CHAPTER V

THE FUNDAMENTAL THEOREMS OF HIGHER K-THEORY

We now restrict our attention to exact categories and Waldhausen categories, where the extra structure enables us to use the following types of comparison theorems: Additivity (1.2), Cofinality (2.3), Approximation (2.4), Resolution (3.1), Devissage (4.1), and Localization (2.1, 2.5, 5.1 and 7.3). These are the extensions to higher K-theory of the corresponding theorems of chapter II. The highlight of this chapter is the so-called "Fundamental Theorem" of K-theory (6.3 and 8.2), comparing $K(R)$ to $K(R[t])$ and $K(R[t,t^{-1}])$, and its analogue (6.13.2 and 8.3) for schemes.

§1. The Additivity theorem

If $F' \to F \to F''$ is a sequence of exact functors $F', F, F'' \colon \mathcal{B} \to \mathcal{C}$ between two exact categories (or Waldhausen categories), the Additivity Theorem tells us when the induced maps $K(\mathcal{B}) \to K(\mathcal{C})$ satisfy $F_* = F'_* + F''_*$. To state it, we need to introduce the notion of a short exact sequence of functors, which was mentioned briefly in II(9.1.8).

Definition 1.1. (a) If $\mathcal{B}$ and $\mathcal{C}$ are exact categories, we say that a sequence $F' \to F \to F''$ of exact functors and natural transformations from $\mathcal{B}$ to $\mathcal{C}$ is a *short exact sequence* of exact functors, and write $F' \rightarrowtail F \twoheadrightarrow F''$, if

$$0 \to F'(B) \to F(B) \to F''(B) \to 0$$

is an exact sequence in $\mathcal{C}$ for every $B \in \mathcal{B}$.
(b) If $\mathcal{B}$ and $\mathcal{C}$ are Waldhausen categories, we say that $F' \rightarrowtail F \twoheadrightarrow F''$ is a *short exact sequence*, or a *cofibration sequence* of exact functors if each $F'(B) \rightarrowtail F(B) \twoheadrightarrow F''(B)$ is a cofibration sequence and if for every cofibration $A \rightarrowtail B$ in $\mathcal{B}$, the evident map $F(A) \cup_{F'(A)} F'(B) \rightarrowtail F(B)$ is a cofibration in $\mathcal{C}$.

When exact categories are regarded as Waldhausen categories, these two notions of "short exact sequence" of exact functors between exact categories are easily seen to be the same.

Universal Example 1.1.1. Recall from chapter II, 9.3, that the extension category $\mathcal{E} = \mathcal{E}(\mathcal{C})$ is the category of all exact sequences $E \colon A \rightarrowtail B \twoheadrightarrow B/A$ in $\mathcal{C}$. If $\mathcal{C}$ is an exact category, or a Waldhausen category, so is $\mathcal{E}$. The source $s(E) = A$, target $t(E) = B$, and quotient $q(E) = C$ of such a sequence are exact functors from $\mathcal{E}$ to $\mathcal{C}$, and $s \rightarrowtail t \twoheadrightarrow q$ is a short exact sequence of functors. This example is universal in the sense that giving an exact sequence of exact functors from $\mathcal{B}$ to $\mathcal{C}$ is the same thing as giving an exact functor $\mathcal{B} \to \mathcal{E}(\mathcal{C})$.

Additivity Theorem 1.2. *Let $F' \rightarrowtail F \twoheadrightarrow F''$ be a short exact sequence of exact functors from $\mathcal{B}$ to $\mathcal{C}$, either between exact categories or between Waldhausen categories. Then $F_* \simeq F'_* + F''_*$ as H-space maps $K(\mathcal{B}) \to K(\mathcal{C})$, and therefore on the homotopy groups we have $F_* = F'_* + F''_* : K_i(\mathcal{B}) \to K_i(\mathcal{C})$.*

Proof of the Additivity Theorem. By universality of $\mathcal{E}$, we may assume that $\mathcal{B}$ is $\mathcal{E}$ and prove that $t_* = s_* + q_*$. The map $s_* + q_*$ is induced by the exact functor $s \coprod q \colon \mathcal{E} \to \mathcal{C}$ which sends $A \rightarrowtail B \twoheadrightarrow C$ to $A \amalg C$, because the H-space structure on $K(\mathcal{C})$ is induced from $\amalg$ (see IV, 6.4 and 8.5.1). The compositions of t and $s \coprod q$ with the coproduct functor $\amalg : \mathcal{C} \times \mathcal{C} \to \mathcal{E}$ agree:

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\amalg} \mathcal{E} \quad \underset{s \vee q}{\overset{t}{\rightrightarrows}} \mathcal{C},$$

and hence give the same map on K-theory. The Extension Theorem 1.3 below proves that $K(\amalg)$ is a homotopy equivalence, from which we conclude that $t \simeq s \coprod q$, as desired. $\square$

Given an exact category (or a Waldhausen category) $\mathcal{A}$, we say that a sequence $0 \to A_n \to \cdots \to A_0 \to 0$ is *admissibly exact* if each map decomposes as $A_{i+1} \twoheadrightarrow B_i \rightarrowtail A_i$, and each $B_i \rightarrowtail A_i \twoheadrightarrow B_{i-1}$ is an exact sequence.

Corollary 1.2.1. *(Additivity for characteristic exact sequences). If*

$$0 \to F^0 \to F^1 \to \cdots \to F^n \to 0$$

is an admissibly exact sequence of exact functors $\mathcal{B} \to \mathcal{C}$, then $\sum (-1)^p F^p_* = 0$ as maps from $K_i(\mathcal{B})$ to $K_i(\mathcal{C})$.

Proof. This follows from the Additivity Theorem 1.2 by induction on n. $\square$

Remark 1.2.2. Suppose that F' and F are the same functor in the Additivity Theorem 1.2. Using the H-space structure we have $F''_* \simeq F_* - F_* \simeq 0$. It follows that the homotopy fiber of $K(\mathcal{B}) \xrightarrow{F''} K(\mathcal{C})$ is homotopy equivalent to $K(\mathcal{B}) \vee \Omega K(\mathcal{C})$.

Example 1.2.3. Let $\mathcal{C}$ be a Waldhausen category with a cylinder functor (IV.8.8). Then the definition of cone and suspension imply that $1 \rightarrowtail \mathrm{cone} \twoheadrightarrow \Sigma$ is an exact sequence of functors: each $A \rightarrowtail \mathrm{cone}(A) \twoheadrightarrow \Sigma A$ is exact. If $\mathcal{C}$ satisfies the Cylinder Axiom (IV.8.8), the cone is null-homotopic. The Additivity Theorem implies that $\Sigma_* + 1 = \mathrm{cone}_* = 0$. It follows that $\Sigma \colon K(\mathcal{C}) \to K(\mathcal{C})$ is a homotopy inverse with respect to the H-space structure on $K(\mathcal{C})$.

The following calculation of $K(\mathcal{E})$, used in the proof of the Additivity Theorem 1.2, is due to Quillen [Q341] for exact categories and to Waldhausen [W1126] for Waldhausen categories. (The K_0 version of this Theorem was presented in II.9.3.1.)

Extension Theorem 1.3. *The exact functor $(s, q) : \mathcal{E} = \mathcal{E}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ induces homotopy equivalences $wS.\mathcal{E} \simeq (wS.\mathcal{C})^2$ and $K(\mathcal{E}) \xrightarrow{\sim} K(\mathcal{C}) \times K(\mathcal{C})$. The coproduct functor $\amalg$, sending (A, B) to the sequence $A \rightarrowtail A \amalg B \twoheadrightarrow B$, is a homotopy inverse.*

Proof for Waldhausen categories. Let $\mathcal{C}^w_m$ denote the category of sequences $A \xrightarrow{\simeq} B \xrightarrow{\simeq} \cdots$ of m weak equivalences; this is a category with cofibrations (defined termwise). The set $s_n\mathcal{C}^w_m$ of sequences of n cofibrations in $\mathcal{C}^w_m$ (IV.8.5.2)

$$
\begin{array}{ccccccc}
A_0 & \rightarrowtail & A_1 & \rightarrowtail & \cdots \rightarrowtail & A_n \\
\downarrow\wr & & \downarrow\wr & & & \downarrow\wr \\
B_0 & \rightarrowtail & B_1 & \rightarrowtail & \cdots \rightarrowtail & B_n \\
\downarrow\wr & & \downarrow\wr & & & \downarrow\wr \\
\cdots & & \cdots & & & \cdots
\end{array}
$$

is naturally isomorphic to the m-simplices in the nerve of the category $wS_n\mathcal{C}$. That is, the bisimplicial sets $wS_.\mathcal{C}$ and $s_.\mathcal{C}^w_.$ are isomorphic. By Ex. IV.8.10 applied to $\mathcal{C}^w_m$, each of the maps $s_.\mathcal{E}(\mathcal{C}^w_m) \to s_.\mathcal{C}^w_m \times s_.\mathcal{C}^w_m$ is a homotopy equivalence. As m varies, we get a bisimplicial map $s_.\mathcal{E}(\mathcal{C}^w_.) \to s_.\mathcal{C}^w_. \times s_.\mathcal{C}^w_.$, which must then be a homotopy equivalence. But we have just seen that this is isomorphic to the bisimplicial map $wS_.\mathcal{E}(\mathcal{C}) \to wS_.\mathcal{C} \times wS_.\mathcal{C}$ of the Extension Theorem. $\square$

We include a proof of the Extension Theorem 1.3 for exact categories, because it is short and uses a different technique.

Proof of 1.3 for Exact Categories. By Quillen's Theorem A (IV.3.7), it suffices to show that, for every pair (A, C) of objects in $\mathcal{C}$, the comma category $\mathcal{T} = (s,q)/(A,C)$ is contractible. A typical object in this category is a triple $T = (u, E, v)$, where E is an extension $A_0 \rightarrowtail B_0 \twoheadrightarrow C_0$ and both $u : A_0 \to A$ and $v : C_0 \to C$ are morphisms in $Q\mathcal{C}$. We will compare $\mathcal{T}$ to its subcategories $\mathcal{T}_A$ and $\mathcal{T}_C$, consisting of those triples T such that: u is an admissible epi, respectively, v is an admissible monic. The contraction of $B\mathcal{T}$ is illustrated in the following diagram.

$$
\begin{array}{rccccccccc}
T: & A & \overset{u}{\leftarrow} & A_0 & \rightarrowtail & B_0 & \twoheadrightarrow & C_0 & \overset{v}{\to} & C \\
\eta_T \downarrow & \| & & i\downarrow & & \downarrow & & \| & & \| \\
p(T): & A & \overset{j}{\twoheadrightarrow} & A_1 & \rightarrowtail & B & \twoheadrightarrow & C_0 & \overset{v}{\to} & C \\
\pi_{p(T)} \uparrow & \| & & \| & & \twoheaduparrow & & \twoheaduparrow & & \| \\
qp(T): & A & \twoheadrightarrow & A_1 & \rightarrowtail & B_1 & \twoheadrightarrow & C_1 & \rightarrowtail & C \\
\uparrow & \| & & \twoheaddownarrow & & \uparrow & & \uparrow & & \| \\
0 & A & \twoheadrightarrow & 0 & \rightarrowtail & 0 & \twoheadrightarrow & 0 & \rightarrowtail & C
\end{array}
$$

Given a triple T, choose a factorization of u as $A_0 \overset{i}{\rightarrowtail} A_1 \overset{j}{\twoheadleftarrow} A$, and let B be the pushout of A_1 and B_0 along A_0; B is in $\mathcal{C}$ and $p(E) : A_1 \rightarrowtail B \twoheadrightarrow C_0$ is an exact sequence by II, Ex. 7.8(2). Thus $p(T) = (j, p(E), v)$ is in the subcategory $\mathcal{T}_A$. The construction shows that p is a functor from $\mathcal{T}$ to $\mathcal{T}_A$, and that there is a natural transformation $\eta_T : T \to p(T)$. This provides a homotopy $B\eta$ between the identity of $B\mathcal{T}$ and the map $Bp : B\mathcal{T} \to B\mathcal{T}_A \subset B\mathcal{T}$.

By duality, if we choose a factorization of v as $C_0 \twoheadleftarrow C_1 \rightarrowtail C$ and let B_1 be the pullback of B_0 and C_1 along C_0, then we obtain a functor $q : \mathcal{T} \to \mathcal{T}_C$, and a natural transformation $\pi_T : q(T) \to T$. This provides a homotopy $B\pi$ between $Bq : B\mathcal{T} \to B\mathcal{T}_C \subset B\mathcal{T}$ and the identity of $B\mathcal{T}$.

The composition of $B\eta$ and the inverse of $B\pi$ is a homotopy $B\mathcal{T} \times I \to B\mathcal{T}$ between the identity and $Bqp : B\mathcal{T} \to B(\mathcal{T}_A \cap \mathcal{T}_C) \subset B\mathcal{T}$. Finally, the category

$(\mathcal{T}_A \cap \mathcal{T}_C)$ is contractible because it has an initial object: $(A \twoheadrightarrow 0, 0, 0 \rightarrowtail C)$. Thus Bqp (and hence the identity of $B\mathcal{T}$) is a contractible map, providing the contraction $B\mathcal{T} \simeq 0$ of the comma category. $\square$

It is useful to have a variant of the Extension Theorem involving two Waldhausen subcategories $\mathcal{A}$, $\mathcal{C}$ of a Waldhausen category $\mathcal{B}$. Recall from II.9.3 that the extension category $\mathcal{E} = \mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of $\mathcal{C}$ by $\mathcal{A}$ is the Waldhausen subcategory of $\mathcal{E}(\mathcal{B})$ consisting of cofibration sequences $A \rightarrowtail B \twoheadrightarrow C$ with A in $\mathcal{A}$ and C in $\mathcal{C}$.

COROLLARY 1.3.1. *Let $\mathcal{A}$ and $\mathcal{C}$ be Waldhausen subcategories of a Waldhausen category $\mathcal{B}$, and $\mathcal{E} = \mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ the extension category. Then $(s, q) : \mathcal{E} \to \mathcal{A} \times \mathcal{C}$ induces a homotopy equivalence $K(\mathcal{E}) \to K(\mathcal{A}) \times K(\mathcal{C})$.*

PROOF. Since (s, q) is a left inverse to $\amalg : \mathcal{A} \times \mathcal{C} \to \mathcal{E}$, it suffices to show that the identity of $K(\mathcal{E})$ is homotopic to $\amalg(s, q)_* = \amalg(s, 0)_* + \amalg(0, t)_*$. This follows from Additivity applied to the short exact sequence of functors $\amalg(s, 0) \rightarrowtail \mathrm{id}_{\mathcal{E}} \twoheadrightarrow \amalg(0, t)$ displayed in the proof of the corresponding Extension Theorem II.9.3.1 for K_0. $\square$

The rest of this section is devoted to applications of the Additivity Theorem.

EXACT SEQUENCES 1.4. Let $\mathcal{A}^{[0,n]}_{\text{exact}}$ denote the category of admissibly exact sequences of length n. If $n = 1$ it is equivalent to the category $\mathcal{A}$; for $n = 2$ it is the category $\mathcal{E}$ of cofibration sequences of 1.1.1. In fact, $\mathcal{A}^{[0,n]}_{\text{exact}}$ is a Waldhausen category in a way which extends the structure in 1.1.1: $A_* \to A'_*$ is a weak equivalence if each $A_i \to A'_i$ is, and is a cofibration if the $B_i \rightarrowtail B'_i$ and $B'_i \cup_{B_i} A_i \rightarrowtail A^i_1$ are all cofibrations. Since $\mathcal{A}^{[0,n]}_{\text{exact}}$ is the extension category $\mathcal{E}$ of $\mathcal{A} \cong \mathcal{A}^{[0,0]}_{\text{exact}}$ by $\mathcal{A}^{[1,n]}_{\text{exact}}$ (1.3.1), the Extension Theorem 1.3 implies that the functors $A_* \mapsto B_i$ ($i = 0, ..., n-1$) from $\mathcal{A}^{[0,n]}_{\text{exact}}$ to $\mathcal{A}$ induce a homotopy equivalence $K\mathcal{A}^{[0,n]}_{\text{exact}} \cong \prod_{i=1}^n K(\mathcal{A})$.

Projective bundles

Let $\mathcal{E}$ be a vector bundle of rank $r+1$ over a quasi-projective scheme X, and consider the projective space bundle $\mathbb{P} = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$. We saw in II.8.5 that $K_0(\mathbb{P})$ is a free $K_0(X)$-module with basis $\{[\mathcal{O}(-i)] : i = 0 \dots, r\}$. The summands of this decomposition arise from the exact functors

$$u_i : \mathbf{VB}(X) \to \mathbf{VB}(\mathbb{P}), \quad u_i(\mathcal{N}) = \pi^*(\mathcal{N})(-i).$$

PROJECTIVE BUNDLE THEOREM 1.5. *Let $\mathbb{P}(\mathcal{E})$ be a projective bundle over a quasi-projective scheme X. Then the u_i induce an equivalence $K(X)^{r+1} \simeq K(\mathbb{P}(\mathcal{E}))$. Thus $K_*(X) \otimes_{K_0(X)} K_0(\mathbb{P}(\mathcal{E})) \to K_*(\mathbb{P}(\mathcal{E}))$ is a ring isomorphism.*

When $\mathcal{E}$ is a trivial bundle, so $\mathbb{P}(\mathcal{E}) = \mathbb{P}^n_X$, we have the following special case.

COROLLARY 1.5.1. *As a ring, $K_*(\mathbb{P}^n_X) \cong K_*(X) \otimes_{\mathbb{Z}} K_0(\mathbb{P}^n_{\mathbb{Z}}) \cong K_*(X)[z]/(z^{r+1})$.*

To prove Theorem 1.5, recall from II.8.7.1 that we call a vector bundle $\mathcal{F}$ *Mumford-regular* if $R^q\pi_*\mathcal{F}(-q) = 0$ for all $q > 0$. We write $\mathbf{MR}$ for the exact category of all Mumford-regular vector bundles. The direct image $\pi_* : \mathbf{MR} \to \mathbf{VB}(X)$ is an exact functor, so the following lemma allows us to define the transfer map $\pi_* : K(\mathbb{P}) \to K(X)$.

LEMMA 1.5.2. $\mathbf{MR} \subset \mathbf{VB}(\mathbb{P})$ *induces an equivalence* $K\mathbf{MR} \simeq K(\mathbb{P})$.

PROOF. Write $\mathbf{MR}(n)$ for the category of all $\mathcal{F}$ for which $\mathcal{F}(-n)$ is Mumford-regular. Then $\mathbf{MR} = \mathbf{MR}(0) \subseteq \mathbf{MR}(-1) \subseteq \cdots$, and $\mathbf{VB}(\mathbb{P}(\mathcal{E}))$ is the union of the $\mathbf{MR}(n)$ as $n \to -\infty$. Thus $K(\mathbb{P}) = \varinjlim K\mathbf{MR}(n)$. Hence it suffices to show that each inclusion $\iota_n : \mathbf{MR}(n) \subset \mathbf{MR}(n-1)$ induces a homotopy equivalence on K-theory. We saw in the proof of II.8.7.10 that the exact functors $\lambda_i : \mathbf{MR}(n-1) \to \mathbf{MR}(n)$, $\lambda_i(\mathcal{F}) = \mathcal{F}(i) \otimes \pi^*(\wedge^i \mathcal{E})$, fit into a Koszul resolution of $\mathcal{F}$:

$$0 \to \mathcal{F} \to \mathcal{F}(1) \otimes \pi^* \mathcal{E} \to \cdots \to \mathcal{F}(r+1) \otimes \pi^* \wedge^{r+1} \mathcal{E} \to 0.$$

By Additivity (1.2.1), ι_n has $\sum_{i=1}^{r} (-1)^i \lambda_i$ as a homotopy inverse. $\square$

PROOF OF THEOREM 1.5. By II.8.7.9 we have exact functors $T_i : \mathbf{MR} \to \mathbf{VB}(X)$, with $T_0 = \pi_*$, which assemble to form a map $t : \mathbf{MR} \to \coprod \mathbf{VB}(X)$. By Quillen's canonical resolution of a Mumford-regular bundle (II.8.7.8) is an exact sequence of exact functors:

$$0 \to \pi^*(T_r)(-r) \xrightarrow{\varepsilon(-r)} \cdots \to \pi^*(T_i)(-i) \xrightarrow{\varepsilon(-i)} \cdots \xrightarrow{\varepsilon(-1)} \pi^*(T_0) \xrightarrow{\varepsilon} \mathcal{F} \to 0.$$

Again by Additivity (1.2.1), $\sum (-1)^i u_i T_i$ is homotopic to the identity on $K\mathbf{MR}$, so u_* is split up to homotopy.

Define $v_i : \mathbf{MR} \to \mathbf{VB}(X)$ by $v_i(\mathcal{F}) = \pi_*(\mathcal{F}(i))$. Then by II.8.7.2:

$$v_i u_j(\mathcal{N}) = \begin{cases} 0, & i < j; \\ \mathcal{N}, & i = j; \\ Sym_{i-j}\mathcal{E} \otimes \pi^*(\mathcal{N}), & i > j. \end{cases}$$

It follows that $v_* \circ u_* : K(X)^{r+1} \to K(X)^{r+1}$ is given by a triangular matrix whose diagonal entries are homotopic to the identity. Thus $v_* \circ u_*$ is a homotopy equivalence, as desired. $\square$

VARIANT 1.5.3. Theorem 1.5 remains valid if X is noetherian instead of quasiprojective, but the proof is more intricate because in this case $K(X)$ is defined to be $K\mathbf{Ch}_{\mathrm{perf}}(X)$ (see 2.7.3). This generalization was proven by Thomason in [TT, 4.1] by (a) replacing $\mathbf{VB}(X)$ by the category of perfect complexes of coherent sheaves; (b) replacing $\mathbf{MR}$ by the category of all Mumford-regular coherent sheaves (II.8.7.1) and passing to the Waldhausen category of bounded perfect complexes of Mumford-regular sheaves (Lemma 1.5.2 remains valid); and (c) observing that Quillen's canonical resolution (II.8.7.8) makes sense for Mumford-regular sheaves. With these modifications, the proof we have given for Theorem 1.5 goes through. By Ex. 1.10, Theorem 1.5 even remains valid for all quasi-compact and quasi-separated schemes X.

The projective line over a ring

Let R be any associative ring. We define $\mathbf{mod}\text{-}\mathbb{P}^1_R$ to be the abelian category of triples $\mathcal{F} = (M_+, M_-, \alpha)$, where $M_\pm$ is in $\mathbf{mod}\text{-}R[t^{\pm 1}]$ and α is is an isomorphism $M_+ \otimes_{R[t]} R[t, 1/t] \xrightarrow{\simeq} M_- \otimes_{R[1/t]} R[t, 1/t]$. It has a full (exact) subcategory $\mathbf{VB}(\mathbb{P}^1_R)$

consisting of triples where $M_\pm$ are finitely generated projective modules, and we write $K(\mathbb{P}^1_R)$ for $K\mathbf{VB}(\mathbb{P}^1_R)$.

If R is commutative, it is well known that $\mathbf{mod}\text{-}\,\mathbb{P}^1_R$ is equivalent to the category of quasi-coherent sheaves on $\mathbb{P}^1_R$, and $\mathbf{VB}(\mathbb{P}^1_R)$ is equivalent to the usual category of vector bundles on the line $\mathbb{P}^1_R$; thus $K(\mathbb{P}^1_R)$ agrees with the definition in IV.6.3.4, and both π_* and $R^1\pi_*$ have their usual meanings.

We define the functors $\pi_*, R^1\pi_* : \mathbf{mod}\text{-}\,\mathbb{P}^1_R \to \mathbf{mod}\text{-}R$ via the exact sequence

$$0 \to \pi_*(\mathcal{F}) \to M_+ \times M_- \xrightarrow{d} M_- \otimes_{R[1/t]} R[t,1/t] \to R^1\pi_*(\mathcal{F}) \to 0.$$

where $d(x,y) = \alpha(x) - y$. If R is commutative, these are the usual functors $\pi_*, R^1\pi_*$.

There are exact functors $u_i : \mathbf{P}(R) \to \mathbf{VB}(\mathbb{P}^1_R)$, sending P to $(P[t], P[1/t], t^i)$; for commutative R the u_i are the functors $u_i(P) = \pi^*(P) \otimes \mathcal{O}(-i)$ of Theorem 1.5.

THEOREM 1.5.4. *The functors u_0, u_1 induce an equivalence $K(R) \oplus K(R) \simeq K(\mathbb{P}^1_R)$. In addition, $(u_{i+1})_* + (u_{i+1})_* \simeq (u_i)_* + (u_{i+2})_*$ for all i.*

PROOF. If $\mathcal{F} = (M_+, M_-, \alpha)$, we define $\mathcal{F}(n)$ to be $(M_+, M_-, t^{-n}\alpha)$, and let $X_0, X_1 : \mathcal{F}(n-1) \to \mathcal{F}(n)$ be the maps $(1, 1/t)$ and $(t, 1)$, respectively. Then we have an exact sequence (the Koszul resolution of $\mathcal{F}$).

$$0 \to \mathcal{F}(-2) \xrightarrow{(X_1, -X_0)} \mathcal{F}(-1)^2 \xrightarrow{(X_0, X_1)} \mathcal{F} \to 0.$$

Applying this to $u_i(P)$ and using $u_i(P)(n) = u_{i-n}(P)$ yields the exact sequence $u_{i+2} \rightarrowtail u_{i+1}^2 \twoheadrightarrow u_i$ of functors, and the relations follow from Additivity 1.2. The proof of Theorem 1.5 now goes through to prove Theorem 1.5.4 (see Ex. 1.3). $\square$

Severi-Brauer schemes

Let A be a central simple algebra over a field k, and ℓ a maximal subfield of A. Then $A \otimes_k \ell \cong M_r(\ell)$ for some r, and the set of minimal left ideals of $M_r(\ell)$ correspond to the ℓ-points of the projective space $\mathbb{P}^{r-1}_\ell$; if I is a minimal left ideal corresponding to a line L of ℓ^r then the rows of matrices in I all lie on L. The Galois group $Gal(\ell/k)$ acts on this set, and it is well known that there is a variety X, defined over k, such that $X_\ell = X \times_k \ell$ is $\mathbb{P}^{r-1}_\ell$ with this Galois action. The variety X is called the *Severi-Brauer* variety of A. For example, the Severi-Brauer variety associated to $A = M_r(k)$ is just $\mathbb{P}^{r-1}_k$.

Historically, these varieties arose in the 1890's (over $\mathbb{R}$) as forms of a complex variety, together with a real structure given by an involution with no fixed points.

EXAMPLE 1.6.1. Let X be a non-singular projective curve over k defined by an irreducible quadratic $aX^2 + bY^2 = Z^2$ $(a, b \in k)$. Then $X \cong \mathbb{P}^1_k$ if and only if X has a k-point, which holds if and only if the quadratic form $q(x,y) = ax^2 + by^2$ has a solution to $q(x,y) = 1$ in k. The associated algebra is the quaternion algebra $A(a,b)$ of III.6.9. For example, if X is the plane curve $X^2 + Y^2 + Z^2 = 0$ over $\mathbb{R}$ then A is the usual quaternions $\mathbb{H}$.

Here are some standard facts about Severi-Brauer varieties. By faithfully flat descent, the vector bundle $\mathcal{O}^r(-1)$ on $\mathbb{P}^{r-1}_\ell$ descends to a vector bundle J on X of rank r, and $A \cong H^0(X, \mathrm{End}_X(J))$ because $\mathrm{End}_{\mathbb{P}^{r-1}_\ell}(\mathcal{O}^r(-1))$ is a sheaf of matrix algebras having $M_r(\ell)$ as its global sections. Moreover, if $\pi : X \to S = \mathrm{Spec}(k)$ is the structure map then $\pi^*(A) \cong \mathrm{End}_X(J)$, as can be checked by pulling back to ℓ.

There is a canonical surjection $\mathcal{O}^r_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}}$ (Ex. II.6.14); by descent it defines a surjection $J \twoheadrightarrow \mathcal{O}_X$. Hence there is a Koszul resolution:

$$0 \to \wedge^r J \to \cdots \to \wedge^2 J \to J \to \mathcal{O}_X \to 0.$$

The n-fold tensor product $A^{\otimes n}$ of A over k is also a central simple algebra, isomorphic to $\mathrm{End}_X(J^{\otimes n})$. Moreover, since $J^{\otimes n}$ is a right module over $A^{\otimes n} = \mathrm{End}_X(J^{\otimes n})$ there is an exact functor $J^{\otimes n}\otimes : \mathbf{P}(A^{\otimes n}) \to \mathbf{VB}(X)$ sending P to $J^{\otimes n} \otimes_{A^{\otimes n}} P$.

Theorem 1.6.2. *(Quillen) If X is the Severi-Brauer variety of A, the functors $J^{\otimes n}\otimes$ define an equivalence $\prod_{n=0}^{r-1} K(A^{\otimes n}) \xrightarrow{\sim} K(X)$, and an isomorphism*

$$\bigoplus_{n=0}^{r-1} K_*(A^{\otimes n}) \xrightarrow{\simeq} K_*(X).$$

Example 1.6.3. If X is the nonsingular curve $aX^2 + bY^2 + Z^2$ associated to the quaternion algebra $A = A(a, b)$, then $K_*(X) \cong K_*(k) \oplus K_*(A)$.

The proof of Theorem 1.6.2 is a simple modification of the proof of Theorem 1.5. First, we define a vector bundle $\mathcal{F}$ to be *regular* if $\mathcal{F} \otimes_k \ell$ is Mumford-regular on $X_\ell = \mathbb{P}^{r-1}_\ell$. The regular bundles form an exact subcategory of $\mathcal{O}_X$-**mod**, and $\mathcal{F} \mapsto \pi_*\mathcal{F} = H^0(X, \mathcal{F})$ is an exact functor from regular bundles to k-modules, as one checks by passing to ℓ and applying II.8.7.4. To get the analogue of the Quillen Resolution Theorem II.8.7.8, we modify Definition II.8.7.6 using J.

Definition 1.6.4 (T_n). Given a regular $\mathcal{O}_X$-module $\mathcal{F}$, we define a natural sequence of k-modules $T_n = T_n(\mathcal{F})$ and $\mathcal{O}_X$-modules $Z_n = Z_n(\mathcal{F})$, starting with $T_0(\mathcal{F}) = \pi_*\mathcal{F}$ and $Z_{-1} = \mathcal{F}$. Let Z_0 be the kernel of the natural map $J \otimes_A \pi_*\mathcal{F} \to \mathcal{F}$. Inductively, we define $T_n(\mathcal{F}) = \pi_* \mathrm{Hom}_X(J^{\otimes n}, Z_{n-1})$ and define Z_n to be the kernel of $J^{\otimes n} \otimes_{A^{\otimes n}} T_n(\mathcal{F}) \to Z_{n-1}(\mathcal{F})$.

These fit together to give a natural sequence of $\mathcal{O}_X$-modules

$$0 \to J^{\otimes r-1} \otimes_{A^{\otimes r-1}} T_{r-1}(\mathcal{F}) \to \cdots \to \mathcal{O}_X \otimes_k T_0(\mathcal{F}) \to \mathcal{F} \to 0. \tag{1.6.5}$$

When lifted to X_ℓ, it is easy to see that these are exactly the functors T_n and Z_n of II.8.7.6 for $\mathcal{F} \otimes_k \ell$. By faithfully flat descent, (1.6.5) is exact, *i.e.*, a resolution of the regular bundle $\mathcal{F}$.

Thus all the tools used in the Projective Bundle Theorem 1.5 are available for Severi-Brauer varieties. The rest of the proof is routine (Exercise 1.4).

Theorem 1.6.2 may be generalized over any base scheme S. Here are the key points. An *Azumaya algebra* over S is a sheaf of rings $\mathcal{A}$ which is locally isomorphic to $M_r(\mathcal{O}_S)$ for the étale topology. That is, there is a faithfully flat map $T \to S$ so that $\mathcal{A}\otimes_S\mathcal{O}_T$ is the sheaf of rings $M_r(\mathcal{O}_T)$. Let $\mathbf{VB}(\mathcal{A})$ denote the exact category of vector bundles on S which are left modules for $\mathcal{A}$, and define

$$K(\mathcal{A})=K\mathbf{VB}(\mathcal{A});\quad K_n(\mathcal{A})=K_n\mathbf{VB}(\mathcal{A}).$$

By a *Severi-Brauer* scheme over S we mean a scheme X which is locally isomorphic to projective space for the étale topology, *i.e.,* such that $X\times_S T\cong\mathbb{P}^{r-1}_T$ for some faithful étale map $T\to S$. In this situation, we may define a vector bundle J on X by faithfully flat descent so that $J\otimes_S T=\mathcal{O}^r_T(-1)$, as above, and then $\mathcal{A}=\pi_*\operatorname{End}_X(J)$ will be an Azumaya algebra over S. Again, each $J^{\otimes n}$ is a right module over $\mathcal{A}^{\otimes n}$ and we have exact functors $J^{\otimes n}\otimes_{\mathcal{A}^{\otimes n}}:\mathbf{VB}(\mathcal{A}^{\otimes n})\to\mathbf{VB}(X)$. Replacing $H^0(X,-)$ with π_*, Definition 1.6.4 makes sense and (1.6.5) is admissibly exact. Therefore the proof still works in this generality, and we have:

THEOREM 1.6.6. *(Quillen) If X is a Severi-Brauer variety over S, with associated Azumaya algebra A, the functors $J^{\otimes n}\otimes_{\mathcal{A}^{\otimes n}}$ define an isomorphism*

$$\coprod_{n=0}^{r-1}K_*(\mathcal{A}^{\otimes n})\xrightarrow{\simeq}K_*(X).$$

Our next application of Additivity was used in IV, 8.5.3–5 to show that $K(\mathcal{C})$ is an infinite loop space. To do this, we defined the relative K-theory space to be $K(f)=\Omega^2|wS.(S.f)|$, and invoked the following result.

PROPOSITION 1.7. *If $f:\mathcal{B}\to\mathcal{C}$ is an exact functor, the following sequence is a homotopy fibration:*

$$\Omega|wS.(S.\mathcal{B})|\to|wS.\mathcal{C}|\to|wS.(S.f)|\to|wS.(S.\mathcal{B})|.$$

PROOF. Each category S_nf of IV.8.5.3 is equivalent to the extension category $\mathcal{E}(\mathcal{B},S_nf,S_n\mathcal{C})$ of $\mathcal{B}$ by $S_n\mathcal{C}$ (see II.9.3). By 1.3.1, the map

$$(s,q):wS.(S_nf.)\xrightarrow{\simeq}wS.\mathcal{B}\times wS.(S_n\mathcal{C})$$

is a homotopy equivalence. That is, $|wS.\mathcal{B}|\to|wS.(S_nf.)|\to|wS.(S_n\mathcal{C})|$ is a (split) fibration of connected spaces for each n. But if $X.\to Y.\to Z.$ is any sequence of simplicial spaces, and each $X_n\to Y_n\to Z_n$ is a fibration with Z_n connected, then $\Omega|Z.|\to|X.|\to|Y.|\to|Z.|$ is a homotopy fibration sequence; see [Wa78, 5.2]. This applies to our situation by realizing in the $wS.$ direction first (so that $X_n=|wS.\mathcal{B}|$ for all n), and the result follows. $\square$

REMARK 1.7.1. As observed in IV.8.5.4, $wS.S.f$ is contractible when f is the identity of $\mathcal{B}$. It follows that $|wS.\mathcal{B}|\simeq\Omega|wS.(S.\mathcal{B})|$, yielding the formulation of 1.7 given in IV.8.5.3.

Let $F:\mathcal{A}\to\mathcal{B}$ be an exact functor between exact categories. An *admissible filtration* of F is a sequence $0\rightarrowtail F_1\rightarrowtail F_2\rightarrowtail\cdots\rightarrowtail F_n=F$ of functors and admissible monomorphisms, sending an object A in $\mathcal{A}$ to the sequence

$$0=F_0(A)\rightarrowtail F_1(A)\rightarrowtail\cdots\rightarrowtail F_n(A)=F(A).$$

It follows that the quotient functors F_p/F_{p-1} exist, but they may not be exact.

PROPOSITION 1.8. *(Admissible Filtrations) If* $0 \rightarrowtail F_1 \rightarrowtail F_2 \rightarrowtail \cdots \rightarrowtail F_n = F$ *is an admissible filtration of* F*, and the quotient functors* F_i/F_{i-1} *are exact, then* $F : K\mathcal{A} \to K\mathcal{B}$ *is homotopic to* $\sum F_i/F_{i-1}$*. In particular,*

$$F_* = \sum (F_i/F_{i-1})_* : K_*(\mathcal{A}) \to K_*(\mathcal{B}).$$

PROOF. Apply the Additivity Theorem 1.2 to $F_{i-1} \rightarrowtail F_i \twoheadrightarrow F_i/F_{i-1}$, and use induction on n. $\square$

Here is a simple application of Proposition 1.8. Let $S = R \oplus S_1 \oplus S_2 \oplus \cdots$ be a graded ring, and consider the category $\mathbf{P}_{gr}(S)$ of finitely generated graded projective S-modules. Its K-groups are naturally modules over $\mathbb{Z}[\sigma, \sigma^{-1}]$, where σ acts by the shift automorphism $\sigma(P) = P[-1]$ of graded modules. If $S = R$, $K_*(R)[\sigma, \sigma^{-1}] \cong K_*(\mathbf{P}_{gr}(R))$ by Ex. IV.6.11. Thus the base change map $\otimes_R S$: $\mathbf{P}_{gr}(R) \to \mathbf{P}_{gr}(S)$ induces a morphism $K_*(R)[\sigma, \sigma^{-1}] \to K_*(\mathbf{P}_{gr}(S))$.

COROLLARY 1.8.1. *If* $S = R \oplus S_1 \oplus S_2 \oplus \cdots$ *is graded then the base change map induces an isomorphism* $K_*(R)[\sigma, \sigma^{-1}] \cong K_*(\mathbf{P}_{gr}(S))$*.*

PROOF. For each $a \le b$, let $\mathbf{P}_{[a,b]}(S)$ denote the (exact) subcategory of $\mathbf{P}_{gr}(S)$ consisting of graded modules P generated by the P_i with $i \le b$, and with $P_i = 0$ for $i < a$. By Ex. 1.9, the identity functor on this category has an admissible filtration: $0 = F_a \rightarrowtail F_{a+1} \rightarrowtail \cdots \rightarrowtail F_b = \mathrm{id}$, where $F_n P$ denotes the submodule of P generated by the P_i with $i \le n$. Moreover, there is a natural isomorphism between F_n/F_{n-1} and the degree n part of the exact functor $\otimes_S R : \mathbf{P}_{gr}(S) \to \mathbf{P}_{gr}(R)$. By Proposition 1.8, the homomorphism $\oplus_{n=a}^{b} K_*(R) \otimes \sigma^n \cong K_* \mathbf{P}_{[a,b]}(R) \to K_* \mathbf{P}_{[a,b]}(S)$ is an isomorphism with inverse $\otimes_S R$. Since $\mathbf{P}_{gr}(S)$ is the filtered colimit of the $\mathbf{P}_{[a,b]}(S)$, the result follows from IV.1.4. $\square$

FLASQUE CATEGORIES 1.9. Call an exact (or Waldhausen) category $\mathcal{A}$ *flasque* if there is an exact functor $\infty\colon \mathcal{A} \to \mathcal{A}$ and a natural isomorphism $\infty(A) \cong A \amalg \infty(A)$, *i.e.,* $\infty \cong 1 \amalg \infty$, 1 being the identity functor. By additivity, $\infty_* = 1_* + \infty_*$, and hence the identity map $1_* : K(\mathcal{A}) \to K(\mathcal{A})$ is null-homotopic. Therefore $K(\mathcal{A})$ is contractible, and $K_i(\mathcal{A}) = 0$ for all i.

THE EILENBERG SWINDLE 1.9.1. For example, the category of *countably* generated R-modules is flasque, so its K-theory is trivial. To see this, let $\infty(M) = M^\infty$ be the direct sum $M \oplus M \oplus \cdots$ of infinitely many copies of M. The isomorphism $M^\infty \cong M \oplus M^\infty$ is the shift

$$(M \oplus M \oplus \cdots) \cong M \oplus (M \oplus M \oplus \cdots).$$

This infinite shifting trick is often called the "Eilenberg swindle" (see I.2.8, II.6.1.4 and II.9.1.4); it is why we restrict to finitely generated modules in defining $K(R)$.

EXAMPLE 1.9.2 (FLASQUE RINGS). Here is another example of a flasque category, due to Karoubi. Recall from II.2.1.3 that a ring R is called *flasque* if there is an R-bimodule M_∞, finitely generated projective as a right module, and a bimodule isomorphism $\theta : R \oplus M_\infty \cong M_\infty$. If R is flasque, then $\mathbf{P}(R)$ and $\mathbf{M}(R)$ are flasque categories in the sense of 1.9, with $\infty(M) = M \otimes_R M_\infty$. The contractibility of $K(R) = K_0(R) \times BGL(R)^+$ for flasque rings, established in Ex. IV.1.17, may be viewed as an alternative proof that $K\mathbf{P}(R) = \Omega BQ\mathbf{P}(R)$ is contractible, via the '$+ = Q$' theorem IV.7.2.

As mentioned in IV.10.4.1, this contractibility was used by Karoubi, Gersten and Wagoner to define deloopings of $K(R)$ in terms of the suspension ring $S(R)$ of IV.1.11.3, forming a nonconnective spectrum $\mathbf{K}^{GW}(R)$ homotopy equivalent to the spectrum $\mathbf{K}^B(R)$ of IV.10.4.

EXERCISES

1.1 In the proof of the Extension Theorem 1.3, show that the functors p and q are left and right adjoint, respectively, to the inclusions of $\mathcal{T}_A$ and $\mathcal{T}_C$ in $\mathcal{T}$. This proves that Bp and Bq are homotopy equivalences.

1.2 Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be a short exact sequence of vector bundles on a scheme (or a ringed space). Show that the map $(-\otimes \mathcal{E})_* : K(X) \to K(X)$ given by the exact functor $\mathcal{F} \to \mathcal{F} \otimes \mathcal{E}$ satisfies $(-\otimes \mathcal{E})_* = (-\otimes \mathcal{E}')_* + (-\otimes \mathcal{E}'')_*$.

1.3 Complete the proof of Theorem 1.5.4, by modifying the proof of 1.5 for $\mathbb{P}^1_R$. (See [Q341, 8.4.1].)

1.4 Complete the proof of Theorems 1.6.2 and 1.6.6, by modifying the proof of 1.5. (See [Q341, 8.4.1].) For extra credit, describe the ring structure on $K_*(X)$ using the pairings $\mathbf{mod}\text{-}A^{\otimes i} \times \mathbf{mod}\text{-}A^{\otimes j} \to \mathbf{mod}\text{-}A^{\otimes i+j}$ (tensor product over k) and the Morita equivalence of $A^{\otimes i}$ and $A^{\otimes i+n}$.

1.5 Given an exact category $\mathcal{A}$ and integers $a \le b$, let $\mathrm{iso}\,\mathbf{Ch}^{[a,b]}(\mathcal{A})$ denote the category of chain complexes $C_b \to \cdots \to C_a$ in $\mathcal{A}$. We may consider it as a Waldhausen category whose cofibrations are degreewise admissible monics, with isomorphisms as the weak equivalences (II.9.1.3). Use the Additivity Theorem 1.2 to show that the "forget differentials" functor $\mathrm{iso}\,\mathbf{Ch}^{[a,b]}(\mathcal{A}) \to \prod_{i=a}^{b} \mathcal{A}$ induces a homotopy equivalence on K-theory.

1.6 If $\mathcal{A}$ is an exact category, the category $\mathcal{A}^{[0,n]}_{\mathrm{exact}}$ of admissibly exact sequences (Example 1.4) may be viewed as a subcategory of the category $\mathrm{iso}\,\mathbf{Ch}^{[0,n]}(\mathcal{A})$ of the previous exercise. Use the Additivity Theorem to show that the "forget differentials" functor $\mathcal{A}^{[0,n]}_{\mathrm{exact}} \to \prod_{i=0}^{n} \mathcal{A}$ and the functor $A_* \mapsto (B_1, B_1 \oplus B_2, ..., B_{n-1} \oplus B_n, B_n)$ induce homotopy equivalent maps on K-theory.

1.7 If $f : \mathcal{B} \to \mathcal{C}$ is exact, show that the composite $wS.\mathcal{B} \to \Omega|wS.S.\mathcal{B}| \to |wS.\mathcal{C}|$ in Proposition 1.7 is the map induced by f. *Hint:* Use $wS.S.\mathrm{id}_{\mathcal{B}} \to wS.S.f$.

1.8 Recall from IV.8.7 that $A(*)$ is the K-theory of the category $\mathcal{R}_f(*)$ of finite based CW complexes. Let $\mathcal{R}^{(2)}_f(*)$ be the subcategory of simply connected complexes. Show that $K\mathcal{R}^{(2)}_f(*) \xrightarrow{\sim} K\mathcal{R}_f(*) = A(*)$ is a homotopy equivalence, with $Y \mapsto \Sigma^2 Y$ as inverse. Then formulate a version for $A(X)$.

