ and ${}_\ell\text{Br}(\mathcal{O}_S) \subseteq K_2(\mathcal{O}_S; \mathbb{Z}/\ell)$; this is immediate from the multiplicative properties of 4.2 described in 4.2.1. Taking the direct limit over $S$, we also have $K_0(F) = \mathbb{Z}/\ell$ and

$$K_n(F; \mathbb{Z}/\ell) \cong \begin{cases} F^\times / F^{\times \ell}, & \text{for } n = 2i-1 \geq 1, \\ \mathbb{Z}/\ell \oplus {}_\ell\text{Br}(F) & \text{for } n = 2i \geq 2. \end{cases}$$

We conclude this section with a comparison to the odd part of $\zeta_F(1-2k)$.

BIRCH-TATE CONJECTURE 8.6. If $F$ is a number field, the zeta function $\zeta_F(s)$ has a pole of order $r_2$ at $s = -1$. Birch and Tate conjectured in 1970 that for totally real number fields ($r_2 = 0$) we have

$$\zeta_F(-1) = (-1)^{r_1}|K_2(\mathcal{O}_F)|/w_2(F).$$

The odd part of this conjecture was proven by Wiles in [Wi], using Tate's calculation that $K_2(\mathcal{O}_S)/m \cong H^2_{\text{et}}(\mathcal{O}_S, \mu_m^{\otimes 2})$ when $1/m \in \mathcal{O}_S$ (see [Tate] or III(6.10.4)).

The two-primary part of the Birch-Tate conjecture is still open, but it is known to be a consequence of the 2-adic Main Conjecture of Iwasawa Theory (see Kolster's appendix to [RW]). This was proven by Wiles for abelian extensions of $\mathbb{Q}$ in *loc. cit.*, so the full Birch-Tate Conjecture holds for all abelian extensions of $\mathbb{Q}$. For example, when $F = \mathbb{Q}$ we have $\zeta_{\mathbb{Q}}(-1) = -1/12$, $|K_2(\mathbb{Z})| = 2$ and $w_2(\mathbb{Q}) = 24$; see the Classical Data 8.1.

To generalize the Birch-Tate Conjecture 8.6, we invoke the following deep result of Wiles [Wi, Thm. 1.6], which is often called the "Main Conjecture" of Iwasawa Theory.

THEOREM 8.7 (WILES). *Let $F$ be a totally real number field. If $\ell$ is odd and $\mathcal{O}_S = \mathcal{O}_F[1/\ell]$, then for all even integers $2k > 0$ there is a rational number $u_k$, prime to $\ell$, such that:*

$$\zeta_F(1-2k) = u_k \, \frac{|H^2_{\text{et}}(\mathcal{O}_S, \mathbb{Z}_\ell(2k)|}{|H^1_{\text{et}}(\mathcal{O}_S, \mathbb{Z}_\ell(2k)|}.$$

The numerator and denominator on the right side are finite (Ex. 8.1–8.2). Note that if $F$ is not totally real then $\zeta_F(s)$ has a pole of order $r_2$ at $s = 1-2k$.

We can now verify a conjecture of Lichtenbaum, made in [Li2, 2.4–2.6], which was only stated up to powers of 2.

THEOREM 8.8. *If $F$ is totally real, and* $\mathrm{Gal}(F/\mathbb{Q})$ *is abelian, then for all $k \geq 1$:*

$$\zeta_F(1-2k) = (-1)^{kr_1}\, 2^{r_1} \frac{|K_{4k-2}(\mathcal{O}_F)|}{|K_{4k-1}(\mathcal{O}_F)|}.$$

PROOF. We first show that the left and right sides of 8.8 have the same power of each odd prime $\ell$. The group $H^2_{\text{et}}(\mathcal{O}_F[1/\ell], \mathbb{Z}_\ell(2k))$ is the $\ell$-primary part of $K_{4k-2}(\mathcal{O}_F)$ by Theorem 8.2. The group $H^1_{\text{et}}(\mathcal{O}_F[1/\ell], \mathbb{Z}_\ell(2k))$ in the numerator of 8.7 is $K_{4k-1}(\mathcal{O}_F)_{(\ell)} \cong \mathbb{Z}/w_{2k}^{(\ell)}(F)$ by the proof of 8.2; the details are left to Exercise 8.3.

By the functional equation, the sign of $\zeta_F(1-2k)$ is $(-1)^{kr_1}$. Therefore it remains to check the power of 2 in Theorem 8.8. By Theorem 9.12 in the next Section, the power of 2 on the right side equals $|H^2_{\text{et}}(\mathcal{O}_F[1/\ell], \mathbb{Z}_2(2k))|/|H^1_{\text{et}}(\mathcal{O}_F[1/\ell], \mathbb{Z}_2(2k))|$. By the 2-adic Main Conjecture of Iwasawa Theory, which is known for abelian $F$, this equals the 2-part of $\zeta_F(1-2k)$. $\square$

EXERCISES

**8.1** Suppose that $\ell$ is odd, or that $F$ is totally imaginary. If $R$ is any ring of integers in $F$ containing $1/\ell$, and $i \geq 2$, show that $H^2_{\text{et}}(R, \mathbb{Z}_\ell(i))$ is a finite group, and conclude that $H^2_{\text{et}}(R, \mathbb{Z}/\ell^\infty(i)) = 0$. *Hint:* Use 4.2 to compare it to $K_{2i-2}(R)$, which is finite by IV, 1.18, 2.9 and 8.8. Then apply Ex. IV.2.6.

**8.2** Supose that $F$ is any number field, and let $R$ be any ring of integers in $F$ containing $1/\ell$. Show that $H^2_{\text{et}}(R, \mathbb{Z}_\ell(i))$ is a finite group for all $i \geq 2$, *Hint:* Let $R'$ be the integral closure of $R$ in $F(\sqrt{-1})$. Use a transfer argument to show that the kernel $A$ of $H^2_{\text{et}}(R, \mathbb{Z}_\ell(i)) \to H^2_{\text{et}}(R', \mathbb{Z}_\ell(i))$ has exponent 2; $A$ is finite because it injects into $H^2_{\text{et}}(R, \mu_2)$. Now apply Exercise 8.1.

**8.3** Let $\ell$ be an odd prime and $F$ a number field. If $i > 1$, show that for every ring $\mathcal{O}_S$ of integers in $F$ containing $1/\ell$,

$$H^1_{\text{et}}(\mathcal{O}_S, \mathbb{Z}_\ell(i)) \cong H^1_{\text{et}}(F, \mathbb{Z}_\ell(i)) \cong \mathbb{Z}_\ell^r \oplus \mathbb{Z}/w_i^{(\ell)}(F),$$

where $r$ is $r_2$ for even $i$ and $r_1 + r_2$ for odd $i$. *Hint:* Compare to $K_{2i-1}(F; \mathbb{Z}_\ell)$, as in the proof of 8.2.

**8.4** It is well known that $\mathbb{Z}[i]$ is a principal ideal domain. Show that the finite group $K_n(\mathbb{Z}[i])$ has odd order for all even $n > 0$. *Hint:* Show that $H^2_{\text{et}}(\mathbb{Z}[\frac{1}{2}, i], \mu_4) = 0$.

**8.5** Show that $K_3(\mathbb{Z}[i]) \cong \mathbb{Z} \oplus \mathbb{Z}/24$, $K_7(\mathbb{Z}[i]) \cong \mathbb{Z} \oplus \mathbb{Z}/240$ and $K_{4k+1}(\mathbb{Z}[i]) \cong \mathbb{Z} \oplus \mathbb{Z}/4$ for all $k > 0$. Note that the groups $w_i(\mathbb{Q}(\sqrt{-1}))$ are given in 2.1.2.

**8.6** Let $F$ be a number field. Recall (Ex. IV.7.10) that there is a canonical involution on $K_*(F)$, and that it is multiplication by $-1$ on $K_1(F) = F^\times$. Show that it is multiplication by $(-1)^i$ on $K_{2i-1}(\mathcal{O}_F)$ and $K_{2i-2}(\mathcal{O}_F)$ for $i > 1$. *Hint:* Pick an odd prime $\ell$ and consider the canonical involution on $K_{2i-1}(F(\zeta_\ell); \mathbb{Z}/\ell)$.

## §9. Real number fields at the prime 2

Let $F$ be a real number field, i.e., $F$ has $r_1 > 0$ embeddings into $\mathbb{R}$. The calculation of the algebraic $K$-theory of $F$ at the prime 2 is somewhat different from the calculation at odd primes, for two reasons. One reason is that a real number field has infinite cohomological dimension, which complicates descent methods. A second reason is that the Galois group of a cyclotomic extension need not be cyclic, so that the $e$-invariant may not split (see 2.1.2). A final reason, explained in IV.2.8, is that, while $K_*(F; \mathbb{Z}/2^\nu)$ is a graded ring for $\ell^\nu = 8$ and a graded-commutative ring for $2^\nu \geq 16$, its graded product may be non-associative and non-commutative for $\ell^\nu = 4$, the groups $K_*(F; \mathbb{Z}/2)$ do not have a natural multiplication.

For the real numbers $\mathbb{R}$, the mod 2 motivic spectral sequence has $E_2^{p,q} = \mathbb{Z}/2$ for all $p, q$ in the octant $q \leq p \leq 0$. In order to distinguish between these terms, it is useful to label the nonzero elements of $H^0_{\text{et}}(\mathbb{R}, \mathbb{Z}/2(i))$ as $\beta_i$, writing 1 for $\beta_0$. Using the multiplicative pairing with the spectral sequence $'E_2^{*,*}$ converging to $K_*(\mathbb{R})$, multiplication by the element $\eta$ of $'E_2^{0,-1} = H^1(\mathbb{R}, \mathbb{Z}(1))$ allows us to write the nonzero elements in the $-i$th column as $\eta^j \beta_i$. (See Table 9.1.1 below)

From Suslin's calculation of $K_n(\mathbb{R})$ in Theorem 3.1, we know that the groups $K_n(\mathbb{R}; \mathbb{Z}/2)$ are cyclic and 8-periodic (for $n \geq 0$) with orders $2, 2, 4, 2, 2, 0, 0, 0$ for $n = 0, 1, \ldots, 7$. The unexpected case $K_2(\mathbb{R}; \mathbb{Z}/2) \cong \mathbb{Z}/4$ is described in IV.2.5.1.

THEOREM 9.1. *In the spectral sequence 4.2 converging to $K_*(\mathbb{R};\mathbb{Z}/2)$, all the $d_2$ differentials with nonzero source on the lines $p \equiv 1,2 \pmod 4$ are isomorphisms. Hence the spectral sequence degenerates at $E_3$. The only extensions are the nontrivial extensions $\mathbb{Z}/4$ in $K_{8a+2}(\mathbb{R};\mathbb{Z}/2)$.*

PROOF. Recall from 4.8.1 that the mod 2 spectral sequence has periodicity isomorphisms $E_r^{p,q} \xrightarrow{\cong} E_r^{p-4,q-4}$, $p \le 0$. Therefore it suffices to work with the columns $-3 \le p \le 0$. These columns are shown in Table 9.1.1.

Because $K_3(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/2$, the differential closest to the origin, from $\beta_2$ to $\eta^3$, must be nonzero. Since the pairing with $'E_2$ is multiplicative and $d_2(\eta) = 0$, we must have $d_2(\eta^j\beta_2) = \eta^{j+3}$ for all $j \ge 0$. Thus the column $p = -2$ of $E_3$ is zero, and every term in the column $p = 0$ of $E_3$ is zero except for $\{1, \eta, \eta^2\}$.