1.9 (Swan) If $S = R \oplus S_1 \oplus \cdots$ is a graded ring and P is a graded projective S-module, show that the map $(P \otimes_S R) \otimes_R S \to P$ is an isomorphism. If F_nP is the submodule of P generated by the P_i with $i \le n$, show that F_nP and P/F_nP are graded projective modules, and that F_n and $P \mapsto P/F_nP$ are exact functors from $\mathbf{P}_{gr}(S)$ to itself. Conclude that $\cdots \rightarrowtail F_nP \rightarrowtail F_{n+1}P \rightarrowtail \cdots$ is an admissible filtration of P. Is there a natural isomorphism $P \cong F_nP \oplus P/F_nP$?

1.10 Let X be a quasi-compact, quasi-separated scheme. Show that the variant 1.5.3 of the Projective Bundle Theorem 1.5 holds for X. *Hint:* X is the inverse limit of an inverse system of noetherian schemes X_α with affine bonding maps by [TT, C.9]. Show that any vector bundle $\mathcal{E}$ on X is the pullback of a vector bundle $\mathcal{E}_\alpha$ over some X_α.

§2. Waldhausen localization and Approximation

Here are two fundamental results about Waldhausen K-theory that, although technical in nature, have played a major role in the development of K-theory.

Waldhausen Localization

The first fundamental result involves a change in the category of weak equivalences, with the same underlying category of cofibrations. The K_0 version of this result, which needed fewer hypotheses, was presented in II.9.6.

WALDHAUSEN LOCALIZATION THEOREM 2.1. *Let $\mathcal{A}$ be a category with cofibrations, equipped with two categories of weak equivalences, $v(\mathcal{A}) \subset w(\mathcal{A})$, such that $(\mathcal{A}, v)$ and (A, w) are both Waldhausen categories. In addition, we suppose that $(\mathcal{A}, w)$ has a cylinder functor satisfying the Cylinder Axiom (IV.8.8), and that $w(\mathcal{A})$ satisfies the Saturation and Extension Axioms (II.9.1.1 and IV.8.2.1). Then*

$$K(\mathcal{A}^w) \to K(\mathcal{A}, v) \to K(\mathcal{A}, w)$$

is a homotopy fibration, where $\mathcal{A}^w$ denotes the Waldhausen subcategory of $(\mathcal{A}, v)$ consisting of all A in $\mathcal{A}$ for which $0 \to A$ is in $w(\mathcal{A})$. In particular, there is a long exact sequence:

$$\cdots \to K_{i+1}(\mathcal{A}, w) \to K_i(\mathcal{A}^w) \to K_i(\mathcal{A}, v) \to K_i(\mathcal{A}, w) \to \cdots,$$

ending in the exact sequence $K_0(\mathcal{A}^w) \to K_0(\mathcal{A}, v) \to K_0(\mathcal{A}, w) \to 0$ of II.9.6.

PROOF. Consider the bicategory $v.w.\mathcal{C}$ (IV.3.10) whose bimorphisms are commutative squares in $\mathcal{C}$

$$\begin{array}{ccc} \cdot & \xrightarrow{w'} & \cdot \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} \\ \cdot & \xrightarrow{w} & \cdot \end{array}$$

in which the vertical maps are in $v\mathcal{C}$ and the horizontal maps are in $w\mathcal{C}$. Considering $w\mathcal{C}$ as a bicategory which is vertically constant, we saw in IV, 3.10.2 and Ex. 3.13 that $w\mathcal{C} \to v.w.\mathcal{C}$ is a homotopy equivalence. Applying this construction to $S_n\mathcal{C}$, we get equivalences $wS_n\mathcal{C} \simeq v.wS_n\mathcal{C}$ and hence $wS.\mathcal{C} \simeq v.wS.\mathcal{C}$.

Let $v.\operatorname{co} w.\mathcal{C}$ denote the sub-bicategory of those squares in $v.w.\mathcal{C}$ whose horizontal maps are also cofibrations. We claim that the inclusions $v.\operatorname{co} w.\mathcal{C} \subset v.w.\mathcal{C}$ are homotopy equivalences. To see this, we fix m and consider the column category $v_m w.\mathcal{C}$ to be the category of weak equivalences in the category $\mathcal{C}(m, v)$ of diagrams $C_0 \xrightarrow{\sim} \cdots \xrightarrow{\sim} C_m$ in $\mathcal{C}$ whose maps are in $v\mathcal{C}$. Because $(\mathcal{C}(m, v), w)$ inherits the saturation and cylinder axioms from $(\mathcal{C}, w)$, it follows from IV, Ex. 8.14 that the inclusion $v_m \operatorname{co} w.\mathcal{C} \subset v_m w.\mathcal{C}$ is a homotopy equivalence. Since this is true for each m, the claim follows.

Now each $S_n\mathcal{C}$ inherits a cylinder functor from $\mathcal{C}$. Replacing $\mathcal{C}$ by the $S_n\mathcal{C}$ shows that the simplicial bicategory $v.w.S.\mathcal{C}$ contains a simplicial bicategory $v.\operatorname{co} w.S.\mathcal{C}$, and that the inclusion $v.\operatorname{co} w.S.\mathcal{C} \subset v.w.S.\mathcal{C}$ is a homotopy equivalence. This means that the right vertical map is a homotopy equivalence in the following diagram; the bottom horizontal map is a homotopy equivalence by the first paragraph of this proof.

$$\begin{tikzcd} vS.\mathcal{C}^w \arrow[r] & vS.\mathcal{C} \arrow[r] \arrow[d] & v.\operatorname{co} w.S.\mathcal{C} \arrow[d, "\simeq"] \\ & wS.\mathcal{C} \arrow[r, "\simeq"] & v.w.S.\mathcal{C} \end{tikzcd}$$

Thus it suffices to show that the top row is a homotopy fibration. We will identify it with the homotopy fibration $vS.\mathcal{C}^w \to vS.\mathcal{C} \to vS.(S.f)$ arising from the relative K-theory space construction (IV.8.5.3), applied to the inclusion $f : (\mathcal{C}^w, v) \to (\mathcal{C}, v)$.

By the extension axiom, a trivial cofibration in $(\mathcal{C}, w)$ is just a cofibration whose quotient lies in $\mathcal{C}^w$. In particular, there is an equivalence $S_1 f \to \operatorname{co} w\mathcal{C}$. Forgetting the choices of the C_i/C_j yields an equivalence $S_n f \to \operatorname{co} w_n\mathcal{C}$, where $\operatorname{co} w_n\mathcal{C}$ is the category of all trivial cofibration sequences $C_0 \overset{\sim}{\rightarrowtail} C_1 \overset{\sim}{\rightarrowtail} \cdots \overset{\sim}{\rightarrowtail} C_n$, and an equivalence between $vS_n f$ and the vertical category $v.\operatorname{co} w_n\mathcal{C}$ of the bicategory $v.\operatorname{co} w.\mathcal{C}$. Similarly, forgetting choices yields an equivalence between the categories $vS_m(S_n f)$ and $v.\operatorname{co} w_n(S_m\mathcal{C})$, and thus a homotopy equivalence $vS.S.f \to v.\operatorname{co} w.S.\mathcal{C}$.

Now $vS_m\mathcal{C} \to v.\operatorname{co} w_n(S_m\mathcal{C})$ factors through $vS_m(S_n f)$, so $vS.\mathcal{C} \to v.\operatorname{co} w.S.\mathcal{C}$ factors through the homotopy equivalence $vS.S.f \to v.\operatorname{co} w.S.\mathcal{C}$, as required. $\square$

For us, the most important application of Waldhausen Localization is the following theorem, which allows us to replace the K-theory of any exact category $\mathcal{A}$ by the K-theory of the category $\mathbf{Ch}^b(\mathcal{A})$ of bounded chain complexes, which is a Waldhausen category with a cylinder functor. This result was first worked out by Waldhausen in special cases, and generalized by Gillet. Our presentation is taken from [TT, 1.11.7].

Let $\mathcal{A}$ be an exact category, and consider the category $\mathbf{Ch}^b(\mathcal{A})$ of bounded chain complexes in $\mathcal{A}$. We saw in II.9.2 that $\mathbf{Ch}^b(\mathcal{A})$ is a Waldhausen category; the cofibrations are degreewise admissible monomorphisms, and the weak equivalences are quasi-isomorphisms (as computed in a specified ambient abelian category). The isomorphism $K_0(\mathcal{A}) \cong K_0\mathbf{Ch}^b(\mathcal{A})$ of Theorem II.9.2.2 generalizes as follows.

THEOREM 2.2 (GILLET-WALDHAUSEN). *Let $\mathcal{A}$ be an exact category, closed under kernels of surjections in an abelian category (in the sense of II.7.0.1.) Then the exact inclusion $\mathcal{A} \subset \mathbf{Ch}^b(\mathcal{A})$ induces a homotopy equivalence $K(\mathcal{A}) \xrightarrow{\sim} K\mathbf{Ch}^b(\mathcal{A})$.*

In particular, $K_n(\mathcal{A}) \cong K_n\mathbf{Ch}^b(\mathcal{A})$ for all n.

PROOF. We will apply Waldhausen's Localization Theorem 2.1 to the following situation. For $a \le b$, let $\mathbf{Ch}^{[a,b]}$ denote the full subcategory of all complexes A_* in $\mathbf{Ch}(\mathcal{A})$ for which the A_i are zero unless $a \le i \le b$. This is a Waldhausen subcategory of $\mathbf{Ch}^b(\mathcal{A})$ with w the quasi-isomorphisms. We write $\operatorname{iso}\mathbf{Ch}^{[a,b]}$ for the Waldhausen category with the same underlying category with cofibrations, but with isomorphisms as weak equivalences. Because $\mathcal{A}$ is closed under kernels of surjections, the subcategory of quasi-isomorphisms in $\operatorname{iso}\mathbf{Ch}^{[a,b]}$ is just the Waldhausen category $\mathcal{A}^{[a,b]}_{\text{exact}}$ of Example 1.4 (see Ex. 2.4). We claim that there is a homotopy fibration

$$K\mathcal{A}^{[a,b]}_{\text{exact}} \to K\operatorname{iso}\mathbf{Ch}^{[a,b]} \xrightarrow{\chi} K(\mathcal{A}).$$

By Example 1.4 and Ex. 1.5, the first two spaces are products of $n = b-a$ and $n+1$ copies of $K(\mathcal{A})$, respectively. By Ex. 1.6, the induced map $\prod_{a+1}^{b} K(\mathcal{A}) \to \prod_{a}^{b} K(\mathcal{A})$ is equivalent to that induced by the exact functor

$$(B_{a+1}, ..., B_b) \mapsto (B_{a+1}, B_{a+1} \oplus B_{a+2}, ..., B_{b-1} \oplus B_b, B_b).$$

The homotopy cofiber of this map is $K(\mathcal{A})$, with the map $\prod_{a}^{b} K(\mathcal{A}) \to K(\mathcal{A})$ being the alternating sum of the factors, *i.e.,* the Euler characteristic χ. This shows that $K\mathbf{Ch}^{[a,b]} \simeq K(\mathcal{A})$ for each a and b.

Taking the direct limit as $a \to -\infty$ and $b \to +\infty$ yields a homotopy fibration

$$K\mathcal{A}_{\text{exact}}^{[-\infty,\infty]} \to K\text{iso}\,\mathbf{Ch}^{[-\infty,\infty]} \xrightarrow{\chi} K(\mathcal{A}),$$

where χ is the Euler characteristic. But by Waldhausen Localization 2.1, the cofiber is $K\mathbf{Ch}^b(\mathcal{A})$. $\square$

Remark 2.2.1. When $\mathcal{A}$ is not closed under kernels in its ambient abelian category, $K_0(\mathcal{A})$ may not equal $K_0\mathbf{Ch}^b(\mathcal{A})$; see Ex. II.9.11. However, the following trick shows that the extra assumption is harmless in Theorem 2.2, provided that we allow ourselves to change the ambient notion of quasi-isomorphism slightly in $\mathbf{Ch}^b(\mathcal{A})$. Consider the Yoneda embedding of $\mathcal{A}$ in the abelian category $\mathcal{L}$ of contravariant left exact functors (Ex. II.7.8). As pointed out in *loc. cit.*, the idempotent completion $\widehat{\mathcal{A}}$ of $\mathcal{A}$ (II.7.3) is closed under surjections in $\mathcal{L}$.

Let $\mathcal{A}'$ be the full subcategory of $\widehat{\mathcal{A}}$ consisting of all B with $[B]$ in the subgroup $K_0(\mathcal{A})$ of $K_0(\widehat{\mathcal{A}})$. We saw in Ex. IV.8.13 that $\mathcal{A}'$ is exact and closed under admissible epis in $\hat{\mathcal{A}}$ (and hence in $\mathcal{L}$), so that Theorem 2.2 applies to $\mathcal{A}'$. By K_0-cofinality (II, 7.2 and 9.4), $K_0(\mathcal{A}) = K_0(\mathcal{A}') = K_0\mathbf{Ch}^b(\mathcal{A}) = K_0\mathbf{Ch}^b(\mathcal{A}')$. By Waldhausen Cofinality (IV.8.9), $K(\mathcal{A}) \simeq K(\mathcal{A}')$ and $K\mathbf{Ch}^b(\mathcal{A}) \simeq K\mathbf{Ch}^b(\mathcal{A}')$. Hence $K(\mathcal{A}) \simeq K\mathbf{Ch}^b(\mathcal{A})$.

Cofinality Theorem 2.3. *Let $(\mathcal{A}, v)$ be a Waldhausen category with a cylinder functor satisfying the cylinder axiom (IV.8.8). Suppose that we are given a surjective homomorphism $\pi\colon K_0(\mathcal{A}) \to G$, and let $\mathcal{B}$ denote the full Waldhausen subcategory of all B in $\mathcal{A}$ with $\pi[B] = 0$ in G.*

Then $vs.\mathcal{B} \to vs.\mathcal{A} \to BG$ and its delooping $K(\mathcal{B}) \to K(\mathcal{A}) \to G$ are homotopy fibrations. In particular, $K_n(\mathcal{B}) \cong K_n(\mathcal{A})$ for all $n > 0$ and (as in II.9.6.2) there is a short exact sequence:

$$0 \to K_0(\mathcal{B}) \to K_0(\mathcal{A}) \xrightarrow{\pi} G \to 0.$$

Proof. (Thomason) As in II.9.6.2, we can form the Waldhausen category $(\mathcal{A}, w)$, where $w(\mathcal{A})$ is the set of maps $A \to A'$ in $\mathcal{A}$ with $\pi[A] = \pi[A']$. It is easy to check that $w(\mathcal{A})$ is saturated (II.9.1.1), $\mathcal{B} = \mathcal{A}^w$, and that $(\mathcal{A}, w)$ satisfies the Extension Axiom IV.8.2.1. By the Waldhausen Localization Theorem 2.1, there is a homotopy fibration

$$K(\mathcal{B}) \to K(\mathcal{A}) \to K(\mathcal{A}, w).$$

By IV.8.10, $ws.(\mathcal{A}, w) \simeq BG$ and hence $K(\mathcal{A}, w) \simeq \Omega(BG) = G$, as required. $\square$

Combining this with the Waldhausen Cofinality Theorem IV.8.9.1, we obtain the following variation. Recall from Theorem II.9.4 that a Waldhausen subcategory $\mathcal{B}$ is said to be *cofinal* in $\mathcal{A}$ if for each A in $\mathcal{A}$ there is an A' so that $A \amalg A'$ is in $\mathcal{B}$, and that this implies that $K_0(\mathcal{B}) \to K_0(\mathcal{A})$ is an injection.

COROLLARY 2.3.1. *Let $\mathcal{B}$ be a cofinal Waldhausen subcategory of $\mathcal{A}$ closed under extensions. Suppose that $\mathcal{A}$ has a cylinder functor satisfying the cylinder axiom (IV.8.8), and restricting to a cylinder functor on $\mathcal{B}$.*

Then for $G = K_0(\mathcal{A})/K_0(\mathcal{B})$ there is a homotopy fibration sequence

$$K(\mathcal{B}) \to K(\mathcal{A}) \to G.$$

PROOF. Clearly $\mathcal{B}$ is contained in the Waldhausen subcategory $\mathcal{A}^w$ associated to $K_0(\mathcal{A}) \to G$. By the Waldhausen Cofinality Theorem IV.8.9.1, $K(\mathcal{B}) \simeq K(\mathcal{A}^w)$. The result now follows from Theorem 2.3. □

Waldhausen Approximation

The second fundamental result is the Approximation Theorem, whose K_0 version was presented in II.9.7. Consider the following "approximate lifting property," which is to be satisfied by an exact functor $F\colon \mathcal{A} \to \mathcal{B}$:

(App) Given any map $b\colon F(A) \to B$ in $\mathcal{B}$, there is a map $a\colon A \to A'$ in $\mathcal{A}$ and a weak equivalence $b' : F(A') \simeq B$ in $\mathcal{B}$ so that $b = b' \circ F(a)$.

Roughly speaking, this axiom says that every object and map in $\mathcal{B}$ lifts up to weak equivalence to $\mathcal{A}$. Note that if we replace A' by the mapping cylinder $T(a)$ of IV.8.8, a by $A \rightarrowtail T(a)$ and b' by $F(T(a)) \simeq F(A') \simeq B$, then we may assume that a is a cofibration. The following result is taken from [W1126, 1.6.7].

WALDHAUSEN APPROXIMATION THEOREM 2.4. *Suppose that $F\colon \mathcal{A} \to \mathcal{B}$ is an exact functor between saturated Waldhausen categories, satisfying the conditions:*

(a) A morphism f in $\mathcal{A}$ is a weak equivalence if and only if $F(f)$ is a w.e. in $\mathcal{B}$.
(b) $\mathcal{A}$ has a cylinder functor satisfying the cylinder axiom.
(c) The approximate lifting property (App) is satisfied.

Then $wS.\mathcal{A} \xrightarrow{\sim} wS.\mathcal{B}$ and $K(\mathcal{A}) \xrightarrow{\sim} K(\mathcal{B})$ are homotopy equivalences. In particular, the groups $K_*\mathcal{A}$ and $K_*\mathcal{B}$ are isomorphic.

PROOF. (Waldhausen) Each of the exact functors $S_n\mathcal{A} \to S_n\mathcal{B}$ also satisfies (App); see Ex. 2.1. Applying Proposition 2.4.1 below to these functors, we see that each $wS_n\mathcal{A} \to wS_n\mathcal{B}$ is also a homotopy equivalence. It follows that the bisimplicial map $wS_.\mathcal{A} \to wS_.\mathcal{B}$ is also a homotopy equivalence, as required. □

PROPOSITION 2.4.1. *(Waldhausen) Suppose that $F\colon \mathcal{A} \to \mathcal{B}$ is an exact functor between Waldhausen categories, satisfying the three hypotheses of Theorem 2.4. Then $wF : w\mathcal{A} \xrightarrow{\sim} w\mathcal{B}$ is a homotopy equivalence.*

PROOF. By Quillen's Theorem A (IV.3.7), it suffices to show that the comma categories wF/B are contractible. The condition (App) states that, given any object (A, b) of $F/\mathcal{B}$ there is a map a in $F/\mathcal{B}$ to an object (A', b') of $wF/\mathcal{B}$. Applying it to $(0, 0)$ shows that $wF/\mathcal{B}$ is nonempty. For any finite set of objects (A_i, b_i) in $wF/\mathcal{B}$, the maps $A_i \to \oplus A_i$ yield maps in $F/\mathcal{B}$ to $(\oplus A_i, b)$, where $b : F(\oplus A_i) \cong$

$\oplus F(A_i) \to B$, and hence maps a_i from each (A_i, b_i) to an object (A', b') in $wF/\mathcal{B}$; the $F(a_i)$ are in $w\mathcal{B}$ by saturation, so the a_i are in $w\mathcal{A}$ and represent maps in $wF/\mathcal{B}$. The same argument shows that if we are given any finite diagram D on these objects in $wF/\mathcal{B}$, an object (A, b) in $F/\mathcal{B}$ and maps $(A_i, b_i) \to (A, b)$ forming a (larger) commutative diagram D_+ in $F/\mathcal{B}$, then by composing with $(A, b) \to (A', b')$, we embed D into a diagram D'_+ in $wF/\mathcal{B}$ with a terminal object. This implies that $|D|$ is contractible in $|wF/\mathcal{B}|$.

The rest of the proof consists of finding such a diagram D_+ for every "non-singular" finite subcomplex of $|wF/\mathcal{B}|$, using simplicial methods. We omit this part of the proof, which is lengthy (5 pages), and does not seem relevant to this book, and refer the reader to [W1126, 1.6.7]. $\square$

Remark 2.4.2. The Approximation Theorem can fail in the absence of a cylinder functor. For example, if $\mathcal{A}$ is an exact category then $\mathcal{A}^{\oplus} \subset \mathcal{A}$ satisfies (App), yet $K_0(\mathcal{A}^{\oplus})$ and $K_0(\mathcal{A})$ are often different; see II.7.1.

Combining Waldhausen Localization 2.1 and Approximation 2.4 yields the following useful result, applicable to exact functors $F : \mathcal{A} \to \mathcal{B}$ which are onto up to weak equivalence. Let $\mathcal{A}^w$ denote the Waldhausen subcategory of $\mathcal{A}$ consisting of all A such that $F(A)$ is weak equivalent to 0 in $\mathcal{B}$.

Proposition 2.5. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between two saturated Waldhausen categories having cylinder functors, with $\mathcal{B}$ extensional (IV.8.2.1). If every object B and every map $F(A) \to B$ in $\mathcal{B}$ lifts to $\mathcal{A}$ up to weak equivalence, then $K(\mathcal{A}^w) \to K(\mathcal{A}, v) \to K(\mathcal{B})$ is a homotopy fibration sequence, and there is a long exact sequence:*

$$\cdots \xrightarrow{F} K_{n+1}(\mathcal{B}) \to K_n(\mathcal{A}^w) \to K_n(\mathcal{A}) \xrightarrow{F} K_n(\mathcal{B}) \to \cdots,$$

ending in the exact sequence $K_0(\mathcal{A}^w) \to K_0(\mathcal{A}) \to K_0(\mathcal{B}) \to 0$.

Proof. (Thomason) Let $v(\mathcal{A})$ and $w(\mathcal{B})$ denote the respective categories of weak equivalences in $\mathcal{A}$ and $\mathcal{B}$, and set $w(\mathcal{A}) = F^{-1}(w(\mathcal{B}))$. Replacing $v(\mathcal{A})$ with $w(\mathcal{A})$ yields a new Waldhausen category, which we write as $(\mathcal{A}, w)$ for clarity. The Approximation Theorem 2.4 states that $K(\mathcal{A}, w) \simeq K(\mathcal{B})$. Since $(\mathcal{A}, w)$ inherits the extension axiom from $\mathcal{B}$, Waldhausen Localization 2.1 applies to give the fibration $K(\mathcal{A}^w) \to K(\mathcal{A}, v) \to K(\mathcal{B})$ and hence the displayed long exact sequence. $\square$

Changing cofibrations 2.5.1. (Hinich-Shektman). Let $\mathcal{A} = (\mathcal{A},\ co\mathcal{A},\ w\mathcal{A})$ be a saturated Waldhausen category with a cylinder functor, satisfying the cylinder axiom. Suppose that $co\mathcal{A} \subset co_1\mathcal{A} \subset \mathcal{A}$ is such that $\mathcal{A}_1 = (\mathcal{A}, co_1\mathcal{A}, w\mathcal{A})$ is also a Waldhausen category. Then $K(\mathcal{A}) \simeq K(\mathcal{A}_1)$, by Waldhausen Approximation.

2.6. Combining Proposition 2.5 with the Gillet-Waldhausen Theorem 2.2 yields several useful localization sequences.

G-theory localization for rings 2.6.1. Localization at a central multiplicatively closed set S in a ring R induces an exact functor $\mathbf{M}(R) \to \mathbf{M}(S^{-1}R)$ satisfying (App). Passing to $\mathbf{Ch}^b\mathbf{M}(R) \to \mathbf{Ch}^b\mathbf{M}(S^{-1}R)$ does not change the K-theory (by 2.2) but does add a cylinder functor, so (App) still holds (see II, Ex. 9.7).

Hence Proposition 2.5 applies, with $\mathcal{A}^w$ being the category $\mathbf{Ch}^b_S\mathbf{M}(R)$ of complexes E such that $S^{-1}E$ is exact.

We define $G(R \text{ on } S)$ to be $K\mathbf{Ch}^b_S\mathbf{M}(R)$, and $G_n(R \text{ on } S) = K_n\mathbf{Ch}^b_S\mathbf{M}(R)$, so that we get a homotopy fibration $G(R \text{ on} S) \to G(R) \to G(S^{-1}R)$, and a long exact sequence

$$\cdots \to G_{n+1}(S^{-1}R) \to G_n(R \text{ on } S) \to G_n(R) \to G_n(S^{-1}R) \to \cdots$$

ending in the surjection $G_0(R) \to G_0(S^{-1}R)$ of II.6.4.1. When R is noetherian, we will identify $G(R \text{ on } S) = K\mathbf{Ch}^b_S\mathbf{M}(R)$ with $K\mathbf{M}_S(R)$ in 6.1 below.

G-theory localization for schemes 2.6.2. If Z is a closed subscheme of a noetherian scheme X, we define $G(X \text{ on } Z)$ to be $K\mathbf{Ch}^b_Z\mathbf{M}(X)$, where $\mathbf{Ch}^b_Z\mathbf{M}(X)$ is the (Waldhausen) category of bounded complexes which are acyclic on $X - Z$.

Now $G(X) = K\mathbf{M}(X)$ by IV.6.3.4, and the localization $\mathbf{M}(X) \to \mathbf{M}(X - Z)$ satisfies (App). Since $\mathcal{A}^w = \mathbf{Ch}^b_Z\mathbf{M}(X)$, Proposition 2.5 and Theorem 2.2 yield a homotopy fibration $G(X \text{ on } Z) \to G(X) \to G(X - Z)$ and a long exact sequence

$$\cdots \to G_{n+1}(X - Z) \to G_n(X \text{ on } Z) \to G_n(X) \to G_n(X - Z) \to \cdots$$

ending in the surjection $G_0(X) \to G_0(X - Z)$ of II.6.4.2. Later on (in 3.10.2, 6.11 and Ex. 4.3), we will identify $G(X \text{ on } Z)$ with $K\mathbf{M}_Z(X)$ and $G(Z)$.

Let S be a central multiplicatively closed set of central elements in a noetherian ring R. If S consists of nonzerodivisors, we will see in Theorem 7.1 that the analogue of $K\mathbf{M}_S(R)$ for projective modules is the K-theory of the category $\mathbf{H}_S(R)$ of S-torsion perfect modules (generalizing II.7.7.4). Otherwise, this is not correct; see Exercises 2.9 and 7.3 below. Instead, as in II.9.8, we define $K(R \text{ on } S)$ to be the K-theory of $\mathbf{Ch}^b_S\mathbf{P}(R)$, the Waldhausen category of bounded complexes P of finitely generated projective modules such that $S^{-1}P$ is exact (II.9.7.5).

Theorem 2.6.3. *If S is a central multiplicatively closed set in a ring R, there is a homotopy fibration $K(R \text{ on } S) \to K(R) \to K(S^{-1}R)$, and hence a long exact sequence*

$$\cdots K_{n+1}(S^{-1}R) \to K_n(R \text{ on } S) \to K_n(R) \to K_n(S^{-1}R) \cdots$$

ending in the exact sequence $K_0(R \text{ on } S) \to K_0(R) \to K_0(S^{-1}R)$ of II.9.8.

Proof. As in the proof of II.9.8, we consider the category $\mathcal{P}$ of $S^{-1}R$-modules of the form $S^{-1}P$ for P in $\mathbf{P}(R)$. We saw in II.9.8.1 that (by clearing denominators in the maps), the localization from $\mathcal{A} = \mathbf{Ch}^b(\mathbf{P}(R))$ to $\mathcal{B} = \mathbf{Ch}^b(\mathcal{P})$ satisfies (App), so Proposition 2.5 applies with $\mathcal{A}^w = \mathbf{Ch}^b_S\mathbf{P}(R)$. Thus we have a homotopy fibration $K\mathbf{Ch}^b_S\mathbf{P}(R) \to K(R) \to K(\mathcal{P})$. By Cofinality (IV.6.4.1), $K(\mathcal{P}) \to K(S^{-1}R) \to G$ is a homotopy fibration, and the result follows. $\square$

We conclude with a few useful models for K-theory, arising from the Waldhausen Approximation Theorem 2.4.

Homologically bounded complexes 2.7.1. If $\mathcal{A}$ is an abelian category, let $\mathbf{Ch}^{hb}(\mathcal{A})$ denote the Waldhausen category of homologically bounded chain complexes of objects in $\mathcal{A}$, and $\mathbf{Ch}^{hb}_{\pm}(\mathcal{A})$ the subcategory of bounded below (resp., bounded above) complexes. We saw in II.9.7.4 that $\mathbf{Ch}^{b}(\mathcal{A}) \subset \mathbf{Ch}^{hb}_{-}(\mathcal{A})$ and $\mathbf{Ch}^{hb}_{+}(\mathcal{A}) \subset \mathbf{Ch}^{hb}(\mathcal{A})$ satisfy (App), by good truncation; dually, $\mathbf{Ch}^{b}(\mathcal{A}) \subset \mathbf{Ch}^{hb}_{+}(\mathcal{A})$ and $\mathbf{Ch}^{hb}_{+}(\mathcal{A}) \subset \mathbf{Ch}^{hb}(\mathcal{A})$ satisfy the dual of (App). By Waldhausen Approximation (and 2.2), this yields

$$K(\mathcal{A}) \simeq K\mathbf{Ch}^{b}(\mathcal{A}) \simeq K\mathbf{Ch}^{hb}_{-}(\mathcal{A}) \simeq K\mathbf{Ch}^{hb}_{+}(\mathcal{A}) \simeq K\mathbf{Ch}^{hb}(\mathcal{A}).$$

The K_0 version of the resulting isomorphism $K_n(\mathcal{A}) \cong K_n\mathbf{Ch}^{b}(\mathcal{A}) \cong K_n\mathbf{Ch}^{hb}(\mathcal{A})$ was given in II.9.7.4. We will see another argument for this in 3.8.1 below.

Perfect complexes 2.7.2. A *perfect complex* of R-modules is a complex M which is quasi-isomorphic to a bounded complex of finitely generated projective R-modules, *i.e.,* to a complex in $\mathbf{Ch}^{b}(\mathbf{P}(R))$. We saw in II.9.7.5 that the perfect complexes of R-modules form a Waldhausen subcategory $\mathbf{Ch}_{\mathrm{perf}}(R)$ of $\mathbf{Ch}(\mathbf{mod}\text{-}R)$, and that (App) holds for the inclusions $\mathbf{Ch}^{b}(\mathbf{P}(R)) \subset \mathbf{Ch}^{-}_{\mathrm{perf}}(R) \subset \mathbf{Ch}_{\mathrm{perf}}(R)$. Thus (invoking Theorems 2.2 and 2.4) we have that

$$K(R) \simeq K\mathbf{Ch}^{b}(\mathbf{P}(R)) \simeq K\mathbf{Ch}^{-}_{\mathrm{perf}}(R) \simeq K\mathbf{Ch}_{\mathrm{perf}}(R).$$

If S is a central multiplicatively closed set in R, then $K(R \text{ on } S) = K\mathbf{Ch}^{b}_{S}\mathbf{P}(R)$ is also the K-theory of the category $\mathbf{Ch}_{\mathrm{perf},S}(R)$ of perfect complexes P with $S^{-1}P$ exact. This follows from Waldhausen Approximation; the Approximation Property for the inclusion $\mathbf{Ch}^{b}_{S}\mathbf{P}(R) \subset \mathbf{Ch}_{\mathrm{perf},S}(R)$ was established in II, Ex. 9.2.

K-theory of schemes 2.7.3. If X is any scheme, we define $K(X)$ to be $K\mathbf{Ch}_{\mathrm{perf}}(X)$. Thus our $K_0(X)$ is the group $K_0^{der}(X)$ of II, Ex. 9.10.

When X is a quasi-projective scheme over a commutative ring, we defined $K(X) = K\mathbf{VB}(X)$ in IV.6.3.4. These definitions agree; in fact they also agree when X is a separated regular noetherian scheme (II.8.2), or more generally a (quasi-compact, quasi-separated) scheme such that every coherent sheaf is a quotient of a vector bundle. Indeed, $K\mathbf{VB}(X) = K\mathbf{Ch}_{\mathrm{perf}}(X)$, by Waldhausen Approximation applied to $\mathbf{Ch}^{b}(\mathbf{VB}(X)) \subset \mathbf{Ch}_{\mathrm{perf}}(X)$. The condition (App) is given in [SGA6, II] or [TT, 2.3.1]: given a map $P \to C$ from a bounded complex of vector bundles to a perfect complex, there is a factorization $P \to Q \xrightarrow{\sim} C$ in these settings.

$G(R)$ and Pseudo-coherent complexes 2.7.4. If R is a noetherian ring, the discussion of 2.7.1 applies to the abelian category $\mathbf{M}(R)$ of finitely generated R-modules. Thus we have:

$$G(R) = K\mathbf{M}(R) \simeq K\mathbf{Ch}^{b}\mathbf{M}(R) \simeq K\mathbf{Ch}^{hb}_{+}\mathbf{M}(R) \simeq K\mathbf{Ch}^{hb}\mathbf{M}(R).$$

Instead of $\mathbf{Ch}^{hb}_{+}\mathbf{M}(R)$, we could consider the (Waldhausen) category $\mathbf{Ch}^{hb}_{+}\mathbf{P}(R)$ of bounded below, homologically bounded chain complexes of finitely generated projective modules, or even the category $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$ of homologically bounded *pseudo-coherent* complexes (R-module complexes which are quasi-isomorphic to a bounded complex of finitely generated modules; see II.9.7.6). By II, Ex. 9.7, Waldhausen Approximation applies to the inclusions of $\mathbf{M}(R)$ and $\mathbf{Ch}^{hb}_{+}\mathbf{P}(R)$ in $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$. Hence we also have $G(R) \simeq K\mathbf{Ch}^{hb}_{+}\mathbf{P}(R) \simeq K\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$.

If R is not noetherian, we can consider the exact category $\mathbf{M}(R)$ of pseudo-coherent modules (II.7.1.4), which we saw is closed under kernels of surjections, and the Waldhausen category $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$ of homologically bounded pseudo-coherent complexes (II.9.7.6). Since (App) holds by Ex. II.9.7, the same proof gives:

$$G(R) = K\mathbf{M}(R) \simeq K\mathbf{Ch}^b\mathbf{M}(R) \simeq K\mathbf{Ch}^{hb}_{+}\mathbf{P}(R) \simeq K\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R).$$

Now suppose that S is a multiplicatively closed set of central elements in R. Anticipating Theorem 5.1 below, we consider the category $\mathbf{M}_S(R)$ of S-torsion modules in $\mathbf{M}(R)$. If R is noetherian, this is an abelian category by II.6.2.8; if not, it is the exact category of pseudo-coherent S-torsion modules (II.7.1.4). By Theorem 2.2, $K\mathbf{M}_S(R) \simeq K\mathbf{Ch}^b\mathbf{M}_S(R)$.

EXERCISES

2.1 If an exact functor $F : \mathcal{A} \to \mathcal{B}$ satisfies the approximate lifting property (App), show (by induction on n) that each $S_nF : S_n\mathcal{A} \to S_n\mathcal{B}$ also satisfies (App).

2.2 If $\mathcal{A}$ is a strictly cofinal exact subcategory of $\mathcal{A}'$, show that $\mathbf{Ch}^b(\mathcal{A}) \subset \mathbf{Ch}^b(\mathcal{A}')$ satisfies (App), and that $K\mathbf{Ch}^b(\mathcal{A}) \simeq K\mathbf{Ch}^b(\mathcal{A}')$.

2.3 Let split$\mathbf{Ch}^b(\mathcal{A})$ denote the category $\mathbf{Ch}^b(\mathcal{A})$, made into a Waldhausen category by restricting the cofibrations to be the degreewise split monomorphisms whose quotients lie in $\mathcal{A}$ (a priori they lie in $\widehat{\mathcal{A}}$; see II.7.3). Generalize II.9.2.4 by showing that split$\mathbf{Ch}^b(\mathcal{A}) \to \mathbf{Ch}^b(\mathcal{A})$ induces a homotopy equivalence on K-theory, so that $K_n(\text{split}\mathbf{Ch}^b(\mathcal{A})) \cong K_n(\mathcal{A})$ for all n.

2.4 If $\mathcal{A}$ is an exact subcategory of an abelian category $\mathcal{M}$, the Waldhausen category $\mathcal{A}^{[0,n]}_{\mathrm{exact}}$ of admissibly exact complexes of length n (Example 1.4) is contained in the category $\mathbf{Ch}^{[0,n]}(\mathcal{A})^{qiso}$ of complexes in $\mathcal{A}$ which are acyclic as complexes in $\mathcal{M}$. If $\mathcal{A}$ is closed under kernels of surjections in $\mathcal{M}$, show that these categories are the same.

2.5 Consider the exact category $\mathbf{F}(R)$ of finite free R-modules (II.5.4.1). Analyze Remark 2.2.1 to show that $K\mathbf{F}(R) \simeq K\mathbf{Ch}^b(\mathbf{F}(R))$. If S is a central multiplicative set in R, compare $K\mathbf{Ch}^b_S(\mathbf{F}(R))$ to $K(R \text{ on } S)$. Is $\mathbf{Ch}^b_S(\mathbf{F}(R))$ cofinal in $\mathbf{Ch}^b_S(\mathbf{P}(R))$?

2.6 Let S be a central multiplicative set in a ring R. Mimick the proof of 2.7.4 to show that $K\mathbf{Ch}^b_S\mathbf{M}(R)$ is equivalent to $K\mathbf{Ch}^{+}_{\mathrm{pcoh},S}(R)$ and $K\mathbf{Ch}^{hb}_{\mathrm{pcoh},S}(R)$. A fancier proof of this equivalence will be given in 3.10.1 below.

2.7 Let X be a noetherian scheme. Show that $G(X) \simeq K\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$, generalizing II, Ex. 9.8. *Hint:* Mimick the proof of II, Ex. 9.7. A fancier proof of this equivalence will be given in 3.10.2 below.

2.8 Let X be a noetherian scheme, and $\mathcal{F}$ the Waldhausen category of bounded above perfect cochain complexes of flat $\mathcal{O}_X$-modules. Show that $\mathbf{VB}(X) \subset \mathcal{F}$ induces an equivalence $K(X) \simeq K(\mathcal{F})$.

2.9 Let $R = k[s,t]/(st)$ and $S = \{s^n\}$, where k is a field, so that $S^{-1}R = k[s, 1/s]$. Show that every R-module M with $s^nM = 0$ for some n has infinite projective dimension, so that the category $\mathbf{H}_S(R)$ consists only of 0. Then use the Mayer-Vietoris sequence III.2.6 to show that $K_0(R \text{ on } S) = \mathbb{Z}$. Conclude that $K\mathbf{H}_S(R)$ is not the homotopy fiber of $K(R) \to K(S^{-1}R)$.

§3. The Resolution Theorems and transfer maps

In this section we establish the Resolution Theorems for exact categories (3.1) and Waldhausen categories of chain complexes (3.9). We first give the version for exact categories, and some of its important applications. The second Resolution Theorem 3.9 requires the properties of derived categories which are listed in 3.8.

The Fundamental Theorem 3.3–4 that $K_* \cong G_*$ for regular rings and schemes (proven for K_0 in II, 7.8 and 8.2), and the existence of transfer maps f_* (3.3.2, 3.5 and 3.7), are immediate consequences of the first Resolution Theorem, as applied to $\mathbf{P}(R) \subseteq \mathbf{H}(R)$ and $\mathbf{VB}(X) \subseteq \mathbf{H}(X)$.

Recall from II.7.0.1 that $\mathcal{P}$ is said to be *closed under kernels of admissible surjections* in an exact category $\mathcal{H}$ if whenever $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{H}$ is an exact sequence with B, C in $\mathcal{P}$ then A is also in $\mathcal{P}$. (A prototype is $\mathcal{P} = \mathbf{H}_n(R)$, $\mathcal{H} \subseteq \mathbf{mod}\text{-}R$.)

Resolution Theorem 3.1. *Let $\mathcal{P}$ be a full exact subcategory of an exact category $\mathcal{H}$, such that $\mathcal{P}$ is closed under extensions and under kernels of admissible surjections in $\mathcal{H}$. Suppose in addition that every object M of $\mathcal{H}$ has a finite $\mathcal{P}$-resolution:*

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

Then $K(\mathcal{P}) \simeq K(\mathcal{H})$, and thus $K_i(\mathcal{P}) \cong K_i(\mathcal{H})$ for all i.

The proof will reduce the theorem to the special case in which objects of $\mathcal{H}$ have a $\mathcal{P}$-resolution of length one, which will be handled in Proposition 3.1.1.

Proof. The category $\mathcal{H}$ is the union of the subcategories $\mathcal{H}_n$ of objects with resolutions of length at most n, and $\mathcal{H}_0 = \mathcal{P}$. Since the kernel of any admissible $P \twoheadrightarrow P'$ is also in $\mathcal{P}$, $\mathcal{H}_{n-1} \subseteq \mathcal{H}_n$ is closed under admissible subobjects and extensions (see Ex. 3.1). Applying 3.1.1, we see that each $K(\mathcal{P}) \to K(\mathcal{H}_{n-1}) \to K(\mathcal{H}_n)$ is a homotopy equivalence. Taking the colimit over n yields the result. $\square$

Proposition 3.1.1. *Let $\mathcal{P} \subset \mathcal{H}$ be as in Theorem 3.1, and suppose that every M in $\mathcal{H}$ fits into an exact sequence $0 \to P_1 \to P_0 \to M \to 0$ with the P_i in $\mathcal{P}$. Then $K(\mathcal{P}) \to K(\mathcal{H})$ is a homotopy equivalence, and $K_*(\mathcal{P}) \cong K_*(\mathcal{H})$.*

Proof. (Quillen) The inclusion $Q\mathcal{P} \subset Q\mathcal{H}$ is not full, so we consider the full subcategory $\mathcal{Q}$ on the objects of $Q\mathcal{P}$, and write i for the inclusion $Q\mathcal{P} \subset \mathcal{Q}$. For each P in $Q\mathcal{P}$, the objects of the comma category i/P are pairs (P_2, u) with u of the form $P_2 \twoheadleftarrow P_1 \rightarrowtail P$ and P/P_1 in $\mathcal{H}$. Set $z(P_2, u) = (P_1, P_1 \rightarrowtail P)$ and note that $P_2 \twoheadleftarrow P_1$ and $0 \rightarrowtail P_1$ are morphisms of $Q\mathcal{P}$. They define natural transformations $(P_2, u) \to z(P_2, u) \leftarrow (0, 0 \rightarrowtail P)$ in i/P. This shows that i/P is contractible, and hence by Theorem A (IV.3.6) that $i : Q\mathcal{P} \to \mathcal{Q}$ is a homotopy equivalence.

It now suffices to show that the inclusion $j : \mathcal{Q} \to Q\mathcal{H}$ is a homotopy equivalence. We shall resort to the dual of Theorem A, so we need to show that for each M in $\mathcal{H}$, the comma category $M\backslash j$ is contractible. The objects of $M\backslash j$ are pairs $(P, u : M \twoheadleftarrow P_1 \rightarrowtail P)$ with P in $\mathcal{P}$; it is nonempty by the assumption that some $P_0 \twoheadrightarrow M$ exists. Let $\mathcal{C}$ denote the full subcategory on the pairs $(P, M \twoheadleftarrow P)$. The inclusion $\mathcal{C} \subset (M\backslash j)$ is a homotopy equivalence because it has a right adjoint, namely $r(P, u) = (P_1, M \twoheadleftarrow P_1)$. And the category $\mathcal{C}$ is contractible because if we fix any $(P_0, u_0 : M \twoheadleftarrow P_0)$, then $p(P, u) = (P \times_M P_0, M \twoheadleftarrow P \times_M P_0)$ is in

$\mathcal{C}$ (because $\mathcal{P}$ is subobject-closed) and there are natural transformations $(P,u) \leftarrow p(P,u) \rightarrow (P_0,u_0)$. $\square$

REMARK 3.1.2. It is not known how to generalize the Resolution Theorem to Waldhausen categories. Other proofs of the Resolution Theorem for exact categories, using Waldhausen K-theory, have been given in [Gra87] and [Staf].

Here is the main application of the Resolution Theorem. It is just the special case in which $\mathcal{P} = \mathbf{P}(R)$ and $\mathcal{H} = \mathbf{H}(R)$.

THEOREM 3.2. *For every ring R, the inclusion of $\mathbf{P}(R)$ in $\mathbf{H}(R)$ induces an equivalence $K(R) = K\mathbf{P}(R) \simeq K\mathbf{H}(R)$, so $K_*(R) = K_*\mathbf{P}(R) \cong K_*\mathbf{H}(R)$.*

If S is a multiplicatively closed set of central nonzero-divisors of R, we introduced the categories $\mathbf{H}_S(R)$ and $\mathbf{H}_{1,S}(R)$ in II.7.7.3. The proof there using Resolution applies verbatim to yield:

COROLLARY 3.2.1. *$K\mathbf{H}_{1,S}(R) \simeq K\mathbf{H}_S(R)$, and $K_*\mathbf{H}_{1,S}(R) \cong K_*\mathbf{H}_S(R)$.*

By definition, a ring R is *regular* if every R-module has a finite projective resolution, *i.e.,* finite projective dimension (see I.3.8.). We say R is *coherent* if the category $\mathbf{M}(R)$ of pseudo-coherent R-modules (II.7.1.4) is abelian.

THEOREM 3.3. *If R is a noetherian (or coherent) regular ring, $K(R) \simeq G(R)$. Thus for every n we have $K_n(R) \cong G_n(R)$.*

PROOF. In either case, $\mathbf{H}(R)$ is the category $\mathbf{M}(R)$. The Resolution Theorem 3.2 gives the identification. $\square$

COROLLARY 3.3.1. *If $f : R \to S$ is a homomorphism, R is regular noetherian and S is finite as an R-module, then there is a transfer map $f_* : K_*(S) \to K_*(R)$, defined by the G-theory transfer map (IV.6.3.3):*

$$K(S) \to G(S) = K\mathbf{M}(S) \xrightarrow{f_*} K\mathbf{M}(R) = G(R) \simeq K(R).$$

TRANSFER MAPS FOR $K_*(R)$ 3.3.2. Let $f\colon R \to S$ be a ring homomorphism such that S has a finite R-module resolution by finitely generated projective R-modules. Then the restriction of scalars defines a functor $\mathbf{P}(S) \to \mathbf{H}(R)$. By 3.2, we obtain a transfer map $f_*\colon K(S) \to K\mathbf{H}(R) \simeq K(R)$, and hence maps $f_* : K_n(S) \to K_n(R)$. If S is projective as an R-module, f_* is the transfer map of IV.6.3.2.

The *projection formula* states that f_* is a $K_*(R)$-module homomorphism when R is commutative. That is, if $x \in K_*(S)$ and $y \in K_*(R)$ then $f_*(x \cdot f^*(y)) = f_*(x) \cdot y$ in $K_*(R)$. To see this, we note that the biexact functor $\mathbf{H}(S) \times \mathbf{P}(R) \to \mathbf{H}(R)$, $(M,P) \mapsto M \otimes_R P$, produces a pairing $K\mathbf{H}(S) \wedge K(R) \to K\mathbf{H}(R)$ representing the right side via Theorem 3.2. Since $M \otimes_R P \cong M \otimes_S (S \otimes_R P)$, it is naturally homotopic to the pairing representing the left side.

We have already seen special cases of the transfer map f_*. It was defined for K_0 in II.7.9, and for K_1 in a special case in III, Ex. 1.11. If S is projective as an R-module then f_* was also constructed for K_2 in III.5.6.3, and for all K_n in IV.1.1.3.

Recall from II.8.2 that a separated noetherian scheme X is *regular* if every coherent $\mathcal{O}_X$-module $\mathcal{F}$ has a finite resolution by vector bundles; see [SGA6, II, 2.2.3 and 2.2.7.1] or II.8.2–3. The Resolution Theorem applies to $\mathbf{VB}(X) \subset \mathbf{M}(X)$, and we have:

THEOREM 3.4. *If X is a separated regular noetherian scheme, then $K(X) = K\mathbf{VB}(X)$ satisfies:*

$$K(X) \simeq G(X) \quad \textit{and} \quad K_*(X) \cong G_*(X).$$

VARIANT 3.4.1. If X is quasi-projective (over a commutative ring), we defined $K(X)$ to be $K\mathbf{VB}(X)$ in IV.6.3.3. We saw in II.8.3.1 that the Resolution Theorem applies to $\mathbf{VB}(X) \subset \mathbf{H}(X)$ so we have $K(X) \simeq K\mathbf{H}(X)$.

REMARK 3.4.2. Theorem 3.4 does not hold for non-separated regular noetherian schemes. This is illustrated when X is the affine line with a double origin over a field, since (as we saw in II, 8.2.4 and Ex. 9.10) $G_0(X) = \mathbb{Z} \oplus \mathbb{Z}$ but $K_0\mathbf{VB}(X) = \mathbb{Z}$. The analogue of Theorem 3.4 for quasi-compact regular schemes is given in Exercise 3.9.

BASE CHANGE MAPS FOR $G_*(R)$ 3.5. Let $f : R \to S$ be a homomorphism of noetherian rings such that S has finite flat dimension $\mathrm{fd}_R\, S$ as a right R-module. Let $\mathcal{F} \subset \mathbf{M}(R)$ be the full subcategory of all R-modules M which are Tor-independent of S in the sense that

$$\mathrm{Tor}_n^R(S, M) = 0 \qquad \text{for } n \neq 0.$$

As observed in II.7.9, the usual properties of Tor show that every finitely generated R-module M has a finite resolution by objects of $\mathcal{F}$, that $\mathcal{F}$ is an exact subcategory closed under kernels, and that $M \mapsto M \otimes_R S$ is an exact functor from $\mathcal{F}$ to $\mathbf{M}(S)$. By the Resolution Theorem 3.1, there is a natural map

$$G(R) \xleftarrow{\simeq} K(\mathcal{F}) \to G(S),$$

giving maps $f^* \colon G_*(R) \to G_*(S)$. If $g : S \to T$ is another map, and T has finite flat dimension over S, then the natural isomorphism $(M \otimes_R S) \otimes_S T \cong M \otimes_R T$ shows that $g^* f^* \simeq (gf)^*$.

Note that if the ring S is finite over R then the forgetful functor $\mathbf{M}(S) \to \mathbf{M}(R)$ is exact and induces a contravariant "finite transfer" map $f_* : G(S) \to G(R)$ (see II.6.2 and IV.6.3.3). The seemingly strange notation (f^* and f_*) is chosen with an eye towards schemes: if $Y = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(S)$ then f maps X to Y, so f^* is contravariant and f_* is covariant as functors on schemes.

EXAMPLE 3.5.1. Let $i : R \to R[s]$ be the inclusion and $f : R[s] \to R$ the map $f(s) = 0$. Since i is flat, we have the flat base change $i^* \colon G(R) \to G(R[s])$. Now f has finite flat dimension, since $\mathrm{Tor}_n^{R[s]}(R, -) = 0$ for $n \geq 2$, so we also have a base change map $f^* \colon G(R[s]) \to G(R)$ by 3.5. Since fi is the identity on R, the composite $f^* i^* : G(R) \to G(R[s]) \to G(R)$ is homotopic to the identity map. The Fundamental Theorem for $G(R)$ (6.2 below) will show that these are inverse homotopy equivalences.

In contrast, the transfer maps $K(R) \xrightarrow{f_*} K(R[s])$ and $G(R) \xrightarrow{f_*} G(R[s])$ are zero. This follows from the Additivity Theorem applied to the sequence of functors $i^* \rightarrowtail i^* \twoheadrightarrow f_*$ sending an R-module M to

$$0 \to M[s] \xrightarrow{s} M[s] \to M \to 0.$$

Example 3.5.2. Suppose that $S = R \oplus S_1 \oplus S_2 \oplus \cdots$ is a graded noetherian ring, and let $\mathbf{M}_{gr}(S)$ be the abelian category of finitely generated graded S-modules. Its K-groups are naturally modules over $\mathbb{Z}[\sigma, \sigma^{-1}]$, where σ acts by the shift automorphism $\sigma(M) = M(-1)$ of graded modules. (See Exercises II.6.12 and II.7.14.)

Now assume that S is flat over R, so that tensoring with S gives a functor from $\mathbf{M}(R)$ to $\mathbf{M}_{gr}(S)$, and hence a $\mathbb{Z}[\sigma, \sigma^{-1}]$-module map

$$\beta \colon G_i(R) \otimes \mathbb{Z}[\sigma, \sigma^{-1}] \to K_i \mathbf{M}_{gr}(S).$$

In the special case $S = R$, $\mathbf{M}_{gr}(R)$ is just a coproduct of copies of $\mathbf{M}(R)$, and the map β is an isomorphism: $G_*(R)[\sigma, \sigma^{-1}] \cong K_* \mathbf{M}_{gr}(R)$. If R has finite flat dimension over S (via $S \to R$) then the Resolution Theorem 3.1, applied to the category $\mathcal{P}_{gr}$ of graded S-modules Tor-independent of R, induces a map

$$K_i \mathbf{M}_{gr}(S) \to K_i \mathbf{M}_{gr}(R) \cong G_i(R) \otimes \mathbb{Z}[\sigma, \sigma^{-1}]$$

which is a left inverse to β, because $\otimes_R$ sends $\mathbf{M}_{gr}(R)$ to $\mathcal{P}_{gr}$ and there is a natural isomorphism $(M \otimes_R S) \otimes_S R \cong M$. In fact, β is an isomorphism (see Ex. 3.3).

Similarly, if $\mathbf{M}_{gr,\geq 0}(S)$ is the subcategory of positively graded S-modules, there is a natural map $\beta : G_i(R) \otimes \mathbb{Z}[\sigma] \to K_i \mathbf{M}_{gr,\geq 0}(S)$. If R has finite flat dimension over S, it is an isomorphism (see Ex. 3.3).

Example 3.5.3. (Projection Formula) Let $f : R \to S$ be a homomorphism of commutative noetherian rings such that S is a finitely generated right R-module of finite projective dimension. Then $G_*(S)$ and $G_*(R)$ are $K_*(R)$-modules by IV.6.6.5. The *projection formula* states that $f_*(x \cdot f^* y) = f_*(x) \cdot y$ in $G_*(R)$, provided that either (i) $x \in G_*(S)$ and $y \in K_*(R)$ or (ii) $x \in K_*(S)$ and $y \in G_*(R)$.

For (i), observe that the functor $\otimes_R : \mathbf{M}(S) \times \mathbf{P}(R) \to \mathbf{M}(S) \to \mathbf{M}(R)$ is biexact, so it induces a pairing $G(S) \wedge K(R) \to G(S) \to G(R)$ representing the right side $f_*(x) \cdot y$. Since $M \otimes_R P \cong M \otimes_S (S \otimes_R P)$, it also represents the left side.

For (ii), let $\mathcal{F} \subset \mathbf{M}(R)$ be as in 3.5 and observe that the functor $\mathbf{P}(S) \times \mathcal{F} \to \mathbf{M}(S) \to \mathbf{M}(R)$, $(P, M) \mapsto P \otimes_R M$ is biexact. Hence it produces a pairing $K(S) \wedge K(\mathcal{F}) \to G(S) \to G(R)$, representing the left side $f_*(x \cdot f^* y)$ of the projection formula. But this pairing also factors through $\mathbf{P}(S) \to \mathbf{H}(R)$ followed by the tensor product pairing $\mathbf{H}(R) \times \mathcal{F} \to \mathbf{M}(R)$ representing the right side.

Base change maps for $G(X)$ 3.6. If $f : X \to Y$ is a morphism of noetherian schemes such that $\mathcal{O}_X$ has finite flat dimension over $f^{-1}\mathcal{O}_Y$, there is also a contravariant map f^* from $G(Y)$ to $G(X)$. This is because every coherent $\mathcal{O}_Y$-module has a finite resolution by coherent modules which are Tor-independent of $f_*\mathcal{O}_X$, locally on X, and f^* is an exact functor on the category $\mathbf{L}(f)$ of these modules. If $g : W \to X$ is another map of finite flat dimension, then $g^* f^* \simeq (fg)^*$ by the natural isomorphism $g^*(f^*\mathcal{F}) \cong (fg)^*\mathcal{F}$.

EXAMPLE 3.6.1. If X is a noetherian scheme we can consider the flat structure map $p : X[s] \to X$ and the zero-section $f : X \to X[s]$, where $X[s] = X \times \mathrm{Spec}(\mathbb{Z}[s])$ as in II.6.5.1. As in 3.5.1, f has finite flat dimension and pf is the identity on X, so $f^* : G(X[s]) \to G(X)$ is defined and the composition f^*p^* is homotopic to the identity on $G(X)$. The Fundamental Theorem 6.13 will show that $p^* : G(X) \simeq G(X[s])$ is a homotopy equivalence.

In contrast the finite transfer map $f_* : G(X) \to G(X[s])$ is zero. This follows from the Additivity Theorem applied to the sequence of functors $p^* \rightarrowtail p^* \twoheadrightarrow f_*$ from $\mathbf{M}(X)$ to $\mathbf{M}(X[s])$, analogous to the one in 3.5.1.

PROPOSITION 3.7. *If $f\colon X \to Y$ is a proper morphism of noetherian schemes, there is a "proper transfer" map $f_*\colon G(X) \to G(Y)$. This induces homomorphisms $f_* : G_n(X) \to G_n(Y)$ for each n. The transfer map makes $G_n(X)$ functorial for proper maps.*

The G_0 version of Proposition 3.7, $f_*([\mathcal{F}]) = \sum(-1)^i[R^if_*(\mathcal{F})]$, is given in II.6.2.6.

PROOF. By Serre's "Theorem B" (see II.6.2.6), the higher direct images $R^if_*(\mathcal{F})$ of a coherent module are coherent, and vanish for i large. They are obtained by replacing $\mathcal{F}$ by a flasque resolution, applying f_* and taking cohomology. The map $f_* : K\mathbf{M}(X) \to K\mathbf{M}(Y)$ exists by Ex. 3.2. $\square$

PROPOSITION 3.7.1. *If $f\colon X \to Y$ is a proper morphism of finite flat dimension. Then there is a "transfer" map $f_*\colon K(X) \to K(Y)$. This induces homomorphisms $f_* : K_n(X) \to K_n(Y)$ for each n. The transfer map makes $K_n(X)$ functorial for projective maps.*

PROOF. As in II.8.4, let $\mathbf{P}(f)$ be the category of vector bundles E on X such that $R^if_*(E) = 0$ for $i > 0$. We saw in *loc. cit.* that $f_* : \mathbf{P}(f) \to \mathbf{H}(Y)$ is an exact functor, *i.e.,* that the $\mathcal{O}_Y$-module $f_*(E)$ is perfect (II, Ex. 9.10). By Ex. 3.6(c) and left exactness of f_*, the hypotheses of the Resolution Theorem 3.1 are satisfied, so we have $K(X) \simeq K\mathbf{P}(f)$. Thus we can define the transfer map to be the composite

$$K(X) \simeq K\mathbf{P}(f) \xrightarrow{f_*} K\mathbf{H}(Y) \simeq K\mathbf{VB}(Y) = K(Y).$$

Given a second map $g : Y \to Z$ of finite flat dimension, we can replace $\mathbf{P}(f)$ by $\mathcal{P} = \mathbf{P}(f \times gf)$, so that E in $\mathcal{P}$ satisfy $R^ig_*(f_*(E)) = R^i(gf)_*(E) = 0$ for $i > 0$. Thus $f_*(\mathcal{P})$ lies in the subcategory $\mathbf{H}(g)$ of perfect g_*-acyclic modules, on which g_* is exact, and $g_*f_* \simeq (gf)_*$ because of the natural isomorphism $(gf)_*(E) \cong g_*f_*(E)$ for E in $\mathcal{P}$. Functoriality is now straightforward (Ex. 3.5). $\square$

BASE CHANGE THEOREM 3.7.2. *Let $f : X \to Y$ be a proper morphism of quasi-projective schemes and $g : Y' \to Y$ a morphism of finite flat dimension,* Tor*-independent of X, and set $X' = X \times_Y Y'$ so there is a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Then $g^*f_* \simeq f'_*{g'}^*$ as maps $G(X) \to G(Y')$.

If in addition f has finite flat dimension, so that $f_*: K(X) \to K(Y)$ is defined then $g^* f_* \simeq f'_* g'^*$ as maps $K(X) \to K(Y')$.

The idea of the proof is to use the following base change formula of [SGA6, IV.3.1.0]: if E is homologically bounded with quasi-coherent cohomology, then

$$Lg^*(Rf_*E) \xrightarrow{\sim} Rf'_* L(g')^* E.$$

PROOF. Let $\mathcal{A}$ be the category of $\mathcal{O}_X$-modules which are f_*-acyclic and Tor-independent of $\mathcal{O}_{X'}$. For E in $\mathcal{A}$, the base change formula implies that for all $i \in \mathbb{Z}$:

$$\mathrm{Tor}_i(f_*E, \mathcal{O}_{Y'}) = L_i g^*(f_*E) = R^{-i} f'_*(g'^* E).$$

These groups must vanish unless $i = 0$, because Tor_i and $R^i f'_*$ vanish for $i < 0$. That is, $f_*(E)$ is Tor-independent of $\mathcal{O}_{Y'}$, and $(g')^*E$ is f'_*-acyclic, and we have $g^* f_*(E) \cong f'_*(g')^*E$. Therefore $g^* f_* = f'_*(g')^*$ as exact functors on $\mathcal{A}$. It remains to apply the Resolution Theorem twice to show that $\mathcal{A} \subset \mathbf{L}(f) \subset \mathbf{M}(Y)$ induce equivalences on K-theory. The second was observed in 3.6, and the first follows from Ex. 3.6(e).

The proof is easier for $K(X) \to K(Y')$ when f has finite flat dimension, using the category $\mathbf{P}(f)$; $f_* : \mathbf{P}(f) \to \mathbf{H}(Y)$ and $(g')^* : \mathbf{P}(f) \to \mathbf{VB}(X')$ are exact, and we saw in the proof of 3.7.1 that $K\mathbf{P}(f) \simeq K(X)$. $\square$

COROLLARY 3.7.3. *(Projection formula) If $f : X \to Y$ is a projective map of finite flat dimension, then for $x \in K_0(X)$ and $y \in G_n(Y)$ we have $f_*(x \cdot f^*y) = f_*(x) \cdot y$ in $G_n(Y)$.*

The G_0 version of this projection formula was given in Ex. II.8.3. (Cf. Ex. 3.10.) We will generalize the projection formula to higher K-theory in 3.12 below.

PROOF. As in the proof of 3.7.1, let $\mathbf{P}(f)$ be the category of f_*-acyclic vector bundles E on X. By Ex. 3.6(c) and left exactness of f_*, the hypotheses of the Resolution Theorem 3.1 are satisfied and we have $K_*(X) \cong K_*\mathbf{P}(f)$. Thus it suffices to show that the projection formula holds when $x = [E]$ for E in $\mathbf{P}(f)$. Let $\mathbf{L}_E$ denote the full subcategory of $\mathbf{M}(Y)$ consisting of modules which are Tor-independent of f_*E and $\mathcal{O}_X$. By the Resolution Theorem, $K(\mathbf{L}_E) \simeq G(Y)$. The functor $\mathbf{L}_E \to \mathbf{M}(Y)$ given by $F \mapsto f_*E \otimes F$ is exact, and induces $y \mapsto f_*(x) \cdot y$.

Similarly, the exact functors $\mathbf{L}_E \to \mathbf{M}(X)$, sending F to $f^*(F)$ and $E \otimes f^*(F)$, induce $y \mapsto f^*(y)$ and $y \mapsto x \cdot f^*(y)$, respectively. The projection formula of [SGA 6, III.3.7] shows that $R^i f_*(E \otimes f^*F) = 0$ for $i > 0$ and that $f_*(E \otimes f^*F) \cong f_*(E) \otimes F$. Hence $F \mapsto f_*(E \otimes f^*F)$ is an exact functor $\mathbf{L}_E \to \mathbf{M}(Y)$, and the projection formula follows. $\square$

Derived Approximation

The third fundamental result for Waldhausen categories is an Approximation Theorem for the K-theory of categories based upon chain complexes, and is proven using Waldhausen Approximation 2.4. Roughly speaking, it says that the K-theory of $\mathcal{C}$ only depends on the derived category of $\mathcal{C}$, defined as localization $w^{-1}\mathcal{C}$ of $\mathcal{C}$ at the set w of quasi-isomorphisms in $\mathcal{C}$.

In order for the statement of this result to make more sense, $\mathcal{C}$ will be a subcategory of $\mathbf{Ch}(\mathcal{M})$ for some abelian category $\mathcal{M}$; recall that the derived category $\mathbf{D}(\mathcal{M})$ is the localization of the category $\mathbf{Ch}(\mathcal{M})$ at the family of quasi-isomorphisms. For basic facts about derived categories and triangulated categories, we refer the reader to the Appendix to chapter II, and to the standard references [Verd], [H20] and [WHomo, 10].

Triangulated and Localizing Categories 3.8. Let $\mathcal{C}$ be a full additive subcategory of $\mathbf{Ch}(\mathcal{M})$ which is closed under all the shift operators $C \mapsto C[n]$ and mapping cones. The localization $w^{-1}\mathcal{C}$ of $\mathcal{C}$ is the category obtained from $\mathcal{C}$ by formally inverting the multiplicatively closed set $w = w(\mathcal{C})$ of all quasi-isomorphisms in $\mathcal{C}$. (The usual construction, detailed in (II.A.5), uses a calculus of fractions to compose maps.) It is a triangulated category by [WHomo, 10.2.5].

Here are two key observations that make it possible for us to better understand these triangulated categories, and to even see that they exist. One is that chain homotopic maps in $\mathcal{C}$ are identified in $w^{-1}\mathcal{C}$; see II, Ex. A.5 or [WHomo, 10.1.2]. Another is that if w is saturated then C is isomorphic to zero in $w^{-1}\mathcal{C}$ if and only if $0 \to C$ is in w; see II.A.3.2 or [WHomo, 10.3.10].

We say that $\mathcal{C}$ is a *localizing subcategory* of $\mathbf{Ch}(\mathcal{M})$ if the natural map $w^{-1}\mathcal{C} \to w^{-1}\mathbf{Ch}(\mathcal{M}) = \mathbf{D}(\mathcal{M})$ is an embedding. This will be the case whenever the following condition holds: given any quasi-isomorphism $C \to B$ with C in $\mathcal{C}$, there is a quasi-isomorphism $B \to C'$ with C' in $\mathcal{C}$. (See II.A.3 or [WHomo, 10.3.13].)

For example, if $\mathcal{B}$ is a Serre subcategory of $\mathcal{M}$, the category $\mathbf{Ch}_{\mathcal{B}}(\mathcal{M})$ of complexes in $\mathbf{Ch}(\mathcal{M})$ with homology in $\mathcal{B}$ is localizing ([WHomo, 10.4.3]), so $\mathbf{D}_{\mathcal{B}}(\mathcal{M}) = w^{-1}\mathbf{Ch}_{\mathcal{B}}(\mathcal{M})$ is a subcategory of $\mathbf{D}(\mathcal{M})$. The functor $\mathbf{D}(\mathcal{B}) \to D_{\mathcal{B}}(\mathcal{A})$ need not be an equivalence; see [WHomo, Ex. 10.4.3].

Example 3.8.1. If $\mathcal{B} \subset \mathcal{C} \subseteq \mathbf{Ch}(\mathcal{M})$, the derived categories $w^{-1}\mathcal{B}$ and $w^{-1}\mathcal{C}$ are equivalent if for every complex C in $\mathcal{C}$ there is a quasi-isomorphism $B \xrightarrow{\sim} C$ with B in $\mathcal{B}$. (See [TT, 1.9.7].)

For example, if $\mathcal{A}$ is an abelian category, the inclusion $\mathbf{Ch}^b(\mathcal{A}) \to \mathbf{Ch}^{hb}(\mathcal{A})$ induces an equivalence of derived categories, where $\mathbf{Ch}^{hb}(\mathcal{A})$ is the category of homologically bounded complexes (II.9.7.4), because (as we saw in 2.7.1) every homologically bounded complex is quasi-isomorphic to a bounded complex.

If R is any ring, $\mathbf{Ch}^b\mathbf{M}(R)$ has the same derived category as $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$ the homologically bounded pseudo-coherent complexes; see [WHomo, Ex. 10.4.6]. Similarly, the inclusions $\mathbf{Ch}^b_{\mathrm{perf}}(R) \subset \mathbf{Ch}^+_{\mathrm{perf}}(R) \subset \mathbf{Ch}_{\mathrm{perf}}(R)$ induce equivalence on derived categories.

Homotopy Commutative Diagrams 3.8.2. As pointed out in II.A.4, the chain homotopy category of $\mathcal{C}$ satisfies a calculus of fractions, allowing us to perform constructions in $w^{-1}\mathcal{C}$. Here is a typical example: any homotopy commutative diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} A' \arrow[r] \arrow[dr] & B_1 \arrow[d, "f"] & B' \arrow[l] \arrow[dl] \\ & B & \end{tikzcd}$$

can be made into a commutative diagram by replacing B_1 by the homotopy pullback of $B_1 \xrightarrow{f} B \xleftarrow{=} B$, which is defined as the shifted mapping cone of $B \to \mathrm{cone}(f)$, and is quasi-isomorphic to B_1.

We are now ready for the Waldhausen version of the Resolution Theorem, which is due to Thomason and Trobaugh [TT]. Fix an ambient abelian category $\mathcal{M}$ and consider the Waldhausen category $\mathbf{Ch}(\mathcal{M})$ of all chain complexes over $\mathcal{M}$.

Thomason-Trobaugh Resolution Theorem 3.9. *Let $\mathcal{A} \subset \mathcal{B}$ be saturated Waldhausen subcategories of $\mathbf{Ch}(\mathcal{M})$, closed under mapping cones and all shift maps $A \mapsto A[n]$. If $w^{-1}\mathcal{A} \xrightarrow{\simeq} w^{-1}\mathcal{B}$ (i.e., the derived categories of $\mathcal{A}$ and $\mathcal{B}$ are equivalent), then $K\mathcal{A} \xrightarrow{\simeq} K\mathcal{B}$ is a homotopy equivalence.*

Remark 3.9.1. A map in $\mathcal{A}$ is a weak equivalence in $\mathcal{A}$ if and only if it is a weak equivalence in $\mathcal{B}$. This is because, by saturation, both conditions are equivalent to the mapping cone being zero in the (common) derived category.

Proof. (Thomason-Trobaugh) Let $\mathcal{A}^+$ be the comma category whose objects are weak equivalences $w : A \xrightarrow{\sim} B$ with A in $\mathcal{A}$ and B in $\mathcal{B}$; morphisms in $\mathcal{A}^+$ are commutative diagrams in $\mathcal{B}$. It is a Waldhausen category in a way that makes $\mathcal{A} \to \mathcal{A}^+ \to \mathcal{B}$ into exact functors; a morphism $(A \xrightarrow{\sim} B) \to (A' \xrightarrow{\sim} B')$ is a cofibration (resp., weak equivalence) if both its component maps $A \to A'$ and $B \to B'$ are. The forgetful functor $\mathcal{A}^+ \to \mathcal{A}$ is right adjoint to the inclusion $\mathcal{A} \to \mathcal{A}^+$, and exact, so $K(\mathcal{A}) \simeq K(\mathcal{A}^+)$.

We will show that the Approximation Theorem 2.4 (or, rather, its dual) applies to the exact functor $\mathcal{A}^+ \to \mathcal{B}$ sending $A \xrightarrow{\sim} B$ to B. This will imply that $K(\mathcal{A}^+) \simeq K(\mathcal{B})$, proving the theorem.

Condition 2.4(a) is satisfied, because given a map $(A \xrightarrow{\sim} B) \to (A' \xrightarrow{\sim} B')$ in $\mathcal{A}^+$ and a weak equivalence $B \xrightarrow{\sim} B'$, the map $A \to A'$ is a weak equivalence in $\mathcal{B}$ by the saturation axiom, and hence is in $w\mathcal{A}$ by 3.9.1. Condition 2.4(b) holds, because the cylinder functor on $\mathcal{B}$ induces one on $\mathcal{A}^+$. Thus it suffices to check that the dual $(\mathrm{App})^{op}$ of the approximation property holds for $\mathcal{A}^+ \to \mathcal{B}$; this will be a consequence of the hypothesis that $\mathcal{A}$ and $\mathcal{B}$ have the same derived category.

Using the Gabriel-Zisman Theorem (II.A.3), the approximation property $(\mathrm{App})^{op}$ states that given $B' \xrightarrow{b} B \xleftarrow{\sim} A$ with A in $\mathcal{A}$, there is a commutative diagram

(3.9.2)
$$\begin{tikzcd} & A'' \arrow[r] \arrow[d, "\sim"'] & A \arrow[d, "\sim"'] \\ B' \arrow[r] & B'' \arrow[r] & B \end{tikzcd}$$

with A'' in $\mathcal{A}$, such that the bottom composite is b.

By assumption, B' is quasi-isomorphic to an object A_1 of $\mathcal{A}$; by calculus of fractions this is represented by a chain $A_1 \xrightarrow{\sim} B_1 \xleftarrow{\sim} B'$. Composing with $B' \to B \xleftarrow{\sim} A$ yields a map from A_1 to A in the derived category, which must be represented by a chain $A_1 \to A_2 \xleftarrow{\sim} A$ with A_2 in $\mathcal{A}$. Composing $B_1 \xleftarrow{\sim} A_1 \to A \to B$ (and using 3.8.2) yields a commutative diagram

(3.9.3)
$$\begin{tikzcd} & A_1 \arrow[r] \arrow[d, "\sim"'] & A_2 \arrow[d, "\sim"'] & A \arrow[l, "\sim"'] \arrow[d, "\sim"'] \\ B' \arrow[r] & B_1 \arrow[r] & B_2 & B \arrow[l, "\sim"'] \end{tikzcd}$$

whose bottom composite is chain homotopic to b. Let A'' and B'' denote the shifted mapping cylinders of $A \oplus A_1 \to A_2$ and $B \oplus B_1 \to B_2$, respectively; by the 5-lemma (the extension axiom), the induced map $A'' \to B''$ is a quasi-isomorphism. By the universal property of mapping cylinders [WHomo, 1.5.3], the chain homotopic maps $B' \xrightarrow{b} B \to B_2$ and $B \to B_1 \to B_2$ lift to a map $B \to B''$. We have now constructed a diagram like (3.9.2), which commutes up to chain homotopy. By 3.8.2, this suffices to construct a commutative diagram of this type. $\square$

Applications

The Thomason-Trobaugh Resolution Theorem 3.9 provides a more convenient criterion than the Waldhausen Approximation Theorem 2.4 in many cases, because of the simplicity of the criterion 3.8.1: every complex in the larger category must be quasi-isomorphic to a complex in the smaller category.

Example 3.10.1 (Homologically bounded complexes). If $\mathcal{A}$ is abelian, we saw in 3.8.1 that $\mathbf{Ch}^b(\mathcal{A}) \subset \mathbf{Ch}^{hb}(\mathcal{A})$ have the same derived categories, so Theorem 3.9 applies applies, and (using Theorem 2.2) we recover the computation of 2.7.1: $K(\mathcal{A}) \simeq K\mathbf{Ch}^b(\mathcal{A}) \simeq K\mathbf{Ch}^{hb}(\mathcal{A})$, and $K_n(\mathcal{A}) \cong K_n\mathbf{Ch}^{hb}(\mathcal{A})$ for all n.

If R is a ring, we saw in 3.8.1 that $\mathbf{Ch}^b\mathbf{M}(R)$ and $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$ have the same derived categories. Again by Theorem 3.9, we recover the computation of 2.7.4: $G(R) \simeq K\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$.

If S is a central multiplicative set in R, it is easy to see by truncating that $\mathbf{Ch}^b_S\mathbf{M}(R) \to \mathbf{Ch}^{hb}_{\mathrm{pcoh},S}(R)$ and $\mathbf{Ch}^b_S\mathbf{P}(R) \to \mathbf{Ch}_{\mathrm{perf},S}(R)$ induce equivalences on derived categories. By Theorem 3.9, we obtain the calculation of 2.7.4 that they induce homotopy equivalences on K-theory.

Example 3.10.2. If X is a noetherian scheme, the discussion of 3.10.1 applies to the abelian category $\mathbf{M}(X)$ of coherent $\mathcal{O}_X$-modules. We saw in Ex. II.9.8 that if a complex E has only finitely many nonzero cohomology sheaves, and these are coherent, then E is pseudo-coherent (*i.e.,* it is quasi-isomorphic to a bounded above complex of vector bundles); by truncating below, it is quasi-isomorphic to a bounded complex of coherent modules. By 3.8.1, this proves that $\mathbf{Ch}^b\mathbf{M}(X)$ and $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$ have the same derived categories, and hence $G(X) \simeq K\mathbf{Ch}^b\mathbf{M}(X) \simeq K\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$.

Let Z be a closed subscheme of X. We saw in 2.6.2 that the relative K-theory of $G(X) \to G(X-Z)$ is the K-theory of the category $\mathbf{Ch}^b_Z\mathbf{M}(X)$ of complexes of coherent modules which are acyclic on $X-Z$. It is contained in the category $\mathbf{Ch}^{hb}_{\mathrm{pcoh},Z}(X)$ of homologically bounded pseudo-coherent complexes acyclic on $X-Z$. The truncation argument in the previous paragraph shows that these two categories have the same derived categories, and hence the same K-theory: $K\mathbf{Ch}^b_Z\mathbf{M}(X) \simeq K\mathbf{Ch}^{hb}_{\mathrm{pcoh},Z}(X)$.

This argument works if X is quasi-compact but not noetherian, provided that we understand $\mathbf{M}(X)$ to be pseudo-coherent modules (see [TT, 3.11]); this is the case when X is quasi-projective over a commutative ring). However, it does not work for general X; see [SGA6, Is]. The following definition generalizes the definition of $G_0(X)$ given in II, Ex. 9.9 and [SGA6, IV(2.2)].

Definition 3.10.3. If a scheme X is not noetherian, then we define $G(X)$ to be $K\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$. If Z is closed in X, we define $G(X \text{ on } Z)$ to be $K\mathbf{Ch}^{hb}_{\mathrm{pcoh},Z}(X)$.

By Proposition 2.5, $G(X \text{ on } Z) \to G(X) \to G(X-Z)$ is a homotopy fibration, and we get a long exact sequence on homotopy groups, exactly as in 2.6.2.

Example 3.10.4. (Thomason) If X is any quasi-compact scheme, the inclusion $\mathbf{Ch}^b_{\text{perf}}(X) \subset \mathbf{Ch}_{\text{perf}}(X)$ induces an equivalence on derived categories by [TT, 3.5]. Comparing with Definition 2.7.3, we see that $K(X) \simeq K\mathbf{Ch}^b_{\text{perf}}(X)$.

If X also has an ample family of line bundles (for example if X is quasi-projective), the inclusion of $\mathbf{Ch}^b\mathbf{VB}(X)$ in $\mathbf{Ch}_{\text{perf}}(X)$ induces an equivalence on derived categories, by [TT, 3.6 and 3.8]. In this case, we get a fancy proof that $K(X) \simeq K\mathbf{VB}(X)$, which was already observed in 2.7.3.

Proper transfer f_* and the Projection Formula

Proper Transfer 3.11. Here is a homological construction of the transfer f_* of 3.7, associated to a proper map $f : X \to Y$ of noetherian schemes. It is based upon the fact (see [SGA4, V.4.9]) that the direct image f_* sends flasque sheaves to flasque sheaves. Let $\mathcal{F}_X$ denote the Waldhausen category of homologically bounded complexes of flasque $\mathcal{O}_X$-modules whose stalks have cardinality at most κ for a suitably large κ; f_* is an exact functor from flasque modules to flasque modules, and from $\mathcal{F}_X$ to $\mathcal{F}_Y$. Moreover, $K(\mathcal{F}_X)$ is independent of κ by Approximation 2.4, and $\mathcal{F}_X \subset \mathbf{Ch}^{hb}_{\text{pcoh}}(X)$ induces a homotopy equivalence on K-theory by Resolution 3.9 (see Ex. 3.8). Hence we obtain the transfer map f_* as the composite

$$G(X) \simeq K\mathbf{Ch}^{hb}_{\text{pcoh}}(X) \simeq K(\mathcal{F}_X) \xrightarrow{f_*} K(\mathcal{F}_Y) \simeq K\mathbf{Ch}^{hb}_{\text{pcoh}}(Y) \simeq G(Y).$$

Given another proper map $g : Y \to Z$ (with Z noetherian), the composition $g_*f_* : \mathcal{F}_X \to \mathcal{F}_Z$ equals $(gf)_*$. Thus $X \mapsto K(\mathcal{F}_X)$ and hence $X \mapsto G(X)$ is a functor on the category of noetherian schemes and proper maps.

Suppose in addition that X has finite flat dimension over Y, so that f_* sends perfect complexes to perfect complexes. (We say that f is a *perfect map.*) Then we have an exact functor $f_* : \mathbf{Ch}_{\text{perf}}(X) \to \mathbf{Ch}_{\text{perf}}(Y)$ and hence a proper transfer $K(X) \simeq K\mathbf{Ch}_{\text{perf}}(X) \xrightarrow{f_*} K\mathbf{Ch}_{\text{perf}}(Y) \simeq K(Y)$. The same argument shows that $X \mapsto K(X)$ is a functor on the category of noetherian schemes and perfect proper maps.

Variant 3.11.1. We can alter f_* using the "Godement resolution" functor T, from complexes of $\mathcal{O}_X$-modules to complexes of flasque sheaves; one takes the direct sum total complex of the Godement resolutions of the individual sheaves, as in [SGA4, XVII.4.2]. Since the direct image f_* is exact on flasque sheaves, the functor $Rf_* = f_* \circ T$ is exact on all complexes of $\mathcal{O}_X$-modules. The resulting map $Rf_* : K\mathbf{Ch}^{hb}_{\text{pcoh}}(X) \to K(\mathcal{F}_Y) \simeq K\mathbf{Ch}^{hb}_{\text{pcoh}}(Y)$ is homotopic to the functorial construction of $G(X) \to G(Y)$ in 3.11, so $f_*(x) = Rf_*(x)$ for all $x \in G_m(X)$.

Projection Formula 3.12. *Let $f : X \to Y$ be a proper morphism of noetherian schemes. Then $f_* : G_*(X) \to G_*(Y)$ is a graded $K_*(Y)$-module homomorphism: for all $x \in G_m(X)$ and $y \in K_n(Y)$:*

$$f_*(x \cdot f^*y) = f_*(x) \cdot y.$$

Suppose in addition that f has finite flat dimension. Then $f_*: K_*(X) \to K_*(Y)$ exists and is a graded $K_*(Y)$-module homomorphism: the same formula holds in $K_{m+n}(Y)$ for all $x \in K_m(X)$, where $f_*(x) \in K_m(Y)$.

Proof. We will express each side as the pairing on K-theory arising from the construction of IV.8.11 applied to a biexact pairing of Waldhausen categories $\mathbf{M}(X) \times \mathbf{VB}(Y) \to \mathcal{A}_X \times \mathcal{F} \to \mathcal{A}_Y$ (II.9.5.2). Let $\mathcal{A}_X$ denote the category $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$ of homologically bounded pseudo-coherent complexes of $\mathcal{O}_X$-modules, and let $\mathcal{F}$ denote the category of bounded above perfect complexes of flat $\mathcal{O}_Y$-modules. Note that $\mathbf{M}(X) \subset \mathcal{A}_X$ induces $G(X) \simeq K\mathcal{A}_X$ by 2.7.4 and $\mathbf{VB}(X) \subset \mathcal{F}$ induces $K(Y) \simeq K(\mathcal{F})$ by Ex. 2.8. Then the functors $(E,F) \mapsto (Rf_*E) \otimes_Y F$ and $(E,F) \mapsto Rf_*(E \otimes_X f^*F)$ are biexact, where Rf_* is defined in 3.11.1. By IV.8.11, they induce maps $K(\mathcal{A}_X) \wedge K(\mathcal{F}) \to K\mathcal{A}_Y$ which on homotopy groups are the pairings sending (x,y) to $f_*(x \cdot f^*(y))$ and $f_*(x) \cdot y$, respectively. The canonical map

$$(Rf_*E) \otimes_Y F \to f_*(TE \otimes_Y f^*F) \to Rf_*(E \otimes_X f^*F)$$

is a natural quasi-isomorphism of pseudo-coherent complexes; see [SGA6, III.3.7]. Hence it induces a homotopy between the two maps, as desired.

If in addition X has finite flat dimension, we merely replace $\mathcal{A}_X$ (resp., $\mathcal{A}_Y$) by the category of perfect complexes on X (resp., on Y), and the same proof works. $\square$

EXERCISES

3.1 Under the hypothesis of the Resolution Theorem 3.1, show that $\mathcal{H}_{n-1}$ is closed under admissible subobjects and extensions in $\mathcal{H}_n$. (It suffices to consider $n=1$.)

3.2 Let $f: \mathcal{B} \to \mathcal{C}$ be a left exact functor between two abelian categories such that the right derived functors R^if exist, and suppose that every object of $\mathcal{B}$ embeds in an f-acyclic object (an object for which R^if vanishes when $i > 0$). Let $\mathcal{A}$ be the full subcategory of f-acyclic objects, and $\mathcal{B}_f$ the full subcategory of objects B such that only finitely many $R^if(B)$ are nonzero.
(a) Show that $\mathcal{A}$ and $\mathcal{B}_f$ are exact subcategories of $\mathcal{B}$, closed under cokernels of admissible monomorphisms in $\mathcal{B}$, and that f is an exact functor on $\mathcal{A}$.
(b) Show that $K(\mathcal{A}) \simeq K(\mathcal{B}_f)$. In this way we can define $f_*: K_n(\mathcal{B}_f) \to K_n(\mathcal{C})$ by $K(\mathcal{B}_f) \simeq K(\mathcal{A}) \to K(\mathcal{C})$.

3.3 Show that β is an isomorphism in Example 3.5.2. To do this, let $F_m(M)$ be the submodule of M generated by $M_{-m} \oplus \cdots \oplus M_m$, and consider the subcategory $\mathbf{M}_m(S)$ of graded B-modules M with $M = F_m(M)$. Show that $K_i\mathbf{M}_m(R)$ is a sum of copies of $G_i(R)$ and use the admissible filtration

$$0 \subset F_0(M) \subset \cdots \subset F_m(M) = M$$

of modules in $\mathbf{M}_m(S)$ (see 1.8) to show that $K_i\mathbf{M}_m(S) \cong K_i\mathbf{M}_m(R)$ for all m.

3.4 If $S = R \oplus S_1 \oplus \cdots$ is a graded noetherian ring, and both S/R and R/S have finite flat dimension, modify the previous exercise to show that Example 3.5.2 still holds. *Hint:* Consider S-modules which are acyclic for both $\otimes_S R$ and $\otimes_S S$.

3.5 Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ are proper morphisms of finite flat dimension. Show that $f^*g^* = (gf)^*$ as maps $G_*(Z) \to G_*(X)$.

3.6 (Quillen) In this exercise, we give another construction of the transfer map $f_* : G(X) \to G(Y)$ associated to a projective morphism $f\colon X \to Y$ of noetherian schemes. Let $\mathcal{A} \subset \mathbf{M}(X)$ denote the (exact) subcategory consisting of all coherent $\mathcal{O}_X$-modules $\mathcal{F}$ such that $Rf^i(\mathcal{F}) = 0$ for $i > 0$. Because X is projective, for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there is an integer n_0 such that, for all $n \ge n_0$, the modules $\mathcal{F}(n)$ are in $\mathcal{A}$, and are generated by global sections; see [Hart, III.8.8].
(a) Show that any coherent $\mathcal{F}$ embeds in $\mathcal{F}(n)^r$ for large n and r. *Hint:* To prove it for $\mathcal{F} = \mathcal{O}_X$, apply $\mathrm{Hom}(\mathcal{O}_X(n), -)$ to a surjection $\mathcal{O}_X^r \to \mathcal{O}_X(n)$; now apply $\otimes\mathcal{F}$.
(b) Show that every coherent module $\mathcal{F}$ has a finite $\mathcal{A}$-resolution, starting with (a).
(c) Show that every vector bundle has a finite resolution by vector bundles in $\mathcal{A}$.
(d) Use Ex. 3.2 to define f_* to be the composition $f_* : G(X) \simeq K(\mathcal{A}) \to G(Y)$. Then show that this definition of f_* is homotopy equivalent to the map defined in 3.7, so that the maps $f_* : G_n(X) \to G_n(Y)$ agree.
(e) Given $g : Z \to X$, let $\mathcal{B} \subset \mathbf{M}(X)$ be the subcategory of modules Tor-independent of $\mathcal{O}_Z$. Show that every $\mathcal{F}$ in $\mathcal{B}$ has a finite $\mathcal{A}\cap\mathcal{B}$-resolution. *Hint:* If $\mathcal{F}$ is in $\mathcal{B}$, so is $\mathcal{F}(n)$.