Similarly, we must have $d_2(\beta_3) = \eta^3\beta_1$ because $K_5(\mathbb{R};\mathbb{Z}/2) = 0$. By multiplicativity, this yields $d_2(\eta^j\beta_3) = \eta^{j+3}\beta_1$ for all $j \ge 0$. Thus the column $p = -3$ of $E_3$ is zero, and every term in the column $p = -1$ of $E_3$ is zero except for $\{\beta_1, \eta\beta_1, \eta^2\beta_1\}$. $\square$

| $p = -3$ | $-2$ | $-1$ | $0$ |
|---|---|---|---|
| | | | $1$ |
| | | $\beta_1$ | $\eta$ |
| | $\beta_2$ | $\eta\beta_1$ | $\eta^2$ |
| $\beta_3$ | $\eta\beta_2$ | $\eta^2\beta_1$ | $\eta^3$ |
| $\eta\beta_3$ | $\eta^2\beta_2$ | $\eta^3\beta_1$ | $\eta^4$ |

The first 4 columns of $E_2$

| $-3$ | $-2$ | $-1$ | $0$ |
|---|---|---|---|
| | | | $1$ |
| | | $\beta_1$ | $\eta$ |
| | $0$ | $\eta\beta_1$ | $\eta^2$ |
| $0$ | $0$ | $\eta^2\beta_1$ | $0$ |
| $0$ | $0$ | $0$ | $0$ |

The same columns of $E_3$

Table 9.1.1. The mod 2 spectral sequence for $\mathbb{R}$.

VARIANT 9.1.2. The spectral sequence 4.2 with coefficients $\mathbb{Z}/2^\infty$ is very similar, except that when $p > q$, $E_2^{p,q} = H_{\text{et}}^{p-q}(\mathbb{R};\mathbb{Z}/2^\infty(-q))$ is: 0 for $p$ even; $\mathbb{Z}/2$ for $p$ odd. If $p$ is odd, the coefficient map $\mathbb{Z}/2 \to \mathbb{Z}/2^\infty$ induces isomorphisms on the $E_2^{p,q}$ terms, so by Theorem 9.1 all the $d_2$ differentials with nonzero source in the columns $p \equiv 1 \pmod 4$ are isomorphisms. Again, the spectral sequence converging to $K_*(\mathbb{R};\mathbb{Z}/2^\infty)$ degenerates at $E_3 = E_\infty$. The only extensions are the nontrivial extensions of $\mathbb{Z}/2^\infty$ by $\mathbb{Z}/2$ in $K_{8a+4}(\mathbb{R};\mathbb{Z}/2^\infty) \cong \mathbb{Z}/2^\infty$.

VARIANT 9.1.3. The analysis of the spectral sequence with 2-adic coefficients is very similar, except that (a) $H^0(\mathbb{R};\mathbb{Z}_2(i))$ is: $\mathbb{Z}_2$ for $i$ even; 0 for $i$ odd and (b) (for $p > q$) $E_2^{p,q} = H_{\text{et}}^{p-q}(\mathbb{R};\mathbb{Z}/2^\infty(-q))$ is: $\mathbb{Z}/2$ for $p$ even; 0 for $p$ odd. All differentials with nonzero source in the column $p \equiv 2 \pmod 4$ are onto. Since there are no extensions to worry about, we omit the details.

We now consider the $K$-theory of the ring $\mathcal{O}_S$ of integers in a number field $F$ with coefficients $\mathbb{Z}/2^\infty$. The $E_2$ terms in the spectral sequence 4.2 are the (étale) cohomology groups $H^n(\mathcal{O}_S;\mathbb{Z}/2^\infty(i))$. Following Tate, the $r_1$ real embeddings of $F$ define natural maps $\alpha_S^n(i)$:

$$(9.2)\quad \alpha_S^n(i)\colon H^n(\mathcal{O}_S;\mathbb{Z}/2^\infty(i)) \to \bigoplus^{r_1} H^n(\mathbb{R};\mathbb{Z}/2^\infty(i)) \cong \begin{cases} (\mathbb{Z}/2)^{r_1}, & i-n \text{ odd} \\ 0, & i-n \text{ even.} \end{cases}$$

This map is an isomorphism for all $n \ge 3$ by Tate-Poitou duality [Milne2, I(4.20)]. It is also an isomorphism for $n = 2$ and $i \ge 2$, as shown in Exercise 9.1.

Write $\widetilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(i))$ for the kernel of $\alpha_S^1(i)$.

LEMMA 9.3. *For even $i$, $H^1(F;\mathbb{Z}/2^\infty(i)) \xrightarrow{\alpha^1(i)} (\mathbb{Z}/2)^{r_1}$ is a split surjection. Hence $H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(i)) \cong (\mathbb{Z}/2)^{r_1} \oplus \widetilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(i))$ for sufficiently large $S$.*

PROOF. By the strong approximation theorem for units of $F$, the left vertical map is a split surjection in the diagram:

$$\begin{array}{ccccc} F^\times/F^{\times 2} & \xrightarrow{\cong} & H^1(F,\mathbb{Z}/2) & \longrightarrow & H^1(F,\mathbb{Z}/2^\infty(i)) \\ {\scriptstyle\text{split onto}}\downarrow{\scriptstyle\oplus\sigma} & & \downarrow & & \downarrow{\scriptstyle\alpha^1(i)} \\ (\mathbb{Z}/2)^{r_1} = \oplus\,\mathbb{R}^\times/\mathbb{R}^{\times 2} & \xrightarrow{\cong} & \oplus H^1(\mathbb{R},\mathbb{Z}/2) & \xrightarrow{\cong} & \oplus H^1(\mathbb{R},\mathbb{Z}/2^\infty(i)). \end{array}$$

Since $F^\times/F^{\times 2}$ is the direct limit (over $S$) of the groups $\mathcal{O}_S^\times/\mathcal{O}_S^{\times 2}$, we may replace $F$ by $\mathcal{O}_S$ for sufficiently large $S$. $\square$

The next result is taken from [RW, 6.9]. When $n \equiv 5 \pmod 8$, we have an unknown group extension; to express it, we write $A \rtimes B$ for an abelian group extension of $B$ by $A$.

THEOREM 9.4. *Let $F$ be a real number field, and let $\mathcal{O}_S$ be a ring of $S$-integers in $F$ containing $\mathcal{O}_F[\frac{1}{2}]$. Then $\alpha^1_S(4k)$ is onto ($k > 0$), and for all $n \geq 0$:*

$$K_n(\mathcal{O}_S;\mathbb{Z}/2^\infty) \cong \begin{cases} \mathbb{Z}/w_{4k}(F) & \text{for } n = 8k, \\ H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(4k+1)) & \text{for } n = 8k+1, \\ \mathbb{Z}/2 & \text{for } n = 8k+2, \\ H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(4k+2)) & \text{for } n = 8k+3, \\ \mathbb{Z}/2w_{4k+2} \oplus (\mathbb{Z}/2)^{r_1-1} & \text{for } n = 8k+4, \\ (\mathbb{Z}/2)^{r_1-1} \rtimes H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(4k+3)) & \text{for } n = 8k+5, \\ 0 & \text{for } n = 8k+6, \\ \widetilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(4k+4)) & \text{for } n = 8k+7. \end{cases}$$

PROOF. Consider the morphism $\alpha_S$ of spectral sequences (4.2) with coefficients $\mathbb{Z}/2^\infty$, from that for $\mathcal{O}_S$ to the direct sum of $r_1$ copies of that for $\mathbb{R}$. By naturality, the morphism in the $E_2^{p,q}$ spot is the map $\alpha_S^{p-q}(-q)$ of (9.2). By Tate-Poitou duality and Ex. 9.1, this is an isomorphism except on the two diagonals $p = q$, where it is the injection of $\mathbb{Z}/w_{-q}^{(2)}(F)$ into $\mathbb{Z}/2^\infty$, and on the critical diagonal $p = q+1$.

When $p \equiv +1 \pmod 4$, we saw in 9.1.2 that $d_2^{p,q}(\mathbb{R})$ is an isomorphism whenever $q \leq p < 0$. It follows that we may identify $d_2^{p,q}(\mathcal{O}_S)$ with $\alpha_S^{p-q}$. Therefore $d_2^{p,q}(\mathcal{O}_S)$ is an isomorphism if $p \geq 2+q$, and an injection if $p = q$. As in 9.1.2, the spectral sequence degenerates at $E_3$, yielding $K_n(\mathcal{O}_S;\mathbb{Z}/2^\infty)$ as proclaimed, except for two points: (a) when $n = 8k+4$, the extension of $\mathbb{Z}/w_{4k+2}$ by $(\mathbb{Z}/2)^{r_1}$ is seen to be nontrivial by comparison with the extension for $\mathbb{R}$, and (b) when $n = 8k+6$, it only shows that $K_n(\mathcal{O}_S;\mathbb{Z}/2^\infty)$ is the cokernel of $\alpha^1_S(4k+4)$.

To resolve (b) we must show that the map $\alpha^1_S(4k+4)$ is onto when $k \geq 0$. Set $n = 8k+6$. Since $K_n(\mathcal{O}_S)$ is finite, $K_n(\mathcal{O}_S;\mathbb{Z}/2^\infty)$ must equal the 2-primary subgroup of $K_{n-1}(\mathcal{O}_S)$, which is independent of $S$ by V.6.8. But for sufficiently large $S$, the map $\alpha^1(4k+4)$ is a surjection by Lemma 9.3, and hence $K_n(\mathcal{O}_S;\mathbb{Z}/2^\infty) = 0$. $\square$

THEOREM 9.5. *Let $\mathcal{O}_S$ be a ring of $S$-integers in a number field $F$. Then for each odd $n \geq 3$, the group $K_n(\mathcal{O}_S) \cong K_n(F)$ is given by:*

a) *If $F$ is totally imaginary, $K_n(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F)$;*
b) *If $F$ has $r_1 > 0$ real embeddings then, setting $i = (n+1)/2$,*

$$K_n(F) \cong \begin{cases} \mathbb{Z}^{r_1+r_2} \oplus \mathbb{Z}/w_i(F), & n \equiv 1 \pmod 8 \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/2w_i(F) \oplus (\mathbb{Z}/2)^{r_1-1}, & n \equiv 3 \pmod 8 \\ \mathbb{Z}^{r_1+r_2} \oplus \mathbb{Z}/\frac{1}{2}w_i(F), & n \equiv 5 \pmod 8 \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F), & n \equiv 7 \pmod 8 \end{cases}$$

Note that these groups are determined only by the number $r_1$, $r_2$ of real and complex places of $F$ and the integers $w_i(F)$.