3.7 Fix a noetherian ring R, and let $\mathbf{Ch}_{\mathbf{M}}^{hb,+}(R)$ be the category of all bounded below chain complexes of R-modules which are quasi-isomorphic to a (bounded) complex in $\mathbf{Ch}^b(\mathbf{M}(R))$. For example, injective resolutions of finitely generated R-modules belong to $\mathbf{Ch}_{\mathbf{M}}^{hb,+}(R)$. Let $\mathcal{I} \subset \mathbf{Ch}_{\mathbf{M}}^{hb,+}(R)$ denote the subcategory of complexes of injective R-modules. Show that the derived categories of $\mathcal{I}$, $\mathbf{Ch}_{\mathbf{M}}^{hb,+}(R)$ and $\mathbf{Ch}^b(\mathbf{M}(R))$ are isomorphic. Conclude that

$$K(\mathcal{I}) \simeq K\mathbf{Ch}_{\mathbf{M}}^{hb,+}(R) \simeq K\mathbf{Ch}^b(\mathbf{M}(R) \simeq G(R).$$

3.8 Fix a noetherian scheme X, and let $\mathbf{Ch}_{\mathbf{M}}^{hb,+}(X)$ be the category of all bounded below chain complexes of $\mathcal{O}_X$-modules which are quasi-isomorphic to a (bounded) complex in $\mathbf{Ch}^b(\mathbf{M}(X))$.
(a) Show that the derived categories of $\mathbf{Ch}_{\mathbf{M}}^{hb,+}(X)$ and $\mathbf{Ch}^b(\mathbf{M}(X))$ are isomorphic, and conclude that $\mathbf{Ch}_{\mathbf{M}}^{hb,+}(X) \simeq K\mathbf{Ch}^b(\mathbf{M}(X) \simeq G(X)$.
(b) Let $\mathcal{I} \subset \mathcal{F} \subset \mathbf{Ch}_{\mathbf{M}}^{hb,+}(X)$ denote the subcategories of complexes of injective and flasque $\mathcal{O}_X$-modules, respectively. Show that $K(\mathcal{I}) \simeq K(\mathcal{F}) \simeq G(X)$ as well.

3.9 If X is a quasi-compact regular scheme (I.5.14), show that every pseudo-coherent module is perfect (II, Ex. 9.10). Then show that $\mathbf{Ch}_{\mathrm{perf}}(X)$ is the same as $\mathbf{Ch}_{\mathrm{pcoh}}^{hb}$. Conclude that $K(X) \simeq G(X)$ by definition (2.7.3 and 3.10.3).

3.10 *(Projection Formula)* Suppose that $f\colon X \to Y$ is a proper map between quasi-projective schemes, both of which have finite flat dimension. Use a modification of the proof of 3.7.3 to show that the proper transfer map $f_* : G_m(X) \to G_m(Y)$ is a $K_0(Y)$-module map, *i.e.*, that $f_*(x\cdot f^*y) = f_*(x)\cdot y$ for $y \in K_0(X)$ and $x \in G_m(X)$. (The conclusion is a special case of 3.12; see [SGA6, IV.2.11.1].)

3.11 (Thomason) Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $g : Y' \to Y$ be a map Tor-independent of f. Show that the Base Change

Theorem 3.7.2 remains valid in this context: if g has finite flat dimension then $g^*f_* \simeq f'_*g'^*$ as maps $G(X) \to G(Y')$, while if f has finite flat dimension then $g^*f_* \simeq f'_*g'^*$ as maps $K(X) \to K(Y')$
(a) First consider the category $\mathcal{C}$ of bounded above pseudo-coherent complexes of flat modules on X. Then g'^* is exact on $\mathcal{C}$ and takes values in $\mathcal{C}'$. Show that the Godement resolution T of 3.11.1 sends $\mathcal{C}$ to itself, and that $E \to TE$ is a quasi-isomorphism. Then show that the inclusion of $\mathcal{C}$ in $\mathbf{Ch}^{hb}_{\text{pcoh}}(X)$ is an equivalence of derived categories, so $G(X) \simeq K(\mathcal{C})$.
(b) Consider the category $\mathcal{A}$ whose objects consist of an E in $\mathcal{C}$, a bounded above complex F of flat modules on Y with a quasi-isomorphism $F \to f_*E$, a bounded below complex G of flasque modules on X' with a quasi-isomorphism $(g')^*E \to G$. Show that $\mathcal{A}$ has the same derived category as $\mathcal{C}$, and conclude that $G(X) \simeq K(\mathcal{A})$.
(c) Show that g^*f_* is represented by the exact functor on $\mathcal{A}$ sending (E, F, G) to g^*F, and that $f'_*(g')^*$ is represented by the exact functor sending (E, F, G) to f'_*G.
(d) The canonical base change of [SGA4, XVII.4.2.12] is the natural isomorphism

$$g^*F \to g^*f_*(E) \to g^*f_*g'_*g'^*E = g^*g_*f'_*g'^*E \to f'_*g'^*E \to f'_*G.$$

Show that this canonical base change induces the desired homotopy $g^*f_* \simeq f'_*g'^*$.

3.12 Suppose that $F : \mathcal{M}^1 \to \mathcal{M}^2$ is an additive functor between abelian categories, and that $\mathcal{A}^i \subset \mathbf{Ch}(\mathcal{M}^i)$ $(i = 1, 2)$ are saturated Waldhausen subcategories, closed under mapping cones and shifts. If F sends $\mathcal{A}^1$ to $\mathcal{A}^2$ and induces an equivalence of derived categories, modify the proof of Theorem 3.9 to show that $K(\mathcal{A}^1) \simeq K(\mathcal{A}^2)$.

3.13 Let $\mathbb{P}^1_R$ denote the projective line over an associative ring R, as in 1.5.4, and let $\mathbf{H}_n$ denote the subcategory of **mod-** $\mathbb{P}^1_R$ of modules having a resolution of length n by vector bundles. Show that $K(\mathbb{P}^1_R) \simeq K\mathbf{H}_n$ for all n.

3.14 Let S be a set of central nonzerodivisors in a ring R.
(a) Let $\mathbf{Ch}_{\text{perf}}(\mathbf{M}_S)$ denote the category of perfect chain complexes of S-torsion R-modules. Show that the inclusion of $\mathbf{Ch}_{\text{perf}}(\mathbf{M}_S)$ into $\mathbf{Ch}_{\text{perf},S}(R)$ induces an equivalence $K\mathbf{Ch}_{\text{perf}}(\mathbf{M}_S) \simeq K(R \text{ on } S)$. (See 2.7.2.) *Hint:* Consider the functor $\varinjlim \operatorname{Hom}(R/sR, -)$ from $\mathbf{Ch}^b_S\mathbf{P}(R)$ to $\mathbf{Ch}_{\text{perf}}(\mathbf{M}_S)$.
(b) Let $\mathcal{H}$ be the additive subcategory of $\mathbf{H}_S(R)$ generated by the projective R/sR-modules. Show that $K\mathbf{Ch}^b(\mathcal{H}) \simeq K\mathbf{H}_S(R)$.
(c) Show that the inclusion of $\mathbf{Ch}^b(\mathcal{H})$ in $\mathbf{Ch}_{\text{perf}}(\mathbf{M}_S)$ satisfies property (App), and conclude that $K(R \text{ on } S) \simeq K\mathbf{H}_S(R)$. By Theorem 2.6.3, this yields a long exact sequence

$$\cdots K_{n+1}(S^{-1}R) \xrightarrow{\partial} K_n\mathbf{H}_S(R) \to K_n(R) \to K_n(S^{-1}R) \xrightarrow{\partial} \cdots .$$

3.15 Let s be a nonzerodivisor in a commutative noetherian ring R, and assume that R is flat over a subring R_0, with R_0 isomorphic to R/sR. Use the projection formula 3.5.3 to show that the transfer map $G_*(R/sR) \to G_*(R)$ is zero.

3.16 (Thomason [TT, 5.7]) Let X be a quasi-projective scheme, and let Z be a subscheme defined by an invertible ideal I of $\mathcal{O}_X$. Let $\mathbf{Ch}_{\text{perf},Z}(X)$ denote the category of perfect complexes on X which are acyclic on $X - Z$. For reasons that will become clear in §7, we write $K(X \text{ on } Z)$ for $K\mathbf{Ch}_{\text{perf},Z}(X)$.

(a) Let $\mathbf{M}_Z(X)$ denote the category of $\mathcal{O}_X$-modules supported on Z, and let $\mathbf{Ch}_{\mathrm{perf}}(\mathbf{M}_Z)$ denote the category of perfect complexes of modules in $\mathbf{M}_Z(X)$. Show that the inclusion into $\mathbf{Ch}_{\mathrm{perf},Z}(X)$ induces an equivalence $K\mathbf{Ch}_{\mathrm{perf}}(\mathbf{M}_Z) \simeq K(X \text{ on } Z)$.
(b) Let $\mathbf{H}_Z(X)$ denote the subcategory of modules in $\mathbf{H}(X)$ supported on Z. If $\mathcal{H}$ is the additive subcategory of $\mathbf{H}_Z(X)$ generated by the $\mathcal{O}_X/I^n\mathcal{O}_X(n)$, show that $K(\mathcal{H}) \cong K\mathbf{H}_Z(X)$.
(c) Show that the inclusion of $\mathbf{Ch}^b(\mathcal{H})$ in $\mathbf{Ch}_{\mathrm{perf}}(\mathbf{M}_Z)$ satisfies (App), so that Waldhausen Approximation 2.4 (and 2.2) imply that $K(X \text{ on } Z) \simeq K\mathbf{H}_Z(X)$.

§4. Devissage

This is a result that allows us to perform calculations like $G_*(\mathbb{Z}/p^r) \cong K_*(\mathbb{F}_p)$, which arise from the inclusion of the abelian category $\mathbf{M}(\mathbb{F}_p)$ of all finite elementary abelian p-groups into the abelian category $\mathbf{M}(\mathbb{Z}/p^r)$ of all finite abelian p-groups of exponent p^r. It is due to Quillen and taken from [Q341].

Devissage Theorem 4.1. *Let $i : \mathcal{A} \subset \mathcal{B}$ be an inclusion of abelian categories such that $\mathcal{A}$ is an exact abelian subcategory of $\mathcal{B}$ (II.6.1.5) and $\mathcal{A}$ is closed in $\mathcal{B}$ under subobjects and quotients. Suppose that every object B of $\mathcal{B}$ has a finite filtration*

$$0 = B_r \subset \cdots \subset B_1 \subset B_0 = B$$

by objects in $\mathcal{B}$ such that every subquotient B_i/B_{i-1} lies in $\mathcal{A}$. Then

$$K(\mathcal{A}) \simeq K(\mathcal{B}) \text{ and } K_*(\mathcal{A}) \cong K_*(\mathcal{B}).$$

Proof. By Quillen's Theorem A (IV.3.7), it suffices to show that the comma categories Qi/B are contractible for every B in $\mathcal{B}$. If B is in $\mathcal{A}$, then $Qi/B \simeq *$ because B is a terminal object. Since B has a finite filtration, it suffices to show that the inclusion $i/B' \to i/B$ is a homotopy equivalence for each $B' \rightarrowtail B$ in $\mathcal{B}$ with B/B' in $\mathcal{A}$.

By Ex. IV.6.1, we may identify the objects of Qi/B, which are pairs $(A, A \to B)$, with admissible layers $u : B_1 \rightarrowtail B_2 \rightarrowtail B$ such that B_2/B_1 is in $\mathcal{A}$. Let J denote the subcategory of Qi/B consisting of all admissible layers of B with $B_1 \subseteq B'$. We have functors $s : Qi/B \to J$, $s(u) = (B_1 \cap B' \rightarrowtail B_2 \rightarrowtail B)$ and $r : J \to Qi/B'$, $r(u) = (B_1 \rightarrowtail B_2 \cap B' \rightarrowtail B)$, because $\mathcal{A}$ is closed under subobjects. The natural transformations $u \to s(u) \leftarrow rs(u)$ defined by

$$(B_1 \rightarrowtail B_2 \rightarrowtail B) \to (B_1 \cap B' \rightarrowtail B_2 \rightarrowtail B) \leftarrow (B_1 \cap B' \rightarrowtail B_2 \cap B' \rightarrowtail B)$$

show that s is left adjoint to the inclusion $J \subset Qi/B$ and that r is right adjoint to the inclusion $Qi/B' \subset J$. It follows (IV.3.2) that $Qi/B' \subset J \subset Qi/B$ are homotopy equivalences, as desired. $\square$

Open Problem 4.1.1. Generalize the Devissage Theorem 4.1 to Waldhausen categories. Such a result should yield the above Devissage Theorem when applied to $\mathbf{Ch}^b(\mathcal{A})$.

COROLLARY 4.2. *If I is a nilpotent ideal in a noetherian ring R, then $G(R/I) \simeq G(R)$ and hence $G_*(R/I) \cong G_*(R)$.*

PROOF. As in II.6.3.1, Devissage applies to $\mathbf{M}(R/I) \subset \mathbf{M}(R)$, because every finitely generated R-module M has a finite filtration by submodules MI^n. $\square$

EXAMPLE 4.2.1. Let R be an artinian local ring with maximal ideal $\mathfrak{m}$ ($\mathfrak{m}^r = 0$) and quotient field $k = R/\mathfrak{m}$. (E.g., $R = \mathbb{Z}/p^r$ and $k = \mathbb{F}_p$). Then $G_*(R) \cong K_*(k)$.

It is instructive to deconstruct the argument slightly. In this case, $\mathbf{M}(k)$ is an exact subcategory of $\mathbf{M}(R)$ and every R-module M has the natural filtration

$$0 = \mathfrak{m}^r M \subset \mathfrak{m}^{r-1} M \subset \cdots \subset \mathfrak{m}M \subset M.$$

Note that the Admissible Filtrations Proposition 1.8 does not apply because for example $F(M) = M/\mathfrak{m}M$ is not an exact functor.

OPEN PROBLEM 4.2.2. Compute the K-groups $K_*(R)$ of an artinian local ring R, assuming $K_*(k)$ is known. If $\mathrm{char}(k) = 0$, this can be done using cyclic homology (Goodwillie's Theorem): the relative groups $K_n(R, \mathfrak{m})$ and $HC_{n-1}(R, \mathfrak{m})$ are isomorphic. If $\mathrm{char}(k) \neq 0$, this can be done using topological cyclic homology (McCarthy's Theorem): $K_n(R, \mathfrak{m})$ and $TC_n(R, \mathfrak{m})$ are isomorphic. In terms of generators and relations, though, we only know K_0, K_1, K_2 and sometimes K_3 at present. (See II.2.2, III.2.4 and III.5.11.1.)

APPLICATION 4.3. Let $\mathcal{A}$ be an abelian category such that every object has finite length. By Devissage, $K(\mathcal{A})$ is equivalent to $K(\mathcal{A}_{ss})$, where $\mathcal{A}_{ss}$ is the subcategory of semisimple objects. By Schur's Lemma, $\mathcal{A}_{ss} \cong \oplus \mathbf{M}(D_i)$, where the D_i are division rings. (The D_i are the endomorphism rings of non-isomorphic simple objects A_i.) It follows from Ex. IV.6.11 that

$$K_*(\mathcal{A}) \simeq \oplus_i K_*(D_i).$$

This applies to finitely generated torsion modules over Dedekind domains and curves, and more generally to finitely generated modules of finite support over any commutative ring or scheme.

APPLICATION 4.4. (R-modules with support) Given a central element s in a noetherian ring R, let $\mathbf{M}_s(R)$ denote the abelian subcategory of $\mathbf{M}(R)$ consisting of all M such that $Ms^n = 0$ for some n. We saw in II.6.3.3 that these modules have finite filtrations with subquotients in $\mathbf{M}(R/sR)$. By Devissage, $K\mathbf{M}_s(R) \simeq K\mathbf{M}(R/sR)$, so we have $K_*\mathbf{M}_s(R) \cong G_*(R/sR)$. More generally, given any ideal I we can form the exact category $\mathbf{M}_I(R)$ of all M such that $MI^n = 0$ for some n. Again by Devissage, $K\mathbf{M}_I(R) \simeq K\mathbf{M}(R/I)$ and we have $K_*\mathbf{M}_I(R) \cong G_*(R/I)$. The case $I = \mathfrak{p}$ ($K_*\mathbf{M}_\mathfrak{p}(R) \cong G_*(R/\mathfrak{p})$) will be useful in section 6 below.

If S is a central multiplicatively closed set in R, the exact category $\mathbf{M}_S(R)$ is the filtered colimit over $s \in S$ of the $\mathbf{M}_s(R)$. By IV.6.4, $K\mathbf{M}_S(R) = \Omega BQ\mathbf{M}_S(R)$ is $\varinjlim K\mathbf{M}_s(R)$ and hence

$$K_n\mathbf{M}_S(R) = \varinjlim K_n\mathbf{M}_s(R) = \varinjlim G_n(R/sR).$$

EXERCISES

4.1 (Jordan-Hölder) Given a ring R, describe the K-theory of the category of R-modules of finite length. (K_0 is given by Ex. II.6.1.)

4.2 If X is a noetherian scheme, show that $G(X_{\text{red}}) \simeq G(X)$ and hence that $G_*(X_{\text{red}}) \cong G_*(X)$. This generalizes the result II.6.3.2 for G_0.

4.3 (Quillen) Let Z be a closed subscheme of a noetherian scheme X, and let $\mathbf{M}_Z(X)$ be the exact category of all coherent X-modules supported on Z. Generalize 4.4 and II.6.3.4 by showing that

$$G(Z) = K\mathbf{M}(Z) \simeq K\mathbf{M}_Z(X).$$

4.4 Let $S = R \oplus S_1 \oplus \cdots$ be a graded noetherian ring, and let $\mathbf{M}^b_{gr}(S)$ denote the category of graded modules M with $M_n = 0$ for all but finitely many n. (Ex. II.6.14) Via $S \to R$, any R-module in $\mathbf{M}^b_{gr}(R) = \mathbf{M}_{gr}(R)$ is a graded S-module in $\mathbf{M}^b_{gr}(S)$.
(a) Use Devissage to show that $\mathbf{M}_{gr}(R) \subset \mathbf{M}^b_{gr}(S)$ induces an equivalence in K-theory, so that (in the notation of Example 3.5.2) $K_*\mathbf{M}^b_{gr}(S) \simeq G_*(R)[\sigma, \sigma^{-1}]$.
(b) If $t \in S_1$, write $\mathbf{M}_{gr,t}(S)$ for the category of t-torsion modules in $\mathbf{M}_{gr}(S)$. Show that $\mathbf{M}_{gr}(S/tS) \subset \mathbf{M}_{gr,t}(S)$ induces $K\mathbf{M}_{gr}(S/tS) \simeq K\mathbf{M}_{gr,t}(S)$.

§5. The Localization Theorem for abelian categories

The K_0 Localization Theorems for abelian categories (II.6.4) and certain exact categories (II.7.7.4) generalize to higher K-theory, in a way we shall now describe. It is also due to Quillen.

Recall from the discussion before II.6.4 that a *Serre subcategory* of an abelian category $\mathcal{A}$ is an abelian subcategory $\mathcal{B}$ which is closed under subobjects, quotients and extensions. The quotient abelian category $\mathcal{A}/\mathcal{B}$ exists; Gabriel's construction of $\mathcal{A}/\mathcal{B}$ using the Calculus of Fractions is also described just before II.6.4.

Abelian Localization Theorem 5.1. *Let $\mathcal{B}$ be a Serre subcategory of a (small) abelian category $\mathcal{A}$. Then*

$$K(\mathcal{B}) \to K(\mathcal{A}) \xrightarrow{\text{loc}} K(\mathcal{A}/\mathcal{B})$$

is a homotopy fibration sequence. Thus there is a long exact sequence of homotopy groups

(5.1.1) $$\cdots \to K_{n+1}(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_n(\mathcal{B}) \to K_n(\mathcal{A}) \xrightarrow{\text{loc}} K_n(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_{n-1}(\mathcal{B}) \to \cdots$$

ending in $K_0(\mathcal{B}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B}) \to 0$, the exact sequence of II.6.4.

Proof. (Quillen) For any A in $\mathcal{A}$, let us write $\bar{A}$ for the object loc(A) of $\mathcal{A}/\mathcal{B}$. Recall from IV.3.2.3 that for any object L in $\mathcal{A}/\mathcal{B}$ the comma category $L\backslash Q$loc consists of pairs (A, u) with A in $\mathcal{A}$ and $u : L \to \bar{A}$ a morphism in $Q\mathcal{A}/\mathcal{B}$. We will deduce the result from Theorem B (IV.3.7), so we need to show that $0\backslash Q$loc is homotopy equivalent to $Q\mathcal{B}$, and that for every $L \to L'$ the map $L'\backslash Q\text{loc} \to L\backslash Q\text{loc}$ is a homotopy equivalence.

To do this, we introduce the full subcategory F_L of $L\backslash Q\mathrm{loc}$ consisting of pairs (A,u) in which u is an isomorphism in $\mathcal{A}/\mathcal{B}$. (By IV.6.1.2, isomorphisms in $Q\mathcal{A}/\mathcal{B}$ are in 1–1 correspondence with isomorphisms in $\mathcal{A}/\mathcal{B}$, and are described in II.A.1.2.) In particular, the category F_0 $(L=0)$ is equivalent to the subcategory $Q\mathcal{B}$ because, by construction (II.6.4), the objects of $\mathcal{B}$ are exactly the objects of $\mathcal{A}$ isomorphic to 0 in $\mathcal{A}/\mathcal{B}$. Thus the functor $Q\mathcal{B}\to Q\mathcal{A}$ factors as $Q\mathcal{B}\cong F_0\hookrightarrow 0\backslash Q\mathrm{loc}\to Q\mathcal{A}$.

Claim 5.1.2: The inclusion $i: F_0\to 0\backslash Q\mathrm{loc}$ is a homotopy equivalence.

This will follow from Quillen's Theorem A (IV.3.7) once we observe that for every A in $\mathcal{A}$ the comma category $i/(A,u:0\to\bar{A})$ is contractible. By IV.6.1.2, the morphism u in $Q\mathcal{A}/\mathcal{B}$ is the same as a subobject L of $\bar{A}$ in $\mathcal{A}/\mathcal{B}$, and this is represented by a subobject of A. By Ex. IV.6.1, we can represent each object of $i/(A,u)$ by an admissible layer $A_1\rightarrowtail A_2\rightarrowtail A$ in $\mathcal{A}$ such that $\bar{A}_1=\bar{A}_2=L$, and it is easy to see that $i/(A,u)$ is equivalent to the poset of such layers. But this poset is directed by the very construction of $\mathcal{A}/\mathcal{B}$: $A_1\rightarrowtail A_2\rightarrowtail A$ and $A_1'\rightarrowtail A_2'\rightarrowtail A$ both map to $(A_1\cap A_1')\rightarrowtail(A_2+A_2')\rightarrowtail A$. This verifies our claim, showing that $Q\mathcal{B}\to 0\backslash Q\mathrm{loc}$ is a homotopy equivalence.

Claim 5.1.3: The inclusion $F_L\xrightarrow{i} L\backslash Q\mathrm{loc}$ is a homotopy equivalence for every L.

This is proven with the same argument used to prove Claim 5.1.2. The only difference is that now the category $i/(A,u)$ is equivalent to the poset of admissible layers $A_1\rightarrowtail A_2\rightarrowtail\bar{A}$ with $\overline{A_2/A_1}\cong L$; the construction works in this context to shows that it is directed.

We now introduce several auxiliary categories. Fix N in $\mathcal{A}$ and let $\mathcal{E}_N$ be the category of pairs $(A,h:A\to N)$ for which $\bar{h}$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. By definition, a morphism from (A,h) to (A',h') is a morphism $A\twoheadleftarrow A''\rightarrowtail A'$ in $Q\mathcal{A}$ such that the two composites $A''\to N$ agree. It is easily checked that there is a well defined functor $k:\mathcal{E}_N\to Q\mathcal{B}$ sending (A,h) to $\ker(h)$.

Let $\mathcal{E}_N'$ denote the full subcategory of $\mathcal{E}_N$ on the (A,h) with h onto. We will show that $\mathcal{E}_N'\to\mathcal{E}_N$ and $\mathcal{E}_N'\to Q\mathcal{B}$ are homotopy equivalences.

Claim 5.1.4: For each N in $\mathcal{A}$, $k':\mathcal{E}_N'\to Q\mathcal{B}$ is a homotopy equivalence.

This will follow from Theorem A once we show that k'/T is contractible for each T in $Q\mathcal{B}$. An object of this comma category is a datum (A,h,u), where (A,h) is in $\mathcal{E}_N$, and $u:\ker(h)\to T$ is in $Q\mathcal{B}$. The subcategory $\mathcal{C}$ of all (A,h,u) with u surjective is contractible because it has $(N,1_N,0\twoheadleftarrow T)$ as initial object. And the inclusion of $\mathcal{C}$ in k'/T is a homotopy equivalence because it has a left adjoint (IV.3.2), sending $(A,h,\ker(h)\rightarrowtail T_0\twoheadleftarrow T)$ to $(A_0,h_0,T_0\twoheadleftarrow T)$, where A_0 is the pushout of $\ker(h)\to A$ along $\ker(h)\rightarrowtail T_0$ and h_0 is the induced map. The claim follows.

Claim 5.1.5: For each N, $\mathcal{E}_N'\hookrightarrow\mathcal{E}_N$ is a homotopy equivalence.

Let I_N denote the partially ordered set of objects N_i of N such that N/N_i is in $\mathcal{B}$, and consider the functor $\mathcal{E}_N\to I_N$ sending (A,h) to the image $h(A)$. This is a fibered functor (IV.3.6.2); the fiber over i is $\mathcal{E}_{N_i}'$, and the base change for $N_j\subset N_i$ sends (A,h) to $(h^{-1}(N_j),h)$. Since $k_N(A,h)=k_{N_i}(A,h)$, it follows from Claim 5.1.4 that the base change maps $\mathcal{E}_{N_i}'\to\mathcal{E}_{N_j}'$ are homotopy equivalences. By Theorem B (IV.3.7), $\mathcal{E}_N'\to\mathcal{E}_N\to I_N$ is a homotopy fibration. Since I_N has a final object (N), it is contractible and we get $\mathcal{E}_N'\simeq\mathcal{E}_N$ as claimed.

Claim 5.1.6: If $g : N \to N'$ is a map in $\mathcal{A}$ which is an isomorphism in $\mathcal{A}/\mathcal{B}$, then $g_* : \mathcal{E}_N \to \mathcal{E}_{N'}$ is a homotopy equivalence.

Indeed, if (A, h) is in $\mathcal{E}_N$ then $\ker(h) \subseteq \ker(gh)$ defines a natural transformation from $k : \mathcal{E}_N \to Q\mathcal{B}$ to $k\, g_* : \mathcal{E}_N \to \mathcal{E}_{N'} \to Q\mathcal{B}$. As k is a homotopy equivalence (by Claims 5.1.4 and 5.1.5), so is g_*.

Now fix L in $\mathcal{A}/\mathcal{B}$, and let I_L be the category of pairs $(N, \bar{N} \xrightarrow{\sim} L)$; morphisms are maps $g : N \to N'$ in $\mathcal{A}$ such that $\bar{N} \cong \bar{N}'$. This category is filtering by II.A.1.2, and there is a functor from I_L to categories sending $(N, \bar{N} \xrightarrow{\sim} L)$ to $\mathcal{E}_N$ and g to g_*.

Claim 5.1.7: F_L is isomorphic to the colimit over I_L of the categories $\mathcal{E}_N$.

For each $n = (N, \bar{N} \xrightarrow{\sim} L)$ we have a functor $p_n : \mathcal{E}_N \to F_L$ sending (A, h) to $L \xrightarrow{\sim} \bar{N} \xrightarrow{\sim} \bar{A}$. Since $p_n = p_{n'} g_*$ for each morphism g, there is a functor $p : \operatorname{colim} \mathcal{E}_N \to F_L$.

Consider the composite functor $\mathcal{E}_N \xrightarrow{k} Q\mathcal{B} \xrightarrow{\sim} F_0 \xrightarrow{\sim} 0\backslash Q\mathrm{loc}$, sending (A, h) to $(\ker(h), 0)$; it is a homotopy equivalence by Claims 5.1.2–5. Using the map $i : 0 \rightarrowtail L$ in $Q(\mathcal{A}/\mathcal{B})$, we have a second functor $\mathcal{E}_N \xrightarrow{p} F_L \hookrightarrow L\backslash Q\mathrm{loc} \xrightarrow{i^*} 0\backslash Q\mathrm{loc}$, sending (A, h) to $(A, 0 \rightarrowtail \bar{A})$. There is a natural transformation between them, given by the inclusion of $\ker(h)$ in A. Hence we have a homotopy commutative diagram:

$$\begin{array}{ccccc}
\mathcal{E}_N & \xrightarrow{p_n} & F_L & \hookrightarrow & L\backslash Q\mathrm{loc} \\
{\scriptstyle k}\big\downarrow & & & & \big\downarrow{\scriptstyle i^*} \\
Q\mathcal{B} & \xrightarrow{\sim} & F_0 & \hookrightarrow & 0\backslash Q\mathrm{loc}.
\end{array}$$

From Claims 5.1.6 and 5.1.7, it follows that each $p_n : \mathcal{E}_N \to F_L$ is a homotopy equivalence. From Claim 5.1.3, $F_L \hookrightarrow L\backslash Q\mathrm{loc}$ is a homotopy equivalence. It follows that $i^* : L\backslash Q\mathrm{loc} \to 0\backslash Q\mathrm{loc}$ is also a homotopy equivalence. This finishes the proof of the Localization Theorem 5.1. $\square$

Taking homotopy groups with coefficients mod ℓ also converts homotopy fibration sequences into long exact sequences (IV.2.1.1), so we immediately obtain a finite coefficient analogue of (5.1.1).

COROLLARY 5.2. *For each ℓ there is also a long exact sequence*

$$\cdots \to K_{n+1}(\mathcal{A}/\mathcal{B}; \mathbb{Z}/\ell) \xrightarrow{\partial} K_n(\mathcal{B}; \mathbb{Z}/\ell) \to K_n(\mathcal{A}; \mathbb{Z}/\ell) \to K_n(\mathcal{A}/\mathcal{B}; \mathbb{Z}/\ell) \xrightarrow{\partial} \cdots$$

OPEN PROBLEM 5.3. Let $\mathcal{B}$ be a Serre subcategory of an abelian category $\mathcal{A}$ (II.6.4), and let $\mathbf{Ch}^b_{\mathcal{B}}(\mathcal{A})$ denote the category of all bounded complexes in $\mathcal{A}$ whose cohomology lies in $\mathcal{B}$ (II, Ex. 9.5). Is $K(\mathbf{Ch}^b_{\mathcal{B}}(\mathcal{A})) \simeq K(\mathcal{B})$? Such a result would make the Localization Theorem 5.1 for abelian categories an immediate consequence of Proposition 2.5 and Theorem 2.2.

EXERCISES

5.1 Suppose that $\alpha : A \to A$ is a morphism in $\mathcal{A}$ which is an isomorphism in $\mathcal{A}/\mathcal{B}$, and so determines an element $[\alpha]$ of $K_1(\mathcal{A}/\mathcal{B})$. Show that $\partial : K_1(\mathcal{A}/\mathcal{B}) \to K_0(\mathcal{B})$

sends $[\alpha]$ to $[\mathrm{coker}(\alpha)]-[\ker(\alpha)]$. *Hint:* Use the representative of $[\alpha]$ in $\pi_2 BQ(\mathcal{A}/\mathcal{B})$ given in IV, Ex. 7.9, and IV.6.2.

5.2 Show that the map $S^n \to P^{n+1}(\mathbb{Z}/\ell)$ of IV.2.1.1 applied to the homotopy fibration of Theorem 5.1 yields a commutative diagram comparing the localization sequences of 5.1 and 5.2:

$$\begin{array}{ccccccc}
K_{n+1}(\mathcal{A}/\mathcal{B};\mathbb{Z}/\ell) & \xrightarrow{\partial} & K_n(\mathcal{B};\mathbb{Z}/\ell) & \rightarrow & K_n(\mathcal{A};\mathbb{Z}/\ell) & \rightarrow & K_n(\mathcal{A}/\mathcal{B};\mathbb{Z}/\ell) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_n(\mathcal{A}/\mathcal{B}) & \xrightarrow{\partial} & K_{n-1}(\mathcal{B}) & \rightarrow & K_{n-1}(\mathcal{A}) & \rightarrow & K_{n-1}(\mathcal{A}/\mathcal{B}).
\end{array}$$

5.3 Suppose that there is a biexact functor $\mathcal{A}\times\mathcal{C}\to\mathcal{A}'$ which induces biexact functors $\mathcal{B}\times\mathcal{C}\to\mathcal{B}'$ and $\mathcal{A}/\mathcal{B}\times\mathcal{C}\to\mathcal{A}'/\mathcal{B}'$. Use Ex. IV.1.23 to show that for $x\in K_j(\mathcal{A}/\mathcal{B})$ and $y\in K_n(\mathcal{C})$ the element $\{x,y\}\in K_{n+i}(\mathcal{A}'/\mathcal{B}')$ satisfies $\partial(\{x,y\}) = \{\partial(x),y\}$ in $K_{n+i}(\mathcal{B}')$.

§6. Applications of the Localization Theorem

In this section, we give two families of applications of the Localization Theorem 5.1, to $G(R)$ and to $G(X)$. All rings and schemes will be noetherian in this section, so that $\mathbf{M}(R)$ and $\mathbf{M}(X)$ are abelian categories.

Application 6.1. Let S be a central multiplicatively closed set in a noetherian ring R. We saw in II.6.4.1 that the category $\mathbf{M}_S(R)$ of finitely generated S-torsion modules is a Serre subcategory of $\mathbf{M}(R)$ with quotient category $\mathbf{M}(S^{-1}R)$. By the Localization Theorem 5.1, there is a homotopy fibration

$$K\mathbf{M}_S(R)\to G(R)\to G(S^{-1}R).$$

We observed in 4.4 that $\mathbf{M}_S(R)$ is the colimit over all $s\in S$ of the $\mathbf{M}(R/sR)$, and that $K_*\mathbf{M}_S(R)\cong \varinjlim G_*(R/sR)$. Comparing this to the Waldhausen localization sequence 2.6.1, we see that $\mathbf{M}_S(R)\to \mathbf{Ch}^b_S\mathbf{M}(R)$ induces a homotopy equivalence $K\mathbf{M}_S(R)\xrightarrow{\sim} K\mathbf{Ch}^b_S\mathbf{M}(R)=G(R \text{ on } S)$.

The prototype is the case when $S=\{s^n\}$. Here the maps $G(R/sR)\to G(R/s^nR)$ are homotopy equivalences by Devissage, so $G(R/sR)\simeq K\mathbf{M}_S(R)$; see 4.4. By inspection, the map $G(R/sR)\to G(R)$ identifying it with the homotopy fiber of $G(R)\to G(R[1/s])$ is the transfer i_* (IV.6.3.3) associated to $i: R\to R/sR$. Thus the long exact Localization sequence (5.1.1) becomes:

$$(6.1.1)\qquad \cdots\to G_{n+1}(R[s^{-1}])\xrightarrow{\partial} G_n(R/sR)\xrightarrow{i_*} G_n(R)\to G_n(R[s^{-1}])\xrightarrow{\partial}\cdots.$$

This is a sequence of $K_*(R)$-modules, because $\mathbf{P}(R)$ acts on the sequence of abelian categories $\mathbf{M}_S(R)\to\mathbf{M}(R)\to\mathbf{M}(S^{-1}R)$.

Example 6.1.2. It is useful to observe that any $s\in S$ determines an element $[s]$ of $K_1(R[1/s])$ and hence $G_1(R[1/s])$, and that $\partial(s)\in G_0(R/sR)$ is $[R/sR]-[I]$, where $I=\{r\in R: sr=0\}$. This formula is immediate from Ex. 5.1. In particular, when R is a domain we have $\partial(s)=[R/sR]$.

EXAMPLE 6.1.3. If $R = \mathbb{Z}[s, 1/f(s)]$ with $f(0) = 1$, the maps i_* are zero in (6.1.1) and hence the maps $G_n(R) \to G_n(R[1/s])$ are injections. Indeed, the vanishing of i_* follows from the projection formula (3.5.3): $i_*(i^*y) = i_*(1 \cdot i^*y) = i_*(1) \cdot y$ together with the observation in 6.1.2 that $i_*([R/sR]) = i_*\partial(s) = 0$. (Cf. Ex. 3.15.)

For the following result, we adopt the notation that f denotes the inclusion of R into $R[s]$, j is $R[s] \hookrightarrow R[s, s^{-1}]$, and $(s = 1)$ denotes the map from either $R[s]$ or $R[s, s^{-1}]$ to R, obtained by sending s to 1. Because R has finite flat dimension over $R[s]$ and $R[s, s^{-1}]$, there are base change maps $(s = 1)^* : G(R[s]) \to G(R)$ and $(s = 1)^* : G(R[s, s^{-1}]) \to G(R)$.

FUNDAMENTAL THEOREM OF $G(R)$ 6.2. *Let R be a noetherian ring. Then:*

(i) The flat base change $f^* : G(R) \to G(R[s])$ is a homotopy equivalence, split by $(s = 1)^*$. Hence

$$f^* : G_n(R) \cong G_n(R[s]) \quad \textit{for all } n.$$

(ii) The flat base change $j^* : G(R[s]) \to G(R[s, s^{-1}])$ induces isomorphisms

$$G_n(R[s, s^{-1}]) \cong G_n(R) \oplus G_{n-1}(R).$$

PROOF. We first observe that (i) implies (ii). Indeed, because $(s = 0)_* : G(R) \to G(R[s])$ is zero by 3.5.1, the localization sequence (6.1.1) for $j : R[s] \to R[s, s^{-1}]$ splits into short exact sequences

$$0 \to G_n(R[s]) \xrightarrow{j^*} G_n(R[s, s^{-1}]) \xrightarrow{\partial} G_{n-1}(R) \to 0.$$

By (i) we may identify $(jf)^* : G_n(R) \to G_n(R[s, s^{-1}])$ with j^*. Because $(s = 1) \circ jf$ is the identity, $(jf)^*(s = 1)^*$ is homotopic to the identity of $G(R)$. Assertion (ii) follows.

To prove (i), we introduce the graded subring $S = R[st, t]$ of $R[s, t]$ where $\deg(s) = 0$, $\deg(t) = 1$. Let $\mathbf{M}^b_{gr}(S)$ denote the Serre subcategory of all graded modules M in $\mathbf{M}_{gr}(S)$ with only finitely many nonzero M_n, *i.e.*, graded t-torsion modules. By Devissage (Ex. 4.4), $\mathbf{M}^b_{gr}(S)$ has the same K-theory as its subcategory $\mathbf{M}_{gr}(S/tS)$. There is also an equivalence of quotient categories

$$\mathbf{M}_{gr}(S)/\mathbf{M}^b_{gr}(S) \cong \mathbf{M}(R[s])$$

induced by the exact functor $M \mapsto M/(t-1)M$ from $\mathbf{M}_{gr}(S)$ to $\mathbf{M}(R[s])$. Note that both S and $S/tS = R[st]$ are flat over R and that R has finite flat dimension over both, so the K-theory of both $\mathbf{M}_{gr}(S)$ and $\mathbf{M}^b_{gr}(S)$ are isomorphic to $G(R)[\sigma, \sigma^{-1}]$ by Example 3.5.2. Hence the localization sequence (5.1.1) gives us the following diagram with exact rows:

$$\begin{tikzcd}
\cdots \to K_n\mathbf{M}_{gr}(S/tS) \arrow[r] & K_n\mathbf{M}_{gr}(S) \arrow[r] & G_n(R[s]) \to \cdots \\
0 \to G_n(R) \otimes \mathbb{Z}[\sigma, \sigma^{-1}] \arrow[r, "h"] \arrow[u, "\simeq" ', "3.5.2"] & G_n(R) \otimes \mathbb{Z}[\sigma, \sigma^{-1}] \arrow[r] \arrow[u, "\simeq" ', "3.5.2"] & G_n(R) \to 0. \arrow[u]
\end{tikzcd}$$

To describe the map h, recall from 3.5.2 that it is induced by the functor from $\mathbf{M}(R)$ to $K\mathbf{M}_{gr}(S)$ sending M to $M \otimes S/tS$. We have an exact sequence of functors from $\mathbf{M}(R)$ to $\mathbf{M}_{gr}(S)$:

$$0 \to M \otimes S(-1) \xrightarrow{t} M \otimes S \to M \otimes S/tS \to 0.$$

As pointed out in Example 3.5.2, the first two maps induce the maps σ and 1, respectively, from $G_n(R)$ to $G(R)[\sigma, \sigma^{-1}]$. Hence $h = 1 - \sigma$, proving (i). $\square$

Because $R[s]$ and $R[s, s^{-1}]$ are regular whenever R is, combining Theorems 6.2 and 3.3 yields the following important consequence.

THEOREM 6.3. *If R is a regular noetherian ring, then the base change $K(R) \to K(R[s])$ is a homotopy equivalence, so $K_n(R) \cong K_n(R[s])$ for all n. In addition,*

$$K_n(R[s, s^{-1}]) \cong K_n(R) \oplus K_{n-1}(R) \quad \textit{for all } n.$$

In particular, regular rings are K_n-regular for all n (in the sense of III.3.4).

COROLLARY 6.3.1. *For any regular ring R, there is a split exact sequence*

$$0 \to K_n(R) \to K_n(R[s]) \oplus K_n(R[s^{-1}]) \to K_n(R[s, s^{-1}]) \underset{\leftarrow}{\xrightarrow{\partial}} K_{n-1}(R) \to 0,$$

in which the splitting is multiplication by $s \in K_1(\mathbb{Z}[s, s^{-1}])$.

PROOF. This is obtained from the direct sum of the split exact sequences (6.1.1) for $R[s] \to R[s, s^{-1}]$ and $R[s^{-1}] \to R[s, s^{-1}]$, and the isomorphism $K_n(R) \cong K_n(R[s])$. By 6.1, the map ∂ in the localization sequence (6.1.1) is $K_*(R)$-linear. Thus for $x \in K_{n-1}(R)$ and $s \in K_1(R[s, s^{-1}])$ we have $\partial\{s, x\} = \{\partial s, x\} = x$. $\square$

The technique used in 6.2 to prove that $G(R) \simeq G(R[s])$ applies more generally to filtered rings. By a *filtered ring* we mean a ring R equipped with an increasing filtration $\{F_i R\}$ such that $1 \in F_0 R$, $F_i R \cdot F_j R \subseteq F_{i+j} R$ and $R = \cup F_i R$. The associated graded ring is $\mathrm{gr}(R) = \oplus F_i R / F_{i-1} R$. It is easy to see that if $\mathrm{gr}(R)$ is noetherian then so is R.

For example, any positively graded ring such as $R[s]$ is filtered with $F_i(R) = R_0 \oplus \cdots \oplus R_i$; in this case $gr(R) = R$.

THEOREM 6.4. *Let R be a filtered ring such that $gr(R)$ is noetherian and of finite flat dimension d over $k = F_0 R$, and k has finite flat dimension over $gr(R)$. Then $k \subset R$ induces $G(k) \simeq G(R)$.*

PROOF. The hypotheses imply that R has flat dimension $\mathrm{fd}_k A \le d$. Indeed, since each of the $F_i R / F_{i-1} R$ has flat dimension at most d it follows by induction that $\mathrm{fd}_k F_i R \le d$ and hence that $\mathrm{fd}_k R \le d$. Thus the map $G(k) \to G(R)$ is defined. Let S denote the graded subring $\oplus (F_i R) t^i$ of $R[t]$. Then $S/tS \cong \mathrm{gr}(R)$, S is noetherian, $\mathrm{fd}_k S \le d$ and k has finite flat dimension over S. Finally, the category $\mathbf{M}_{gr}(S)$ is abelian (as S is noetherian) and we have $\mathbf{M}_{gr}(S)/\mathbf{M}^b_{gr}(S) \cong \mathbf{M}(R)$.

It follows that $K\mathbf{M}_{gr}(S)$ and $K\mathbf{M}_{gr}(S/tS) \cong K\mathbf{M}^b_{gr}(S)$ are both isomorphic to $G(k)[\sigma, \sigma^{-1}]$ by Example 3.5.2 (and Ex. 3.4). The rest of the proof is the same as the proof of the Fundamental Theorem 6.2. (See Ex. 6.12). $\square$

REMARK 6.4.1. If in addition k is regular, then so is R. (This follows from the fact that for $M \in \mathbf{M}(R)$, $\mathrm{pd}_R(M) = \mathrm{fd}_R(M)$; see [Q341, p. 36].) It follows from Theorem 3.3 that $K(k) \simeq K(R)$.

EXAMPLE 6.4.2. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k. Then the universal enveloping algebra $R = U(\mathfrak{g})$ is filtered, with $F_1R = k \oplus \mathfrak{g}$, and its associated graded algebra is a polynomial ring (the symmetric algebra of the vector space $\mathfrak{g}$). Thus Theorem 6.4 implies that $K(k) \simeq K(U(\mathfrak{g}))$.

THEOREM 6.5. *(Gersten) Let $A = k\{X\}$ be a free k-algebra on a set X, where k is noetherian regular. Then $K_*(k\{X\}) \cong G_*(k\{X\}) \cong K_*(k)$.*

PROOF. It is known (see [CLL]) that A is a coherent regular ring (3.3), *i.e.,* that the category $\mathbf{M}(A)$ of pseudo-coherent modules is abelian. Hence $K_*(A) \cong G_*(A)$ by Theorem 3.3. Replacing k by $k[t]$, we see that the same is true of $A[t]$.

Now A has the set of all words as a k-basis; we grade A by word length and consider the category of graded pseudo-coherent A-modules. The proof of 6.4 goes through (Exercise 6.13) to prove that $G_*(k) \cong G_*(A)$, as desired. $\square$

As remarked in IV.1.9(iv), the calculation $K_*(\mathbb{Z}\{X\}) = K_*(\mathbb{Z})$ was used by Anderson to show that Swan's definition $K^{Sw}(R)$ of the higher K-theory of R agrees with Quillen's definitions $K_0(R) \times BGL(R)^+$ and $\Omega BQ\mathbf{P}(R)$ (which we saw were equivalent in IV.7.2).

Dedekind Domains

Suppose that R is a Dedekind domain with fraction field F. Then R and F and regular, as are the residue fields $R/\mathfrak{p}$, so $K_*(R) \cong G_*(R)$, etc. Hence the localization sequence of 6.1 with $S = R - \{0\}$ becomes the long exact sequence:

$$(6.6) \qquad \cdots \to K_{n+1}(F) \xrightarrow{\partial} \oplus_{\mathfrak{p}} K_n(R/\mathfrak{p}) \xrightarrow{\oplus(i_{\mathfrak{p}})_*} K_n(R) \to K_n(F) \xrightarrow{\partial} \cdots .$$

Here $\mathfrak{p}$ runs over the nonzero prime ideals of R, and the maps $(i_{\mathfrak{p}})_* : K_n(R/\mathfrak{p}) \to K_n(R)$ are the transfer maps of 3.3.2.

Writing $K_1(R) = R^\times \oplus SK_1(R)$ (see III.1.1.1), the formula 6.1.2 allows us to identify the ending with the sequence $1 \to R^\times \to F^\times \xrightarrow{\text{div}} D(R) \to K_0(R) \to \mathbb{Z} \to 0$ of I.3.6. Therefore we may extract the following exact sequence, which we have already studied in III.6.5:

$$(6.6.1) \qquad \oplus_{\mathfrak{p}} K_2(R/\mathfrak{p}) \to K_2(R) \to K_2(F) \xrightarrow{\partial} \oplus_{\mathfrak{p}} (R/\mathfrak{p})^\times \xrightarrow{\oplus(i_{\mathfrak{p}})_*} SK_1(R) \to 1.$$

We claim that ∂ is the tame symbol of III.6.3 and that the above continues the sequence of III.6.5. Since the $\mathfrak{p}$-component of ∂ factors through the localization $K_2(R) \to K_2(R_{\mathfrak{p}})$ and the localization sequence for $R_{\mathfrak{p}}$, we may suppose that R is a DVR with parameter π. In this case, we know that ∂ in $K_*(R)$-linear, so if $u \in R^\times$ has image $\bar{u} \in R/\mathfrak{p}$ then $\partial\{\pi, u\} = [\bar{u}]$ in $R/\mathfrak{p}^\times$. Similarly, ∂ sends $\{\pi, \pi\} = \{\pi, -1\}$ to $\{\partial\pi, -1\} = [R/\pi] \cdot [-1]$, which is the class of the unit -1. Since every element of $K_2(F)$ is a product of such terms modulo $K_2(R)$, it follows that ∂ is indeed the tame symbol, as claimed.

Here are two special cases of (6.6) which arose in chapter III. First, if all of the residue fields $R/\mathfrak{p}$ are finite, then $K_2(R/\mathfrak{p}) = 0$ and we obtain the exact sequence:

$$0 \to K_2(R) \to K_2(F) \xrightarrow{\partial} \oplus_{\mathfrak{p}} (R/\mathfrak{p})^\times \xrightarrow{\oplus(i_{\mathfrak{p}})_*} SK_1(R) \to 1.$$

Corollary 6.6.2. *If R is any semilocal Dedekind domain, $K_3(F) \xrightarrow{\partial} \oplus K_2(R/\mathfrak{p})$ is onto, and we obtain the exact sequence*

$$0 \to K_2(R) \to K_2(F) \xrightarrow{\partial} \oplus_{\mathfrak{p}}(R/\mathfrak{p})^\times \to 1.$$

Proof. It suffices to lift a symbol $\{\bar{a}, \bar{b}\} \in K_2(R/\mathfrak{p})$. We can lift $\bar{a}, \bar{b}$ to units a, b of R as R is semilocal. Choose $s \in R$ so that $R/sR = R/\mathfrak{p}$. Since ∂ is a $K_*(R)$-module homomorphism, we have:

$$\partial(\{a, b, s\}) = \{a, b\}\partial(s) = \{a, b\} \cdot [R/sR] = (i_{\mathfrak{p}})_*\{a, b\} = \{\bar{a}, \bar{b}\}. \qquad \square$$

Suppose that $R \subset R'$ is an inclusion of Dedekind domains, with R' finitely generated as an R-module. Then the fraction field F' of R' is finite over F, so the exact functors $\mathbf{M}(R') \to \mathbf{M}(R)$ and $\mathbf{M}(F') \to \mathbf{M}(F)$ inducing the transfer maps (IV.6.3.3) are compatible. Thus we have a homotopy commutative diagram

$$\text{(6.6.3)} \qquad \begin{array}{ccccc} K(\mathbf{M}_{\text{tors}}R') & \longrightarrow & G(R') & \longrightarrow & G(F') \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathbf{M}_{\text{tors}}R) & \longrightarrow & G(R) & \longrightarrow & G(F). \end{array}$$

Taking homotopy groups yields the morphism of localization sequences (6.6):

$$\text{(6.6.4)} \qquad \begin{array}{ccccccc} K_n(R') & \longrightarrow & K_n(F') & \xrightarrow{\oplus\partial_{\mathfrak{p}'}} & \oplus_{\mathfrak{p}'}K_{n-1}(R'/\mathfrak{p}') & \longrightarrow & K_{n-1}(R') \\ \downarrow{\scriptstyle N_{R'/R}} & & \downarrow{\scriptstyle N_{F'/F}} & & \downarrow{\scriptstyle \bar{N}=\oplus N_{\mathfrak{p}'/\mathfrak{p}}} & & \downarrow \\ K_n(R) & \longrightarrow & K_n(F) & \xrightarrow{\partial} & \oplus_{\mathfrak{p}}K_{n-1}(R/\mathfrak{p}) & \longrightarrow & K_{n-1}(R). \end{array}$$

If R is a discrete valuation ring with fraction field F, parameter s and residue field $R/sR = k$, we define the *specialization map* $\lambda_s : K_n(F) \to K_n(k)$ by $\lambda_s(a) = \partial(\{s, a\})$. A different choice of parameter will yield a different specialization map: if $u \in R^\times$ is such that $s' = us$ then $\lambda_{s'}(a) = \lambda_s(a) + \{\bar{u}, a\}$.

Theorem 6.7. *If a discrete valuation ring R contains a field k_0, and $[k : k_0]$ is finite, then the localization sequence (6.6) breaks up into split exact sequences:*

$$0 \to K_n(R) \to K_n(F) \underset{\leftarrow}{\xrightarrow{\partial}} K_{n-1}(k) \to 0.$$

Moreover, the canonical map $K_n(R) \to K_n(k)$ factors through the specialization map $K_n(F) \to K_n(k)$. A similar assertion holds for K-theory with coefficients.

Proof. Suppose first that $k \subset R$, so that (6.1) is a sequence of $K_*(k)$-modules. Consider the map $K_n(k) \to K_{n+1}(F)$ sending a to $\{s, a\}$; we have $\partial(\{s, a\}) = \{\partial(s), a\} = [k] \cdot a = a$. Hence ∂ is a split surjection, and the result follows.

In general, there is a finite field extension F' of F so that the integral closure R' of R contains k, and $R \to k$ extends to a map $R' \to k$ with kernel $\mathfrak{p} = tR'$. Then the map $\partial' : K_{n+1}(R'[1/t]) \to K_n(k)$ is a surjection, split by $a \mapsto \{t, a\}$, by the same argument. Now the composition $\mathbf{M}(R'/tR') \to \mathbf{M}(R') \to \mathbf{M}(R)$ is just $\mathbf{M}(k) \to \mathbf{M}(R)$ so, as in (6.6.4), we have a morphism of localization sequences

$$\begin{array}{ccccc} K_n(R') & \longrightarrow & K_n(R'[1/t]) & \underset{\leftarrow}{\overset{\partial'}{\longrightarrow}} & K_{n-1}(k) \\ \downarrow & & \downarrow & & \| \\ K_n(R) & \longrightarrow & K_n(F) & \overset{\partial}{\longrightarrow} & K_{n-1}(k). \end{array}$$

It follows that ∂ is also a split surjection. $\square$

Corollary 6.7.1. *If k is a field, the localization sequence (6.6) for $k[t] \subset k(t)$ breaks up into split short exact sequences:*

$$0 \to K_n(k) \underset{\leftarrow}{\longrightarrow} K_n(k(t)) \overset{\partial}{\to} \oplus_{\mathfrak{p}} K_{n-1}(k[t]/\mathfrak{p}) \to 0.$$

Proof. When $R = k[t]_{(t)}$, 6.7 says that the localization sequence for $R \subset k(t)$ breaks up into split exact sequences $0 \to K_n(R) \to K_n(k(t)) \to K_{n-1}(k) \to 0$. In addition, the map $K_n(k) \cong K_n(k[t]) \to K_n(R)$ is split by $(t = 0)^*$, so the localization sequence for $k[t] \to R$ also breaks up into the split exact sequences $0 \to K_n(k) \to K_n(R) \overset{\partial}{\to} \oplus_{\mathfrak{p}\neq 0} K_{n-1}(k[t]/\mathfrak{p}) \to 0$. Combining these split exact sequences yields the result. $\square$

We remark that if R contains any field then it always contains a field k_0 so that k is algebraic over k_0. Thus the argument used to prove 6.7 actually proves more.

Corollary 6.7.2. *If a discrete valuation ring R contains a field k_0, then each $K_{n+1}(F) \overset{\partial}{\to} K_n(k)$ is onto, and the localization sequence (6.6) breaks up into short exact sequences. (The sequences may not split.)*

Proof. We may assume that k is algebraic over k_0, so that every element $a \in K_n(k)$ comes from $a \in K_n(k')$ for some finite field extension k' of k_0. Passing to a finite field extension F' of F containing k', such that $R \to k$ extends to R' as above, consider the composite $\gamma : K_n(k') \to K_{n+1}(F') \to K_{n+1}(F)$ sending a' to the transfer of $\{t, a'\}$. As in the proof of Theorem 6.7, $\partial\gamma : K_n(k') \to K_n(k)$ is the canonical map, so $a = \partial\gamma(a')$. $\square$

Write i and π for the maps $R \to F$ and $R \to k$, respectively.

Lemma 6.7.3. *The composition of $i^* : K_n(R) \to K_n(F)$ and λ_s is the natural map $\pi^* : K_n(R) \to K_n(k)$. A similar assertion holds for K-theory with coefficients.*

Proof. Because the localization sequence (6.6) is a sequence of $K_*(R)$-modules, for every $a \in K_n(R)$ we have

$$\lambda_s(i^* a) = \partial(\{s, a\}) = \{\partial(s), a\} = [k] \cdot a = \pi^*(a). \qquad \square$$

COROLLARY 6.7.4. *If k is an algebraically closed field, and A is any commutative k-algebra, the maps $K_n(k) \to K_n(A)$ and $K_n(k;\mathbb{Z}/m) \to K_n(A;\mathbb{Z}/m)$ are injections.*

PROOF. Choosing a map $A \to F$ to a field, we may assume that $A = F$. Since F is the union of its finitely generated subfields F_α and $K_*(F) = \varinjlim K_*(F_\alpha)$, we may assume that F is finitely generated over k. We may now proceed by induction on the transcendence degree of F over k. It is a standard fact that F is the fraction field of a discrete valuation ring R, with residue field $E = R/sR$. By Lemma 6.7.3, the composition of $K_*(k) \to K_*(F)$ with the specialization map λ_s is the natural map $K_n(k) \to K_n(E)$, which is an injection by the inductive hypothesis. It follows that $K_*(k) \to K_*(F)$ is an injection. $\square$

To prove that $K_*(\mathbb{Z})$ injects into $K_*(\mathbb{Q})$ it is useful to generalize to integers in arbitrary global fields. Recall that a *global field* is either a number field or the function field of a curve over a finite field.

THEOREM 6.8. *(Soulé) Let R be a Dedekind domain whose field of fractions F is a global field. Then $K_n(R) \cong K_n(F)$ for all odd $n \geq 3$; for even $n \geq 2$ the localization sequence breaks up into exact sequences:*

$$0 \to K_n(R) \to K_n(F) \to \bigoplus\nolimits_{\mathfrak{p}} K_{n-1}(R/\mathfrak{p}) \to 0.$$

PROOF. Let $SK_n(R)$ denote the kernel of $K_n(R) \to K_n(F)$; from (6.6), it suffices to prove that $SK_n(R) = 0$ for $n \geq 1$. For $n = 1$ this is the Bass-Milnor-Serre Theorem III.2.5 (and III.2.5.1). From the computation of $K_n(\mathbb{F}_q)$ in IV.1.13 and the fact that $K_n(R)$ is finitely generated (IV.6.9), we see that $SK_n(R)$ is 0 for $n > 0$ even, and is finite for n odd. Fixing $n = 2i-1 \geq 1$, we may choose a positive integer ℓ annihilating the (finite) torsion subgroup of $K_n(R)$. Thus $SK_n(R)$ injects into the subgroup $K_n(R)/\ell$ of $K_n(R;\mathbb{Z}/\ell)$, which in turn injects into $K_n(F;\mathbb{Z}/\ell)$ by Proposition 6.8.1 below. Since $SK_n(R)$ vanishes in $K_n(F)$ and hence in $K_n(F;\mathbb{Z}/\ell)$, this forces $SK_n(R)$ to be zero. $\square$

PROPOSITION 6.8.1. *(Soulé) Let R be a Dedekind domain whose field of fractions F is a global field. Then for each ℓ and each even $n \geq 2$, the boundary map $\partial : K_n(F;\mathbb{Z}/\ell) \to \oplus K_{n-1}(R/\mathfrak{p};\mathbb{Z}/\ell)$ is onto in the localization sequence with coefficients $\mathbb{Z}/\ell$.*

The conclusion of 6.8.1 is false for $n = 1$. Indeed, the kernel of $K_0(R) \to K_0(F)$ is the finite group $\mathrm{Pic}(R)$, so $K_1(F;\mathbb{Z}/\ell) \to \oplus K_0(R/\mathfrak{p};\mathbb{Z}/\ell)$ is not onto.

For $n = 2$, it is easy to see that the map $\partial : K_2(F;\mathbb{Z}/\ell) \to \oplus K_1(R/\mathfrak{p};\mathbb{Z}/\ell)$ is onto, because $K_1(R;\mathbb{Z}/\ell)$ injects into $K_1(F;\mathbb{Z}/\ell)$ by Ex. IV.2.3.

PROOF. Since $K_{n-1}(R/\mathfrak{p})/2\ell = K_{n-1}(R/\mathfrak{p};\mathbb{Z}/2\ell)$ surjects onto $K_{n-1}(R/\mathfrak{p})/\ell = K_{n-1}(R/\mathfrak{p};\mathbb{Z}/\ell)$ for each $\mathfrak{p}$, we may increase ℓ to assume that $\ell \not\equiv 2 \pmod 4$, so that the product (IV.2.8) is defined on $K_*(R;\mathbb{Z}/\ell)$.

Suppose first that R contains a primitive ℓ^{th} root of unity, ζ_ℓ. If $\beta \in K_2(R;\mathbb{Z}/\ell)$ is the Bott element (IV.2.5.2), then multiplication by β^{i-1} induces an isomorphism $\oplus\mathbb{Z}/\ell \cong \oplus K_1(R/\mathfrak{p})/\ell \to \oplus K_{2i-1}(R/\mathfrak{p})$ by IV.1.13. That is, every element of

$\oplus K_{2i-1}(R/\mathfrak{p})$ has the form $\beta^{i-1}a$ for a in $\oplus K_1(R/\mathfrak{p})$. Lifting a to $s \in K_2(F)$, the element $x = \beta^{i-1}s$ of $K_{2i}(F;\mathbb{Z}/\ell)$ satisfies $\partial(x) = \beta^{i-1}\partial(x) = \beta^{i-1}a$, as desired.

In the general case, we pass to the integral closure R' of R in the field $F' = F(\zeta_\ell)$. Every prime ideal $\mathfrak{p}$ of R has a prime ideal $\mathfrak{p}'$ of R' lying over it, and the transfer maps $K_{2i-1}(R'/\mathfrak{p}') \to K_{2i-1}(R/\mathfrak{p})$ are all onto by IV.1.13. The surjectivity of ∂ follows from the commutative diagram

$$\begin{tikzcd} K_{2i}(F';\mathbb{Z}/\ell) \arrow[r, "\partial'", "\text{onto}"'] \arrow[d] & \oplus_{\mathfrak{p}'} K_{2i-1}(R'/\mathfrak{p}';\mathbb{Z}/\ell) \arrow[d, "\text{onto}"] \\ K_{2i}(F;\mathbb{Z}/\ell) \arrow[r, "\partial"] & \oplus_{\mathfrak{p}} K_{2i-1}(R/\mathfrak{p};\mathbb{Z}/\ell). \end{tikzcd} \qquad \square$$

Wild Kernels 6.8.2. For even n, the map $K_n(R;\mathbb{Z}/\ell) \to K_n(F;\mathbb{Z}/\ell)$ need not be an injection. In fact the subgroup $\mathrm{div}K_n(F)$ of elements of $K_n(F)$ which map to zero in *each* quotient $K_n(F)/\ell \cdot K_n(F)$ lies in the torsion subgroup T of $K_n(R)$ (by 6.8.1) and if $\ell \cdot T = 0$ then $\mathrm{div}K_n(F)$ is the kernel of $K_n(R;\mathbb{Z}/\ell) \to K_n(F;\mathbb{Z}/\ell)$. Tate observed that $\mathrm{div}K_n(F)$ can be nonzero even for K_2. In fact, $\mathrm{div}K_{2i}(F)$ is isomorphic to the *wild kernel*, defined as the intersection (over all valuations v on F) of the kernels of all maps $K_{2i}(F) \to K_{2i}(F_v)$. This is proven in [We06].

Gersten's DVR Conjecture 6.9. Suppose R is a discrete valuation domain with maximal ideal $\mathfrak{m} = sR$, residue field $k = R/\mathfrak{m}$ and field of fractions $F = R[s^{-1}]$. The localization sequence (6.6) becomes

$$\cdots \xrightarrow{\partial} K_n(k) \xrightarrow{i_*} K_n(R) \to K_n(F) \xrightarrow{\partial} K_{n-1}(k) \xrightarrow{i_*} K_{n-1}(R) \to \cdots .$$

Gersten conjectured that this sequence splits up (for every discrete valuation ring) into short exact sequences

$$0 \to K_n(R) \to K_n(F) \to K_{n-1}(k) \to 0.$$

This conjecture is known for $n = 0, 1, 2$ (see 6.6.2), when $\mathrm{char}(F) = \mathrm{char}(k)$ (6.7 ff) or when k is algebraic over $\mathbb{F}_p$ (6.9.2 and Ex. 6.11). It is not known in the general mixed characteristic case, *i.e.*, when $\mathrm{char}(F) = 0$ and $\mathrm{char}(k) = p$. However, it does hold for K-theory with coefficients $\mathbb{Z}/\ell$, as we now show.

Theorem 6.9.1. *If R is a discrete valuation ring with residue field k, then the localization sequence with coefficients breaks up:*

$$0 \to K_n(R;\mathbb{Z}/\ell) \to K_n(F;\mathbb{Z}/\ell) \xrightarrow{\partial} K_{n-1}(k;\mathbb{Z}/\ell) \to 0.$$

Proof. We begin with the case when $1/\ell \in k$, which is due to Gillet. The *henselization* R^h of R is a union of localizations $R'_{\mathfrak{m}'}$ over semilocal Dedekind domains R' which are étale over R such that $R'/\mathfrak{m}' \cong k$; see [Milne, I.4.8]. By Gabber Rigidity IV.2.10, $K_*(k;\mathbb{Z}/\ell) \cong K_*(R^h;\mathbb{Z}/\ell) \cong \varinjlim K_*(R'_{\mathfrak{m}'};\mathbb{Z}/\ell)$. Hence for each $a \in K_{n-1}(k;\mathbb{Z}/\ell)$ there is an $(R', \mathfrak{m}')$ with $R'/\mathfrak{m}' \cong k$ and an $a' \in K_{n-1}(R'_{\mathfrak{m}'};\mathbb{Z}/\ell)$

mapping to a. If $s' \in R'$ generates $\mathfrak{m}'$ then $F' = R'[1/s']$ is the field of fractions of R', and the product $b = \{a', s'\}$ is an element of $K_n(F';\mathbb{Z}/\ell)$. By (6.6.3), the functor $\mathbf{M}(R') \to \mathbf{M}(R)$ induces a map of localization sequences

$$\begin{array}{ccccc} K_n(R';\mathbb{Z}/\ell) & \longrightarrow & K_n(F';\mathbb{Z}/\ell) & \xrightarrow{\oplus\partial_{\mathfrak{p}}} & \oplus_{\mathfrak{p}} K_{n-1}(R'/\mathfrak{p};\mathbb{Z}/\ell) \\ \downarrow{\scriptstyle N_{R'/R}} & & \downarrow{\scriptstyle N_{F'/F}} & & \downarrow{\scriptstyle \bar{N}=\oplus N_{(R'/\mathfrak{p})/k}} \\ K_n(R;\mathbb{Z}/\ell) & \longrightarrow & K_n(F;\mathbb{Z}/\ell) & \xrightarrow{\partial} & K_{n-1}(k;\mathbb{Z}/\ell). \end{array}$$

Because the top row is a sequence of $K_*(R;\mathbb{Z}/\ell)$-modules (6.1), we have $\partial_{\mathfrak{p}}\{a', s'\} = \{a', \partial_{\mathfrak{p}} s'\}$ for all $\mathfrak{p}$. Since $\partial(s') = [R'/s'R'] = [k]$ by III.1.1, we see that $\partial_{\mathfrak{m}'}\{a', s'\} = \{a', [k]\} = a' \cdot 1 = a$, while if $\mathfrak{p} \neq \mathfrak{m}'$ we have $\partial_{\mathfrak{p}}\{a', s'\} = 0$. Hence $\bar{N} \oplus \partial_{\mathfrak{p}}$ sends $\{a', s'\}$ to a. By commutativity of the right square in the above diagram, we see that $b = N_{F'/F}(\{a', s'\})$ is an element of $K_n(F;\mathbb{Z}/\ell)$ with $\partial(b) = a$. This shows that ∂ is onto, and hence that the sequence breaks up as stated.

Next we consider the case that $\ell = p^\nu$ and k has characteristic p, which is due to Geisser and Levine [GeiL, 8.2]. We will need the following facts from VI.4.7 below, which were also proven in *op. cit.*: the group $K_{n-1}(k)$ has no ℓ-torsion, and $K_n^M(k)/\ell^\nu \to K_n(k;\mathbb{Z}/\ell^\nu)$ is an isomorphism for all ν. By the Universal Coefficient Theorem IV.2.5, this implies that every element of $K_n(k,\ell)$ is the image of a symbol $\{a_1, \dots, a_n\}$ with a_i in $k^\times$. If $a_i' \in R^\times$ is a lift of a_i and $s \in R$ is a parameter then $\partial : K_1(F) \to K_0(k)$ sends s to $[k]$ by III.1.1. It follows that $\partial : K_{n+1}(F;\mathbb{Z}/\ell) \to K_n(k;\mathbb{Z}/\ell)$ is onto, because it sends $\{s, a_1', \dots, a_n'\}$ to $\{a_1, \dots, a_n\}$. $\square$

We can now prove Gersten's DVR Conjecture 6.9 when k is finite; the same proof works if k is algebraic over a finite field (Ex. 6.11).

COROLLARY 6.9.2. *If R is a discrete valuation domain whose residue field k is finite, then for all $i > 0$: $K_{2i-1}(R) \cong K_{2i-1}(F)$ and there is a split exact sequence*

$$0 \to K_{2i}(R) \to K_{2i}(F) \underset{\leftarrow}{\longrightarrow} K_{2i-1}(k) \to 0.$$

PROOF. It suffices to show that the map $K_{2i}(F) \xrightarrow{\partial} K_{2i-1}(k)$ is a split surjection in the localization sequence (6.6), because $K_{2i}(k) = 0$ by IV.1.13. Set $\ell = |k|^i - 1$ and recall from IV.1.13 that $K_{2i-1}(k) \cong \mathbb{Z}/\ell$, and hence $K_{2i}(k;\mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$. Pick a generator b of this group; using the surjection $K_{2i}(F;\mathbb{Z}/\ell) \to K_{2i}(F;\mathbb{Z}/\ell)$ of 6.9.1, lift b to an element a of $K_{2i}(F;\mathbb{Z}/\ell)$. By compatibility of localization sequences (Ex. 5.2), the image of a under $K_{2i}(F;\mathbb{Z}/\ell) \to K_{2i-1}(F)$ is sent by ∂ to b. $\square$

We can now determine the K-theory with coefficients of local fields.

PROPOSITION 6.10. *Let E be a local field, finite over $\mathbb{Q}_p$, and let π be a parameter for the ring of integers in E. If the residue field is $\mathbb{F}_q$ and $q \equiv 1 \pmod{m}$, then $K_*(E;\mathbb{Z}/m)$ is a free $\mathbb{Z}/m[\beta]$-module on generators 1 and z, where β is the Bott element and z is the class of π in $K_1(E;\mathbb{Z}/m) = E^\times/E^{\times m}$.*

Since $\{\pi, \pi\} = \{-1, \pi\}$ in $K_2(E)$ (III.5.10.2), the ring structure is given by $z^2 = [-1] \cdot z$. If $q = 2^\nu$ or if $q \equiv 1 \pmod{2m}$ then $[-1] = 0$ in $K_1(E)/m$ so $z^2 = 0$.

PROOF. Since the ring R of integers in E is a complete DVR and $R/\pi R \cong \mathbb{F}_q$, Gabber rigidity (IV.2.9) implies that $K_*(R;\mathbb{Z}/m) \cong K_*(\mathbb{F}_q;\mathbb{Z}/m) \cong \mathbb{Z}/m[\beta]$. By 6.9.2, $K_*(E;\mathbb{Z}/m)$ is a free $\mathbb{Z}/m[\beta]$-module on generators 1 and z. $\square$

REMARK 6.10.1. We will see in VI.7.4.1 that when $n = p^\nu$ we have

$$K_{2i-1}(E;\mathbb{Z}/n) \cong (\mathbb{Z}/n)^d \oplus \mathbb{Z}/m_i \oplus \mathbb{Z}/m_{i-1},$$

where $d = [E : \mathbb{Q}_p]$, $m_i = \min\{n, w_i^{(p)}\}$ and the numbers $w_i^{(p)}$ are defined in VI, 2.2 and 2.3. By Moore's Theorem III.6.2.4, the torsion subgroup of $K_2(E)$ is the group of roots of unity in E, cyclic of order w_1, so $K_3(E)$ must be the direct sum of $\mathbb{Z}/w_2^{(p)}$ and an extension of a p-divisible group by $\mathbb{Z}_{(p)}^d$.

I do not know how to reconstruct the other groups $K_*(E)$ from the above information; there might be a $\mathbb{Z}_{(p)}^d$ in all odd degrees, or there might be divisible p-torsion in even degrees.

EXAMPLE 6.10.2. Consider the union E_q of the local fields E over $\mathbb{Q}_p$ whose ring of integers R has residue field $\mathbb{F}_q$. For each such E, $K_*(E;\mathbb{Z}/m)$ is described by Proposition 6.10. If $E \subset E'$ is a finite field extension, with ramification index e, the map $K_*(E;\mathbb{Z}/m) \to K_*(E';\mathbb{Z}/m)$ sends z_E to $e\, z_{E'}$. If m divides e, the map sends z_E to 0. Taking the direct limit over all E, we see that $K_*(E_q;\mathbb{Z}/m) = \mathbb{Z}/m[\beta]$.

Localization for Schemes

EXAMPLE 6.11. Let X be a noetherian scheme, $j : U \subset X$ an open subscheme, and $Z = X - U$ the closed complement. In this case we take $\mathcal{A} = \mathbf{M}(X)$ and $\mathcal{B} = \mathbf{M}_Z(X)$ the category of coherent X-modules supported on Z, *i.e.,* modules whose restriction to U is zero. Gabriel has shown that $\mathbf{M}(X)/\mathbf{M}_Z(X)$ is the category $\mathbf{M}(U)$ of coherent U-modules; see II.6.4.2. By Devissage (Ex. 4.3), $K\mathbf{M}_Z(X) \simeq G(Z)$. Hence we have a homotopy fibration $G(Z) \to G(X) \to G(U)$, and the localization sequence becomes:

$$\cdots \xrightarrow{\partial} G_n(Z) \to G_n(X) \xrightarrow{j^*} G_n(U) \xrightarrow{\partial} G_{n-1}(Z) \to \cdots$$

ending in the exact sequence $G_0(Z) \to G_0(X) \to G_0(U) \to 0$ of II.6.4.2. This is a sequence of $K_*(X)$-modules, because $\otimes$ is a biexact pairing of $\mathbf{VB}(X)$ with the sequence $\mathbf{M}(Z) \to \mathbf{M}(X) \to \mathbf{M}(U)$.

EXAMPLE 6.11.1. If R is commutative noetherian, $U = \mathrm{Spec}(R[s])$ is an open subset of the line $\mathbb{P}^1_R$ with complement $\infty : \mathrm{Spec}(R) \hookrightarrow \mathbb{P}^1_R$. Since $\pi : \mathbb{P}^1_R \to \mathrm{Spec}(R)$ is flat, the map $\pi^* : G(R) \to G(\mathbb{P}^1_R)$ exists and $j^*\pi^* \simeq p^*$ is the homotopy equivalence of Theorem 6.2. It follows that π^* splits the localization sequence $G(R) \xrightarrow{\infty_*} G(\mathbb{P}^1_R) \xrightarrow{j^*} G(R)$ and hence we have $G_n(\mathbb{P}^1_R) \cong G_n(R) \oplus G_n(R)$.

MAYER-VIETORIS SEQUENCES 6.11.2. If $X = U \cup V$ then $Z = X - U$ is contained in V and $V - Z = U \cap V$. Comparing the two localization sequences, we see that the square

$$\begin{array}{ccc} G(X) & \longrightarrow & G(U) \\ \downarrow & & \downarrow \\ G(V) & \longrightarrow & G(U \cap V). \end{array}$$

is homotopy cartesian, *i.e.,* there is a homotopy fibration sequence $G(X) \to G(U) \times G(V) \xrightarrow{\pm} G(U \cap V)$ which on homotopy groups yields the long exact "Mayer-Vietoris" sequence

$$\cdots \to G_{n+1}(U \cap V) \xrightarrow{\partial} G_n(X) \xrightarrow{\Delta} G_n(U) \times G_n(V) \xrightarrow{\pm} G_n(U \cap V) \xrightarrow{\partial} \cdots .$$

Smooth Curves 6.12. Suppose that X is an irreducible curve over a field k, with function field F. For each closed point $x \in X$, the field $k(x)$ is a finite field extension of k. The category $\mathbf{M}_0(X)$ of coherent torsion modules (modules of finite length) is a Serre subcategory of $\mathbf{M}(X)$, and $\mathbf{M}(F) \cong \mathbf{M}(X)/\mathbf{M}_0(X)$ (II.6.4.2). By Devissage (4.3), $K_*\mathbf{M}_0(X) \cong \oplus_x K_*(k(x)) = \oplus K_*(x)$. In this case, the Localization sequence (5.1.1) becomes:

$$\cdots \to K_{n+1}(F) \xrightarrow{\partial} \oplus_x K_n(k(x)) \xrightarrow{\oplus(i_x)_*} G_n(X) \to K_n(F) \xrightarrow{\partial} \cdots .$$

Here the maps $(i_x)_* : K_n(x) \to G_n(X)$ are the finite transfer maps of 3.6 associated to the inclusion of $x = \mathrm{Spec}(k(x))$ into X. If X is regular, then $K_n(X) \cong G_n(X)$ and this sequence tells us about $K_n(X)$.

The residue fields $k(x)$ of X are finite field extensions of k. As such, we have transfer maps $N_{k(x)/k} : K_n(k(x)) \to K_n(k)$. The following result, due to Gillet, generalizes the Weil Reciprocity of III.6.5.3 for symbols $\{f, g\} \in K_2(F)$. We write ∂_x for the component $K_{n+1}(F) \to K_n(x)$ of the map ∂ in 6.12.

Weil Reciprocity 6.12.1. *Let X be a projective curve over a field k, with function field F. For every $a \in K_{n+1}(F)$ we have the following formula in $K_n(k)$:*

$$\sum\nolimits_{x \in X} N_{k(x)/k} \partial_x(a) = 0.$$

Proof. Consider the proper transfer $\pi_* : G_n(X) \to G_n(k)$ of 3.7 associated to the structure map $\pi : X \to \mathrm{Spec}(k)$. By functoriality, $\pi_*(i_x)_* = N_{k(x)/k}$ for each closed point $x \in X$. Because $\oplus_x (i_x)_* \partial_x = (\oplus(i_x)_*)\partial = 0$, we have:

$$\sum N_{k(x)/k} \partial_x(a) = \sum \pi_*(i_x)_* \partial_x(a) = \pi_* \sum (i_x)_* \partial_x(a) = 0. \qquad \square$$

Fundamental Theorem for $G(X)$ 6.13. *If X is a noetherian scheme, the flat maps $X[s, s^{-1}] \overset{j}{\hookrightarrow} X[s] \xrightarrow{p} X$ induce a homotopy equivalence $p^* : G(X) \simeq G(X[s])$ and isomorphisms*

$$G_n(X[s, s^{-1}]) \cong G_n(X) \oplus G_{n-1}(X).$$

PROOF. If $z : X \to X[s]$ is the zero-section, we saw in 3.6.1 that $z_* = 0$. Hence the Localization Sequence 6.11 splits into short exact sequences

$$0 \xrightarrow{z_*} G_n(X[s]) \xrightarrow{j^*} G_n(X[s,s^{-1}]) \xrightarrow{\partial} G_{n-1}(X) \xrightarrow{z_*} 0.$$

If $f : X \to X[s,s^{-1}]$ is the section $s = 1$, we saw in 3.6.1 that f defines a map f^* and that $f^*j^*p^*$ is homotopic to the identity of $G(X)$. Hence it suffices to prove that $p^* : G(X) \simeq G(X[s])$.

We first suppose that X is separated, so that the intersection of affine opens is affine, and proceed by induction on the number of affine opens. If X is affine, this is Theorem 6.2, so suppose that $X = U_1 \cup U_2$ with U_2 affine. The inductive hypothesis applies to U_1 and $U_{12} = U_1 \cap U_2$, and we have a map of Mayer-Vietoris sequences (6.11.2):

$$\begin{tikzcd}
* \arrow[r] \arrow[d, "\simeq"] & G_{n+1}(U_{12}) \arrow[r, "\partial"] \arrow[d, "\simeq"'] & G_n(X) \arrow[r] \arrow[d] & G_n(U_1)\times G_n(U_2) \arrow[r] \arrow[d, "\simeq"'] & G_n(U_{12}) \arrow[d, "\simeq"'] \\
* \arrow[r] & G_{n+1}(U_{12}[s]) \arrow[r, "\partial"] & G_n(X[s]) \arrow[r] & G_n(U_1[s])\times G_n(U_2[s]) \arrow[r] & G_n(U_{12}[s]).
\end{tikzcd}$$

The 5-lemma implies that $G(X) \simeq G(X[s])$ for X separated. Another induction establishes the result for non-separated X, since any open subscheme of a separated scheme is separated. $\square$

COROLLARY 6.13.1. *If X is noetherian, then $X \times \mathbb{P}^1 \xrightarrow{\pi} X$ and $X \xrightarrow{\infty} X \times \mathbb{P}^1$ induce homotopy equivalences $(\pi^*, \infty_*) : G(X) \oplus G(X) \xrightarrow{\simeq} G(X \times \mathbb{P}^1)$.*

PROOF. As in 6.11.1, the complement of $\infty(X)$ is $X \times \mathbb{A}^1$ and the map π^* is split by $j^* : G(X \times \mathbb{P}^1) \to G(X \times \mathbb{A}^1) \simeq G(X)$. Hence π^* splits the localization sequence $G(X) \xrightarrow{\infty_*} G(X \times \mathbb{P}^1) \xrightarrow{j^*} G(X)$. $\square$

COROLLARY 6.13.2. *If X is regular noetherian then for all n: $K_n(X) \cong K_n(X[s])$, $K_n(X[s,s^{-1}]) \cong K_n(X) \oplus K_{n-1}(X)$ and $K_n(X \times \mathbb{P}^1) \cong K_n(X) \oplus K_n(X)$.*

We now generalize 6.11.1 and 6.13.1 from $\mathbb{P}^1$ to $\mathbb{P}^r$; the G_0 version of the following result was given in II, Ex. 6.14. If X is regular, Theorem 6.14 and Ex. 6.7 provide another proof of the Projective Bundle Theorem 1.5 and 1.5.1.

THEOREM 6.14. *If X is a noetherian scheme, the functors $\mathbf{M}(X) \xrightarrow{\mathcal{O}(i)} \mathbf{M}(\mathbb{P}^r_X)$ sending M to $M \otimes \mathcal{O}(i)$ induce an isomorphism $\sum \mathcal{O}(i) \colon \coprod_{i=0}^r G(X) \xrightarrow{\simeq} G(\mathbb{P}^r_X)$. That is, $G_n(\mathbb{P}^r_X) \cong G_n(X) \otimes K_0(\mathbb{P}^r_{\mathbb{Z}})$.*

PROOF. By localization, we may assume $X = \mathrm{Spec}(R)$; set $S = R[x_0, \cdots, x_r]$. The standard description of coherent sheaves on $\mathbb{P}^r_R$ ([Hart, Ex. II.5.9]) amounts to the equivalence of quotient categories

$$\mathbf{M}(\mathbb{P}^r_R) \cong \mathbf{M}_{gr}(S)/\mathbf{M}^b_{gr}(S),$$

where $\mathbf{M}^b_{gr}(S)$ is the Serre subcategory of graded modules M with $M_m = 0$ for all but finitely many m. By Devissage (Ex. 4.4), $\mathbf{M}^b_{gr}(S)$ has the same K-theory as its subcategory $\mathbf{M}_{gr}(R)$, *i.e.*, $G_*(R)[\sigma,\sigma^{-1}]$. Thus the localization sequence (5.1.1) and Example 3.5.2 give us the following diagram with exact rows

$$\begin{array}{ccccccccc}
\cdots \to & K_n(\mathbf{M}_{gr}(R)) & \to & K_n(\mathbf{M}_{gr}(R[x_0,\cdots,x_r])) & \to & G_n(\mathbb{P}^r_R) & \to & \cdots \\
& \simeq\uparrow & & \simeq\uparrow & & \uparrow & & \\
0 \to & G_n(R)\otimes\mathbb{Z}[\sigma,\sigma^{-1}] & \xrightarrow{h} & G_n(R)\otimes\mathbb{Z}[\sigma,\sigma^{-1}] & \to & G_n(R) & \to & 0.
\end{array}$$

Now σ is induced by $\otimes_S S(-1)$. Setting $E = S^{r+1}$, consider the Koszul resolution for R (I.5.3):

$$0 \to \wedge^{r+1}E(-r-1) \to \cdots \to \wedge^2 E(-2) \to E(-1) \to S \to R \to 0.$$

The Additivity Theorem for $\otimes_S S : \mathbf{M}_{gr}(S) \to \mathbf{M}_{gr}(S)$ shows that h is multiplication by the map $\sum \binom{r+1}{i}(-\sigma)^i = (1-\sigma)^{r+1}$. Since this is an injection with the prescribed cokernel, we have proven the first assertion. The second is immediate from II.8.6. $\square$

EXERCISES

6.1 Suppose that R is a 1-dimensional commutative noetherian domain with fraction field F. Show that there is a long exact sequence

$$\cdots \to K_{n+1}(F) \xrightarrow{\partial} \oplus_{\mathfrak{p}} K_n(R/\mathfrak{p}) \to G_n(R) \to K_n(F) \xrightarrow{\partial} \cdots .$$

ending in the Heller-Reiner sequence of Ex. II.6.8. If R is regular, i.e., a Dedekind domain, this is the sequence (6.6).

Now let R be the local ring of $\mathbb{R} + x\mathbb{C}[x]$ at the maximal ideal (x, ix). Show that $G_0(R) \cong \mathbb{Z}\oplus\mathbb{Z}/2$, and conclude that the localization sequence (6.1.1) does not split up into short exact sequences.

6.2 If S is the seminormalization of R (I, Ex. 3.15–16), and S is finite over R, show that the transfer $G(S) \to G(R)$ is an equivalence.

6.3 If R is noetherian, show that the splitting in Theorem 6.2 is given by the map $G_n(R) \to G_{n+1}(R[s,s^{-1}])$ which is multiplication by $s \in K_1(\mathbb{Z}[s,s^{-1}])$. *Hint:* Use Ex. 5.3 with $\mathcal{C} = \mathbf{M}(R)$.

6.4 Let φ be an automorphism of a commutative noetherian ring R, and let $R_\varphi[t]$ be the twisted polynomial ring of all polynomials $\sum r_i t^i$ with multiplication determined by $tr = \varphi(r)t$. It is well known that $R_\varphi[t]$ is a regular ring when R is. Show that the hypotheses of Theorem 6.4 are satisfied, so that $G(R) \simeq G(R_\varphi[t])$. Then show that $\{t^n\}$ is a multiplicative system (II.A.1.1), so that the localization $R_\varphi[t, 1/t]$ is a well defined ring. Then show that there is an exact sequence:

$$\cdots \to G_n(R) \xrightarrow{\varphi_*} G_n(R) \to G_n(R_\varphi[t,1/t]) \to \cdots .$$

6.5 Let R be commutative noetherian, and $S = \mathrm{Sym}(P)$ the symmetric algebra (I.5.8) of a finitely generated projective R-module P. Show that $R \subset S$ induces an equivalence $G(R) \simeq G(S)$. *Hint:* Locally, $S \cong R[s_1,...,s_n]$.

6.6 *Jouanolou's Trick.* Let X be a noetherian scheme. If $p : E \to X$ is a vector bundle, show that $p^* : G(X) \to G(E)$ is an equivalence. More generally, show that $p^* : G(X) \to G(E)$ is an equivalence for any flat map $p : E \to X$ whose fibers are affine spaces (such as a torsor under a vector bundle E).

Jouanolou proved that if X is quasi-projective over a field then there is an affine scheme $E = \mathrm{Spec}(R)$ and a torsor $E \to X$ under a vector bundle. It follows that $G(X) \simeq G(R)$. This trick reduces the study of $G(X)$ for quasi-projective X to the study of $G(R)$.

6.7 (Quillen) If E is any vector bundle over X of rank $r+1$, we can form the projective bundle $\mathbb{P}(E) = \mathrm{Proj}(\mathrm{Sym}(E))$. Modify the proof of Theorem 6.14 to show that once again the maps $M \mapsto M \otimes \mathcal{O}(i)$ induce an isomorphism

$$G_n(\mathbb{P}(E)) \cong K_0(\mathbb{P}(E)) \otimes G_n(X) \cong \coprod_{i=0}^{r} G_n(X).$$

6.8 Let X be the affine plane with a double origin over a field k, the standard example of a (quasi-compact) scheme which is not separated. Generalize II.8.2.4 by showing that $G(X) \simeq G(k) \times G(k)$. Then show that every vector bundle is trivial and $\mathbf{VB}(X) \cong \mathbf{VB}(\mathbb{A}^2)$, so $K\mathbf{VB}(X) \simeq K(k)$. Since $G(X) \simeq K(X)$ (by Ex. 3.9), this shows that $K\mathbf{VB}(X) \neq K(X)$. Note that $G(X) = K\mathbf{VB}(X)$ for separated regular schemes by Resolution (see 3.4.2).

6.9 (Roberts) If $\pi : X' \to X$ is a finite birational map, there is a closed $Z \subset X$ (nowhere dense) such that $\pi : X' - Z' \cong X - Z$, where $Z' = Z \times_X X'$.
(a) Show that there is a fibration $G(Z') \to G(Z) \times G(X') \to G(X)$. This yields a Mayer-Vietoris sequence

$$G_{n+1}(X) \xrightarrow{\partial} G_n(Z') \xrightarrow{\Delta} G_n(Z) \times G_n(X') \xrightarrow{\pm} G_n(X) \xrightarrow{\partial} G_{n-1}(Z').$$

(b) If X is the node $\mathrm{Spec}(k[x,y]/(y^2 - x^3 - x))$, show that $G_n(X) \cong G_n(k) \oplus G_{n-1}(k)$ for all n. This contrasts with $K_0(X) \cong K_0(k) \oplus K_1(k)$; see II.2.9.1.