PROOF. Part (a), when $F$ is totally imaginary, is given by Theorem 8.4. In case (b), since the rank is classically known (see 8.1), and $K_n(\mathcal{O}_S) \cong K_n(F)$ by V.6.8, it suffices to determine the torsion subgroup of $K_n(\mathcal{O}_S)$. The odd torsion is given by Theorem 8.2, so we need only worry about the 2-primary torsion. Since $K_{n+1}(\mathcal{O}_S)$ is finite, it follows from Ex. IV.2.6 that the 2-primary subgroup of $K_n(\mathcal{O}_S)$ is $K_{n+1}(\mathcal{O}_S;\mathbb{Z}/2^\infty)$, which we can read off from Theorem 9.4, recalling from 2.3(b) that $w_i^{(2)}(F) = 2$ for odd $i$. $\square$

EXAMPLE 9.5.1. $K_n(\mathbb{Q}) \cong \mathbb{Z}$ for all $n \equiv 5 \pmod 8$ as $w_i(\mathbb{Q}) = 2$; see 2.1.2. More generally, if $F$ has a real embedding and $n \equiv 5 \pmod 8$, then $K_n(F)$ has no 2-primary torsion, because $\frac{1}{2}w_i(F)$ is an odd integer when $i$ is odd; see 2.3(b).

*The narrow Picard group*

To determine the 2-primary torsion in $K_n(\mathcal{O}_S)$ when $n$ is even, we need to introduce the narrow Picard group and the signature defect of the ring $\mathcal{O}_S$. We begin with some terminology.

Each real embedding $\sigma_i : F \to \mathbb{R}$ determines a map $F^\times \to \mathbb{R}^\times \to \mathbb{Z}/2$, detecting the sign of units of $F$ under that embedding. The sum of these is the *sign map* $\sigma \colon F^\times \to (\mathbb{Z}/2)^{r_1}$; it is surjective by the strong approximation theorem for $F$. The kernel $F_+^\times$ of $\sigma$ is called the group of *totally positive units* in $F$, since it consists of units which are positive under every real embedding.

If $R = \mathcal{O}_S$ is a ring of integers in $F$, we write $R_+^\times$ for $R^\times \cap F_+^\times$, the subgroup of totally positive units in $R$. Since the sign map $\sigma$ factors through $F^\times/F^{\times 2} = H^1_{\text{et}}(F,\mathbb{Z}/2)$, the restriction to $R^\times$ also factors through $\alpha^1 : H^1_{\text{et}}(R,\mathbb{Z}/2) \to (\mathbb{Z}/2)^{r_1}$. This map is part of a family of maps

$$\alpha^n : H^n_{\text{et}}(R,\mathbb{Z}/2) \to \oplus_{r_1} H^n_{\text{et}}(\mathbb{R},\mathbb{Z}/2) = (\mathbb{Z}/2)^{r_1} \tag{9.6}$$

related to the maps $\alpha^n(i)$ in (9.2). By Tate-Poitou duality, $\alpha^n$ is an isomorphism for all $n \geq 3$; it is a surjection for $n = 2$ (see Ex. 9.2). We will be interested in $\alpha^1$. The following classical definitions are due to Weber; see [Co, 5.2.7] or [Neu, VI.1].

DEFINITION 9.6.1. The *signature defect* $j(R)$ of $R$ is defined to be the dimension of the cokernel of $\alpha^1$. Since the sign of $-1$ nontrivial, we have $0 \le j(R) < r_1$. Note that $j(F) = 0$, and that $j(\mathcal{O}_S) \le j(\mathcal{O}_F)$ for all $S$.

The *narrow Picard group* $\mathrm{Pic}_+(R)$ is defined to be the cokernel of the restricted divisor map $F_+^\times \to \bigoplus_{\wp \notin S} \mathbb{Z}$ of I.3.5; it is a finite group. Some authors call $\mathrm{Pic}_+(\mathcal{O}_S)$ the *ray class group* and write it as $Cl_F^S$.

The kernel of the restricted divisor map is clearly $R_+^\times$, and it is easy to see from this that there is an exact sequence

$$0 \to R_+^\times \to R^\times \xrightarrow{\sigma} (\mathbb{Z}/2)^{r_1} \to \mathrm{Pic}_+(R) \to \mathrm{Pic}(R) \to 0.$$

For simplicity we write $H^n(R, \mathbb{Z}/2)$ for $H^n_{\mathrm{et}}(R, \mathbb{Z}/2)$ and, as in (9.2), we define $\widetilde{H}^n(R; \mathbb{Z}/2)$ to be the kernel of $\alpha^n$. A diagram chase (left to Ex. 9.3) shows that there is an exact sequence

$$(9.6.2)\quad 0 \to \widetilde{H}^1(R;\mathbb{Z}/2) \to H^1(R;\mathbb{Z}/2) \xrightarrow{\alpha^1} (\mathbb{Z}/2)^{r_1} \to \mathrm{Pic}_+(R)/2 \to \mathrm{Pic}(R)/2 \to 0.$$

Thus the signature defect $j(R)$ is also the dimension of the kernel of $\mathrm{Pic}_+(R)/2 \to \mathrm{Pic}(R)/2$. If we let $t$ and $u$ denote the dimensions of $\mathrm{Pic}(R)/2$ and $\mathrm{Pic}_+(R)/2$, respectively, then this means that $u = t + j(R)$. If $s$ denotes the number of finite places of $\mathcal{O}_S$, then $\dim H^1(\mathcal{O}_S; \mathbb{Z}/2) = r_1 + r_2 + s + t$ and $\dim H^2(\mathcal{O}_S; \mathbb{Z}/2) = r_1 + s + t - 1$. This follows from 8.1 and (8.1.1), using Kummer theory.

LEMMA 9.6.3. *Suppose that* $\frac{1}{2} \in \mathcal{O}_S$. *Then* $\dim \widetilde{H}^1(\mathcal{O}_S, \mathbb{Z}/2) = r_2 + s + u$, *and* $\dim \widetilde{H}^2(\mathcal{O}_S, \mathbb{Z}/2) = t + s - 1$.

PROOF. The first assertion is immediate from (9.6.2). Since $\alpha^2$ is onto, the second assertion follows. $\square$

THEOREM 9.7. *Let $F$ be a real number field, and $\mathcal{O}_S$ a ring of integers containing $\frac{1}{2}$. If $j = j(\mathcal{O}_S)$ is the signature defect, then the mod 2 algebraic $K$-groups of $\mathcal{O}_S$ are given (up to extensions) for $n > 0$ as follows:*

$$K_n(\mathcal{O}_S; \mathbb{Z}/2) \cong \begin{cases} \widetilde{H}^2(\mathcal{O}_S;\mathbb{Z}/2) \oplus \mathbb{Z}/2 & \text{for } n = 8k, \\ H^1(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n = 8k+1, \\ H^2(\mathcal{O}_S;\mathbb{Z}/2) \rtimes \mathbb{Z}/2 & \text{for } n = 8k+2, \\ (\mathbb{Z}/2)^{r_1-1} \rtimes H^1(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n = 8k+3, \\ (\mathbb{Z}/2)^j \rtimes H^2(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n = 8k+4, \\ (\mathbb{Z}/2)^{r_1-1} \rtimes \widetilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n = 8k+5, \\ (\mathbb{Z}/2)^j \oplus \widetilde{H}^2(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n = 8k+6, \\ \widetilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n = 8k+7. \end{cases}$$

| $p=-3$ | $-2$ | $-1$ | $0$ |
|---|---|---|---|
| | | | $1$ |
| | | $\beta_1$ | $H^1$ |
| | $0$ | $H^1$ | $H^2$ |
| $0$ | $\widetilde{H}^1$ | $H^2$ | $(\mathbb{Z}/2)^{r_1-1}$ |
| $\widetilde{H}^1$ | $\widetilde{H}^2$ | $(\mathbb{Z}/2)^{r_1-1}$ | $(\mathbb{Z}/2)^j$ |
| $\widetilde{H}^2$ | $0$ | $(\mathbb{Z}/2)^j$ | $0$ |
| $0$ | $0$ | $0$ | $0$ |

The first 4 columns of $E_3 = E_\infty$

Table 9.7.1. The mod 2 spectral sequence for $\mathcal{O}_S$.

PROOF. (Cf. [RW, 7.8].) As in the proof of Theorem 9.4, we compare the spectral sequence for $R = \mathcal{O}_S$ with the sum of $r_1$ copies of the spectral sequence for $\mathbb{R}$. For $n \geq 3$ we have $H^n(R;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{r_1}$. It is not hard to see that we may identify the differentials $d_2 : H^n(R,\mathbb{Z}/2) \to H^{n+3}(R,\mathbb{Z}/2)$ with the maps $\alpha^n$. Since these maps are described in 9.6, we see from periodicity 4.8.1 that the columns $p \leq 0$ of $E_3$ are 4-periodic, and all nonzero entries are described by Figure 9.7.1.

As in Example 4.6, the $E_2$ page of the spectral sequence 4.2 has only one nonzero entry for $p > 0$, namely $E_3^{+1,-1} = \mathrm{Pic}(R)/2$, and it only affects $K_0(R;\mathbb{Z}/2)$. By inspection, $E_3 = E_\infty$, yielding the desired description of the groups $K_n(R,\mathbb{Z}/2)$ in terms of extensions. The proof that the extensions split for $n \equiv 0, 6 \pmod 8$ is left to Exercises 9.4 and 9.5. □

The case $F = \mathbb{Q}$ has historical importance, because of its connection with the image of $J$ (see 2.1.3 or [Q5]) and classical number theory. The following result was first established in [We2]; the groups are not truly periodic only because the order of $K_{8k-1}(\mathbb{Z})$ depends upon $k$.

COROLLARY 9.8. *For $n \geq 0$, the 2-primary subgroups of $K_n(\mathbb{Z})$ and $K_n(\mathbb{Z}[1/2])$ are essentially periodic, of period eight, and are given by the following table for $n \geq 2$. (When $n \equiv 7 \pmod 8$, we set $k = (n+1)/8$.)*

| $n \pmod 8$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $K_n(\mathbb{Z})\{2\}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/16$ | 0 | 0 | 0 | $\mathbb{Z}/16k$ | 0 |

In particular, $K_n(\mathbb{Z})$ and $K_n(\mathbb{Z}[1/2])$ have odd order for all $n \equiv 4, 6, 8 \pmod 8$, and the finite group $K_{8k+2}(\mathbb{Z})$ is the sum of $\mathbb{Z}/2$ and a finite group of odd order. We will say more about the odd torsion in the next section.

PROOF. When $n$ is odd, this is Theorem 9.5; $w_{4k}^{(2)}$ is the 2-primary part of $16k$ by 2.3(c). For $R = \mathbb{Z}[1/2]$ we have $s = 1$ and $t = u = j = 0$. By Lemma 9.6.3 we have $\dim \widetilde{H}^1(R;\mathbb{Z}/2) = 1$ and $\widetilde{H}^2(R;\mathbb{Z}/2) = 0$. By 9.7, the groups $K_n(\mathbb{Z}[1/2];\mathbb{Z}/2)$ are periodic of orders 2, 4, 4, 4, 2, 2, 1, 2 for $n \equiv 0, 1, ..., 7$ respectively. The groups $K_n(\mathbb{Z}[1/2])$ for $n$ odd, given in 9.5, together with the $\mathbb{Z}/2$ summand in $K_{8k+2}(\mathbb{Z})$ provided by topology (see 2.1.3), account for all of $K_n(\mathbb{Z}[1/2];\mathbb{Z}/2)$, and hence must contain all of the 2-primary torsion in $K_n(\mathbb{Z}[1/2])$. □

Recall that the 2-rank of an abelian group $A$ is the dimension of $\mathrm{Hom}(\mathbb{Z}/2, A)$. We have already seen (in Theorem 9.5) that for $n \equiv 1, 3, 5, 7 \pmod 8$ the 2-ranks of $K_n(\mathcal{O}_S)$ are: 1, $r_1$, 0 and 1, respectively.

COROLLARY 9.9. *For $n \equiv 2, 4, 6, 8 \pmod 8$, $n > 0$, the respective 2-ranks of the finite groups $K_n(\mathcal{O}_S)$ are: $r_1 + s + t - 1$, $j + s + t - 1$, $j + s + t - 1$ and $s + t - 1$.*

Here $j$ is the signature defect of $\mathcal{O}_S$ (9.6.1), $s$ is the number of finite places of $\mathcal{O}_S$ and $t$ is the rank of $\mathrm{Pic}(\mathcal{O}_S)/2$.

PROOF. Since $K_n(R;\mathbb{Z}/2)$ is an extension of $\mathrm{Hom}(\mathbb{Z}/2, K_{n-1}R)$ by $K_n(R)/2$, and the dimensions of the odd groups are known, we can read this off from the list given in Theorem 9.7, using Lemma 9.6.3. □

EXAMPLE 9.9.1. Consider $F = \mathbb{Q}(\sqrt{p})$, where $p$ is prime. When $p \equiv 1 \pmod 8$, it is well known that $t = j = 0$ but $s = 2$. It follows that $K_{8k+2}(\mathcal{O}_F)$ has 2-rank 3, while the two-primary summand of $K_n(\mathcal{O}_F)$ is nonzero and cyclic when $n \equiv 4, 6, 8 \pmod 8$.

When $p \equiv 7 \pmod 8$, we have $j = 1$ for both $\mathcal{O}_F$ and $R = \mathcal{O}_F[1/2]$. Since $r_1 = 2$ and $s = 1$, the 2-ranks of the finite groups $K_n(R)$ are: $t + 2$, $t + 1$, $t + 1$ and $t$ for $n \equiv 2, 4, 6, 8 \pmod 8$ by 9.9. For example, if $t = 0$ ($\mathrm{Pic}(R)/2 = 0$) then $K_n(R)$ has odd order for $n \equiv 8 \pmod 8$, but the 2-primary summand of $K_n(R)$ is $(\mathbb{Z}/2)^2$ when $n \equiv 2$ and is cyclic when $n \equiv 4, 6$.

EXAMPLE 9.9.2. (2–regular fields) A number field $F$ is said to be *2–regular* if there is only one prime over 2 and the narrow Picard group $\mathrm{Pic}_+(\mathcal{O}_F[\frac{1}{2}])$ is odd (*i.e.*, $t = u = 0$ and $s = 1$). In this case, we see from 9.9 that $K_{8k+2}(\mathcal{O}_F)$ is the sum of $(\mathbb{Z}/2)^{r_1}$ and a finite odd group, while $K_n(\mathcal{O}_F)$ has odd order for all $n \equiv 4, 6, 8 \pmod 8$ ($n > 0$). In particular, the map $K_4^M(F) \to K_4(F)$ must be zero, since it factors through the odd order group $K_4(\mathcal{O}_F)$, and $K_4^M(F) \cong (\mathbb{Z}/2)^{r_1}$.

Browkin and Schinzel [BrwS] and Rognes and Østvær [ROst] have studied this case. For example, when $F = \mathbb{Q}(\sqrt{m})$ and $m > 0$ ($r_1 = 2$), the field $F$ is 2-regular exactly when $m = 2$, or $m = p$ or $m = 2p$ with $p \equiv 3, 5 \pmod 8$ prime. (See [BrwS].)

A useful example is $F = \mathbb{Q}(\sqrt{2})$. Note that $K_4^M(F) \cong (\mathbb{Z}/2)^2$ is generated by the Steinberg symbols $\{-1,-1,-1,-1\}$ and $\{-1,-1,-1,1+\sqrt{2}\}$. Both symbols must vanish in $K_4(\mathbb{Z}[\sqrt{2}])$, since this group has odd order. This is the case $j = 0$, $r_1 = 2$ of the following result.

Let $\rho$ denote the rank of the image of the group $K_4^M(F) \cong (\mathbb{Z}/2)^{r_1}$ in $K_4(F)$.

COROLLARY 9.10. *Let $F$ be a real number field. Then $j(\mathcal{O}_F[1/2]) \le \rho \le r_1 - 1$. The image $(\mathbb{Z}/2)^\rho$ of $K_4^M(F) \to K_4(F)$ lies in the subgroup $K_4(\mathcal{O}_F)$, and its image in $K_4(\mathcal{O}_S)/2$ has rank $j(\mathcal{O}_S)$ whenever $S$ contains all primes over 2.*

*In particular, the image $(\mathbb{Z}/2)^\rho$ of $K_4^M(F)$ lies in $2 \cdot K_4(F)$.*

PROOF. By Ex. IV.1.12(d), $\{-1,-1,-1,-1\}$ is nonzero in $K_4^M(F)$ but zero in $K_4(F)$. Since $K_4^M(F) \cong (\mathbb{Z}/2)^{r_1}$ by III.7.2(d), we have $\rho < r_1$. The assertion that $K_4^M(F) \to K_4(F)$ factors through $K_4(\mathcal{O}_F)$ follows from the fact that $K_3(\mathcal{O}_F) = K_3(F)$ (see IV.6.8), by multiplying $K_3^M(F)$ and $K_3(\mathcal{O}_F) \cong K_3(F)$ by $[-1] \in K_1(\mathbb{Z})$. We saw in 4.3 that the edge map $H^n(F, \mathbb{Z}(n)) \to K_n(F)$ in the motivic spectral sequence agrees with the usual map $K_n^M(F) \to K_n(F)$. By Theorem 4.1 (due to Voevodsky), $K_n^M(F)/2^\nu \cong H^n(F,\mathbb{Z}(n))/2^\nu \cong H^n(F,\mathbb{Z}/2^\nu(n))$. For $n = 4$, the image of the edge map from $H^4(\mathcal{O}_S,\mathbb{Z}/2^\nu(4)) \cong H^4(F,\mathbb{Z}/2^\nu(4)) \to K_4(\mathcal{O}_S;\mathbb{Z}/2)$ has rank $j$ by Table 9.7.1; this implies the assertion that the image in $K_4(\mathcal{O}_S)/2 \subset K_4(\mathcal{O}_S;\mathbb{Z}/2)$ has rank $j(\mathcal{O}_S)$. Finally, taking $\mathcal{O}_S = \mathcal{O}_F[1/2]$ yields the inequality $j(\mathcal{O}_S) \le \rho$. □

EXAMPLE 9.10.1. ($\rho = 1$) Consider $F = \mathbb{Q}(\sqrt{7})$, $\mathcal{O}_F = \mathbb{Z}[\sqrt{7}]$ and $R = \mathcal{O}_F[1/2]$; here $s = 1$, $t = 0$ and $j(R) = \rho = 1$ (the fundamental unit $u = 8 + 3\sqrt{7}$ is totally positive). Hence the image of $K_4^M(F) \cong (\mathbb{Z}/2)^2$ in $K_4(\mathbb{Z}[\sqrt{7}])$ is $\mathbb{Z}/2$ on the symbol $\sigma = \{-1,-1,-1,\sqrt{7}\}$, and this is all of the 2-primary torsion in $K_4(\mathbb{Z}[\sqrt{7}])$ by 9.9.

On the other hand, $\mathcal{O}_S = \mathbb{Z}[\sqrt{7}, 1/7]$ still has $\rho = 1$, but now $j = 0$, and the 2-rank of $K_4(\mathcal{O}_S)$ is still one by 9.9. Hence the extension $0 \to K_4(\mathcal{O}_F) \to K_4(\mathcal{O}_S) \to \mathbb{Z}/48 \to 0$ of V.6.8 cannot be split, implying that the 2-primary subgroup of $K_4(\mathcal{O}_S)$ must then be $\mathbb{Z}/32$.

In fact, the nonzero element $\sigma$ is *divisible* in $K_4(F)$. This follows from the fact that if $p \equiv 3 \pmod{28}$ then there is an irreducible $q = a + b\sqrt{7}$ whose norm is $-p = q\bar{q}$. Hence $R' = \mathbb{Z}[\sqrt{7}, 1/2q]$ has $j(R') = 0$ but $\rho = 1$, and the extension $0 \to K_4(\mathcal{O}_F) \to K_4(\mathcal{O}_S) \to \mathbb{Z}/(p^2-1) \to 0$ of V.6.8 is not split. If in addition $p \equiv -1 \pmod{2^\nu}$ — there are infinitely many such $p$ for each $\nu$ — then there is an element $v$ of $K_4(R')$ such that $2^{\nu+1}v = \sigma$ See [We3] for details.

QUESTION 9.10.2. Can $\rho$ be less than the minimum of $r_1 - 1$ and $j + s + t - 1$?

As in (9.2), when $i$ is even we define $\widetilde{H}^2(R;\mathbb{Z}_2(i))$ to be the kernel of $\alpha^2(i)$ : $H^2(R;\mathbb{Z}_2(i)) \to H^2(\mathbb{R};\mathbb{Z}_2(i))^{r_1} \cong (\mathbb{Z}/2)^{r_1}$. By Lemma 9.6.3, $\widetilde{H}^2(R;\mathbb{Z}_2(i))$ has 2-rank $s + t - 1$. The following result is taken from [RW, 0.6].

THEOREM 9.11. *Let $F$ be a number field with at least one real embedding, and let $R = \mathcal{O}_S$ denote a ring of integers in $F$ containing $1/2$. Let $j$ be the signature defect of $R$, and write $w_i$ for $w_i^{(2)}(F)$.*

*Then there is an integer $\rho$, $j \le \rho < r_1$, such that, for all $n \ge 2$, the two-primary subgroup $K_n(\mathcal{O}_S)\{2\}$ of $K_n(\mathcal{O}_S)$ is isomorphic to:*

$$K_n(\mathcal{O}_S)\{2\} \cong \begin{cases} H^2_{\text{et}}(R;\mathbb{Z}_2(4k+1)) & \text{for } n = 8k, \\ \mathbb{Z}/2 & \text{for } n = 8k+1, \\ H^2_{\text{et}}(R;\mathbb{Z}_2(4k+2)) & \text{for } n = 8k+2, \\ (\mathbb{Z}/2)^{r_1-1} \oplus \mathbb{Z}/2w_{4k+2} & \text{for } n = 8k+3, \\ (\mathbb{Z}/2)^{\rho} \rtimes H^2_{\text{et}}(R;\mathbb{Z}_2(4k+3)) & \text{for } n = 8k+4, \\ 0 & \text{for } n = 8k+5, \\ \widetilde{H}^2_{\text{et}}(R;\mathbb{Z}_2(4k+4)) & \text{for } n = 8k+6, \\ \mathbb{Z}/w_{4k+4} & \text{for } n = 8k+7. \end{cases}$$