6.10 Suppose that a curve X is a union of n affine lines, meeting in a set Z of m rational points (isomorphic to $\mathrm{Spec}(k)$), and that no three lines meet in a point. Show that $G_n(X) \cong G_n(k)^n \oplus G_{n-1}(k)^m$.

6.11 Let R be a DVR whose residue field k is infinite, but algebraic over $\mathbb{F}_p$. Modify the proof of 6.9.2 to show that Gersten's DVR conjecture 6.9 holds for R.

6.12 Complete the proof of Theorem 6.4 for filtered rings.

6.13 Let $R = k \oplus R_1 \oplus \cdots$ be a graded ring such that R and $R[t]$ are coherent (3.3) and of finite flat dimension over k, and k has finite flat dimension over R. Modify the proof of Theorem 6.4 to show that $k \subset R$ induces $G(k) \simeq G(R)$. *Hint:* $\mathbf{M}_{gr}(S)$ is still an abelian category.

6.14 The formula $\lambda_s(a) = \partial(\{s, a\})$ defines a specialization map for K-theory with coefficients $\mathbb{Z}/m$. Now suppose that k is algebraically closed, and show that: (i) the specialization $\lambda_s : K_n(F; \mathbb{Z}/m) \to K_n(k; \mathbb{Z}/m)$ is independent of the choice of $s \in R$; (ii) if $F = k(t)$ and then the specialization $\lambda_s : K_n(F; \mathbb{Z}/m) \to K_n(k; \mathbb{Z}/m)$ is independent of R and s. *Hint:* If $f \in F$ then $\lambda(f) = 0$ as $K_1(k; \mathbb{Z}/m) = 0$.

§7. Localization for $K_*(R)$ and $K_*(X)$.

When R is a ring and $s \in R$ is a central element, we would like to say something about the localization map $K(R) \to K(R[\frac{1}{s}])$. If R is regular, we know the third term in the long exact sequence from the localization theorem for G_*:

$$\cdots \xrightarrow{\partial} G_n(R/sR) \xrightarrow{i_*} K_n(R) \to K_n(R[1/s]) \xrightarrow{\partial} \cdots .$$

Because R is regular, every R/sR-module has finite projective dimension over R, and i_* is induced from the inclusion $\mathbf{M}(R/sR) \subset \mathbf{M}(R) = \mathbf{H}(R)$. More generally, suppose that S is a central multiplicatively closed set of nonzerodivisors in R, and consider the category $\mathbf{H}_S(R)$ of all S-torsion R-modules M in $\mathbf{H}(R)$.

We saw in II.7.7.4 that the sequence $K_0\mathbf{H}_S(R) \to K_0(R) \to K_0(S^{-1}R)$ is exact. This K_0 exact sequence has the following extension to higher K-theory.

Theorem 7.1. (Localization for Nonzerodivisors). *Let S be a central multiplicatively closed subset of R consisting of nonzerodivisors. Then*

$$K\mathbf{H}_S(R) \to K(R) \to K(S^{-1}R)$$

is a homotopy fibration. Thus there is a long exact sequence

$$\cdots \to K_{n+1}(S^{-1}R) \xrightarrow{\partial} K_n\mathbf{H}_S(R) \to K_n(R) \to K_n(S^{-1}R) \xrightarrow{\partial} \cdots$$

ending in $K_0\mathbf{H}_S(R) \to K_0(R) \to K_0(S^{-1}R)$, the sequence of II.7.7.4.

In this section, we will give a direct proof of Theorem 7.1, following Quillen. We note that an indirect proof is given in Exercise 3.14, using Theorem 2.6.3. In addition, we saw in Ex. 2.9 that the nonzerodivisor hypothesis is necessary. In the general case, the third term $K(R \text{ on } S)$ is more complicated; see Theorem 2.6.3.

Caveat 7.1.1. The map $K_0(R) \to K_0(S^{-1}R)$ is not onto. Instead, the sequence continues with $K_{-1}\mathbf{H}_S(R)$ etc., using negative K-groups. In order to get a spectrum-level fibration, therefore, one needs to use the nonconnective spectra $\mathbf{K}^B(R)$ of IV.10.4 to get nontrivial negative homotopy groups, or else replace $K(S^{-1}R)$ with $K(\mathcal{P})$, as we shall now do (and did in the proofs of II.9.8 and 2.6.3).

Definition 7.2. Let $\mathcal{P}$ denote the exact subcategory of $\mathbf{P}(S^{-1}R)$ consisting of projective modules which are localizations of projective R-modules.

By Cofinality (IV.6.4.1), $K_n\mathcal{P} \cong K_n(S^{-1}R)$ for $n > 0$, and $K_0(R) \to K_0(S^{-1}R)$ factors as a surjection $K_0(R) \to K_0\mathcal{P}$ followed by an inclusion $K_0\mathcal{P} \to K_0(S^{-1}R)$. Hence we may replace $K(S^{-1}R)$ by $K\mathcal{P}$ in Theorem 7.1.

Let $\mathcal{E}$ denote the category of admissible exact sequences in $\mathcal{P}$, as defined in IV.7.3; morphisms are described in (IV.7.3.1), and the target of $A \rightarrowtail B \twoheadrightarrow C$ yields a functor $t : \mathcal{E} \to Q\mathcal{P}$. We write $\mathcal{F}$ for the pullback category $Q\mathbf{P}(R) \times_{Q\mathcal{P}} \mathcal{E}$ whose objects are pairs $(P, A \rightarrowtail B \twoheadrightarrow S^{-1}P)$ and whose morphisms are pairs of compatible morphisms. Since t is a fibered functor (by Ex. IV.7.2), so is $\mathcal{F} \to Q\mathbf{P}(R)$.

The monoidal category $T = \operatorname{iso}\mathbf{P}(R)$ acts fiberwise on $\mathcal{E}$ via the inclusion of T in $\mathcal{E}$ given by IV.7.4.1. Hence T also acts on $\mathcal{F}$, and we may localize at T (IV.4.7.1).

Lemma 7.2.1. *The sequence $T^{-1}\mathcal{F} \to Q\mathbf{P}(R) \to Q\mathcal{P}$ is a homotopy fibration.*

Proof. By IV, Ex. 3.11, the fibers of $T^{-1}\mathcal{F} \to Q\mathbf{P}(R)$ and $T^{-1}\mathcal{E} \to \mathcal{P}$ are equivalent. By Quillen's Theorem B (IV.3.8.1), we have a homotopy cartesian square:

$$\begin{array}{ccc} T^{-1}\mathcal{F} & \longrightarrow & T^{-1}\mathcal{E} \text{ (contractible)} \\ \downarrow & & \downarrow t \\ Q\mathbf{P}(R) & \longrightarrow & Q\mathcal{P}. \end{array}$$

We saw in the proof of Theorem IV.7.1 that $T^{-1}\mathcal{E}$ is contractible (because $\mathcal{E}$ is, by Ex. IV.7.3), whence the result. $\square$

To prove Theorem 7.1, we need to identify $T^{-1}\mathcal{F}$ and $K\mathbf{H}_S(R)$. We first observe that we may work with the exact subcategory $\mathbf{H}_{1,S}$ of $\mathbf{H}_S(R)$ consisting of S-torsion modules of projective dimension ≤ 1. Indeed, the Resolution Theorem (see 3.2.1) implies that $K_*\mathbf{H}_S(R) \cong K_*\mathbf{H}_{1,S}$.

Next, we construct a diagram of categories on which T acts, of the form

$$Q\mathbf{H}_{1,S} \xleftarrow{h} \mathcal{G} \xrightarrow{f} \mathcal{F}$$

and whose localization is $Q\mathbf{H}_{1,S} \xleftarrow{T^{-1}h} T^{-1}\mathcal{G} \xrightarrow{T^{-1}f} T^{-1}\mathcal{F}$.

Definition 7.3. Let $\mathcal{G}$ denote the category whose objects are exact sequences $0 \to K \rightarrowtail P \twoheadrightarrow M \oplus Q$ with P, Q in $\mathbf{P}(R)$ and M in $\mathbf{H}_{1,S}$. The morphisms in $\mathcal{G}$ are isomorphism classes of diagrams (in which the maps in the bottom row are direct sums of maps):

$$\begin{array}{ccccc} K' & \leftarrowtail & K & = & K \\ \downarrow & & \downarrow & & \downarrow \\ P' & = & P' & \rightarrowtail & P \\ \downarrow & & \downarrow & & \downarrow \\ M' \oplus Q' & \twoheadleftarrow & M_1 \oplus Q_1 & \rightarrowtail & M \oplus Q. \end{array}$$

We let $T = \operatorname{iso}\mathbf{P}(R)$ act on $\mathcal{G}$ by $T\square(P \twoheadrightarrow M \oplus Q) = T \oplus P \twoheadrightarrow P \twoheadrightarrow M \oplus Q$.

The functor $f : \mathcal{G} \to \mathcal{F}$ sends $K \rightarrowtail P \twoheadrightarrow M \oplus Q$ to $(Q, T{-}1P \twoheadrightarrow T^{-1}M)$. By inspection, f is compatible with the action of T, so $T^{-1}f$ is defined.

The functor $h : \mathcal{G} \to Q\mathbf{H}_{1,S}$ sends $K \rightarrowtail P \twoheadrightarrow M \oplus Q$ to M. The action of T is fiberwise for h (IV, Ex. 4.11) so h induces a functor $T^{-1}h : T^{-1}\mathcal{G} \to Q\mathcal{P}$.

Proof of Theorem 7.1. The maps $Q\mathbf{H}_{1,S} \xleftarrow{\simeq} T^{-1}\mathcal{G} \xrightarrow{\simeq} T^{-1}\mathcal{F}$ are homotopy equivalences by Lemmas 7.3.1–2. Let g be the composite of these equivalences with $T^{-1}\mathcal{F} \to Q\mathbf{P}(R)$, so that $Q\mathbf{H}_{1,S} \xrightarrow{g} Q\mathbf{P}(R) \to Q\mathcal{P}$ is a homotopy fibration by 7.2.1. In Lemma 7.4, we identify $-g$ with the canonical map, proving the theorem. $\square$

Lemma 7.3.1. *The functor $h : \mathcal{G} \to Q\mathbf{H}_{1,S}$ is a homotopy equivalence. It follows that $\mathcal{G} \to T^{-1}\mathcal{G}$ and $T^{-1}\mathcal{G} \to Q\mathbf{H}_{1,S}$ are also homotopy equivalences.*

Proof. For each M, let $\mathcal{G}_M$ denote the category whose objects are surjections $P \twoheadrightarrow M$, and whose morphisms are admissible monics $P' \rightarrowtail P$ in $\mathbf{P}(R)$ compatible with the maps to M. Choosing a basepoint $P_0 \twoheadrightarrow M$, $P \times_M P'$ is projective and there are natural transformations $(P \twoheadrightarrow M) \leftarrow (P \times_M P' \twoheadrightarrow M) \rightarrow (P_0 \twoheadrightarrow M)$ giving a contracting homotopy for each $\mathcal{G}_M$.

The Segal subdivision $Sub(\mathcal{G}_M)$ of $\mathcal{G}_M$ (IV, Ex. 3.9) is equivalent to the fiber $h^{-1}(M)$ (by Ex. 7.2), so the fibers of h are contractible. Since h is fibered (by Ex. 7.1), Quillen's Theorem A (variation IV.3.7.4) applies to show that h is a homotopy equivalence. In particular, since T acts trivially on $Q\mathbf{H}_{1,S}$ it acts invertibly on $\mathcal{G}$ in the sense of IV.4.7. It follows from Ex. IV.4.6 that $\mathcal{G} \rightarrow T^{-1}\mathcal{G}$ is also a homotopy equivalence. $\square$

For the following lemma, we need the following easily checked fact: if P is a projective R-module, then P is a submodule of $S^{-1}P$, because S consists of central nonzerodivisors. In addition, $S^{-1}P$ is the union of the submodules $\frac{1}{s}P$, $s \in S$.

LEMMA 7.3.2. *The functor $f : \mathcal{G} \rightarrow \mathcal{F}$ is a homotopy equivalence, and hence $T^{-1}f : T^{-1}\mathcal{G} \rightarrow T^{-1}\mathcal{F}$ is too.*

PROOF. We will show that for each Q in $\mathbf{P}(R)$ the map $(pf)^{-1}(Q) \rightarrow p^{-1}(Q)$ is a homotopy equivalence of fibers. Since $p : \mathcal{F} \rightarrow Q\mathbf{P}(R)$ and $pf : \mathcal{G} \rightarrow Q\mathbf{P}(R)$ are fibered (Ex. 7.1), the result will follow from Quillen's Theorem B (IV.3.8.1).

Fix Q and let T_Q be the category whose objects are maps $P \twoheadrightarrow Q$ with P in $\mathbf{P}(R)$, and whose morphisms are module injections $P' \rightarrowtail P$ over Q whose cokernel is S-torsion. Then the functor $Sub(T_Q) \rightarrow (pf)^{-1}(Q)$ which sends $P' \rightarrowtail P$ over Q to $P \twoheadrightarrow Q \oplus (P/P')$ is an equivalence of categories (by Ex. 7.3).

Set $W = S^{-1}Q$, so that $p^{-1}(Q) = \mathcal{E}_W$. Then $Sub(T_Q) \xrightarrow{\sim} (pf)^{-1}(Q) \rightarrow \mathcal{E}_W$ sends $P' \rightarrowtail P$ over Q to $S^{-1}P \rightarrow W$; this factors through the target functor $Sub(T_Q) \rightarrow T_Q$ (which is a homotopy equivalence by Ex. IV.3.9), and the functor $w : T_Q \rightarrow \mathcal{E}_W$ sending $P \twoheadrightarrow Q$ to $S^{-1}P \twoheadrightarrow W$. Thus we are reduced to showing that w is an equivalence. By Quillen's Theorem A (IV.3.7), it suffices to fix V in $\mathcal{P}$ and $E : V \twoheadrightarrow W$ in $\mathcal{E}_W$ and show that w/E is contractible.

Consider the poset Λ of projective R-submodules P of V such that $S^{-1}P \cong V$, and whose image under $V \rightarrow W$ is Q. The evident map from Λ to w/E sending P to $(P, S^{-1}P \twoheadrightarrow W)$ is an equivalence, so we are reduced to showing that Λ is contractible. Fix P and P' in Λ and let K, K' denote the kernels of $P \twoheadrightarrow Q$ and $P' \twoheadrightarrow Q$. Then $S^{-1}K = S^{-1}K'$ and $S^{-1}K + P = S^{-1}K' + P'$ as submodules of V. Thus for some $s \in S$, both P and P' are contained in the submodule $P'' = P + \frac{1}{s}K$ of V. But P'' is easily seen to be in Λ, proving that Λ is a filtering poset. Thus Λ and hence w/E are contractible. $\square$

LEMMA 7.4. *The canonical map $Q\mathbf{H}_{1,S} \rightarrow Q\mathbf{P}(R)$ is homotopic to the additive inverse of $g : Q\mathbf{H}_{1,S} \xleftarrow{T^{-1}h} T^{-1}\mathcal{G} \xrightarrow{T^{-1}f} T^{-1}\mathcal{F} \xrightarrow{p} Q\mathbf{P}(R)$.*

PROOF. Recall from II.7.7 that $\mathbf{H}_1 = \mathbf{H}_1(R)$ denotes the category of R-modules M having a resolution $0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ with the P_i in $\mathbf{P}(R)$. By the Resolution Theorem 3.1.1, $K(R) \simeq K\mathbf{H}_1$ and $Q\mathbf{P}(R) \simeq Q\mathbf{H}_1$. We claim that the following diagram commutes up to sign.

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{p \circ f} & Q\mathbf{P}(R) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \simeq} \\ Q\mathbf{H}_{1,S}(R) & \longrightarrow & Q\mathbf{H}_1. \end{array}$$

The two functors from $\mathcal{G}$ to $Q\mathbf{H}_1$ in this diagram send $\pi = (K \rightarrowtail P \twoheadrightarrow M \oplus Q)$ to M and Q, respectively. By Additivity 1.2, their sum is homotopic to the target functor t. It suffices to show that t is null homotopic.

Let s be the functor $\mathcal{G} \to Q\mathbf{P}(R)$ sending π to P; since s maps all morphisms to injections, $0 \rightarrowtail P$ defines a natural transformation $0 \Rightarrow s$. Since $P \twoheadrightarrow M \oplus Q$ is a natural transformation $t \Rightarrow s$, we have a homotopy $t \sim s \sim 0$, as desired. $\square$

The following result generalizes Exercises III.3.10 and III.4.11.

Proposition 7.5. *(Karoubi) Let $f : A \to B$ be a ring homomorphism and S a central multiplicatively closed set of nonzerodivisors in A such that $f(S)$ is a central set of nonzerodivisors in B. Assume that $f : A/sA \cong B/sB$ for all $s \in S$. Then:*
(1) the functor $\mathbf{H}_S(A) \to \mathbf{H}_S(B)$ is an equivalence;
(2) The square

$$\begin{array}{ccc} \mathbf{K}^B(A) & \longrightarrow & \mathbf{K}^B(B) \\ \downarrow & & \downarrow \\ \mathbf{K}^B(S^{-1}A) & \longrightarrow & \mathbf{K}^B(S^{-1}B) \end{array}$$

is homotopy cartesian, and there is a Mayer-Vietoris sequence

$$\cdots \to K_{n+1}(S^{-1}B) \xrightarrow{\partial} K_n(A) \xrightarrow{\Delta} K_n(B) \times K_n(S^{-1}A) \xrightarrow{\pm} K_n(S^{-1}B) \xrightarrow{\partial} \cdots .$$

Proof. The assumption implies that $\hat{A} = \varprojlim A/sA$ and $\hat{B} = \varprojlim B/sB$ are isomorphic. Therefore, in order to prove part (1) we may assume that $B = \hat{A}$. Then B is faithfully flat over A, and $\otimes_A B$ is a faithful exact functor from $\mathbf{H}_S(A)$ to $\mathbf{H}_S(B)$. To show that $\mathbf{H}_S(A) \to \mathbf{H}_S(B)$ is an equivalence, it suffices to show that every M in $\mathbf{H}_S(B)$ is isomorphic to a module coming from $\mathbf{H}_S(A)$.

Choose a B-module resolution $0 \to Q_1 \xrightarrow{f} Q_0 \to M \to 0$, and $s \in S$ so that $sM = 0$. Then the inclusion $sQ_0 \subset Q_0$ factors through a map $f' : sQ_0 \to Q_1$. The cokernel $M' = Q_1/sQ_0$ is also in $\mathbf{H}_S(B)$, and if we choose Q' so that $Q_0 \oplus Q_1 \oplus Q' \cong B^n$ then we have a resolution

$$0 \to B^n \xrightarrow{\gamma=(f,f',1)} B^n \to M \oplus M' \to 0.$$

Replacing M by $M \oplus M'$, we may assume that $Q_1 = Q_0 = B^n$. Then there is a matrix γ' over B such that $\gamma\gamma' = sI$. Now the assumption that $A/sA \cong B/sB$ implies that there is a matrix $\alpha : A^n \to A^n$ of the form $\alpha = \gamma + s^2\beta$. Since $\alpha = \gamma(I + s\gamma'\beta)$ and $I + s\gamma'\beta$ is invertible over $B = \hat{A}$, we see that the A-module $\operatorname{coker}(\alpha)$ is in $\mathbf{H}_S(A)$ and that $\operatorname{coker}(\alpha) \otimes_A B \cong \operatorname{coker}(\gamma) = M$.

Part (2) follows from Part(1) and Theorem 7.1; the Mayer-Vietoris sequence follows formally from the square (as in 6.11.1). $\square$

Example 7.5.1. The proposition applies to $S = \{p^n\}$ and the rings $\mathbb{Z} \to \mathbb{Z}_{(p)} \to \hat{\mathbb{Z}}_p$ (p-adics). If G is a group, it also applies to $\mathbb{Z}[G] \to \mathbb{Z}_{(p)}[G] \to \hat{\mathbb{Z}}_p[G]$. That is, $K_*(\mathbb{Z}[G])$ fits into a Mayer-Vietoris sequence involving $K_*(\mathbb{Z}_{(p)}[G])$, $K_*(\mathbb{Z}[\frac{1}{p}][G])$ and $K_*(\mathbb{Q}[G])$, as well as a Mayer-Vietoris sequence of the form:

$$K_{n+1}(\hat{\mathbb{Q}}_p[G]) \xrightarrow{\partial} K_n(\mathbb{Z}[G]) \to K_n(\hat{\mathbb{Z}}_p[G]) \oplus K_n(\mathbb{Z}[\frac{1}{p}][G]) \to K_n(\hat{\mathbb{Q}}_p[G]) \xrightarrow{\partial} \cdots .$$

More generally, let $S \subset \mathbb{Z}$ be generated by a finite set P of primes; Proposition 7.5 applies to $\mathbb{Z}[G] \to \prod_{p\in P} \hat{\mathbb{Z}}_p[G]$, and there is a similar Mayer-Vietoris sequence relating $K_*(\mathbb{Z}[G])$ to the product of the $K_*(\hat{\mathbb{Z}}_p[G])$. This sequence is particularly useful when G is finite and P is the set of primes dividing $|G|$, because the rings $S^{-1}\mathbb{Z}[G]$ and $\hat{\mathbb{Q}}_p[G]$ are semisimple.

Localization for vector bundles

Here is the analogue of Theorem 7.1 for vector bundles on a scheme X, which generalizes the exact sequence in Ex. II.8.1. As discussed in II.8.3, we assume that X is quasi-compact in order that every module in $\mathbf{H}(X)$ has a finite resolution by vector bundles. We define $K(X \text{ on } Z)$ to be the K-theory space of the Waldhausen category $\mathbf{Ch}_{\mathrm{perf},Z}(X)$ of perfect complexes on X which are exact on U.

THEOREM 7.6 (THOMASON-TROBAUGH [TT, 5.1]). *Let X be a quasi-compact, quasi-separated scheme, and U a quasi-compact open in X with complement Z. Then $K(X \text{ on } Z) \to K(X) \xrightarrow{j^*} K(U)$ is a homotopy fibration, and there is a long exact sequence*

$$\cdots \to K_{n+1}(U) \xrightarrow{\partial} K_n(X \text{ on } Z) \to K_n(X) \to K_n(U) \xrightarrow{\partial} \cdots$$

ending in $K_0(X \text{ on } Z) \to K_0(X) \to K_0(U)$, the sequence of Ex. II.8.1.

Recall that $\mathbf{H}_Z(X)$ denotes the category of modules in $\mathbf{H}(X)$ supported on Z.

COROLLARY 7.6.1. *If X is quasiprojective and Z is defined by an invertible ideal, then $\mathbf{H}_Z(X) \subset \mathbf{Ch}_{\mathrm{perf},Z}(X)$ induces an equivalence on K-theory. Thus there is a long exact sequence*

$$\cdots \to K_{n+1}(U) \xrightarrow{\partial} K_n\mathbf{H}_Z(X) \to K_n(X) \to K_n(U) \xrightarrow{\partial} \cdots .$$

PROOF. (Thomason) The Approximation Theorem implies that $K\mathbf{H}_Z(X) \simeq K(X \text{ on } Z)$, as we saw in Ex. 3.16 $\square$

REMARK 7.6.2. Since the model $K\mathbf{Ch}_{\mathrm{perf},Z}(X)$ for the fiber $K(X \text{ on } Z)$ of $K(X) \to K(U)$ is complicated, it would be nice to have a simpler model. A naïve guess for such a model would be the K-theory of the category $\mathbf{H}_Z(X)$. This is correct if X is regular by the localization sequence 6.11 for G-theory, if Z is a divisor (7.6.1), and even if Z is locally a complete intersection in X [TT, 5.7]. Exercise 7.4 shows that this cannot be right in general, even if $X = \mathrm{Spec}(A)$.

DEFINITION 7.6.3. Let us define the (non-connective) spectrum $\mathbf{K}^B(X \text{ on } Z)$ to be the homotopy fiber of the morphism $\mathbf{K}^B(X) \to \mathbf{K}^B(X - Z)$. Then Theorem 7.6 states that the K-theory spectrum $\mathbf{K}(X \text{ on } Z)$ of $\mathbf{Ch}_{\mathrm{perf},Z}(X)$ is the (-1)-connected cover of $\mathbf{K}^B(X \text{ on } Z)$. In particular, $K_n(X \text{ on } Z)$ is $\pi_n\mathbf{K}^B(X \text{ on } Z)$ for all $n \geq 0$. For $n < 0$ we define $K_n(X \text{ on } Z) = \pi_n\mathbf{K}^B(X \text{ on } Z)$; since $K_n(X) = \pi_n\mathbf{K}^B(X)$ in this range (by IV.10.6), the sequence of Theorem 7.6 may be continued:

$$K_0(X) \to K_0(U) \to K_{-1}(X \text{ on } Z) \to K_{-1}(X) \to K_{-1}(U) \to \cdots .$$

Proof of 7.6. (Thomason) For simplicity, we write $\mathcal{A}$ for the Waldhausen category $\mathbf{Ch}_{\mathrm{perf}}(X)$ of II.9.2, in which weak equivalences are quasi-isomorphisms. Let w be the class of weak equivalences such that $\mathcal{F} \xrightarrow{\sim}_w \mathcal{G}$ if and only if $\mathcal{F}|_U \xrightarrow{\sim} \mathcal{G}|_U$. By the Waldhausen Localization Theorem 2.1, we have a fibration $K(X \text{ on } Z) \to K(X) \to K(w\mathcal{A})$. Let G be the cokernel of $K_0(X) \to K_0(U)$ and let $\mathcal{B}$ denote the full Waldhausen category of all perfect complexes $\mathcal{F}$ on U with $[\mathcal{F}] = 0$ in G. By the Cofinality Theorem 2.3, $K_n(\mathcal{B}) \simeq K_n(U)$ for all $n > 0$ and $K_0(\mathcal{B})$ is the image of $K_0(X) \to K_0(U)$. Thus the proof reduces to showing that $K(w\mathcal{A}) \to K(\mathcal{B})$ is a homotopy equivalence. By the Approximation Theorem 2.4, this reduces to showing that $\mathcal{A} \to \mathcal{B}$ induces an equivalence of derived categories. This is the conclusion of the following theorem of Thomason and Trobaugh. $\square$

Theorem 7.7. *$w^{-1}\mathbf{Ch}_{\mathrm{perf}}(X) \to w^{-1}\mathcal{B}$ is an equivalence of triangulated categories. In more detail:*
(a) For every perfect complex $\mathcal{F}$ on U with $[\mathcal{F}]$ in the image of $K_0(X) \to K_0(U)$, there is a perfect complex $\mathcal{E}$ on X such that $\mathcal{E}|_U \simeq \mathcal{F}$ in $\mathbf{D}(U)$.
(b) Given perfect complexes $\mathcal{E}$, $\mathcal{E}'$ on X and a map $b : \mathcal{E}|_U \to \mathcal{E}'|_U$ in $\mathbf{D}(U)$, there is a diagram $\mathcal{E} \xleftarrow{a} \mathcal{E}'' \xrightarrow{a'} \mathcal{E}'$ of perfect complexes on X so that $a|_U$ is an isomorphism and $b = a'|_U(a|_U)^{-1}$ in $\mathbf{D}(U)$.
(c) If $a : \mathcal{E} \to \mathcal{E}'$ is a map of perfect complexes on X which is 0 in $\mathbf{D}(U)$, there is a perfect complex $\mathcal{E}''$ and a map $s : \mathcal{E}'' \to \mathcal{E}$ which is a quasi-isomorphism on U such that $a\,s = 0$ in $\mathbf{D}(X)$.

Proof. The proof of this theorem is quite deep, and beyond the level of this book; we quote [TT, 5.2.2–4] for the proof. As a side-note, it is part (a) that was suggested to Thomason by the Trobaugh simulacrum. $\square$

7.8 We shall now give an elementary proof (due to Quillen) of Corollary 7.6.1 in the easier case when Z is a divisor and U is affine. We need assume that X is quasi-projective (over a commutative ring R), in order to use the definition $K(X) = K\mathbf{VB}(X)$ (IV.6.3.4). We write $\mathbf{H}_{1,Z}(X)$ for $\mathbf{H}_Z(X) \cap \mathbf{H}_1(X)$, the (exact) subcategory of modules in $\mathbf{H}(X)$ with a length one resolution by vector bundles, which are supported on Z.

For our proof of 7.6.1, let $\mathcal{P}$ denote the exact category of vector bundles on U of the form $j^*(P)$, where P is a vector bundle on X. The assumption that U is affine guarantees $\mathcal{P}$ is cofinal in $\mathbf{VB}(U)$. Therefore, as with the $\mathcal{P}$ of 8.2, we may replace $K(U)$ by $K\mathcal{P}$. We also need the following analogue of the easily checked fact about the relation between P and $S^{-1}P$. If P is a vector bundle on X then $j^*(P)$ is a vector bundle on U, and its direct image $j_*j^*(P)$ is a quasicoherent module on X. If $X = \mathrm{Spec}(R)$ and $I = sR$ then $j_*j^*(P)$ is the R-module $P[1/s]$.

Lemma 7.8.1. *If P is a vector bundle on X, then P is a submodule of $j_*j^*(P)$, and $j_*j^*(P)$ is the union of its submodules $I^{-n}P$.*

Proof of Corollary 7.6.1. The proof of Theorem 7.1 goes through formally, replacing $\mathbf{P}(R)$ by $\mathbf{VB}(X)$, $S^{-1}R$ by U and $\mathbf{H}_{1,S}$ by $\mathbf{H}_{1,Z}(X)$, the (exact) subcategory of modules in $\mathbf{H}(X)$ with a length one resolution by vector bundles, which are supported on Z. The definitions of $\mathcal{P}$, $\mathcal{F}$ and $\mathcal{G}$ make sense in this context, and every exact sequence in $\mathcal{P}$ splits because U is affine. Now everything goes through immediately except for the proof that Λ is a filtering poset. That proof used the structure of $S^{-1}P$; the argument still goes through formally if we use the corresponding structure of $j_*j^*(P)$ which is given by Lemma 7.8.1. $\square$

PROPOSITION 7.9. *(Excision) Let X and Z be as in Theorem 7.6, and let $i : V \subset X$ be the inclusion of an open subscheme containing Z. Then the restriction $i^* : \mathbf{Ch}_{\mathrm{perf},Z}(X) \to \mathbf{Ch}_{\mathrm{perf},Z}(V)$ is exact and induces a homotopy equivalence*

$$\mathbf{K}^B(X \text{ on } Z) \xrightarrow{\simeq} \mathbf{K}^B(V \text{ on } Z).$$

PROOF. ([TT, 3.19]) Let $T(\mathcal{F})$ denote the functorial Godement resolution of $\mathcal{F}$, and represent the functor Ri_* as $i_* \circ T : \mathbf{Ch}_{\mathrm{perf},Z}(V) \to \mathbf{Ch}_{\mathrm{perf},Z}(X)$. This is an exact functor, taking values in the Waldhausen subcategory $\mathcal{A}$ of perfect complexes which are strictly zero on $U = X - Z$. The left exact functor Γ_Z (subsheaf with supports in Z) induces an exact functor $\Gamma_Z \circ T$ from $\mathbf{Ch}_{\mathrm{perf},Z}(X)$ to $\mathcal{A}$ which is an inverse of the inclusion up to natural quasi-isomorphism. Thus we have a homotopy equivalence $K\mathcal{A} \simeq K\mathbf{Ch}_{\mathrm{perf},Z}(X)$.

If $\mathcal{F}$ is a perfect complex on V, the natural map $i^*Ri_*\mathcal{F} \to \mathcal{F}$ is a quasi-isomorphism since $\mathcal{F} \to T(\mathcal{F})$ is. If $\mathcal{G}$ is a perfect complex on X, strictly zero on U, then $\mathcal{G} \to Ri_*(i^*\mathcal{G})$ is a quasi-isomorphism because it is an isomorphism at points of Z and a quasi-isomorphism at points of U. Thus $Ri_* : K\mathbf{Ch}_{\mathrm{perf},Z}(V) \to K(\mathcal{A})$ is a homotopy inverse of i^*. This proves that $K(X \text{ on } Z) \xrightarrow{\simeq} K(V \text{ on } Z)$. Given the Fundamental Theorem for schemes (8.3 below), it is a routine matter of bookkeeping to deduce the result for the non-connective Bass spectra $\mathbf{K}^B$. $\square$

COROLLARY 7.10. *(Mayer-Vietoris) Let X be a quasi-compact, quasi-separated scheme, and U, V quasi-compact open subschemes with $X = U \cup V$. Then the square*

$$\begin{array}{ccc} \mathbf{K}^B(X) & \longrightarrow & \mathbf{K}^B(U) \\ \downarrow & & \downarrow \\ \mathbf{K}^B(V) & \longrightarrow & \mathbf{K}^B(U \cap V) \end{array}$$

is homotopy cartesian, i.e., there is a homotopy fibration sequence

$$\mathbf{K}^B(X) \to \mathbf{K}^B(U) \times \mathbf{K}^B(V) \xrightarrow{\pm} \mathbf{K}^B(U \cap V).$$

On homotopy groups, this yields the long exact "Mayer-Vietoris" sequence:

$$\cdots \to K_{n+1}(U \cap V) \xrightarrow{\partial} K_n(X) \xrightarrow{\Delta} K_n(U) \times K_n(V) \xrightarrow{\pm} K_n(U \cap V) \xrightarrow{\partial} \cdots .$$

PROOF. Take $Z = X - U$ in Theorem 7.6, and apply Proposition 7.9. $\square$

COROLLARY 7.11. *Let X be a quasi-compact, quasi-separated scheme, and U, V quasi-compact open subschemes with $X = U \cup V$. Then the square*

$$\begin{array}{ccc} KH(X) & \longrightarrow & KH(U) \\ \downarrow & & \downarrow \\ KH(V) & \longrightarrow & KH(U \cap V) \end{array}$$

is homotopy cartesian. On homotopy groups, this yields the long exact "Mayer-Vietoris" sequence:

$$\cdots \to KH_{n+1}(U\cap V) \xrightarrow{\partial} KH_n(X) \xrightarrow{\Delta} KH_n(U)\times KH_n(V) \xrightarrow{\pm} KH_n(U\cap V) \xrightarrow{\partial} \cdots .$$

Proof. Recall from IV.12.7 that $KH(X)$ is defined to be the realization of $\mathbf{K}^B(X\times\Delta^\cdot)$. The square of simplicial spectra is degreewise homotopy cartesian by 7.10, and hence homotopy cartesian. □

EXERCISES

7.1 Show that the functors $h:\mathcal{G}\to Q\mathcal{P}$ and $pf:\mathcal{G}\to Q\mathbf{P}(R)$ of Definition 7.3 are fibered, with base change ϕ^* constructed as in Lemma IV.6.7. (The proofs for h and pf are the same.)
7.2 Show that there is a functor from the Segal subdivision $Sub(\mathcal{G}_M)$ of $\mathcal{G}_M$ (IV, Ex. 3.9) to the fiber $h^{-1}(M)$, sending $P'\rightarrowtail P$ to $P\to(P/P')\oplus M$. Then show that it is an equivalence of categories.
7.3 Show that the functor $Sub(T)\to(pf)^{-1}(Q)$ which sends $P'\rightarrowtail P\twoheadrightarrow Q$ to $P\twoheadrightarrow Q\oplus(P/P')$ is an equivalence of categories. (See IV, Ex. 7.3.)

7.4 (Gersten) Let A be the homogeneous coordinate ring of a smooth projective curve X over an algebraically closed field. Then A is a 2-dimensional graded domain such that the "punctured spectrum" $U=\mathrm{Spec}(A)-\{\mathfrak{m}\}$ is regular, where $\mathfrak{m}$ is the maximal ideal at the origin. The blowup of $\mathrm{Spec}(A)$ at the origin is a line bundle over X, and U is isomorphic to the complement of the zero-section of this bundle.
(a) Show that $K_n(U)\cong K_n(X)\oplus K_{n-1}(X)$.
(b) Suppose now that A is not a normal domain. We saw in II, Ex. 8.1 that $\mathbf{H}_{\mathfrak{m}}(A)\cong 0$. Show that the image of $K_1(A)\to K_1(U)$ is $k^\times$, and conclude that $K_*\mathbf{H}_{\mathfrak{m}}(A)$ cannot be the third term $K_*(\mathrm{Spec}(A)\text{ on }U)$ in the localization sequence for $K_*(A)\to K_*(U)$.

7.5 Let $\mathbb{P}^1_R$ denote the projective line over an associative ring R, as in 1.5.4, let $\mathbf{H}_1$ denote the subcategory of modules which have a resolution of length 1 by vector bundles, as in Ex. 3.13, and let $\mathbf{H}_{1,t}$ denote the subcategory $\mathbf{H}_1$ consisting of modules $\mathcal{F}=(M,0,0)$.
(a) Show that $\mathbf{H}_{1,T}(R[t])\to\mathbf{H}_{1,t}$, $M\mapsto(M,0,0)$, is an equivalence of categories.
(b) Show that there is an exact functor $\mathbf{VB}(\mathbb{P}^1_R)\xrightarrow{j^*}\mathbf{P}(R[1/t])$, $j^*(\mathcal{F})=M_-$.
(c) Using Ex. 3.13, show that the inclusion $\mathbf{H}_{1,t}\subset\mathbf{H}_1$ induces a homotopy fibration sequence $K\mathbf{H}_T(R[t])\to K(\mathbb{P}^1_R)\xrightarrow{i^*}K(R[s])$.
(d) If R is commutative, show that the fibration sequence in (c) is the same as the sequence in 7.6.1 with Z the origin of $X=\mathbb{P}^1_R$.

For the next few exercises, we fix an automorphism φ of a ring R, and form the twisted polynomial ring $R_\varphi[t]$ and its localization $R_\varphi[t,1/t]$, as in Ex. 6.4. Similarly, there is a twisted polynomial ring $R_{\varphi^{-1}}[s]$ and an isomorphism $R_\varphi[t,1/t]\cong R_{\varphi^{-1}}[s,1/s]$ obtained by identifying $s=1/t$. Let $\mathbf{H}_{1,t}(R_\varphi[t])$ denote the exact category of t-torsion modules in $\mathbf{H}_1(R[t])$.

7.6 Show that $K\mathbf{H}_{1,t}(R_\varphi[t]) \to K(R_\varphi[t]) \to K(R_\varphi[t,1/t])$ is a homotopy fibration. This generalizes Theorem 7.1 and parallels Ex. 6.4. (This result is due to Grayson; the induced K_1–K_0 sequence was discovered by Farrell and Hsiang.)

7.7 (Twisted projective line $\mathbb{P}^1_{R,\varphi}$) We define $\mathbf{mod}\text{-}\,\mathbb{P}^1_{R,\varphi}$ to be the abelian category of triples $\mathcal{F} = (M_t, M_s, \alpha)$, where M_t (resp., M_s) is a module over $R_\varphi[t]$ (resp. $R_{\varphi^{-1}}[s]$) and α is is an isomorphism $M_t[1/t] \xrightarrow{\simeq} M_s[1/s]$. It has a full (exact) subcategory $\mathbf{VB}(\mathbb{P}^1_{R,\varphi})$ consisting of triples where M_t and M_s are finitely generated projective modules, and we write $K(\mathbb{P}^1_{R,\varphi})$ for $K\mathbf{VB}(\mathbb{P}^1_{R,\varphi})$.
(a) Show that $\mathcal{F}(-1) = (M_t, \varphi^* M_s, \alpha t)$ is a vector bundle whenever $\mathcal{F}$ is.
(b) Generalize Theorem 1.5.4 to show that $K(R) \times K(R) \simeq K(\mathbb{P}^1_{R,\varphi})$.
(c) Show that there is an exact functor $\mathbf{VB}(\mathbb{P}^1_{R,\varphi}) \xrightarrow{j^*} \mathbf{P}(R_{\varphi^{-1}}[s])$, $j^*(\mathcal{F}) = M_s$.
(d) Show that $\mathbf{H}_{1,t}(R_\varphi[t])$ is equivalent to the category of modules $\mathcal{F}$ having a resolution of length 1 by vector bundles, and with $j^*(\mathcal{F}) = 0$. (Cf. Ex. 7.5(a).)
(e) Generalize Ex. 7.5(c) to show that there is a homotopy fibration

$$K\mathbf{H}_T(R[t]) \to K(\mathbb{P}^1_{R,\varphi}) \xrightarrow{j^*} K(R[s]).$$

§8. The Fundamental Theorem for $K_*(R)$ and $K_*(X)$.

The main goal of this section is to prove the Fundamental Theorem, which gives a decomposition of $K_*(R[t,1/t])$. For regular rings, the decomposition simplifies to the formulas $NK_n(R) = 0$ and $K_n(R[t,1/t]) \cong K_n(R) \oplus K_{n-1}(R)$ of Theorem 6.3.

Let R be a ring, set $T = \{t^n\} \subset R[t]$, and consider the category $\mathbf{H}_{1,T}$ of t-torsion $R[t]$-modules M in $\mathbf{H}_1(R[t])$. On the one hand, we know from II.7.8.2 that $\mathbf{H}_{1,T}$ is equivalent to the category $\mathbf{Nil}(R)$ of nilpotent endomorphisms of projective R-modules (II.7.4.4). We also saw in IV.6.7 that the forgetful functor induces a decomposition $K\mathbf{Nil}(R) \simeq K(R) \times \mathrm{Nil}(R)$.

Theorem 8.1. *For every R and every n,* $\mathrm{Nil}_n(R) \cong NK_{n+1}(R)$

Theorem 8.1 was used in IV.6.7 to derive several properties of the group $NK_*(R)$, including: if R is a $\mathbb{Z}/p\mathbb{Z}$-algebra then each $NK_n(R)$ is a p-group, and if $\mathbb{Q} \subset R$ then $NK_n(R)$ is a uniquely divisible abelian group.

Proof. We know from Ex. 7.5 that $\mathbf{Nil}(R)$ is equivalent to the category $\mathbf{H}_{1,t}$ of modules $\mathcal{F}$ on the projective line $\mathbb{P}^1_R$ with $j^*\mathcal{F} = 0$, and which have a length 1 resolution by vector bundles. Substituting this into Corollary 7.6.1 (or Ex. 7.5 if R is not commutative) yields the exact sequence

$$(8.1.1)\quad \cdots K_{n+1}(R[1/t]) \to K_n(R) \oplus \mathrm{Nil}_n(R) \to K_n(\mathbb{P}^1_R) \xrightarrow{j^*} K_n(R[1/t]) \to \cdots .$$

Now the composition $\mathbf{P}(R) \subset \mathbf{Nil}(R) \to \mathbf{H}(\mathbb{P}^1_R)$ sends P to $(P,0,0)$, and there is an exact sequence $u_1(P) \rightarrowtail u_0(P) \twoheadrightarrow (P,0,0)$ obtained by tensoring P with the standard resolution $0 \to \mathcal{O}(-1) \to \mathcal{O} \to (R,0,0) \to 0$. By Additivity 1.2, the corresponding map $K(R) \to K(\mathbb{P}^1_R)$ in (8.1.1) is $u_0 - u_1$. By the Projective Bundle Theorem 1.5 (or 1.5.4 if R is not commutative), the map $(u_0, u_0 - u_1)$: $K(R) \times K(R) \to K(\mathbb{P}^1_R)$ is an equivalence. Since $j^*u_0(P) \cong P \otimes_R R[1/t]$, the composition $j^* \circ u_0 : K(R) \to K(R[1/t])$ is the standard base change map inducing the decomposition $K_n(R[1/t]) \cong K_n(R) \oplus NK_n(R)$. Thus (8.1.1) splits into the split extension $0 \to K_n(R) \xrightarrow{u_0-u_1} K_n(\mathbb{P}^1_R) \to K_n(R) \to 0$ and the desired isomorphism $NK_{n+1}(R) \cong K_{n+1}(R[t])/K_{n+1}(R) \xrightarrow{\simeq} \mathrm{Nil}_n(R)$. $\square$

Theorem 8.2. *There is a canonically split exact sequence*

$$0 \to K_n(R) \xrightarrow{\Delta} K_n(R[t]) \oplus K_n(R[1/t]) \xrightarrow{\pm} K_n(R[t,1/t]) \underset{\leftarrow}{\xrightarrow{\partial}} K_{n-1}(R) \to 0.$$

in which the splitting of ∂ is given by multiplication by $t \in K_1(\mathbb{Z}[t,t^{-1}])$.