PROOF. When $n = 2i - 1$ is odd, this is Theorem 9.5, since $w_i^{(2)}(F) = 2$ when $n \equiv 1 \pmod 4$ by 2.3(b). When $n = 2$ it is III.6.9.3. To determine the two-primary subgroup $K_n(\mathcal{O}_S)\{2\}$ of the finite group $K_{2i+2}(\mathcal{O}_S)$ when $n = 2i + 2$, we use the universal coefficient sequence

$$0 \to (\mathbb{Z}/2^\infty)^r \to K_{2i+3}(\mathcal{O}_S;\mathbb{Z}/2^\infty) \to K_{2i+2}(\mathcal{O}_S)\{2\} \to 0,$$

where $r$ is the rank of $K_{2i+3}(\mathcal{O}_S)$ and is given by 8.1 ($r = r_1 + r_2$ or $r_2$). To compare this with Theorem 9.4, we note that $H^1(\mathcal{O}_S, \mathbb{Z}/2^\infty(i))$ is the direct sum of $(\mathbb{Z}/2^\infty)^r$ and a finite group, which must be $H^2(\mathcal{O}_S, \mathbb{Z}_2(i))$ by universal coefficients; see [RW, 2.4(b)]. Since $\alpha_S^1(i) : H^1(R; \mathbb{Z}_2(i)) \to (\mathbb{Z}/2)^{r_1}$ must vanish on the divisible group $(\mathbb{Z}/2^\infty)^r$, it induces the natural map $\alpha_S^2(i)\colon H^2_{\mathrm{et}}(\mathcal{O}_S; \mathbb{Z}_2(i)) \to (\mathbb{Z}/2)^{r_1}$ and

$$\widetilde{H}^1(\mathcal{O}_S, \mathbb{Z}/2^\infty(i)) \cong (\mathbb{Z}/2^\infty)^r \oplus \widetilde{H}^2(\mathcal{O}_S, \mathbb{Z}_2(i)).$$

This proves all of the theorem, except for the description of $K_n(\mathcal{O}_S)$, $n = 8k+4$. By mod 2 periodicity 4.8.1, the integer $\rho$ of 9.10 equals the rank of the image of $H^4(\mathcal{O}_S, \mathbb{Z}/2(4)) \cong H^4(\mathcal{O}_S, \mathbb{Z}/2(4k+4)) \cong (\mathbb{Z}/2)^{r_1}$ in $\mathrm{Hom}(\mathbb{Z}/2, K_n(\mathcal{O}_S))$, considered as a quotient of $K_{n+1}(\mathcal{O}_S; \mathbb{Z}/2)$. $\square$

We can combine the 2-primary information in 9.11 with the odd torsion information in 8.2 and 8.8 to relate the orders of $K$-groups to the orders of étale cohomology groups. Up to a factor of $2^{r_1}$, they were conjectured by Lichtenbaum in [Li2]. Let $|A|$ denote the order of a finite abelian group $A$.

THEOREM 9.12. *Let $F$ be a totally real number field, with $r_1$ real embeddings, and let $\mathcal{O}_S$ be a ring of integers in $F$. Then for all even $i > 0$*

$$2^{r_1} \cdot \frac{|K_{2i-2}(\mathcal{O}_S)|}{|K_{2i-1}(\mathcal{O}_S)|} = \frac{\prod_\ell |H^2_{\mathrm{et}}(\mathcal{O}_S[1/\ell]; \mathbb{Z}_\ell(i))|}{\prod_\ell |H^1_{\mathrm{et}}(\mathcal{O}_S[1/\ell]; \mathbb{Z}_\ell(i))|}.$$

PROOF. Since $2i-1 \equiv 3 \pmod 4$, all groups involved are finite (see 8.1, Ex. 8.2 and Ex. 8.3.) Write $h^{n,i}(\ell)$ for the order of $H^n_{\mathrm{et}}(\mathcal{O}_S[1/\ell]; \mathbb{Z}_\ell(i))$. By Ex. 8.3, $h^{1,i}(\ell) = w_i^{(\ell)}(F)$. By 9.5, the $\ell$-primary subgroup of $K_{2i-1}(\mathcal{O}_S)$ has order $h^{1,i}(\ell)$ for all odd $\ell$ and all even $i > 0$, and also for $\ell = 2$ with the exception that when $2i-1 \equiv 3 \pmod 8$ then the order is $2^{r_1} h^{1,i}(2)$.

By Theorems 8.2 and 9.11, the $\ell$-primary subgroup of $K_{2i-2}(\mathcal{O}_S)$ has order $h^{2,i}(\ell)$ for all $\ell$, except when $\ell = 2$ and $2i-2 \equiv 6 \pmod 8$ when it is $h^{1,i}(2)/2^{r_1}$. Combining these cases yields the formula asserted by the theorem. $\square$

Theorem 9.12 was used in the previous section (Theorem 8.8) to equate the ratio of orders of the finite groups $K_{4k-2}(\mathcal{O}_F)$ and $K_{4k-1}(\mathcal{O}_F)$ with $|\zeta_F(1-2k)|/2^{r_1}$.

EXERCISES

**9.1** Suppose that $F$ has $r_1 > 0$ embeddings into $\mathbb{R}$. Show that

$$H^2_{\mathrm{et}}(\mathcal{O}_S; \mathbb{Z}/2^\infty(i)) \cong H^2_{\mathrm{et}}(F; \mathbb{Z}/2^\infty(i)) \cong \begin{cases} (\mathbb{Z}/2)^{r_1}, & i \geq 3 \text{ odd} \\ 0, & i \geq 2 \text{ even.} \end{cases}$$

Using (8.1.1), determine $H^2_{\mathrm{et}}(\mathcal{O}_S; \mathbb{Z}/2^\infty(1))$. *Hint:* Compare $F$ with $F(\sqrt{-1})$, and use Exercise 8.1 to see that $H^2_{\mathrm{et}}(\mathcal{O}_S; \mathbb{Z}/2^\infty(i))$ has exponent 2. Hence the Kummer sequence is:

$$0 \to H^2_{\mathrm{et}}(\mathcal{O}_S; \mathbb{Z}/2^\infty(i)) \to H^3_{\mathrm{et}}(\mathcal{O}_S; \mathbb{Z}/2) \to H^3_{\mathrm{et}}(\mathcal{O}_S; \mathbb{Z}/2^\infty(i)) \to 0.$$

Now plug in the values of the $H^3$ groups, which are known by (9.2).

**9.2** Show that $\alpha^2$ is onto. *Hint:* Use Ex. 9.1 and the coefficient sequence for $\mathbb{Z}/2 \subset \mathbb{Z}/2^\infty(4)$ to show that the map $H^2_{\mathrm{et}}(R;\mathbb{Z}/2) \to H^2_{\mathrm{et}}(R;\mathbb{Z}/2^\infty(4))$ is onto.

**9.3** Establish the exact sequence (9.6.2). (This is taken from [RW, 9.6].)

**9.4** The stable homotopy group $\pi_{8k}(QS^0;\mathbb{Z}/2)$ contains an element $\beta_{8k}$ of exponent 2 which maps onto the generator of $K_{8k}(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/2$; see [RW, 5.1]. Use it to show the extension $K_{8k}(\mathcal{O}_S;\mathbb{Z}/2)$ of $\mathbb{Z}/2$ by $\widetilde{H}^2(\mathcal{O}_S,\mathbb{Z}/2)$ splits in Theorem 9.7.

**9.5** Show that the extension $K_{8k+6}(\mathcal{O}_S;\mathbb{Z}/2)$ splits in Thm. 9.7. Conclude that $K_{8k+6}(\mathcal{O}_S)/2 \cong \widetilde{H}^2(\mathcal{O}_S,\mathbb{Z}/2) \oplus (\mathbb{Z}/2)^j$. *Hint:* use Example 9.5.1.

**9.6** Let $R = \mathcal{O}_F[1/2]$, where $F$ is a real number field. Show that $K_{8k+4}(R;\mathbb{Z}/2)$ is an extension of ${}_2\mathrm{Br}(R)$ by $\mathrm{Pic}_+(R)/2$.

Let $\mathrm{Br}_+(R)$ denote the kernel of the canonical map $\mathrm{Br}(R) \to (\mathbb{Z}/2)^{r_1}$ induced by (8.1.1). Show that $K_{8k+7}(R;\mathbb{Z}/2) \cong \mathrm{Pic}_+(R)/2 \oplus {}_2\mathrm{Br}_+(R)$. (See [RW, 7.8].)

## §10. The $K$-theory of $\mathbb{Z}$

The determination of the groups $K_n(\mathbb{Z})$ has been a driving force in the development of $K$-theory. We saw in Chapters II and III that the groups $K_0(\mathbb{Z})$, $K_1(\mathbb{Z})$ and $K_2(\mathbb{Z})$ are related to very classical mathematics. In the 1970's, homological methods led to the calculation of the rank of $K_n(\mathbb{Z})$ by Borel (see 8.1) and $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ by Lee and Szczarba (see Example 2.1.2 or [LSz]).

In order to describe the groups $K_n(\mathbb{Z})$, we use the Bernoulli numbers $B_k$. We let $c_k$ denote the numerator of $B_k/4k$; $c_k$ is a product of irregular primes (see 2.4.1). We saw in Lemma 2.4 that the denominator of $B_k/4k$ is $w_{2k}$, so $B_k/4k = c_k/w_{2k}$.

Theorem 10.1. *For $n \not\equiv 0 \pmod 4$ and $n > 1$, we have:*

(1) *If $n = 8k+1$, $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$;*
(2) *If $n = 8k+2$, $|K_n(\mathbb{Z})| = 2c_{2k+1}$;*
(3) *If $n = 8k+3$, $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q}) \cong \mathbb{Z}/2w_{4k+2}$;*
(4) *If $n = 8k+5$, $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q}) \cong \mathbb{Z}$;*
(5) *If $n = 8k+6$, $|K_n(\mathbb{Z})| = c_{2k+1}$;*
(6) *If $n = 8k+7$, $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q}) \cong \mathbb{Z}/c_{4k+4}$.*

*As a consequence, for $k \geq 1$ we have:* $\dfrac{|K_{4k-2}(\mathbb{Z})|}{|K_{4k-1}(\mathbb{Z})|} = \dfrac{B_k}{4k} = \dfrac{(-1)^k}{2}\zeta(1-2k)$.

Proof. The equality $B_k/4k = (-1)^k\zeta(1-2k)/2$ comes from 2.4.2. The equality of this with $|K_{4k-2}(\mathbb{Z})|/|K_{4k-1}(\mathbb{Z})|$ comes from Theorem 8.8 (using 9.12). This gives the displayed formula.

When $n$ is odd, the groups $K_n(\mathbb{Z})$ were determined in Theorem 9.5, and $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q})$ by IV.6.8. Thus we may suppose that $n = 4k-2$. Since the 2-primary torsion in $K_n(\mathbb{Z})$ was determined in Corollary 9.8, we can ignore factors of 2. But up to a factor of 2, $|K_{4k-1}(\mathbb{Z})| = w_{2k}(\mathbb{Q})$ so the displayed formula yields $|K_{4k-2}(\mathbb{Z})|/w_{2k} = B_k/4k$ and hence $|K_{4k-2}(\mathbb{Z})| = c_k$. $\square$

The groups $K_n(\mathbb{Z})$ are much harder to determine when $n \equiv 0 \pmod 4$. The group $K_4(\mathbb{Z})$ was proven to be zero in the late 1990's (see Remark 10.1.3 or [Rognes]). If $n = 4i \geq 8$, the groups $K_{4i}(\mathbb{Z})$ are known to be products of irregular primes $\ell$, with $\ell > 10^8$, and are conjectured to be zero; this conjecture follows from, and implies, Vandiver's conjecture (stated in 10.8 below).