Proof. Because the base change $\mathbf{mod}\text{-}\,\mathbb{P}^1_R \to \mathbf{mod}\text{-}R[t]$ is exact, there is a map between the localization sequences for T in Theorem 7.1 and (8.1.1), yielding the commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & K_n\mathbf{H}_{1,T} \arrow[r] \arrow[d, equal] & K_n(\mathbb{P}^1_R) \arrow[r, "j^*"] \arrow[d] & K_n(R[1/t]) \arrow[r] \arrow[d] & K_{n-1}\mathbf{H}_{1,T} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & K_n\mathbf{H}_{1,T} \arrow[r, "f_*\oplus 0"] & K_n(R[t]) \arrow[r, "j^*"] & K_n(R[t,1/t]) \arrow[r] & K_{n-1}\mathbf{H}_{1,T} \arrow[r] & \cdots .
\end{tikzcd}$$

Now $K_n(R) \to K_n\mathbf{Nil}(R) \to K_n(R[t])$ is the transfer map f_* of (3.3.2), induced from the ring map $f : R[t] \to R$, and $\mathrm{Nil}_n(R) \to K_n(\mathbb{P}^1_R)$ is zero by the proof of Theorem 8.1, Since f_* is zero by 3.5.1, the diagram yields the exact sequence $Seq(K_n, R)$ displayed in the Theorem, for $n \geq 1$. The exact sequence $Seq(K_n, R)$ for $n \leq 0$ was constructed in III.4.1.2.

To see that $Seq(K_n, R)$ is split exact, we only need to show that ∂ is split by the cup product with $[t] \in K_1(R[t,1/t])$. But the maps in the localization sequence commute with multiplication by $K_*(R)$, by Exercise 8.1 (or Ex. IV.1.23). Hence we have the formula: $\partial(\{t,x\}) = \partial(t) \cdot x = [R[t]/tR[t]] \cdot x$. This shows that $x \mapsto \{t,x\}$ is a right inverse to ∂; the maps $t \mapsto 1$ from $K_n(R[t])$ and $K_n(R[1/t])$ to $K_n(R)$ yield the rest of the splitting. $\square$

There is of course a variant of the Fundamental Theorem 8.2 for schemes. For every scheme X, let $X[t]$ and $X[t,t^{-1}]$ denote the schemes $X \times \mathrm{Spec}(\mathbb{Z}[t])$ and $X \times \mathrm{Spec}(\mathbb{Z}[t,t^{-1}])$ respectively.

Theorem 8.3. *For every quasi-projective scheme X we have canonically split exact sequences for all n, where the splitting of ∂ is by multiplication by t.*

$$0 \to K_n(X) \xrightarrow{\Delta} K_n(X[t]) \oplus K_n(X[1/t]) \xrightarrow{\pm} K_n(X[t,1/t]) \underset{\leftarrow}{\xrightarrow{\partial}} K_{n-1}(X) \to 0.$$

in which the splitting of ∂ is given by multiplication by $t \in K_1(\mathbb{Z}[t,t^{-1}])$.

Proof. Consider the closed subscheme $X_0 = X \times 0$ of $\mathbb{P}^1_X$. The open inclusion $X[t] \hookrightarrow \mathbb{P}^1_X$ is a flat map, and induces a morphism of homotopy fibration sequences for $n \geq 1$ from Corollary 7.6.1:

(8.3.1)
$$\begin{tikzcd}
K_n\mathbf{H}_{X_0}(\mathbb{P}^1_X) \arrow[r] \arrow[d, equal] & K_n(\mathbb{P}^1_X) \arrow[r, "j^*"] \arrow[d] & K_n(X[1/t]) \arrow[r, "\partial"] \arrow[d] & K_{n-1}\mathbf{H}_{X_0}(\mathbb{P}^1_X) \arrow[d, equal] \\
K_n\mathbf{H}_{X_0}(X[t]) \arrow[r] & K_n(X[t]) \arrow[r, "j^*"] & K_n(X[t,1/t]) \arrow[r, "\partial"] & K_{n-1}\mathbf{H}_{X_0}(X[t]).
\end{tikzcd}$$

As in the proof of Theorem 8.1, there is an exact sequence $u_1 \rightarrowtail u_0 \twoheadrightarrow i_*$ of functors from $\mathbf{VB}(X)$ to $\mathbf{H}(\mathbb{P}^1_X)$, where $u_i(E) = E \otimes \mathcal{O}(i)$ and i_* is the restriction of scalars associated to $i : X_0 \hookrightarrow \mathbb{P}^1_X$. By Additivity 1.2, $i_* = u_0 - u_1$. Since j^*u_0 is the base change, we see that the top row splits in the same way that (8.1.1) does. Since the map $K_n(X) \to K_n\mathbf{H}_{X_0}(X[t]) \to K_n(X[t])$ is zero by 3.6.1, the bottom left map is zero on homotopy groups. Via a diagram chase on (8.3.1), using the Projective Bundle Theorem 1.5, the exact sequence of Theorem 8.3 follows formally for $n \geq 1$, and for $n = 0$ provided we define $K_{-1}(X)$ to be the cokernel of the displayed map '$\pm$'.

Since the maps in (8.3.1) are $K_*(X)$-module maps (by Ex. 8.2) we have $\partial(\{t, x\}) = \partial(t) \cdot x$. But the base change $X[t] \xrightarrow{\pi} \mathrm{Spec}(\mathbb{Z}[t])$ induces a morphism of localization sequences, and t lifts to $K_1(\mathbb{Z}[t, 1/t])$, so by naturality we have $\partial_X(t) = \pi^*\partial_{\mathbb{Z}}(t) = \pi^*([\mathbb{Z}[t]/t]) = 1$ in the subgroup $K_0(X)$ of $K_0\mathbf{H}_{X_0}(X[t])$. Hence $\partial(\{t, x\}) = x$, regarded as an element of the subgroup $K_*(X)$ of $K_*\mathbf{H}_{X_0}(X[t])$.

Finally, given the result for $n = 1$, the result follows formally for $n \leq 0$, where $K_{n-1}(X)$ is given by Definition IV.10.6; see Ex. 8.3. $\square$

REMARK 8.3.2. $K_n(X)$ was defined for $n \leq -1$ to be $\pi_n\mathbf{K}^B(X)$ in Definition IV.10.6, where $\mathbf{K}^B(X)$ was defined using Theorem 8.4 below. Unravelling that definition, we see that the groups $K_n(X)$ may also be inductively defined to be the cokernel $LK_{n+1}(X)$ of $K_{n+1}(X[t]) \oplus K_{n+1}(X[1/t]) \to K_{n+1}(R[t, 1/t])$. As with Definition III.4.1.1, this definition is concocted so that Theorem 8.3 remains true for $n = 0$ and also for all negative n; see Ex. 8.3.

Theorems 8.2 and 8.3 have versions which involve the (connective) spectra $\mathbf{K}(R)$ and $\mathbf{K}(X)$ associated to $K(R)$ and $K(X)$. Recall from IV.10.1 that the spectrum $\Lambda\mathbf{K}(R)$ is defined so that we have a cofibration sequence

$$\Lambda\mathbf{K}(R) \to \mathbf{K}(R[t]) \vee_{\mathbf{K}(R)} \mathbf{K}(R[1/t]) \xrightarrow{f_0} \mathbf{K}(R[t, 1/t]) \xrightarrow{\partial} \Omega^{-1}\Lambda\mathbf{K}(R).$$

Replacing R by X in Definition IV.10.1 yields a spectrum $\Lambda\mathbf{K}(X)$ fitting into the cofibration sequence obtained from this one by replacing R with X throughout. Fixing a spectrum map $S^1 \to \mathbf{K}(\mathbb{Z}[t, 1/t])$ representing $[x] \in [S^1, \mathbf{K}(\mathbb{Z}[t, 1/t])] = K_1(\mathbb{Z}[t, 1/t])$, the product yields a map $\mathbf{K}(R)$ to $\mathbf{K}(R[t, 1/t])$; composed with $\Omega\partial$, it yields morphisms of spectra $\mathbf{K}(R) \to \Lambda\mathbf{K}(R)$, and $\mathbf{K}(X) \to \Lambda\mathbf{K}(X)$. The following result was used in section IV.10 to define the non-commutative "Bass K-theory spectra" $\mathbf{K}^B(R)$ and $\mathbf{K}^B(X)$ as the homotopy colimit of the iterates $\Lambda^k\mathbf{K}(R)$ and $\Lambda^k\mathbf{K}(X)$.

THEOREM 8.4. *For any ring R, the map $\mathbf{K}(R) \to \Lambda\mathbf{K}(R)$ induces a homotopy equivalence between $\mathbf{K}(R)$ and the (-1)-connective cover of the spectrum $\Lambda\mathbf{K}(R)$. In particular, $K_n(R) \cong \pi_n\Lambda\mathbf{K}(R)$ for all $n \geq 0$.*

Similarly, for any quasi-projective scheme X, the map $\mathbf{K}(X) \to \Lambda\mathbf{K}(X)$ induces a homotopy equivalence between $\mathbf{K}(X)$ and the (-1)-connective cover of the spectrum $\Lambda\mathbf{K}(X)$. In particular, $K_n(X) \cong \pi_n\Lambda\mathbf{K}(X)$ for all $n \geq 0$.

PROOF. We give the proof for R; the proof for X is the same. By Theorem 8.3, there is a morphism $\mathbf{K}(R) \to \Lambda\mathbf{K}(R)$ which is an isomorphism on π_n for all $n \geq 0$, and the only other nonzero homotopy group is $\pi_{-1}\Lambda\mathbf{K}(R) = K_{-1}(R)$. The theorem is immediate. $\square$

COROLLARY 8.4.1. *For every ring R, the spectrum $\mathbf{K}^B(R[t])$ decomposes as $\mathbf{K}^B(R) \vee N\mathbf{K}^B(R)$ and the spectrum $\mathbf{K}^B(R[t,1/t])$ splits as*

$$\mathbf{K}^B(R[t,1/t]) \simeq \mathbf{K}^B(R) \vee N\mathbf{K}^B(R) \vee N\mathbf{K}^B(R) \vee \Omega^{-1}\mathbf{K}^B(R).$$

K_n-regularity

The following material is due to Vorst [Vo] and van der Kallen. Let s be a central nonzerodivisor in R, and write $[s] : R[x] \to R[x]$ for the substitution $f(x) \mapsto f(sx)$. Let $NK_*(R)_{[s]}$ denote the colimit of the directed system

$$NK_*(R) \xrightarrow{[s]} NK_*(R) \xrightarrow{[s]} NK_*(R) \xrightarrow{[s]} \cdots .$$

LEMMA 8.5. *For any central nonzerodivisor s, $NK_*(R)_{[s]} \xrightarrow{\simeq} NK_*(R[1/s])$. In particular, if $NK_n(R) = 0$ then $NK_n(R[1/s]) = 0$.*

PROOF. Let C be the colimit of the directed system $R[x] \xrightarrow{[s]} R[x] \xrightarrow{[s]} R[x] \cdots$ of ring homomorphisms. There is an evident map $C \to R$ splitting the inclusion, and the kernel is the ideal $I = xR[1/s][x] \subset R[1/s][x]$. By IV.6.4, $K_*(C,I)$ is the filtered colimit of the groups $NK_n(R) = K_n(R[x],x)$ along the maps $[s]$, *i.e.*, $NK_*(R)_{[s]}$. Moreover, $C[1/s] \cong R[1/s][x]$.

We can apply Proposition 7.5 to $R \to C$, since $I/sI = 0$ implies $R/s^iR \cong C/s^iC$, to obtain the Mayer-Vietoris sequence

$$\cdots \to K_n(R) \xrightarrow{\Delta} K_n(C) \times K_n(R[1/s]) \xrightarrow{\pm} K_n(R[1/s][x]) \xrightarrow{\partial} \cdots .$$

Splitting off $K_n(R)$ from $K_n(C)$ and $K_n(R[1/s])$ from $K_n(R[1/s][x])$ yields the desired isomorphism of $NK_n(R)_{[s]} \cong K_n(C,I)$ with $NK_n(R[1/s])$. $\square$

EXAMPLE 8.5.1. If R is commutative and reduced, then $NK_n(R) = 0$ implies that $NK_n(R_P) = 0$ for every prime ideal P of R. Vorst [Vo] also proved the converse: if $NK_n(R_\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ then $NK_n(R) = 0$. Van der Kallen proved the stronger result that NK_n is a Zariski sheaf on $\mathrm{Spec}(R)$.

Recall from III.3.4 that R is called *K_n-regular* if $K_n(R) \cong K_n(R[t_1,\dots,t_m])$ for all m. By Theorem 6.3, every noetherian regular ring is K_n-regular for all n.

THEOREM 8.6. *If R is K_n-regular, then it is K_{n-1}-regular. More generally, if $K_n(R) \cong K_n(R[s,t])$ then $K_{n-1}(R) \cong K_{n-1}(R[s])$.*

PROOF. It suffices to suppose that $K_n(R) \cong K_n(R[s,t])$, so that $NK_n(R[s]) = 0$, and prove that $NK_{n-1}(R) = 0$. By 8.5, $NK_n(R[s,1/s]) = 0$. But $NK_{n-1}(R)$ is a summand of $NK_n(R[s,1/s])$ by the Fundamental Theorem 8.2. $\square$

The following partial converse was proven in [CHWW].

THEOREM 8.7. *Let R be a commutative ring containing $\mathbb{Q}$. If R is K_{n-1}-regular and $NK_n(R) = 0$, then R is K_n-regular.*

REMARK 8.7.1. There are rings R for which $NK_n(R) = 0$ but R is not K_n-regular. For example, the ring $R = \mathbb{Q}[x,y,z]/(z^2 + y^3 + x^{10} + x^7y)$ has $K_0(R) \cong K_0(R[t])$ but $K_0(R) \ncong K_0(R[s,t])$, and $K_{-1}(R) \ncong K_{-1}(R[t])$; see [CHWW].

EXERCISES

8.1 For any central nonzerodivisor $s \in R$, multiplication by $[s] \in K_1(R[1/s])$ yields a map $K_n(R) \to K_{n+1}(R[1/s])$. Show that the boundary map $\partial : K_{n+1}(R[1/s]) \to K_n\mathbf{H}_s(R)$ satisfies $\partial(\{s,x\}) = \bar{x}$ for every $x \in K_n(R)$, where $\bar{x}$ is the image of x under the natural map $K_n(R) \to K_n(R/sR) \to K_n\mathbf{H}_s(R)$. *Hint:* Use Exercises 5.3 and IV.1.23; the ring map $\mathbb{Z}[t] \to R$, $t \mapsto s$, induces compatible pairings $\mathbf{P}(\mathbb{Z}[t]) \times \mathbf{P}(R) \to \mathbf{P}(R)$ and $\mathbf{P}(\mathbb{Z}[t,1/t]) \times \mathbf{P}(R) \to \mathbf{P}(R[1/s])$.

8.2 Show that $\mathbf{VB}(X)$ acts on the terms in the localization sequence of Theorem 8.7 in the sense of IV.6.6.4. Deduce that the map $\partial : K_n(U) \to K_{n-1}\mathbf{H}_Z(X)$ satisfies $\partial(\{u,x\}) = \partial(u)\cdot x$ for $u \in K_*(U)$ and $x \in K_*(X)$. *Hint:* use Ex. IV.1.23 and mimick Ex. 5.3.

8.3 Given a scheme X, show that $F_n(R) = K_n(X \times \operatorname{Spec} R)$ is a contracted functor in the sense of III.4.1.1. Then show that the functor $K_{-n}(X)$ of 8.3.1 is the contracted functor $L^nF_0(\mathbb{Z})$.

8.4 *Twisted* Nil *groups*, Let φ be an automorphism of a ring R, and consider the category $\mathbf{Nil}(\varphi)$ whose objects are pairs (P,ν), where P is a finitely generated projective R-module and ν is a nilpotent endomorphism of P which is semi-linear in the sense that $\nu(xr) = \nu(x)\varphi(r)$. If $\varphi = \mathrm{id}_R$, this is the category $\mathbf{Nil}(R)$ of II.7.4.4. As in *loc. cit.*, we define $\mathrm{Nil}_n(\varphi)$ to be the kernel of the map $K_n\mathbf{Nil}(\varphi) \to K_n(R)$ induced by $(P,\nu) \mapsto P$. Similarly, we define $NK_n(\varphi)$ to be the cokernel of the natural map $K_n(R) \to K_n(R_\varphi[t])$.

(a) Show that $K_n\mathbf{Nil}_\varphi(R) \cong K_n(R) \oplus \mathrm{Nil}_n(\varphi)$, and $K_n(R_\varphi[t]) \cong K_n(R) \oplus NK_n(\varphi)$.

(b) Show that $\mathbf{Nil}_\varphi(R)$ is equivalent to the category $\mathbf{H}_{1,t}(R_\varphi[t])$ of Ex. 7.6.

(c) Show that $NK_n(\varphi) \cong NK_n(\varphi^{-1})$. *Hint:* $\mathbf{P}(R^{op}) \cong \mathbf{P}(R)^{op}$ by IV.6.4.

(d) Prove the twisted analogue of Theorem 8.1: $\mathrm{Nil}_n(\varphi) \cong NK_{n+1}(\varphi)$ for all n.

8.5 (Grayson) Let $K^\varphi(R)$ be the homotopy fiber of $K(R) \xrightarrow{1-\varphi^*} K(R)$, and set $K_n^\varphi(R) = \pi_n K^\varphi(R)$. If R is regular, use Ex. 6.4 to show that $K_*(R_\varphi[t,1/t]) \cong K_*^\varphi(R)$. Then show that there is a canonical isomorphism for any R:

$$K_n(R_\varphi[t,1/t]) \cong K_n^\varphi(R) \oplus \mathrm{Nil}_{n-1}(\varphi) \oplus \mathrm{Nil}_{n-1}(\varphi^{-1}).$$

§9. The coniveau spectral sequence of Gersten and Quillen

In this section we give another application of the Localization Theorem 5.1, which reduces the calculation of $G_n(X)$ to a knowledge of the K-theory of fields, up to extensions. The prototype of the extension problem is illustrated by the exact sequences in 6.6 and 6.12, for Dedekind domains and smooth curves.

Suppose first that R is a finite-dimensional commutative noetherian ring. We let $\mathbf{M}^i(R)$ denote the subcategory of $\mathbf{M}(R)$ consisting of those R-modules M whose associated prime ideals all have height $\geq i$. We saw in II.6.4.3 and Ex. II.6.9 that each $\mathbf{M}^i(R)$ is a Serre subcategory of $\mathbf{M}(R)$ and that $\mathbf{M}^i/\mathbf{M}^{i+1}(R) \cong \bigoplus_{ht(\mathfrak{p})=i} \mathbf{M}_\mathfrak{p}(R_\mathfrak{p})$, where $\mathbf{M}_\mathfrak{p}(R_\mathfrak{p})$ is the category of $R_\mathfrak{p}$-modules of finite length.

By Devissage (Application 4.4) we have $K\mathbf{M}_\mathfrak{p}(R_\mathfrak{p}) \simeq G(k(\mathfrak{p})) \simeq K(k(\mathfrak{p}))$, where $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, so $K_*\mathbf{M}^i/\mathbf{M}^{i+1}(R) \cong \oplus_{ht(\mathfrak{p})=i} G_*(k(\mathfrak{p}))$. The Localization Theorem yields long exact sequences

$$\xrightarrow{\partial} K_n\mathbf{M}^{i+1}(R) \to K_n\mathbf{M}^i(R) \to \bigoplus_{ht(\mathfrak{p})=i} K_n(k(\mathfrak{p})) \xrightarrow{\partial} K_{n-1}\mathbf{M}^{i+1}(R) \to \cdots$$

and, writing $\mathbf{M}^{i-1}/\mathbf{M}^{i+1}$ for $\mathbf{M}^{i-1}(R)/\mathbf{M}^{i+1}(R)$,

$$(9.1)\qquad \bigoplus_{ht(\mathfrak{p})=i-1} K_{n+1}(k(\mathfrak{p})) \xrightarrow{\partial} \bigoplus_{ht(\mathfrak{p})=i} K_n(k(\mathfrak{p})) \to K_n(\mathbf{M}^{i-1}/\mathbf{M}^{i+1}) \to \bigoplus_{ht(\mathfrak{p})=i-1} K_n(k(\mathfrak{p})) \xrightarrow{\partial}$$

ending in an extension to K_1 of the K_0 sequence of II.6.4.3:

$$\bigoplus_{ht(\mathfrak{p})=i-1} k(\mathfrak{p})^\times \xrightarrow{\Delta} D^i(R) \to K_0(\mathbf{M}^{i-1}/\mathbf{M}^{i+1}) \to D^{i-1}(R) \to 0.$$

Here $D^i(R)$ is the free abelian group on the height i primes, and Δ sends $r/s \in k(\mathfrak{p})^\times$ to $[R/(r,\mathfrak{p})]-[R/(s,\mathfrak{p})]$ by Example 6.1.2. (This is the formula of II, Ex. 6.8.)

Recall from II.6.4.3 that the *generalized Weil divisor class group* $CH^i(R)$ is defined to be the image of $D^i(R) \to K_0\mathbf{M}^{i-1}/\mathbf{M}^{i+1}(R)$. As the kernel of this map is the image of Δ, this immediately gives the interpretation, promised in II.6.4.3:

LEMMA 9.1.1. *$CH^i(R)$ is the quotient of $D^i(R)$ by the relations that $\Delta(r/s) = 0$ for each $r/s \in k(\mathfrak{p})^\times$ and each prime ideal $\mathfrak{p}$ of height $i-1$.*

This equivalence relation, that the length of $R_\mathfrak{p}/(r,\mathfrak{p})$ is zero in $D^i(R)$ for each $r \in R$ and each prime $\mathfrak{p}$ of height $i-1$, is called *rational equivalence*; see 9.4.1.

For general R, the localization sequences (9.1) cannot break up. Indeed, we saw in I.3.6 and II.6 that even the map $\oplus k(y)^\times \xrightarrow{\Delta} D^i(X)$ can be nonzero. Instead, the sequences assemble to form a spectral sequence converging to $G_*(R)$.

PROPOSITION 9.2. *If R is noetherian and* $\dim(R) < \infty$*, there is a convergent 4^{th} quadrant cohomological spectral sequence*

$$E_1^{p,q} = \bigoplus_{ht(\mathfrak{p})=p} K_{-p-q}(k(\mathfrak{p})) \Rightarrow G_{-p-q}(R).$$

The edge maps $G_n(R) \to E_1^{0,-n} = \oplus G_n(k(\mathfrak{p}))$ associated to the minimal primes $\mathfrak{p}$ of R are induced by the localizations $R \to R_\mathfrak{p}$ followed by $G(k(\mathfrak{p})) \simeq G(R_\mathfrak{p})$ of 4.2.1.
Along the line $p+q=0$, we have $E_1^{p,-p} \cong D^p(R)$ and $E_2^{p,-p} \cong CH^p(R)$.

$$\begin{array}{|cccc}
\hline
\mathbb{Z} & 0 & 0 & 0 \\
F^\times & \to\ D^1(R) & 0 & 0 \\
K_2(F) & \to \oplus k(x_1)^\times & \to D^2(R) & 0 \\
K_3(F) & \to \oplus K_2(x_1) & \to \oplus k(x_2)^\times & \to D^3(R)
\end{array}$$

The E_1 page of the spectral sequence

PROOF. Setting $D_1^{p,q} = \oplus_i K_{-p-q}\mathbf{M}^i(R)$, the localization sequences (9.1) yield an exact couple (D_1, E_1). Because $\mathbf{M}(R) = \mathbf{M}^0(R)$ and $\mathbf{M}^p(R) = \emptyset$ for $p > \dim(R)$, the resulting spectral sequence is bounded and converges to $K_*(R)$ (see [WHomo, 5.9.7]). Since $E_1^{p,-p}$ is the divisor group $D^p(R)$, and $d_1^{p-1,p}$ is the map $\oplus k(x)^\times \xrightarrow{\Delta} D^p(R)$ of 9.1, the group $E_2^{p,-p}$ is isomorphic to $CH^p(R)$ by 9.1.1. Finally, the edge maps are given by $G_n(R) = K_n\mathbf{M}(R) \to K_n\mathbf{M}(R)/\mathbf{M}^1(R) \cong \oplus K_n(k(\eta))$; the component maps are induced by $\mathbf{M}(R) \to \mathbf{M}(k(\eta))$. $\square$

Coniveau Filtration 9.2.1. The p^{th} term in the filtration on the abutment $G_n(R)$ is defined to be the image of $K_n\mathbf{M}^p(R) \to K_n\mathbf{M}(R)$. In particular, the filtration on $G_0(R)$ is the coniveau filtration of II.6.4.3.

Remark 9.2.2. If $f: R \to S$ is flat then $\otimes_R S$ sends $\mathbf{M}^i(R)$ to $\mathbf{M}^i(S)$. It follows that the spectral sequence 9.2 is covariant for flat maps.

Remark 9.2.3. If $\dim(R) = \infty$, it follows from [WHomo, Ex. 5.9.2] that the spectral sequence 9.2 converges to $\varprojlim K_*\mathbf{M}(R)/\mathbf{M}^i(R)$.

Motivated by his work with Brown, Gersten made the following conjecture for regular local rings. (If $\dim(R) = 1$, this is Gersten's conjecture 6.9.) His conjecture was extended to semilocal rings by Quillen in [Q341], who then established the important special case when R is essentially of finite type over a field (9.6 below).

Gersten-Quillen Conjecture 9.3. If R is a semilocal regular ring, the maps $K_n\mathbf{M}^{i+1}(R) \to K_n\mathbf{M}^i(R)$ are zero for every n and i.

This conjecture implies that, for R a regular semilocal domain, (9.1) breaks into short exact sequences $0 \to K_n\mathbf{M}^i(R) \to \oplus_{ht=i} K_n(k(\mathfrak{p})) \to K_{n-1}\mathbf{M}^{i+1}(R) \to 0$, which splice to yield exact sequences (with F the field of fractions of R):

$$(9.3.1) \quad 0 \to K_n(R) \to K_n(F) \to \bigoplus_{ht(\mathfrak{p})=1} K_{n-1}(k(\mathfrak{p})) \to \cdots \to \bigoplus_{ht(\mathfrak{p})=i} K_{n-i}(k(\mathfrak{p})) \to \cdots$$

It follows of course that the spectral sequence 9.2 collapses at E_2, with $E_2^{p,q} = 0$ for $p > 0$. Hence the Gersten-Quillen conjecture implies that $K_n(R)$ is the kernel of $K_n(F) \to \oplus K_{n-1}(k(\mathfrak{p}))$ for all n.

The coniveau filtration for schemes

Of course, the above discussion extends to modules over a noetherian scheme X. Here we let $\mathbf{M}^i(X)$ denote the category of coherent $\mathcal{O}_X$-modules whose support has codimension $\geq i$. If $i < j$ we write $\mathbf{M}^i/\mathbf{M}^j$ for the quotient abelian category $\mathbf{M}^i(X)/\mathbf{M}^j(X)$. Then $\mathbf{M}^i/\mathbf{M}^{i+1}$ is equivalent to the direct sum, over all points x of codimension i in X, of the $\mathbf{M}_x(\mathcal{O}_{X,x})$; by Devissage this category has the same K-theory as its subcategory $\mathbf{M}(k(x))$. Thus the Localization Theorem yields a long exact sequence

$$(9.4) \qquad \bigoplus_y K_{n+1}(k(y)) \xrightarrow{\partial} \bigoplus_x K_n(k(x)) \to K(\mathbf{M}^{i-1}/\mathbf{M}^{i+1}) \to \bigoplus_y K_n(k(y)),$$

where y runs over all points of codimension $i-1$ and x runs over all points of codimension i. This ends in the K_1–K_0 sequence of II.6.4.3:

$$\bigoplus_{\mathrm{codim}(y)=i-1} k(y)^\times \xrightarrow{\Delta} D^i(X) \to K_0(\mathbf{M}^{i-1}/\mathbf{M}^{i+1}) \to D^{i-1}(X) \to 0,$$

where $D^i(X)$ denotes the free abelian group on the set of points of X having codimension i (see II.6.4.3). If $r/s \in k(\mathfrak{p})^\times$, the formula for Δ on $k(y)^\times$ is determined by the formula in 9.1: choose an affine open $\mathrm{Spec}(R) \subset X$ containing y; then

$\Delta(r/s)$ is $[R/(r,\mathfrak{p})] - [R/(s,\mathfrak{p})]$ in $D^i(R) \subset D^i(X)$. This gives a presentation for the Weil divisor class group $CH^i(X)$ of II.6.4.3, defined as the image of $D^i(X)$ in $K_0\mathbf{M}^{i-1}/\mathbf{M}^{i+1}$: it is the cokernel of Δ.

Now the usual Chow group $A^i(X)$ of codimension i cycles on X modulo rational equivalence, as defined in [Fulton], is the quotient of $D^i(X)$ by the following relation: for every irreducible subvariety W of $X \times \mathbb{P}^1$ having codimension i, meeting $X \times \{0,\infty\}$ properly, the cycle $[W \cap X \times 0]$ is equivalent to $[W \cap X \times \infty]$. We have the following identification, which was promised in II.6.4.3.

Lemma 9.4.1. *$CH^i(X)$ is the usual Chow group $A^i(X)$.*

Proof. The projection $W \to \mathbb{P}^1$ defines a rational function, *i.e.,* an element $t \in k(W)$. Let Y denote the image of the projection $W \to X$, and y its generic point; the proper intersection condition implies that Y has codimension $i+1$ and W is finite over Y. Hence the norm $f \in k(y)$ of t exists. Let $x \in X$ be a point of codimension i; it defines a discrete valuation ν on $k(y)$, and it is well known [Fulton] that the multiplicity of x in the cycle $[W \cap X \times 0] - [W \cap X \times \infty]$ is $\nu(f)$. $\square$

As observed in 9.1 (and II.6.6), the sequences in (9.4) do not break up for general X. The proof of Proposition 9.2 generalizes to this context to prove the following:

Proposition 9.5 (Gersten). *If X is noetherian and $\dim(X) < \infty$, there is a convergent 4^{th} quadrant cohomological spectral sequence (zero unless $p+q \le 0$):*

$$E_1^{p,q} = \bigoplus\nolimits_{codim(x)=p} K_{-p-q}(k(x)) \Rightarrow G_{-p-q}(X).$$

If X is reduced, the components $G_n(X) \to G_n(k(\eta))$ of the edge maps $G_n(X) \to E_1^{0,-n}$ associated to the generic points η of X are induced by the flat inclusions $\operatorname{Spec}(k(\eta)) \to X$.

Along the line $p+q=0$, we have $E_2^{p,-p} \cong CH^p(X)$.

Coniveau Filtration 9.5.1. The p^{th} term in the filtration on the abutment $G_n(X)$ is defined to be the image of $K_n\mathbf{M}^p(X) \to K_n\mathbf{M}(X)$. In particular, the filtration on $G_0(X)$ is the coniveau filtration of II.6.4.3.

Remark 9.5.2. The spectral sequence in 9.5 is covariant for flat morphisms, for the reasons given in Remark 9.2.2. If X is noetherian but not finite-dimensional, it converges to $\varprojlim K_*\mathbf{M}(X)/\mathbf{M}^i(X)$.

We shall now prove the Gersten-Quillen Conjecture 9.3 for algebras over a field.

Theorem 9.6. *(Quillen) Let R be an algebra of finite type over a field, and let $A = S^{-1}R$ be the semilocal ring of R at a finite set of prime ideals. Then Conjecture 9.3 holds for A: for each i the map $K\mathbf{M}^{i+1}(A) \to K\mathbf{M}^i(A)$ is zero.*

Proof. We may replace R by $R[1/f]$, $f \in S$, to assume that R is smooth. Because $\mathbf{M}^{i+1}(S^{-1}R)$ is the direct limit (over $s \in S$) of the $\mathbf{M}^{i+1}(R[1/s])$, which in turn is the direct limit (over nonzerodivisors t) of the $\mathbf{M}^i(R[1/s]/tR[1/s])$, it suffices to show that for every nonzerodivisor t of R that there is an $s \in S$ so that the functor $\mathbf{M}^i(R/tR) \to \mathbf{M}^i(R[1/s])$ is null homotopic on K-theory spaces. This is the conclusion of Proposition 9.6.1 below. $\square$

PROPOSITION 9.6.1. *(Quillen) Let R be a smooth domain over a field and $S \subset R$ a multiplicative set so that $S^{-1}R$ is semilocal. Then for each $t \neq 0$ in R with $t \notin S$ there is an $s \in S$ so that each base change $\mathbf{M}^i(R/tR) \to \mathbf{M}^i(R[1/s])$ induces a null-homotopic map on K-spaces.*

PROOF. Suppose first that R contains a subring B mapping isomorphically onto R/tR, and that R is smooth over B. We claim that the kernel I of $R \to R/tR$ is locally principal. To see this, we may assume that B is local, and even (by Nakayama's Lemma) that B is a field. But then R is a Dedekind domain and every ideal is locally principal, whence the claim. We choose s so that $I[1/s] \cong R[1/s]$.

Now for any B-module M we have the characteristic split exact sequence of $R[1/s]$-modules:

$$0 \to I[1/s] \otimes_B M \to R[1/s] \otimes_B M \to M[1/s] \to 0.$$

Since R is flat, if M is in $\mathbf{M}^i(B)$ then this is an exact sequence in $\mathbf{M}^i(R[1/s])$. That is, we have a short exact sequence of exact functors $\mathbf{M}^i(B) \to \mathbf{M}^i(R[1/s])$. By the Additivity Theorem 1.2, we see that $K\mathbf{M}^i(B) \to K\mathbf{M}^i(R[1/s])$ is null homotopic.

For the general case of 9.6.1, we need the following algebraic lemma, due to Quillen, and we refer the reader to [Q341, 5.12] for the proof.

LEMMA 9.6.2. *Suppose that $X = \operatorname{Spec}(R)$, for a finitely generated ring R over an infinite field k. If $Z \subset X$ is closed of dimension r and $T \subset X$ is a finite set of closed points, then there is a projection $X \to \operatorname{Spec}(k[t_1, ..., t_r])$ which is finite on Z and smooth at each point of T.*

Resuming the proof of 9.6, we next assume that k is an infinite field. Let $A = k[t_1, ..., t_r]$ be the polynomial subalgebra of R given by Lemma 9.6.1 so that $B = R/tR$ is finite over A and R is smooth over A at the primes of R not meeting S. Set $R' = R \otimes_A B$; then R'/R is finite and hence $S^{-1}R'$ is also semilocal, and R' is smooth over B at the primes of R' not meeting S. In particular, there is an $s \in S$ so that $R'[1/s]$ is smooth (hence flat) over B. For such s, each $\mathbf{M}^i(B) \to \mathbf{M}^i(R[1/s])$ factors through exact functors $\mathbf{M}^i(B) \to \mathbf{M}^i(R'[1/s]) \to \mathbf{M}^i(R[1/s])$. But $K\mathbf{M}^i(B) \to K\mathbf{M}^i(R'[1/s])$ is zero by the first part of the proof. It follows that $K\mathbf{M}^i(B) \to K\mathbf{M}^i(R[1/s])$ is zero as well.

Finally, if k is a finite field we invoke the following standard transfer argument. Given $x \in K_n\mathbf{M}^i(R/tR)$ and a prime p, let k'' denote the infinite p-primary algebraic extension of k; as the result is true for $R \otimes_k k''$ then there is a finite subextension k' with $[k' : k] = p^r$ so that $p^r x$ maps to zero over $R \otimes_k k'$. Applying the transfer from $K_n\mathbf{M}^i(R \otimes k')$ to $K_n\mathbf{M}^i(R)$, it follows that $F^i_*(x)$ is killed by a power of p. Since this is true for all p, we must have $F^i_*(x) = 0$. $\square$

PURE EXACTNESS 9.6.3. A subgroup B of an abelian group A is said to be *pure* if $B/nB \to A/nA$ is an injection for every n. More generally, an exact sequence A_* of abelian groups is *pure exact* if the image of each A_{n+1} is a pure subgroup of A_n; it follows that each sequence A_*/nA_* is exact. If A^i_* is a filtered family of pure exact sequences, it is easy to see that $\operatorname{colim} A^i_*$ is pure exact.

COROLLARY 9.6.4. *If R is a semilocal ring, essentially of finite type over a field, then the sequence (9.3.1) is also pure exact.*

In particular, we have exact sequences (see Ex. 9.4):

$$0 \to K_n(R)/\ell \to K_n(F)/\ell \to \bigoplus_{ht(\mathfrak{p})=1} K_{n-1}(k(\mathfrak{p}))/\ell \to \cdots,$$

$$0 \to {}_\ell K_n(R) \to {}_\ell K_n(F) \to \bigoplus_{ht(\mathfrak{p})=1} {}_\ell K_{n-1}(k(\mathfrak{p})) \to \cdots.$$

PROOF. (Grayson) Because the maps in 9.6.1 are null homotopic, we see from 1.2.2 that if $F^i_{s,t}$ is the homotopy fiber of $\mathbf{M}^i(R/tR) \to \mathbf{M}^i(R[1/s])$ the the exact sequences $0 \to K_{n+1}\mathbf{M}^i(R[1/s]) \to \pi_n F^i_{s,t} \to K_n\mathbf{M}^i(R/tR) \to 0$ splits. Taking the direct limit, we see that the sequences $0 \to K_{n+1}\mathbf{M}^i(S^{-1}R) \to \pi_n F^i \to K_n\mathbf{M}^{i+1}(R) \to 0$ are pure exact. Splicing these sequences together yields the assertion. $\square$

In mixed characteristic, Gillet and Levine proved the following result; we refer the reader to [GL] for the proof. Let Λ be a discrete valuation domain with parameter π, residue field $k = \Lambda/\pi$ of characteristic $p > 0$ and field of fractions of characteristic 0.

THEOREM 9.7. *Let A be a smooth algebra of finite type over Λ and $S \subset A$ a multiplicative set so that $R = S^{-1}A$ is semilocal. If $t \in A$ and A/tA is flat over Λ then every base change $K\mathbf{M}^i(A/tA) \to K\mathbf{M}^i(R)$ is null-homotopic.*

Now the set $T = \{t \in A : R/tR \text{ is flat over } \Lambda\}$ is multiplicatively closed, generated by the height 1 primes other than πR. Hence the localization $D = T^{-1}R$ is the discrete valuation ring $D = R_{(\pi R)}$ whose residue field is the quotient field of $R/\pi R$. By Theorem 9.7 with $i = 0$, $K_n(R) \to K_n(D)$ is an injection.

COROLLARY 9.7.1. *Let R be a regular semilocal Λ-algebra, as in 9.7. Then:*

(a) For $i \geq 1$, the transfer map $K\mathbf{M}^{i+1}(R) \to K\mathbf{M}^i(R)$ is null homotopic.

(b) Each $K_0\mathbf{M}^i(R)$ is generated by the classes $[R/\mathbf{x}R]$, where $\mathbf{x} = (x_1, \dots, x_i)$ is a regular sequence in R of length i.

(c) Sequence (9.3.1) is exact except possibly at $K_n(R)$ and $\oplus_{ht(\mathfrak{p})=1} K_{n-1}(k(\mathfrak{p}))$.

(d) If Gersten's DVR conjecture 6.9 is true for D, then the Gersten-Quillen conjecture 9.3 is true for R.

PROOF. (a) As $i+1 > 1 = \dim(D)$, each module in $K\mathbf{M}^{i+1}(R)$ vanishes over D, so it is killed by a t for which R/tR is flat. Thus $K\mathbf{M}^{i+1}(R)$ is the direct limit over $t \in T$ and $s \in S$ of the $K\mathbf{M}^i(A[1/s]/t)$; since the maps $K\mathbf{M}^i(A[1/s]/t) \to K\mathbf{M}^i(R)$ are zero by Theorem 9.7, (a) follows. We prove (b) by induction on i, the case $i = 1$ being immediate from the localization sequence

$$R^\times \to \mathrm{frac}(R)^\times \xrightarrow{\Delta} K_0\mathbf{M}^1(R) \to 0,$$

together with the formula $\Delta(r/s) = [R/rR] - [R/sR]$ of 6.1.2. For $i > 1$ the localization sequence (9.1), together with (a), yield an exact sequence

$$\oplus_{ht(\mathfrak{p})=i-1} k(\mathfrak{p})^\times \xrightarrow{\partial} K_0\mathbf{M}^i(R) \xrightarrow{0} K_0(R) \xrightarrow{\cong} K_0(\mathrm{frac}(R)) \to 0.$$

If $0 \neq r, s \in R/\mathfrak{p}$ then the formula (Ex. 5.1) for $\partial_{\mathfrak{p}} : k(\mathfrak{p})^{\times} \to K_0\mathbf{M}^i(R)$ yields $\partial_{\mathfrak{p}}(r/s) = [R/\mathfrak{p} + rR] - [R/\mathfrak{p} + sR]$. Thus it suffices to show that each $[R/\mathfrak{p} + rR]$ is a sum of terms $R/\mathbf{x}R$. Since the height 1 primes of R are principal we have $\mathfrak{p} = xR$ and $R/\mathfrak{p} + aR = R/(x,r)R$ when $i = 2$. By induction there are regular sequences $\mathbf{x}$ of length $i-1$ such that $[R/\mathfrak{p}] = \sum [R/\mathbf{x}R]$ in $K_0\mathbf{M}^{i-1}(R)$. Given $r \in R - \mathfrak{p}$ we can choose (by prime avoidance) an r' so that $\mathfrak{p} + rR = \mathfrak{p} + r'R$ and $(\mathbf{x}, r')$ is a regular sequence in R. Because $R/\mathbf{x}$ is Cohen-Macaulay, all its associated primes have height $i-1$. Hence we have the desired result (by Ex. II.6.16): $[R/\mathfrak{p} + rR] = \sum [R/(\mathbf{x}, r')R]$ in $K_0\mathbf{M}^i(R)$.

Part (c) is immediate from exactness of $K_n(R) \to K_n(E) \to K_{n-1}\mathbf{M}^1(R)$, where F is the field of fractions of R, and part (a), which yields the exact sequence:

$$0 \to K_n\mathbf{M}^1(R) \to \bigoplus_{ht(\mathfrak{p})=1} K_{n-1}(k(\mathfrak{p})) \to \cdots \to \bigoplus_{ht(\mathfrak{p})=i} K_{n-i}(k(\mathfrak{p})) \to \cdots$$

It follows that (9.3.1) is exact for R (the Gersten-Quillen conjecture 9.3 holds for R) if and only if each $K_n(R) \to K_n(F)$ is an injection. Suppose now that $K_*(D) \to K_*(F)$ is an injection (conjecture 6.9 holds for D). Then the composition $G_n(R) \hookrightarrow G_n(D) \hookrightarrow G_n(F)$ is an injection, proving (d). $\square$

K-cohomology

Fix a noetherian scheme X. For each point x, let us write i_x for the inclusion of x in X. Then we obtain a skyscraper sheaf $(i_x)_*A$ on X (for the Zariski topology) for each abelian group A. If we view the coniveau spectral sequence 9.5 as a presheaf on X and sheafify, the rows of the E_1 page assemble to form the following chain complexes of sheaves on X:

$$(9.8) \qquad 0 \to \mathcal{K}_n \to \oplus_{\mathrm{cd}=1}(i_x)_*K_n(x) \to \oplus_{\mathrm{cd}=2}(i_y)_*K_n(y) \to \cdots \to 0.$$

Here $\mathcal{K}_n$ is the sheaf associated to the presheaf $U \mapsto K_n(U)$; its stalk at $x \in X$ is $K_n(\mathcal{O}_{X,x})$. Since the stalk sequence of (9.8) at $x \in X$ is the row $q = -n$ of the coniveau spectral sequence for $R = \mathcal{O}_{X,x}$, we see that if (9.3.1) is exact for every local ring of X then (9.8) is an exact sequence of sheaves. Since each $(i_y)_*K_n(y)$ is a flasque sheaf, (9.8) is a flasque resolution of the sheaf $\mathcal{K}_n$. In summary, we have proven:

Proposition 9.8.1. *Assume that X is a regular quasi-projective scheme, or more generally that the Gersten-Quillen conjecture 9.3 holds for the local rings of X. Then (9.8) is a flasque resolution of $\mathcal{K}_n$, and the E_2 page of the coniveau spectral sequence (9.5) is*

$$E_2^{p,q} \cong H^p(X, \mathcal{K}_{-q}).$$

In addition, we have $H^p(X, \mathcal{K}_p) \cong CH^p(X)$ for all $p > 0$.

The isomorphism $H^p(X, \mathcal{K}_p) \cong CH^p(X)$ is often referred to as *Bloch's formula*, since it was first discovered for $p = 2$ by Spencer Bloch in [Bl74].

EXAMPLE 9.8.2. When X is the projective line $\mathbb{P}^1_k$ over a field k, the direct image $\pi_* : K_n(\mathbb{P}^1_k) \to K_n(k)$ is compatible with a morphism of spectral sequences (see Ex. 9.2); the only nontrivial maps are from $E_1^{1,-n-1}(\mathbb{P}^1_k) = \oplus K_n(k(x))$ to $E_1^{0,-n}(k) = K_n(k)$. As these are the transfer maps for the field extensions $k(x)/k$, this is a split surjection. Comparing with Corollary 1.5.1, we see that

$$H^0(\mathbb{P}^1_k, \mathcal{K}_n) \cong K_n(k), \quad H^1(\mathbb{P}^1_k, \mathcal{K}_n) \cong K_{n-1}(k).$$

EXERCISES

9.1 Let k be a field and S a multiplicative set in a domain $R = k[x_1, \dots, x_m]/J$ so that $S^{-1}R$ is regular. Modify the proof of Proposition 9.6 to show that for each $t \neq 0$ in R there is an $s \in S$ so that each $\mathbf{M}^i(R/tR) \to \mathbf{M}^i(R[1/s])$ induces the zero map on K-groups.