In Table 10.1.1, we have summarized what we know for $n < 20,000$; conjecturally the same pattern holds for all $n$ (see Theorem 10.2).

| | | | |
|---|---|---|---|
| $K_0(\mathbb{Z}) = \mathbb{Z}$ | $K_8(\mathbb{Z}) = (0?)$ | $K_{16}(\mathbb{Z}) = (0?)$ | $K_{8k}(\mathbb{Z}) = (0?)$, $k \geq 1$ |
| $K_1(\mathbb{Z}) = \mathbb{Z}/2$ | $K_9(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$ | $K_{17}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$ | $K_{8k+1}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$ |
| $K_2(\mathbb{Z}) = \mathbb{Z}/2$ | $K_{10}(\mathbb{Z}) = \mathbb{Z}/2$ | $K_{18}(\mathbb{Z}) = \mathbb{Z}/2$ | $K_{8k+2}(\mathbb{Z}) = \mathbb{Z}/2c_{2k+1}$ |
| $K_3(\mathbb{Z}) = \mathbb{Z}/48$ | $K_{11}(\mathbb{Z}) = \mathbb{Z}/1008$ | $K_{19}(\mathbb{Z}) = \mathbb{Z}/528$ | $K_{8k+3}(\mathbb{Z}) = \mathbb{Z}/2w_{4k+2}$ |
| $K_4(\mathbb{Z}) = 0$ | $K_{12}(\mathbb{Z}) = (0?)$ | $K_{20}(\mathbb{Z}) = (0?)$ | $K_{8k+4}(\mathbb{Z}) = (0?)$ |
| $K_5(\mathbb{Z}) = \mathbb{Z}$ | $K_{13}(\mathbb{Z}) = \mathbb{Z}$ | $K_{21}(\mathbb{Z}) = \mathbb{Z}$ | $K_{8k+5}(\mathbb{Z}) = \mathbb{Z}$ |
| $K_6(\mathbb{Z}) = 0$ | $K_{14}(\mathbb{Z}) = 0$ | $K_{22}(\mathbb{Z}) = \mathbb{Z}/691$ | $K_{8k+6}(\mathbb{Z}) = \mathbb{Z}/c_{2k+2}$ |
| $K_7(\mathbb{Z}) = \mathbb{Z}/240$ | $K_{15}(\mathbb{Z}) = \mathbb{Z}/480$ | $K_{23}(\mathbb{Z}) = \mathbb{Z}/65520$ | $K_{8k+7}(\mathbb{Z}) = \mathbb{Z}/w_{4k+4}$. |

**Table 10.1.1**. The groups $K_n(\mathbb{Z})$, $n < 20,000$. The notation '(0?)' refers to a finite group, conjecturally zero, whose order is a product of irregular primes $> 10^8$.

Relation to $\pi_n^s$ 10.1.2. Using homotopy-theoretic techniques, the torsion subgroups of $K_n(\mathbb{Z})$ had been detected by the late 1970's, due to the work of Quillen [Qlet], Harris-Segal [HS], Soulé [Sou] and others.

As pointed out in Remark 2.1.3, the image of the natural maps $\pi_n^s \to K_n(\mathbb{Z})$ capture most of the Harris-Segal summands 2.5. When $n$ is $8k+1$ or $8k+2$, there is a $\mathbb{Z}/2$-summand in $K_n(\mathbb{Z})$, generated by the image of Adams' element $\mu_n$. (It is the 2-torsion subgroup by 9.8.) Since $w_{4k+1}(\mathbb{Q}) = 2$, we may view it as the Harris-Segal summand when $n = 8k+1$. When $n = 8k+5$, the Harris-Segal summand is zero by Example 9.5.1. When $n = 8k+7$ the Harris-Segal summand of $K_n(\mathbb{Z})$ is isomorphic to the subgroup $J(\pi_n O) \cong \mathbb{Z}/w_{4k+4}(\mathbb{Q})$ of $\pi_n^s$.

When $n = 8k+3$, the subgroup $J(\pi_n O) \cong \mathbb{Z}/w_{4k+2}(\mathbb{Q})$ of $\pi_n^s$ is contained in the Harris-Segal summand $\mathbb{Z}/(2w_i)$ of $K_n(\mathbb{Z})$; the injectivity was proven by Quillen in [Qlet], and Browder showed that the order of the summand was $2w_i(\mathbb{Q})$.

The remaining calculations of $K_*(\mathbb{Z})$ depend upon the development of motivic cohomology, via the tools described in Section 4, and date to the period 1997–2007. The 2-primary torsion was resolved in 1997 using [V-MC] (see Section 9), while the order of the odd torsion (conjectured by Lichtenbaum) was only determined using the Norm Residue Theorem 4.1 of Rost and Voevodsky.

Homological methods 10.1.3. Lee-Szczarba [LSz] and Soulé [So78] used homological methods in the 1970s to show that $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ and that there is no $p$-torsion in $K_4(\mathbb{Z})$ or $K_5(\mathbb{Z})$ for $p > 3$. Much later, Rognes [Rognes] and Elbaz-Vincent–Gangl–Soulé [EGS] refined this to show that $K_4(\mathbb{Z}) = 0$, $K_5(\mathbb{Z}) = \mathbb{Z}$, and that $K_6(\mathbb{Z})$ has at most 3-torsion. This used the calculation in [RW] (using [V-MC]) that there is no 2-torsion in $K_4(\mathbb{Z})$, $K_5(\mathbb{Z})$ or $K_6(\mathbb{Z})$.

Our general description of $K_*(\mathbb{Z})$ is completed by the following assertion, which follows immediately from Theorems 10.1,10.9 and 10.10 below, It was observed independently by Kurihara [Kur] and Mitchell [Mit].

Theorem 10.2. *If Vandiver's conjecture holds, then the groups $K_n(\mathbb{Z})$ are given by Table 10.2.1, for all $n \geq 2$. Here $k$ is the integer part of $1 + \frac{n}{4}$.*

| $n \pmod 8$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $K_n(\mathbb{Z})$ | $\mathbb{Z} \oplus \mathbb{Z}/2$ | $\mathbb{Z}/2c_k$ | $\mathbb{Z}/2w_{2k}$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}/c_k$ | $\mathbb{Z}/w_{2k}$ | 0 |

Table 10.2.1. The $K$-theory of $\mathbb{Z}$, assuming Vandiver's Conjecture.

When $n$ is at most 20,000 and $n \equiv 2 \pmod 4$, we show that the finite groups $K_n(\mathbb{Z})$ are cyclic in Examples 10.3 and 10.3.2. (The order is $c_k$ or $2c_k$, where $k = (n+2)/4$, by Theorem 10.1.)

Examples 10.3. For $n$ at most 450, the group $K_n(\mathbb{Z})$ is cyclic because its order is squarefree. For $n \leq 30$ we need only consult 2.1.2 to see that the groups $K_2(\mathbb{Z})$, $K_{10}(\mathbb{Z})$, $K_{18}(\mathbb{Z})$ and $K_{26}(\mathbb{Z})$ are isomorphic to $\mathbb{Z}/2$, while $K_6(\mathbb{Z}) = K_{14}(\mathbb{Z}) = 0$. Since $c_6 = 691$, $c_8 = 3617$, $c_9 = 43867$ and $c_{13} = 657931$ are all prime, we have $K_{22}(\mathbb{Z}) \cong \mathbb{Z}/691$, $K_{30}(\mathbb{Z}) \cong \mathbb{Z}/3617$, $K_{34}(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/43867$ and $K_{50} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/657931$.

The next hundred values of $c_k$ are also squarefree: $c_{10} = 283 \cdot 617$, $c_{11} = 131 \cdot 593$, $c_{12} = 103 \cdot 2294797$, $c_{14} = 9349 \cdot 362903$ and $c_{15} = 1721 \cdot 1001259881$ are all products of two primes, while $c_{16} = 37 \cdot 683 \cdot 305065927$ is a product of 3 primes. Hence

$K_{38}(\mathbb{Z}) = \mathbb{Z}/c_{10}$, $K_{42}(\mathbb{Z}) = \mathbb{Z}/2c_{11}$, $K_{46} = \mathbb{Z}/c_{12}$, $K_{54}(\mathbb{Z}) = \mathbb{Z}/c_{14}$, $K_{58}(\mathbb{Z}) = \mathbb{Z}/2c_{15}$ and $K_{62}(\mathbb{Z}) = \mathbb{Z}/c_{16} = \mathbb{Z}/37 \oplus \mathbb{Z}/683 \oplus \mathbb{Z}/305065927$.

Thus the first occurrence of the smallest irregular prime (37) is in $K_{62}(\mathbb{Z})$; it also appears as a $\mathbb{Z}/37$ summand in $K_{134}(\mathbb{Z})$, $K_{206}(\mathbb{Z})$, ... , $K_{494}(\mathbb{Z})$. In fact, there is 37-torsion in every group $K_{72a+62}(\mathbb{Z})$ (see Ex. 10.2). This direct method fails for $K_{454}(\mathbb{Z})$, because its order $2c_{114}$ is divisible by $103^2$.

To go further, we need to consider the torsion in the groups $K_{4k-2}(\mathbb{Z})$ on a prime-by-prime basis. Since the 2-torsion has order at most 2 by 9.8, we may suppose that $\ell$ is an odd prime. Our method is to consider the cyclotomic extension $\mathbb{Z}[\zeta_\ell]$ of $\mathbb{Z}$, $\zeta_\ell = e^{2\pi i/\ell}$. Because $K_n(\mathbb{Z}) \to K_n(\mathbb{Z}[1/\ell])$ is an isomorphism on $\ell$-torsion (by the Localization Sequence V.6.6), and similarly for $K_n(\mathbb{Z}[\zeta_\ell]) \to K_n(\mathbb{Z}[\zeta_\ell, 1/\ell])$, it suffices to work with $\mathbb{Z}[1/\ell]$ and $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$.

The usual transfer argument 10.3.1. The ring extension $\mathbb{Z}[1/\ell] \subset R$ is Galois and its Galois group $G = \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ is cyclic of order $\ell - 1$. The map $i^* : K_*(\mathbb{Z}) \to K_*(R)$ identifies $K_n(\mathbb{Z}[1/\ell])_{(\ell)}$ with $K_n(R)^G_{(\ell)}$ for all $n$, because $i_* i^*$ is multiplication by $|G|$ on $K_*(\mathbb{Z})$ and $i^* i_*$ is multiplication by $|G|$ on $K_n(R)^G$ (see Ex. IV.6.13). This style of argument is called the *usual transfer argument.*

Example 10.3.2. The group $K_{4k-2}(\mathbb{Z})$ is cyclic (of order $c_k$ or $2c_k$) for all $k \le 5000$. To see this, we observe that $K_{4k-2}(\mathbb{Z})_{(\ell)}$ is cyclic if $\ell^2$ does not divide $c_k$, and in this range only seven of the $c_k$ are not square-free; see [OEIS, A090943]. The numerator $c_k$ is divisible by $\ell^2$ only for the following pairs $(k, \ell)$: $(114, 103)$, $(142, 37)$, $(457, 59)$, $(717, 271)$, $(1646, 67)$, $(2884, 101)$ and $(3151, 157)$. In each of these cases, we note that $\mathrm{Pic}(\mathbb{Z}[\zeta_\ell]) = \mathrm{Pic}(R)/\ell \cong \mathbb{Z}/\ell$. By Example 8.3.1, $K_{4k-2}(R)/\ell \cong \mathrm{Pic}(R) \cong \mathbb{Z}/\ell$. The usual transfer argument (10.3.1) now shows that $K_{4k-2}(\mathbb{Z})/\ell$ is either 0 or $\mathbb{Z}/\ell$ for all $k$. Since $c_k$ is divisible by $\ell^2$ but not $\ell^3$, $K_{4k-2}(\mathbb{Z})_{(\ell)} \cong \mathbb{Z}/\ell^2$.

Representations of $G$ over $\mathbb{Z}/\ell$ 10.4. When $G$ is the cyclic group of order $\ell - 1$, a $\mathbb{Z}/\ell[G]$-module is just a $\mathbb{Z}/\ell$-vector space on which $G$ acts linearly. By Maschke's theorem, $\mathbb{Z}/\ell[G] \cong \prod_{i=0}^{\ell-2} \mathbb{Z}/\ell$ is a simple ring, so every $\mathbb{Z}/\ell[G]$-module has a unique decomposition as a sum of its $\ell - 1$ irreducible modules. Since $\mu_\ell$ is an irreducible $G$-module, it is easy to see that the irreducible $G$-modules are $\mu_\ell^{\otimes i}$, $i = 0, 1, ..., \ell - 2$. The "trivial" $G$-module is $\mu_\ell^{\otimes \ell-1} = \mu_\ell^{\otimes 0} = \mathbb{Z}/\ell$. By convention, $\mu_\ell^{\otimes i} = \mu_\ell^{\otimes i+a(\ell-1)}$ for all integers $a$.

For example, the $G$-submodule $\langle \beta^i \rangle$ of $K_{2i}(\mathbb{Z}[\zeta]; \mathbb{Z}/\ell)$ generated by $\beta^i$ is isomorphic to $\mu_\ell^{\otimes i}$. It is a trivial $G$-module only when $(\ell - 1)|i$.

If $A$ is any $\mathbb{Z}/\ell[G]$-module, it is traditional to decompose $A = \oplus A^{[i]}$, where $A^{[i]}$ denotes the sum of all $G$-submodules of $A$ isomorphic to $\mu_\ell^{\otimes i}$.

Example 10.4.1. Set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$. It is known that the torsionfree part $R^\times/\mu_\ell \cong \mathbb{Z}^{\frac{\ell-1}{2}}$ of the units of $R$ is isomorphic as a $G$-module to $\mathbb{Z}[G] \otimes_{\mathbb{Z}[c]} \mathbb{Z}$, where $c$ is complex conjugation. (This is sometimes included as part of Dirichlet's theorem on units.) It follows that as a $G$-module,

$$R^\times/R^{\times\ell} \cong \mu_\ell \oplus (\mathbb{Z}/\ell) \oplus \mu_\ell^{\otimes 2} \oplus \cdots \oplus \mu_\ell^{\otimes \ell-3}.$$

The first two terms $\mu_\ell$ and $\mathbb{Z}/\ell$ are generated by the root of unity $\zeta_\ell$ and the class of the unit $\ell$ of $R$. It will be convenient to choose units $x_0 = \ell$, $x_1, \dots, x_{(\ell-3)/2}$ of $R$ such that $x_i$ generates the summand $\mu_\ell^{-2i}$ of $R^\times/R^{\times\ell}$; the notation is set up so that $x_i \otimes \zeta_\ell^{\otimes 2i}$ is a $G$-invariant element of $R^\times \otimes \mu_\ell^{\otimes 2i}$.

EXAMPLE 10.4.2. The $G$-module decomposition of $M = R^\times \otimes \mu_\ell^{\otimes i-1}$ is obtained from Example 10.4.1 by tensoring with $\mu_\ell^{\otimes i-1}$. If $i$ is even, $\mathbb{Z}/\ell$ occurs only when $i \equiv 0 \pmod{\ell-1}$, corresponding to $\zeta^{\otimes i}$. If $i$ is odd, exactly one term of $M$ is $\mathbb{Z}/\ell$; $M^G$ is $\mathbb{Z}/\ell$ on the generator $x_j \otimes \zeta_\ell^{i-1}$, where $i \equiv 1 + 2j \pmod{\ell-1}$.

*Torsion for odd regular primes*

Suppose that $\ell$ is an odd regular prime. By definition, $\mathrm{Pic}(\mathbb{Z}[\zeta])$ has no $\ell$-torsion, and $K_1(R)/\ell \cong R^\times/R^{\times\ell}$ by III.1.3.6. Kummer showed that $\ell$ cannot divide the order of any numerator $c_k$ of $B_k/k$ (see 2.4.1). Therefore the case $2i = 4k-2$ of the following result follows from Theorem 10.1.

PROPOSITION 10.5. *When $\ell$ is an odd regular prime, the group $K_{2i}(\mathbb{Z})$ has no $\ell$-torsion. Thus the only $\ell$-torsion subgroups of $K_*(\mathbb{Z})$ are the Harris-Segal subgroups $\mathbb{Z}/w_i^{(\ell)}(\mathbb{Q})$ of $K_{2i-1}(\mathbb{Z})$ when $i \equiv 0 \pmod{\ell-1}$.*

PROOF. Since $\ell$ is regular, we saw in Example 8.3.2 that the group $K_{2i}(\mathbb{Z}[\zeta])$ has no $\ell$-torsion. Hence the same is true for its $G$-invariant subgroup, $K_{2i}(\mathbb{Z})$. The restriction on $i$ comes from Example 2.1.2. □

We can also describe the algebra structure of $K_*(\mathbb{Z};\mathbb{Z}/\ell)$. For this we set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$ and $G = \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, noting that $K_*(\mathbb{Z}[1/\ell];\mathbb{Z}/\ell) \cong K_*(R;\mathbb{Z}/\ell)^G$ by the usual transfer argument (10.3.1). Recall from Example 8.5 that $K_*(R;\mathbb{Z}/\ell)$ is a free graded $\mathbb{Z}/\ell[\beta]$-module on $\frac{\ell+1}{2}$ generators: the $x_i$ of $R^\times/R^{\times\ell} = K_1(R;\mathbb{Z}/\ell)$, together with $1 \in K_0(R;\mathbb{Z}/\ell)$.

Thus $K_{2i}(R;\mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$ is generated by $\beta^i$, and is isomorphic to $\mu_\ell^{\otimes i}$ as a $G$-module. It follows that $K_{2i}(R;\mathbb{Z}/\ell)^G$ is zero unless $i = a(\ell-1)$, when it is $\mathbb{Z}/\ell$ on the generator $\beta^i$. By abuse of notation, we shall write $\beta^{\ell-1}$ for the element of $K_{2(\ell-1)}(\mathbb{Z};\mathbb{Z}/\ell)$ corresponding to $\beta^{\ell-1}$; if $i = a(\ell-1)$ we shall write $\beta^i$ for the element $(\beta^{\ell-1})^a$ of $K_{2i}(\mathbb{Z};\mathbb{Z}/\ell)$ corresponding to $\beta^i \in K_{2i}(R;\mathbb{Z}/\ell)^G$.

By Example 8.5, $K_{2i-1}(R;\mathbb{Z}/\ell)$ is just $R^\times \otimes \mu_\ell^{\otimes i-1}$ when $\ell$ is regular. The $G$-module structure was determined in Example 10.4.2: if $i$ is even, exactly one term is $\mathbb{Z}/\ell$; if $i$ is odd, $\mathbb{Z}/\ell$ occurs only when $i \equiv 1 \pmod{\ell-1}$.

Multiplying $[\zeta] \in K_1(R;\mathbb{Z}/\ell)$ by $\beta^{\ell-2}$ yields the $G$-invariant element $v = [\zeta]\,\beta^{\ell-2}$ of $K_{2\ell-3}(R;\mathbb{Z}/\ell)$. Again by abuse of notation, we write $v$ for the corresponding element of $K_{2\ell-3}(\mathbb{Z};\mathbb{Z}/\ell)$.

Similarly, multiplying $x_k \in R^\times = K_1(R)$ by $\beta^{2k} \in K_{4k}(R;\mathbb{Z}/\ell)$ gives a $G$-invariant element $y_k = x_k\beta^{2k}$ of $K_{4k+1}(R;\mathbb{Z}/\ell)$ with $y_0 = [\ell]$ in $K_1(R;\mathbb{Z}/\ell)$. Again by abuse of notation, we write $y_k$ for the corresponding element of $K_{4k+1}(\mathbb{Z};\mathbb{Z}/\ell)$.

THEOREM 10.6. *If $\ell$ is an odd regular prime then $K_* = K_*(\mathbb{Z}[1/\ell];\mathbb{Z}/\ell)$ is a free graded module over the polynomial ring $\mathbb{Z}/\ell[\beta^{\ell-1}]$. It has $(\ell+3)/2$ generators: $1 \in K_0$, $v \in K_{2\ell-3}$, and the elements $y_k \in K_{4k+1}$ $(k = 0, \dots, \frac{\ell-3}{2})$ described above.*

*Similarly, $K_*(\mathbb{Z};\mathbb{Z}/\ell)$ is a free graded module over $\mathbb{Z}/\ell[\beta^{\ell-1}]$; a generating set is obtained from the generators of $K_*$ by replacing $y_0$ by $y_0\beta^{\ell-1}$.*

*The $\mathbb{Z}/\ell[\beta^{\ell-1}]$-submodule generated by $v$ and $\beta^{\ell-1}$ comes from the Harris-Segal summands of $K_{2i-1}(\mathbb{Z})$. The submodule generated by the $y$'s comes from the $\mathbb{Z}$ summands in $K_n(\mathbb{Z})$, $n \equiv 1 \pmod 4$.*

Proof. $K_*(\mathbb{Z}[1/\ell];\mathbb{Z}/\ell)$ is the $G$-invariant subalgebra of $K_*(R;\mathbb{Z}/\ell)$. Given 10.5, it is not very hard to check that this is just the subalgebra described in the theorem. Since $\ell-1$ is even, the elements $y_k\beta^{a(\ell-1)}$ are in $K_n(\mathbb{Z};\mathbb{Z}/\ell)$ for some $n \equiv 1 \pmod 4$. Since $K_{n-1}(\mathbb{Z})$ has no $\ell$-torsion by Proposition 10.5, $K_n(\mathbb{Z};\mathbb{Z}/\ell) = K_n(\mathbb{Z})/\ell$. Since $1 \le 4k+1 \le 2\ell-4$, we have $n \equiv 4k+1 \not\equiv 0 \pmod{2\ell-2}$ and hence $K_n(\mathbb{Z})$ has no $\ell$-torsion (combine 10.1 with 2.1.2). Hence the element $y_k\beta^{a(\ell-1)}$ must come from the $\mathbb{Z}$-summand of $K_n(\mathbb{Z}[1/\ell])$. $\square$

Examples 10.6.1. When $\ell = 3$, the groups $K_n = K_n(\mathbb{Z}[1/3];\mathbb{Z}/3)$ are 4-periodic of ranks $1,1,0,1$, generated by an appropriate power of $\beta^2 \in K_4$ times one of $\{1,[3],v\}$. Here $v \in K_3$.

When $\ell = 5$, the groups $K_n = K_n(\mathbb{Z}[1/5];\mathbb{Z}/5)$ are 8-periodic, with respective ranks $1,1,0,0,0,1,0,1$ ($n = 0,...,7$), generated by an appropriate power of $\beta^4 \in K_8$ times one of $\{1,[5],y_1,v\}$. Here $y_1 \in K_5$ ($x_1$ is the golden mean) and $v \in K_7$.

*Torsion for irregular primes*

Now suppose that $\ell$ is an irregular prime, so that $\mathrm{Pic}(R)$ has $\ell$-torsion for $R = \mathbb{Z}[\zeta, 1/\ell]$. Then $H^1_{\mathrm{et}}(R,\mu_\ell)$ is $R^\times/\ell \oplus {}_\ell\,\mathrm{Pic}(R)$ and $H^2_{\mathrm{et}}(R,\mu_\ell) \cong \mathrm{Pic}(R)/\ell$ by Kummer theory and (8.1.1). This yields $K_*(R;\mathbb{Z}/\ell)$ by Example 8.5.

Set $P = \mathrm{Pic}(R)/\ell$. When $\ell$ is irregular, the $G$-module structure of $P$ is not fully understood; see Vandiver's conjecture 10.8 below.

Lemma 10.7. *For $i = 0,-1,-2,-3$, $P$ contains no summands isomorphic to $\mu_\ell^{\otimes i}$, i.e., $P^{[i]} = 0$.*

Proof. The usual transfer argument shows that $P^G \cong \mathrm{Pic}(\mathbb{Z}[1/\ell])/\ell = 0$. Hence $P$ contains no summands isomorphic to $\mathbb{Z}/\ell$. By III.6.9.3, there is a $G$-module isomorphism $(P\otimes\mu_\ell) \cong K_2(R)/\ell$. Since $K_2(R)/\ell^{\,G} \cong K_2(\mathbb{Z}[1/\ell])/\ell = 0$, $(P\otimes\mu_\ell)$ has no $\mathbb{Z}/\ell$ summands — and hence $P$ contains no summands isomorphic to $\mu_\ell^{\otimes -1}$.

Finally, we have $(P\otimes\mu_\ell^{\otimes 2}) \cong K_4(R)/\ell$ and $(P\otimes\mu_\ell^{\otimes 3}) \cong K_6(R)/\ell$ by 8.3. Again, the transfer argument shows that $K_n(R)/\ell^{\,G} \cong K_n(\mathbb{Z}[1/\ell])/\ell$ for $n = 4,6$. The groups $K_4(\mathbb{Z})$ and $K_6(\mathbb{Z})$ are known to be zero by [Rognes] and [EGS]; see 10.1.3. It follows that $P$ contains no summands isomorphic to $\mu_\ell^{\otimes -2}$ or $\mu_\ell^{\otimes -3}$. $\square$

Vandiver's conjecture 10.8. If $\ell$ is an irregular prime then $\mathrm{Pic}(\mathbb{Z}[\zeta_\ell+\zeta_\ell^{-1}])$ has no $\ell$-torsion. Equivalently, the natural representation of $G = \mathrm{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})$ on $\mathrm{Pic}(\mathbb{Z}[\zeta_\ell])/\ell$ is a sum of $G$-modules $\mu_\ell^{\otimes i}$ with $i$ odd.

This means that complex conjugation $c$ acts as multiplication by $-1$ on the $\ell$-torsion subgroup of $\mathrm{Pic}(\mathbb{Z}[\zeta_\ell])/\ell$, because $c$ is the unique element of $G$ of order 2.

As partial evidence for this conjecture, we mention that Vandiver's conjecture has been verified for all primes up to 163 million; see [BH]. We also known from Lemma 10.7 that $\mu_\ell^{\otimes i}$ does not occur as a summand of $\mathrm{Pic}(R)/\ell$ for $i = 0,-2$.

REMARK 10.8.1. The Herbrand-Ribet theorem [Wash, 6.17–18] states that $\ell | B_k$ if and only if $(\mathrm{Pic}\ R/\ell)^{[\ell-2k]} \neq 0$. Among irregular primes $< 4000$, this happens for at most 3 values of $k$. For example, $37|c_{16}$ (see 10.3), so $(\mathrm{Pic}\ R/\ell)^{[5]} = \mathbb{Z}/37$ and $(\mathrm{Pic}\ R/\ell)^{[k]} = 0$ for $k \neq 5$.

HISTORICAL REMARK 10.8.2. What we now call "Vandiver's conjecture" was actually discussed by Kummer and Kronecker in 1849–1853; Harry Vandiver was not born until 1882 and only started using this assumption circa 1920 (e.g., in [Van29] and [Van34]), but only retroactively claimed to have conjectured it "about 25 years ago" in the 1946 paper [Van46, p. 576].

In 1849, Kronecker asked if Kummer conjectured that a certain lemma ([Wash, 5.36]) held for all $p$, and that therefore $p$ never divided $h^+$ (*i.e.,* Vandiver's conjecture holds). Kummer's reply [Kum, pp.114–115] pointed out that the Lemma could not hold for irregular $p$, and then referred to the assertion [Vandiver's conjecture] as the *noch zu beweisenden Satz* (theorem still to be proven). Kummer also pointed out some of its consequences. In an 1853 letter (see [Kum, p.123]), Kummer wrote to Kronecker that in spite of months of effort, the assertion [now called Vandiver's conjecture] was still unproven.

For the rest of this paper, we set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$, and $P = \mathrm{Pic}(R)/\ell$.

THEOREM 10.9. *(Kurihara [Kur]) Let $\ell$ be an irregular prime number. Then the following are equivalent for every integer $k$ between* 1 *and* $\frac{\ell-1}{2}$*:*

1. $\mathrm{Pic}(\mathbb{Z}[\zeta])/\ell^{[-2k]} = 0$.
2. $K_{4k}(\mathbb{Z})$ *has no $\ell$-torsion;*
3. $K_{2a(\ell-1)+4k}(\mathbb{Z})$ *has no $\ell$-torsion for all* $a \geq 0$*;*
4. $H^2_{\mathrm{et}}(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k+1}) = 0$.

*In particular, Vandiver's conjecture for $\ell$ is equivalent to the assertion that $K_{4k}(\mathbb{Z})$ has no $\ell$-torsion for all $k < \frac{\ell-1}{2}$, and implies that $K_{4k}(\mathbb{Z})$ has no $\ell$-torsion for all $k$.*

PROOF. By Kummer theory and (8.1.1), $P \cong H^2_{\mathrm{et}}(R, \mu_\ell)$. Hence $P \otimes \mu_\ell^{\otimes 2k} \cong H^2_{\mathrm{et}}(R, \mu_\ell^{\otimes 2k+1})$ as $G$-modules. Taking $G$-invariant subgroups shows that

$$H^2_{\mathrm{et}}(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k+1}) \cong (P \otimes \mu_\ell^{\otimes 2k})^G \cong P^{[-2k]}.$$

Hence (1) and (4) are equivalent.

By 8.3, $K_{4k}(\mathbb{Z})/\ell \cong H^2_{\mathrm{et}}(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k+1})$ for all $k > 0$. Since $\mu_\ell^{\otimes b} = \mu_\ell^{\otimes a(\ell-1)+b}$ for all $a$ and $b$, this shows that (2) and (3) are separately equivalent to (4). $\square$

THEOREM 10.10. *If Vandiver's conjecture holds for $\ell$ then the $\ell$-primary torsion subgroup of $K_{4k-2}(\mathbb{Z})$ is cyclic for all $k$.*

*If Vandiver's conjecture holds for all $\ell$, the groups $K_{4k-2}(\mathbb{Z})$ are cyclic for all $k$.*

(We know that the groups $K_{4k-2}(\mathbb{Z})$ are cyclic for all $k < 5000$, by 10.3.2.)

PROOF. Vandiver's conjecture also implies that each of the "odd" summands $P^{[1-2k]} = P^{[\ell-2k]}$ of $P$ is cyclic; see [Wash, 10.15]. Taking the $G$-invariant subgroups of $\mathrm{Pic}(R) \otimes \mu_\ell^{\otimes 2k-1} \cong H^2_{\mathrm{et}}(R, \mu_\ell^{\otimes 2k})$, yields $P^{[1-2k]} \cong H^2_{\mathrm{et}}(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k})$. By Corollary 8.3, this group is the $\ell$-torsion in $K_{4k-2}(\mathbb{Z}[1/\ell])/\ell$. $\square$

REMARK 10.11. The elements of $K_{2i}(\mathbb{Z})$ of odd order become divisible in the larger group $K_{2i}(\mathbb{Q})$. (The assertion that an element $a$ is divisible in $A$ means that for every $m$ there is an element $b$ so that $a = mb$.) This was proven by Banaszak and Kolster for $i$ odd (see [Ban, Thm. 2]), and for $i$ even by Banaszak and Gajda [BaGj, Proof of Prop. 8]. It is an open question whether there are any divisible elements of even order.

For example, recall from 10.3 that $K_{22}(\mathbb{Z}) = \mathbb{Z}/691$ and $K_{30}(\mathbb{Z}) \cong \mathbb{Z}/3617$. Banaszak observed [Ban] that these groups are divisible in $K_{22}(\mathbb{Q})$ and $K_{30}(\mathbb{Q})$, *i.e.,* that the inclusions $K_{22}(\mathbb{Z}) \subset K_{22}(\mathbb{Q})$ and $K_{30}(\mathbb{Z}) \subset K_{30}(\mathbb{Q})$ do not split.

Let $t_j$ (resp., $s_j$) be generators of the summand of $\mathrm{Pic}(R)/\ell$ (resp. $K_1(R;\mathbb{Z}/\ell)$) isomorphic to $\mu_\ell^{\otimes -j}$. The following result follows easily from Examples 8.5 and 10.4.1, using the proof of 10.6, 10.9 and 10.10. It was originally proven in [Mit].

THEOREM 10.12. *If $\ell$ is an irregular prime for which Vandiver's conjecture holds, then $K_* = K_*(\mathbb{Z};\mathbb{Z}/\ell)$ is a free module over $\mathbb{Z}/\ell[\beta^{\ell-1}]$ on 1, $v \in K_{2\ell-3}$, the $(\ell-3)/2$ generators $y_k \in K_{4k+1}$ described in Theorem 10.6, together with the generators $t_j\beta^j \in K_{2j}$ and $s_j\beta^j \in K_{2j+1}$ $(j = 3, 5, ..., (\ell-8))$.*

EXERCISES

**10.1** Let $\ell$ be an irregular prime and suppose that $K_n(\mathbb{Z})$ has no $\ell$-torsion for some positive $n \equiv 0 \pmod 4$. Show that $K_{4k}(\mathbb{Z})$ has no $\ell$-torsion for every $k$ satisfying $n \equiv 4k \pmod{2\ell-2}$.

**10.2** Show that $K_n(\mathbb{Z})$ has nonzero 37-torsion for all positive $n \equiv 62 \pmod{72}$, and that $K_n(\mathbb{Z})$ has nonzero 103-torsion for all positive $n \equiv 46 \pmod{204}$.

**10.3** Give a careful proof of Theorem 10.12, by using Examples 8.5 and 10.4.1 for $\mathbb{Z}[\zeta_\ell, 1/\ell]$ to modify the proof of Theorem 10.6.

**10.4** The Bockstein operation $b : K_n(R;\mathbb{Z}/\ell) \to K_{n+1}(R;\mathbb{Z}/\ell)$ is the boundary map in the long exact sequence associated to the coefficient sequence $0 \to \mathbb{Z}/\ell \to \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell \to 0$. Show that when $R = \mathbb{Z}$ the Bockstein sends $v$ to $\beta^{\ell-1}$, $t_j$ to $s_j$ and $t_j\beta^j$ to $s_j\beta^j$ in Theorems 10.6 and 10.12.