9.2 (Quillen) Let $R = k[[x_1, ..., x_n]]$ be a power series ring over a field. Modify the proof of Theorem 9.6 to show that the maps $K\mathbf{M}^{i+1}(R) \to K\mathbf{M}^i(R)$ are zero, so that (9.3.1) is exact.

If k is complete with respect to a nontrivial valuation, show that the above holds for the subring A of convergent power series in R.

9.3 (Gillet) Let $f : X \to Y$ be a proper map of relative dimension d. Show that $f_* : \mathbf{M}^i(X) \to \mathbf{M}^{i-d}(Y)$ for all i, and deduce that there is a homomorphism of spectral sequences $f_* : E_r^{p,q}(X) \to E_r^{p-d,q+d}(Y)$, compatible with the proper transfer map $f_* : G_*(X) \to G_*(Y)$ of 3.7, and the pushforward maps $CH^i(X) \to CH^{i-d}(Y)$.

9.4 Let $F : \mathbf{Ab} \to \mathbf{Ab}$ be any additive functor which commutes with filtered direct limits. Show that F sends pure exact sequences to pure exact sequences. In particular, this applies to $F(A) = {}_\ell A = \{a \in A : \ell a = 0\}$ and $F(A) = A \otimes B$.

§10. Descent and Mayer-Vietoris properties

In recent decades, the fact that K-theory has the Mayer-Vietoris property with respect to special cartesian squares of schemes has played an important role in understanding its structure. For cartesian squares describing open covers (see 10.1 below), it is equivalent to the assertion that K-theory satisfies "Zariski descent," and this in turn is related to features such as the coniveau spectral sequence of Proposition 9.2. These notions generalize from K-theory to other presheaves, either of simplicial sets or of spectra, on a finite-dimensional noetherian scheme X.

As in 10.6 below, a presheaf F is said to satisfy descent if the fibrant replacement $F(U) \to \mathbb{H}_{\text{zar}}(U, F)$ is a weak equivalence for all U, where "fibrant replacement" is with respect to the local injective model structure (defined in 10.5 below). From a practical viewpoint, the recognition criterion for descent is the Mayer-Vietoris property.

Definition 10.1. Let F be a presheaf of simplicial sets (or spectra) on a scheme X. We say that F has the *Mayer-Vietoris property* (for the Zariski topology on X) if for every pair of open subschemes U and V the following square is homotopy cartesian.

$$\begin{array}{ccc} F(U \cup V) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U \cap V). \end{array}$$

Before defining Zariski descent, we motivate it by stating the following theorem, due to Brown and Gersten. We postpone the proof of this theorem until later in this section.

Theorem 10.2. *Let F be a presheaf of simplicial sets (or spectra) on X. Then F satisfies Zariski descent if and only if it has the Mayer-Vietoris property.*

Examples 10.3. Recall that $G(X)$ denotes the K-theory space for the category of coherent sheaves on X. Since restriction of sheaves is functorial, $U \mapsto G(U)$ is a presheaf on X for the Zariski topology. Theorem 6.11.2 states that G has the Mayer-Vietoris property on any noetherian scheme X. This was the original example of a presheaf satisfying Zariski descent, discovered in 1972 by Brown and Gersten in [BG].

On the other hand, let $\mathbf{K}^B$ denote the non-connective K-theory spectrum of vector bundles (IV.10.6). As above, $U \mapsto \mathbf{K}^B(U)$ is a presheaf on X. Using big vector bundles IV.10.5, we may even arrange that it is a presheaf on schemes of finite type over X. Corollary 7.10 states that $\mathbf{K}^B$ has the Mayer-Vietoris property on any noetherian scheme X. (This fails for the connective spectrum $\mathbf{K}$ because the map $K_0(U) \oplus K_0(V) \to K_0(U \cap V)$ may not be onto. Corollary 7.11 states that KH has the Mayer-Vietoris property on any noetherian scheme X.

Model categories

In order to define Zariski descent in the category of presheaves on X, we need to consider two model structures on this category. Recall that a map $i : A \to B$ is said to have the *left lifting property* relative to a class $\mathcal{F}$ of morphisms if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{b} & Y \end{array}$$

with p in $\mathcal{F}$ there is a lift $h : B \to X$ such that $hi = a$ and $ph = b$.

DEFINITION 10.4. Let $\mathcal{C}$ be a complete and cocomplete category. A *model structure* on $\mathcal{C}$ consists of three subcategories whose morphisms are called *weak equivalences*, *fibrations* and *cofibrations*, together with two functorial factorizations of each morphism, satisfying the following properties.

(1) (2-out-of-3) If f and g are composable morphisms in $\mathcal{C}$ and two of f, g, fg are weak equivalences, so is the third.
(2) (Retracts) If f is a retract of g, and g is a weak equivalence (resp., a fibration, resp., a cofibration) then so is f.

A map which is both a cofibration and a weak equivalence is called a *trivial cofibration*, and a map which is both a fibration and a weak equivalence is called a *trivial fibration*. These notions are used in the next two axioms.

(3) (Lifting) A trivial cofibration has the left lifting property with respect to fibrations. Similarly, cofibrations have the left lifting property with respect to trivial fibrations.
(4) (Factorization) One of the two functorial factorizations of a morphism is as a trivial cofibration followed by a fibration; the other is as a cofibration followed by a trivial fibration.

A *model category* is a category $\mathcal{C}$ with a model structure. The *homotopy category* of $\mathcal{C}$, $\mathrm{Ho}\,\mathcal{C}$, is defined to be the localization of $\mathcal{C}$ with respect to the class of weak equivalences. (This kind of localization is described in the Appendix to Chapter II.)

We refer the reader to [Hovey] for more information about model categories. The main point is that a model structure provides us with a calculus of fractions for the homotopy category of $\mathcal{C}$.

EXAMPLES 10.4.1. The standard example is the model structure on the category of simplicial sets, in which cofibrations are injections, fibrations are Kan fibrations; a map $X \to Y$ is a weak equivalence if its geometric realization $|X| \to |Y|$ is a homotopy equivalence. ($|X|$ is defined in IV.3.1.4.)

There is also a model structure on the category of spectra. Since the details depend on the exact definition of spectrum used, we will not dwell on this point. The model category of symmetric spectra is introduced and studied in [HSS].

DEFINITION 10.5. A morphism $A \to B$ of presheaves (of either simplicial sets or spectra) is called a *global weak equivalence* if each $A(U) \to B(U)$ is a weak equivalence; it is called a (Zariski) *local weak equivalence* if it induces an isomorphism on (Zariski) sheaves of homotopy groups (resp., stable homotopy groups).

We say that $A(U) \to B(U)$ is a *cofibration* if each $A(U) \to B(U)$ is a cofibration; a *local injective fibration* is a map which has the right lifting property with respect to cofibrations which are local weak equivalences. Jardine showed that the Zariski-local weak equivalences, cofibrations and local injective fibrations determine model structures on the categories of presheaves of simplicial sets, and of spectra (see [J, 2.3]). We shall call these the *local injective* model structures, to distinguish them from other model structures in the literature.

In any model structure, an object B is called *fibrant* if the terminal map $B \to *$ is a fibration. A *fibrant replacement* of an object A is a trivial cofibration $A \to B$ with B fibrant. By the Factorization axiom, there is a functorial fibrant replacement. For the local injective model structure on presheaves, we write the fibrant replacement as $A \to \mathbb{H}_{\mathrm{zar}}(-, A)$, and write $\mathbb{H}^n_{\mathrm{zar}}(X, A)$ for $\pi_{-n}\mathbb{H}_{\mathrm{zar}}(X, A)$.

Definition 10.6. Let A be a presheaf of either simplicial presheaves or spectra. We say that A satisfies *Zariski descent* on a scheme X if the fibrant replacement $A \to \mathbb{H}_{\mathrm{zar}}(-, A)$ is a global weak equivalence. That is, if $A(U) \xrightarrow{\simeq} \mathbb{H}_{\mathrm{zar}}(U, A)$ is a weak equivalence for every open U in X.

Note that although inverse homotopy equivalences $\mathbb{H}_{\mathrm{zar}}(U, A) \to A(U)$ exist, they are only natural in U up to homotopy (unless A is fibrant).

Given any abelian group A and a positive integer n, the *Eilenberg-Mac Lane spectrum* $K(A, n)$ is a functorial spectrum satisfying $\pi_n K(A, n) = A$, and $\pi_i K(A, n) = 0$ for $i \neq n$. More generally, given a chain complex A_* of abelian groups, the *Eilenberg-Mac Lane spectrum* $K(A_*, n)$ is a functorial spectrum satisfying $\pi_k K(A_*, n) = H_{k-n}(A_*)$. See [WHomo, 8.4.1, 10.9.19 and Ex. 8.4.4] for one construction.

Example 10.6.1. Let $\mathcal{A}$ be a Zariski sheaf on X. The *Eilenberg-Mac Lane spectrum* $K(\mathcal{A}, n)$ is the sheaf $U \mapsto K(\mathcal{A}(U), n)$. If $\mathcal{A} \to \mathcal{I}$ is an injective resolution, then $\mathbb{H}(-, \mathcal{A})$ is weak equivalent to the sheaf of Eilenberg-Mac Lane spectra $K(\mathcal{I}, 0)$, so $\mathbb{H}^n(X, \mathcal{A}) = \pi_{-n}\mathbb{H}(X, \mathcal{A}) \cong \pi_{-n}K(\mathcal{I}, 0)(X) = H^n\mathcal{I}(X)$ is the usual sheaf cohomology group $H^n(X, \mathcal{A})$. This observation, that the fibrant replacement is the analogue of an injective resolution, is due to Brown-Gersten [BG] and dubbed the "Great Enlightenment" by Thomason.

Remark 10.6.2. Given any Grothendieck topology t, we obtain the notion of *t-local weak equivalence* of presheaves by replacing Zariski sheaves by t-sheaves in the above definition. Jardine also showed that the t-local weak equivalences and cofibrations determine a model structure on the category of presheaves, where the t-local injective fibrations are defined by the right lifting property. Thus we have other fibrant replacements $A \to \mathbb{H}_t(-, A)$.

We say that A *satisfies t-descent* (on the appropriate site) if the fibrant replacement $A \to \mathbb{H}_t(-, A)$ is a global weak equivalence. The most commonly used versions are: Zariski descent, étale descent, Nisnevich descent and *cdh* descent.

If A is a presheaf of simplicial sets, we can sheafify it to form a Zariski sheaf, $a_{\mathrm{zar}}A$. Since $A \to a_{\mathrm{zar}}A$ is a weak equivalence, their fibrant replacements are weak equivalent. This is the idea behind our next lemma.

Lemma 10.7. *Let A, B be presheaves of simplicial sets on schemes. If B satisfies Zariski descent, then any natural transformation $\eta_R : A(\mathrm{Spec}\ R) \to B(\mathrm{Spec}\ R)$ (from commutative rings to simplicial sets) extends to a natural transformation $\eta_U : A(U) \to \mathbb{H}(U, B)$.*

The composite $A(U) \to B(U)$ with a homotopy equivalence $\mathbb{H}(U,B) \xrightarrow{\simeq} B(U)$ exists, but (unless B is fibrant) is only well defined up to weak equivalence.

PROOF. Define the presheaf A_{aff} by $A_{\mathrm{aff}}(U) = A(\mathrm{Spec}\, \mathcal{O}(U))$, and define B_{aff} similarly; the natural map $A_{\mathrm{aff}} \to A$ is a weak equivalence because it is so locally. Clearly η extends to $\eta_{\mathrm{aff}} : A_{\mathrm{aff}} \to B_{\mathrm{aff}}$. The composite map in the diagram

$$\begin{tikzcd} A(U) \arrow[r] & \mathbb{H}_{\mathrm{zar}}(U,A) & \mathbb{H}_{\mathrm{zar}}(U,A_{\mathrm{aff}}) \arrow[l, "\simeq"', "\to"] \arrow[d, "\eta_{\mathrm{aff}}"] \\ B(U) \arrow[r, "\simeq"] & \mathbb{H}_{\mathrm{zar}}(U,B) & \mathbb{H}_{\mathrm{zar}}(U,B_{\mathrm{aff}}) \arrow[l, "\simeq"'] \end{tikzcd}$$

is the natural transformation we desire. Note that the top right horizontal map is a trivial fibration, evaluated at U; the existence of a natural inverse, *i.e.,* a presheaf map $\mathbb{H}_{\mathrm{zar}}(-,A) \to \mathbb{H}_{\mathrm{zar}}(-,A_{\mathrm{aff}})$ splitting it, follows from the lifting axiom 10.4(3). Thus the map $A(U) \to \mathbb{H}_{\mathrm{zar}}(U,B)$ is natural in U. $\square$

REMARK 10.7.1. In Lemma 10.7, η only needs to be defined on a category of commutative rings containing the rings $\mathcal{O}(U)$ and maps $\mathcal{O}(U) \to \mathcal{O}(V)$.

Zariski descent and the Mayer-Vietoris property

If F is a presheaf of spectra which satisfies Zariski descent, then F has the Mayer-Vietoris property, by Exercise 10.5. In order to show that the converse holds, we need the following result, originally proven for simplicial presheaves in [BG, Thm. 1′].

PROPOSITION 10.8. *Let X be a finite-dimensional noetherian space, and F a presheaf of spectra on X which has the Mayer-Vietoris property (for the Zariski topology). If all the presheaves $\pi_q F$ have zero as associated sheaves, then $\pi_q F(X) = 0$ for all q.*

PROOF. It suffices to prove the following assertion for all $d \geq 0$: for all open $X' \subseteq X$, all q and all $a \in \pi_q F(X')$, there exists an open $U \subseteq X'$ with $a|_U = 0$ and $\mathrm{codim}_X(X-U) \geq d$. Indeed, when $d > \dim(X)$ we must have $U = X'$ and hence $a|_{X'} = 0$ as required.

The assertion is clear for $d = 0$ (take $U = \emptyset$), so suppose that it holds for d and consider $a \in \pi_q F(X')$ as in the assertion. By induction, $a|_U = 0$ for an open U whose complement Z has codimension $\geq d$. Let $x_1, \dots, x_n$ be the generic points of Z of codimension d. By assumption, there is a neighborhood V of these points such that $a|_V = 0$, and $X - V$ has codimension at least d. The Mayer-Vietoris property gives an exact sequence

$$\pi_{q+1}F(U \cap V) \xrightarrow{\partial} \pi_q F(U \cup V) \to \pi_q F(U) \oplus \pi_q F(V).$$

If $a|_{U \cup V}$ vanishes we are done. If not, there is a $z \in \pi_{q+1}F(U \cap V)$ with $a = \partial(z)$. By induction, with $X'' = U \cap V$, there is an open W in X'' whose complement has codimension $\geq d$ and such that $z|_W = 0$. Let $y_1, \dots$ be the generic points of $X'' - W$, Y the closure of these points in V and set $V' = V - Y$. Looking at codimensions, we see that V' is also a neighborhood of the x_i and that $U \cap V' = W$. Mapping the above sequence to the exact sequence

$$\pi_{q+1}F(W) \xrightarrow{\partial} \pi_q F(U \cup V') \to \pi_q F(U) \oplus \pi_q F(V'),$$

we see that $a|_{U\cup V'} = \partial(z_| W) = 0$. As $U \cup V'$ contains all the x_i, its complement has codimension $> d$. This completes the inductive step, proving the desired result. $\square$

PROOF OF THEOREM 10.2. We have already noted that the 'only if' direction holds by Ex. 10.5. In particular, since $\mathbb{H}(-, E)$ is fibrant, it always has the Mayer-Vietoris property. Now assume that E has the Mayer-Vietoris property, and let $F(U)$ denote the homotopy fiber of $E(U) \to \mathbb{H}(U, E)$. Then F is a presheaf, and F has the Mayer-Vietoris property by Exercise 10.1. Since $E \to \mathbb{H}(-, E)$ is a local weak equivalence, the stalks of the presheaves $\pi_q F$ are zero. By Proposition 10.8, $\pi_q F(U) = 0$ for all U. From the long exact homotopy sequence of a fibration, this implies that $\pi_* E(U) \cong \pi_* \mathbb{H}(U, E)$ for all U, *i.e.,* $E \to \mathbb{H}(-, E)$ is a global weak equivalence. $\square$

Nisnevich descent

To discuss descent for the Nisnevich topology, we need to introduce some terminology. A commutative square of schemes of the form

$$\begin{array}{ccc} U_Y & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow f \\ U & \xrightarrow{i} & X \end{array}$$

is called *upper distinguished* if $U_Y = U \times_X Y$, U is open in X, f is étale and $(Y - U_Y) \to (X - U)$ is an isomorphism of the underlying reduced closed subschemes. The *Nisnevich topology* on the category of schemes of finite type over X is the Grothendieck topology generated by coverings $\{U \to X, Y \to X\}$ for the upper distinguished squares. In fact, a presheaf F is a sheaf if and only if F takes upper distinguished squares to cartesian (*i.e.,* pullback) squares; see [MVW, 12.7].

For each point $x \in X$, there is a canonical map from the hensel local scheme $\operatorname{Spec} \mathcal{O}^h_{X,x}$ to X; these form a conservative family of points for the Nisnevich topology. Thus a sheaf $\mathcal{F}$ is zero if and only if its stalks are zero at these points, and if $a \in \mathcal{F}(X)$ is zero then for every point x there is a point y in an étale $Y \to X$ such that $k(x) \cong k(y)$ and $a|_Y = 0$.

DEFINITION 10.9. A presheaf of simplicial sets on X is said to have the Mayer-Vietoris property for the Nisnevich topology if it sends upper distinguished squares to homotopy cartesian squares. (Taking $Y = V$ and $X = U \cup V$, this implies that it also has the Mayer-Vietoris property for the Zariski topology.)

THEOREM 10.10. *A presheaf has the Mayer-Vietoris property for the the Nisnevich topology if and only if it satisfies Nisnevich descent.*

PROOF. (Thomason) The proof of Theorem 10.2 goes through, replacing Ex. 10.5 by Ex. 10.6 and Proposition 10.8 by Ex. 10.7 $\square$

EXAMPLES 10.10.1. As in 10.3, the localization sequence 6.11 for $G(X) \to G(U)$ and devissage (Ex. 4.2) show that G satisfies Nisnevich descent.

The functor $\mathbf{K}^B$ satisfies Nisnevich descent, by [TT, 10.8]. It follows formally from this, and the definition of KH (IV.12.1) that homotopy K-theory KH also satisfies Nsnevich descent.

The descent spectral sequence

The descent spectral sequence, due to Brown and Gersten [BG], is a method for computing the homotopy groups of a presheaf F of spectra on X. It is convenient to write F_n for the presheaf of abelian groups $U \mapsto \pi_n F(U)$, and let $\mathcal{F}_n$ denote the Zariski sheaf associated to the presheaf F_n.

THEOREM 10.11. *Let X be a noetherian scheme with $\dim(X) < \infty$. Then for every presheaf of spectra F having the Mayer-Vietoris property (for the Zariski topology on X) there is a spectral sequence:*

$$E_2^{pq} = H^p_{\text{zar}}(X, \mathcal{F}_{-q}) \Rightarrow F_{-p-q}(X).$$

PROOF. For each U, we have a Postnikov tower $\cdots P_nF(U) \xrightarrow{p_n} P_{n-1}F(U) \cdots$ for $F(U)$; each p_n is a fibration, $\pi_q P_n F(U) = \pi_q F(U)$ for $n \le q$ and $\pi_q P_n F(U) = 0$ for all $n > q$. It follows that the fiber of p_n is an Eilenberg-Mac Lane space $K(\pi_n F(U), n)$. Since the Postnikov tower is functorial, we get a tower of presheaves of spectra. Since the fibrant replacement is also functorial, we have a tower of spectra $\cdots \mathbb{H}(-, P_n F) \xrightarrow{p'_n} \mathbb{H}(-, P_{n-1}F) \cdots$. By Ex. 10.2, the homotopy fiber of p'_n is $K(\pi_n \mathcal{F}, n)$. Thus we have an exact couple with $D^1_{pq} = \oplus \pi_{p+q}\mathbb{H}(X, P_p F)$ and $E^1_{pq} = \oplus \pi_{p+q} K(\pi_p \mathcal{F}, p)(X) = \oplus H^{-q}(X, \mathcal{F}_p)$. Since the spectral sequence is bounded, it converges to $\varinjlim D^1_* = \pi_* F(X)$; see [WHomo, 5.9.7]. Re-indexing as in [WHomo, 5.4.3], and rewriting E^2 as E_2 yields the desired cohomological spectral sequence. $\square$

REMARK 10.11.1. The proof of Theorem 10.11 goes through for the Nisnevich topology. That is, if F has the Mayer-Vietoris property for the Nisnevich topology, there is a spectral sequence:

$$E_2^{pq} = H^p_{\text{nis}}(X, \mathcal{F}_{-q}) \Rightarrow F_{-p-q}(X).$$

EXAMPLE 10.12. As in Example 10.2, the G-theory presheaf $\mathbf{G}$ satisfies the Mayer-Vietoris property, so there is a fourth quadrant spectral sequence with $E_2^{p,q} = H^p(X, \mathcal{G}_{-q})$ converging to $G_*(X)$. Here $\mathcal{G}_n$ is the sheaf associated to the presheaf G_n. This is the original Brown-Gersten spectral sequence of [BG].

Similarly, the presheaf $U \mapsto \mathbf{K}^B$ satisfies the Mayer-Vietoris property, so there is a spectral sequence with $E_2^{p,q} = H^p(X, \mathcal{K}_{-q})$ converging to $K_*(X)$. This spectral sequence lives mostly in the fourth quadrant. For example for $F = \mathbf{K}$ we have $\mathcal{K}_0 = \mathbb{Z}$ and $\mathcal{K}_1 = \mathcal{O}_X^\times$ (see Section II.2 and III.1.4), so the rows $q = 0$ and $q = -1$ are the cohomology groups $H^p(X, \mathbb{Z})$ and $H^p(X, \mathcal{O}_X^\times)$. This spectral sequence first appeared in [TT].

If X is regular, of finite type over a field, then Gillet and Soulé proved in [GiS, 2.2.4] that the descent spectral sequence 10.11 is isomorphic to the spectral sequence of Proposition 9.8.1, which arises from the coniveau spectral sequence of 9.2. This assertion holds more generally if X is regular and the Gersten-Quillen Conjecture 9.3 holds for the local rings of X.

EXERCISES

10.1 Let $F \to E \to B$ be a sequence of presheaves which is a homotopy fibration sequence when evaluated at any U. If E and B have the Mayer-Vietoris property, for either the Zariski or Nisnevich topology, show that F does too.

10.2 Suppose that $F \to E \to B$ is a sequence of presheaves such that each $F(U) \to E(U) \to B(U)$ is a fibration sequence. Show that each $\mathbb{H}(U, F) \to \mathbb{H}(U, E) \to \mathbb{H}(U, B)$ is a homotopy fibration sequence.

10.3 Let F be a simplicial presheaf which is fibrant for the local injective model structure 10.5. Show that $F(V) \to F(U)$ is a Kan fibration for every $U \subset V$.

10.4 Let F be a simplicial presheaf which is fibrant for the local injective model structure 10.5. Show that F has the Mayer-Vietoris property. *Hint:* ([MV, 3.3.1]) If F is fibrant, $\hom(-, F)$ takes homotopy cocartesian squares to homotopy cartesian squares of simplicial sets. Apply this to the square involving U, V and $U \cap V$.

10.5 (Jardine) Let F be a presheaf of spectra which satisfies Zariski descent on X. Show that F has the Mayer-Vietoris property. *Hint:* We may assume that F is fibrant for the local injective model structure of 10.5. In that case, F is a sequence of simplicial presheaves F^n, fibrant for the local injective model structure on simplicial presheaves, such that all the bonding maps $F^n \to \Omega F^{n+1}$ are global weak equivalences; see [J].

10.6 Let F be a presheaf of spectra satisfying Nisnevich descent. Show that F has the Mayer-Vietoris property for the Nisnevich topology.

10.7 Show that the Nisnevich analogue of Proposition 10.8 holds: if F has the Mayer-Vietoris property for the Nisnevich topology, and the Nisnevich sheaves associated to the $\pi_q F$ are zero, then $\pi_q F(X) = 0$ for all q. *Hint:* Consider the $f^{-1}(x_i) \subset Y$.

10.8 (Gillet) Let G be a sheaf of groups on a scheme X, and $\rho : G \to \mathrm{Aut}(\mathcal{F})$ a representation of G on a locally free $\mathcal{O}_X$-module $\mathcal{F}$ of rank n. Show that ρ determines a homotopy class of maps from $B_\bullet G$ to $B_\bullet GL_n(\mathcal{O}_X)$. *Hint:* Use the Mayer-Vietoris property for $\mathbb{H}(-, B_\bullet GL_n(\mathcal{O}_X))$ and induction on the number of open subschemes in a cover trivializing $\mathcal{F}$.

10.9 Suppose that X is a 1-dimensional noetherian scheme, with singular points $x_1, \dots, x_s$. If A_i is the local ring of X at x_i, show that $K_{-1}(X) \cong \oplus K_{-1}(A_i)$. (Note that $K_n(X) = K_n(A_i) = 0$ for all $n \leq -2$, by Ex. III.4.4.)

Using the fact (III.4.4.3) that K_{-1} vanishes on hensel local rings, show that $K_{-1}(X) \cong H^1_{\mathrm{nis}}(X, \mathbb{Z})$. This group is isomorphic to $H^1_{\mathrm{et}}(X, \mathbb{Z})$; see III.4.1.4.

10.10 Suppose that X is an irreducible 2-dimensional noetherian scheme, with isolated singular points $x_1, \dots, x_s$. If A_i is the local ring of X at x_i, show that there is an exact sequence

$$0 \to H^2_{\mathrm{zar}}(X, \mathcal{O}_X^\times) \to K_{-1}(X) \to \oplus K_{-1}(A_i) \to 0.$$

If X is normal, it is well known that $H^1_{\mathrm{nis}}(X,\mathbb{Z})=0$. In this case, use III.4.4.2 to show that $K_{-1}(X)\cong H^2_{\mathrm{nis}}(X,\mathcal{O}_X^\times)$. (This can be nonzero, as Ex. III.4.13 shows.)

§11. Chern classes

The machinery involved in defining Chern classes on higher K-theory can be overwhelming, so we begin with two simple constructions over rings.

DENNIS TRACE MAP 11.1. Let R be a ring. It is a classical fact that $H_*(G;R)$ is a direct summand of $HH_*(R[G])$ for any group G; see [WHomo, 9.1.2]. Using the ring maps $R[GL_m(R)] \to M_m(R)$ and Morita invariance of Hochschild homology [WHomo, 9.5], we have natural maps

$$HH_*(R[GL_m(R)]) \to HH_*(M_m(R)) \xrightarrow{\simeq} HH_*(R).$$

Via the Hurewicz map $\pi_n BGL(R)^+ \xrightarrow{h} H_n(BGL(R)^+, \mathbb{Z})$, this yields homomorphisms:

$$K_n(R) \xrightarrow{h} H_n(GL(R), \mathbb{Z}) = \varinjlim_m H_n(GL_m(R), \mathbb{Z}) \to HH_n(R).$$

They are called the *Dennis trace maps*, having been discovered by R. K. Dennis around 1975. In fact, they are ring homomorphisms; see [Igusa, p. 133].

Given any representation $\rho : G \to GL_m(R)$, there is a natural map

$$H_*(G, \mathbb{Z}) \to HH_*(R[G]) \xrightarrow{\rho} HH_*(R[GL_m(R)]) \to HH_*(M_m(R)) \xrightarrow{\simeq} HH_*(R).$$

This map, regarded as an element of $\mathrm{Hom}(H_*(G), HH_*(R))$, naturally lifts to an element $c_1(\rho)$ of $\mathbb{H}^0(G, HH)$. The verification that the $c_1(\rho)$ form Chern classes (in the sense of 11.2 below) is left to Exercise 11.4.

Let Ω^*_R denote the exterior algebra of Kähler differentials of R over $\mathbb{Z}$ [WHomo, 9.4.2]. If $\mathbb{Q} \subset R$, there is a projection $HH_n(R) \to \Omega^n_R$ sending $r_0 \otimes \cdots \otimes r_n$ to $(r_0/n!)\, dr_1 \wedge \cdots \wedge dr_n$. Gersten showed that, up to the factor $(-1)^{n-1}n$, this yields Chern classes $c_n : K_n(R) \to \Omega^n_R$. We encountered them briefly in Chapter III (Ex.6.10 and 7.7): if x, y are units of R then $c_2(\{x,y\}) = -\frac{dx}{x} \wedge \frac{dy}{y}$.

Chern classes for rings

Suppose that $A = A(0) \oplus A(1) \oplus \cdots$ is a graded-commutative ring. Then for any group G, the group cohomology $\oplus H^i(G, A(i))$ is a graded-commutative ring (see [WHomo, 6.7.11]). More generally we may suppose that each $A(i)$ is a chain complex, so that $A = A(0) \oplus A(1) \oplus \cdots$ is a graded dg ring; the group hypercohomology $\oplus H^i(G, A(i))$ is still a graded-commutative ring

DEFINITION 11.2. A *theory of Chern classes* for a ring R with coefficients A is a rule assigning to every group G and every representation $G \xrightarrow{\rho} \mathrm{Aut}_R(P)$ with P in $\mathbf{P}(R)$ elements $c_i(\rho) \in H^i(G, A(i))$ with $c_0(\rho) = 1$ and satisfying the following axioms. The formal power series $c_t(\rho) = \sum c_i(\rho) t^i$ in $1 + \prod H^i(G, A(i))$ is called the *total Chern class* of ρ.

(1) *Functoriality.* For each homomorphism $\phi : H \to G$, $c_i(\rho \circ \phi) = \phi^* c_i(\rho)$.
(2) *Triviality.* The trivial representation ε of G on R has $c_i(\varepsilon) = 0$ for $i > 0$. If $\rho : G \to \mathrm{Aut}(P)$ then $c_i(\rho) = 0$ for $i > \mathrm{rank}(P)$.

(3) *Sum Formula.* For all ρ_1 and ρ_2, $c_n(\rho_1 \oplus \rho_2) = \sum_{i+j=n} c_i(\rho_1)c_j(\rho_2)$. Alternatively, $c_t(\rho_1 \oplus \rho_2) = c_t(\rho_1)c_t(\rho_2)$.
(4) *Multiplicativity.* For all ρ_1 and ρ_2, $c_t(\rho_1 \otimes \rho_2) = c_t(\rho) * c_t(\rho_2)$. Here $*$ denotes the product in the (non-unital) λ-subring $1 + \prod_{i>0} H^i(G, A(i))$ of $W(H^*(G,A))$; see II.4.3. (We ignore this axiom if R is non-commutative.)

EXAMPLE 11.2.1. Taking $G = 1$, the Chern classes $c_i(\rho)$ belong to the cohomology $H^iA(i)$ of the complex $A(i)$. Since $\rho : 1 \to \mathrm{Aut}(P)$ depends only on P, we write $c_i(P)$ for $c_i(\rho)$. The Sum Formula for $c_t(P \oplus Q)$ shows that the total Chern class c_t factors through $K_0(R)$, with $c_t([P]) = c_t(P)$. Comparing the above axioms to the axioms given in II.4.11, we see that a theory of Chern classes for R induces Chern classes $c_i : K_0(R) \to H^iA(i)$ in the sense of II.4.11.

VARIANT 11.2.2. A common trick is to replace $A(i)$ by its cohomological shift $B(i) = A(i)[i]$. Since $H^nB(i) = H^n\mathcal{A}(i)[i] = H^{i+n}A(i)$, a theory of Chern classes for B is a rule assigning elements $c_i(\rho) \in H^{2i}(G, A(i))$.

UNIVERSAL ELEMENTS 11.2.3. The tautological representations $\mathrm{id}_n : GL_n(R) \to \mathrm{Aut}(R^n)$ play a distinguished role in any such theory. They are interconnected by the natural inclusions $\rho_{mn} : GL_n(R) \hookrightarrow GL_m(R)$ for $m \geq n$, and if $m \geq n$ then $c_n(\mathrm{id}_n) = \rho^*_{mn}c_n(\mathrm{id}_m)$. These elements form an inverse system, and we set

$$c_n(\mathrm{id}) = \varprojlim_m \ c_n(\mathrm{id}_m) \in \varprojlim_m \ H^n(GL_m(R), A(n)).$$

The elements $c_n(\mathrm{id})$ are universal in the sense that, if $P \oplus Q \cong R^m$ and we extend $G \xrightarrow{\rho} \mathrm{Aut}(P)$ to $\rho \oplus 1_Q : G \to GL_m(R)$, then we have $c_n(\rho) = (\rho \oplus 1_Q)^* c_n(\mathrm{id}_m)$.

To get Chern classes, we need to recall how the cohomology $H^*(G,A)$ is computed. Let B_* denote the bar resolution of G (it is a projective $\mathbb{Z}[G]$-module resolution of $\mathbb{Z}$), and set $C = B_* \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, so that $H_*(G,\mathbb{Z}) = H_*C$. Since G acts trivially on A, $H^*(G,A)$ is the cohomology of $\mathrm{Hom}_{\mathbb{Z}[G]}(B_*, A) = \mathrm{Hom}_{\mathbb{Z}}(C, A)$. The i^{th} cohomology of this Hom complex is the group of chain homotopy equivalence classes of maps $C \to A[i]$, and each such class determines a map from $H_n(G,\mathbb{Z}) = H_n(C)$ to $H_{n-i}A = H^{i-n}A$; see [WHomo, 2.7.5].

When A is a complex of $\mathbb{Z}/m$-modules, we could replace the bar construction B_* with B_*/mB_* in the above: $H^*(G,A)$ is the cohomology of $\mathrm{Hom}(C/mC, A)$. Since $H_*(G, \mathbb{Z}/m) = H_*(C/mC)$, we get maps $H_*(G,\mathbb{Z}/m) \to H^{i-n}A$.

LEMMA 11.3. *A theory of Chern classes for a ring R with coefficients A yields homomorphisms for all $n, i > 0$, called Chern classes:*

$$c_{i,n} : K_n(R) \to H_{n-i}A(i) = H^{i-n}A(i).$$

If A is a graded ring, the maps are zero for $i \neq n$ and we simply have classes

$$c_i : K_i(R) \to A(i).$$

When R is commutative, we have the product rule: if $a_m \in K_m(R)$ and $a_n \in K_n(R)$ then

$$c_{m+n}(a_m a_n) = \frac{-(m+n-1)!}{(m-1)!(n-1)!} c_m(a_m)c_n(a_n).$$

In particular, $c_n(\{x_1,\dots,x_n\}) = (-1)^{n-1}(n-1)!c_1(x_1)\cdots c_1(x_n)$ *for all* $x_1,\dots,x_n$ *in* $K_1(R)$.

PROOF. By the above remarks, each element $c_i(\mathrm{id}_m)$ determines a homomorphism $H_*(GL_m(R),\mathbb{Z}) \to H_*A(i)$. By functoriality, these maps are compatible as m varies and yield a homomorphism c_i from $H_*(GL(R),\mathbb{Z}) = \varinjlim H_*(GL_m(R),\mathbb{Z})$ to $H_*A(i)$. Via the Hurewicz map h, this yields homomorphisms for $n > 0$:

$$c_{i,n} : K_n(R) \xrightarrow{h} H_n(GL(R),\mathbb{Z}) \xrightarrow{c_i} H^{i-n}A(i).$$

In the special case that A is a ring, the target vanishes for $i \neq n$. The product rule follows from Exercise 11.3, which states that the right side is $c_m(a_m) * c_n(a_n)$. □

CONSTRUCTION 11.3.1. Here is a homotopy-theoretic construction for the maps $c_{i,n}$ of Lemma 11.3, due to Quillen. Recall that the Dold-Kan construction applied to the chain complex $A(i)[i]$ produces an H-space $K(A(i),i)$, called a *generalized Eilenberg-Mac Lane space*, with $\pi_n K(A(i),i) = H_{n-i}A(i)$. It classifies cohomology in the sense that for any topological space X, elements of $H^n(X,A(i))$ are in 1–1 correspondence with homotopy classes of maps $X \to K(A(i),n)$; for $X = BG$, elements of $H^n(G,A(i)) = H^n(BG,A(i))$ correspond to maps $BG \to K(A(i),n)$. See [WHomo, 6.10.5 and 8.6.4].

In this way, each $c_i(\mathrm{id})$ determines a system of maps $BGL_m(R) \to K(A(i),i)$, compatible up to homotopy. As $K(A(i),i)$ is an H-space, these maps factor through the spaces $BGL_m(R)^+$; via the telescope construction, they determine a map $BGL(R)^+ \to K(A(i),i)$. The Chern classes of 11.3 may also be defined as:

$$c_{i,n} : K_n(R) = \pi_n BGL(R)^+ \to \pi_n K(A(i),i) = H^{i-n}A(i).$$

The map $BGL(R)^+ \to K(A(i),i)$ can be made functorial in R by the following trick. Apply the integral completion functor $\mathbb{Z}_\infty$ (IV.1.9.ii) to get a functorial map $\mathbb{Z}_\infty BGL(R) \to \mathbb{Z}_\infty K(A(i),i)$, and choose a homotopy inverse for the homotopy equivalence $K(A(i),i) \to \mathbb{Z}_\infty K(A(i),i)$.

FINITE COEFFICIENTS 11.3.2. If A is a complex of $\mathbb{Z}/m$-modules, then by the remarks before 11.3, the $c_i(\mathrm{id}_m)$ determine a homomorphism from $H_*(GL(R),\mathbb{Z}/m)$ to $H_*A(i)$. Via the Hurewicz map h (IV.2.4), this yields homomorphisms for $n > 0$:

$$c_{i,n} : K_n(R;\mathbb{Z}/m) \xrightarrow{h} H_n(GL(R),\mathbb{Z}/m) \xrightarrow{c_i} H^{i-n}A(i).$$

The product rule holds in this case if m is odd or if $8|m$; see IV.2.8.

EXAMPLE 11.4. Suppose that $\mathcal{A}$ is a complex of Zariski sheaves on $X = \mathrm{Spec}(R)$. If $\mathcal{A} \to \mathcal{I}$ is an injective resolution (or the total complex of a Cartan-Eilenberg resolution if $\mathcal{A}$ is a bounded below complex), then $H^*\mathcal{I}(X) = H^*(X,\mathcal{A})$. If $\mathcal{A} = \oplus\mathcal{A}(i)$, a theory of Chern classes for R with coefficients $A = \oplus\mathcal{I}(i)(X)$ yields Chern classes $c_{i,n} : K_n(R) \to H^{i-n}(X,\mathcal{A}(i))$. We will see many examples of this construction below.

Chern classes for schemes

One way to get a theory of Chern classes for a scheme X is to consider equivariant cohomology groups $H^*(X,G,\mathcal{A})$. The functors $\mathcal{F}\mapsto H^n(X,G,\mathcal{F})$ are defined to be the right derived functors of the functor $H^0(X,G,\mathcal{F}) = \mathrm{Hom}_{X,G}(\mathbb{Z}_X,\mathcal{F}) = H^0(X,\mathcal{F}^G)$, where $\mathcal{F}$ belongs to the abelian category of sheaves of G-modules on X. Since it is useful to include open subschemes of X, we want to extend the definition in a natural way over a category of schemes including X.

DEFINITION 11.5. Let $\mathcal{V}$ be a category of schemes, and $\mathcal{A}=\oplus\mathcal{A}(i)$ a graded complex of Zariski sheaves of abelian groups on $\mathcal{V}$, forming a graded sheaf of dg rings. A *theory of Chern classes* on $\mathcal{V}$ is a rule associating to every X in $\mathcal{V}$, every sheaf G of groups on X and every representation ρ of G in a locally free $\mathcal{O}_X$-module $\mathcal{F}$, elements $c_i(\rho)\in H^i(X,G,\mathcal{A}(i))$ for $i\geq 0$, with $c_0(\rho)=1$, satisfying the following axioms (resembling those of 11.2). Here $c_t(\rho)$ denotes the total Chern class $\sum c_i(\rho)t^i$ in $1+\prod H^i(X,G,\mathcal{A}(i))$.

(1) *Functoriality.* For each compatible system ϕ of morphisms $X\xrightarrow{f}X'$, $G\to f^*G'$ and $\mathcal{F}\to f^*\mathcal{F}'$ we have $c_i(\phi^*\rho)=\phi^*c_i(\rho)$.
(2) *Triviality.* The trivial representation ε of G on $\mathcal{O}_X$ has $c_i(\varepsilon)=0$ for $i>0$. If $\rho: G\to \mathrm{Aut}(\mathcal{F})$ then $c_i(\rho)=0$ for $i>\mathrm{rank}(\mathcal{F})$.
(3) *Whitney sum formula.* If $0\to\rho'\to\rho\to\rho''\to 0$ is a short exact sequence of representations of G in locally free $\mathcal{O}_X$-modules, then $c_t(\rho)=c_t(\rho')c_t(\rho'')$.
(4) *Multiplicativity.* Given representations ρ_1 and ρ_2 of G on locally free $\mathcal{O}_X$-modules, $c_t(\rho_1\otimes\rho_2)=c_t(\rho)*c_t(\rho_2)$. Here $*$ denotes the product in the non-unital λ-subring $\prod_{i>0}H^i(X,G,\mathcal{A}(i))$ of $W(H^*(X,G,\mathcal{A}))$; see II.4.3.

EXAMPLE 11.5.1. Taking $G=1$, the Chern classes $c_i(\rho)$ belong to the cohomology $H^i(X,\mathcal{A}(i))$ of the complex $\mathcal{A}(i)$. By the Whitney sum formula, the total Chern class c_t factors through $K_0(X)$; cf. 11.2.1. Comparing the above axioms to the axioms given in II.4.11, we see that a theory of Chern classes for X induces Chern classes $c_i: K_0(X)\to H^i(X,\mathcal{A}(i))$ in the sense of II.4.11.

LEMMA 11.6. *Let G be a sheaf of groups and $\mathcal{A}$ a complex of Zariski sheaves on a scheme X, with injective resolution $\mathcal{A}\to\mathcal{I}$. Then there are natural maps*

$$H^n(X,G,\mathcal{A})\to H^n(G,\mathcal{I}(X))\to\bigoplus_{i+j=n}\mathrm{Hom}\left(H_i(G,\mathbb{Z}),H^j(X,\mathcal{A})\right).$$

PROOF. Since G acts trivially on $\mathcal{A}$, we can compute the equivariant cohomology of $\mathcal{A}$ as hyperExt groups. Take an injective sheaf resolution $\mathcal{A}\to\mathcal{I}$ and let B_* denote the bar resolution of G, with $C=B_*\otimes_{\mathbb{Z}[G]}\mathbb{Z}$ as before. Then

$$H^n(X,G,A)=H^n\,\mathrm{Hom}_{\mathbb{Z}[G]}(B_*,\mathbb{Z})\otimes\mathcal{I}(X)=H^n\,\mathrm{Hom}_{\mathbb{Z}}(C,\mathbb{Z})\otimes\mathcal{I}(X).$$

The cohomology of the natural map $\mathrm{Hom}(C,\mathbb{Z})\otimes\mathcal{I}(X)\to\mathrm{Hom}(C,\mathcal{I}(X))$ is the desired map $H^n(X,G,A)\to H^n(G,\mathcal{I}(X))$. The second map in the Lemma comes from the discussion before Lemma 11.3, since $H^*\mathcal{I}(X)=H^*(X,\mathcal{A})$. $\square$

COROLLARY 11.7. *A theory of Chern classes for* $\mathrm{Spec}(R)$ *with coefficients* $\mathcal{A}$ *(in the sense of Definition 11.5) determines a theory of Chern classes for the ring* R *with coefficients* $\mathcal{I}(X)$ *(in the sense of Definition 11.2), and hence (by 11.3) Chern class homomorphisms*

$$c_{i,n} : K_n(R) \to H^{i-n}(\mathrm{Spec}(R), \mathcal{A}(i)).$$

PROPOSITION 11.8. *A theory of Chern classes for* X *with coefficients* $\mathcal{A}$ *determines Chern class homomorphisms*

$$c_{i,n} : K_n(X) \to H^{i-n}(X, \mathcal{A}(i)).$$

PROOF. By 11.7, there is a theory of Chern classes for rings, and by Construction 11.3.1 there are natural transformations $c_i : K_0(R) \times BGL(R)^+ \to K(\mathcal{A}(i)(R), i))$. Let $K(\mathcal{O}_X)$ denote the presheaf sending X to $K_0(R) \times BGL(R)^+$, where $R = \mathcal{O}(X)$; the map $K \to K(\mathcal{O}_X)$ is a local weak equivalence (10.5). By construction, the presheaf $\mathbb{H}(-, K(\mathcal{A}(i), i))$ satisfies Zariski descent (10.6). By Lemma 10.7, each of the c_i extends to a morphism of presheaves $c_i : \mathbb{H}_{\mathrm{zar}}(-, K(\mathcal{O}_X)) \to \mathbb{H}(-, K(\mathcal{A}(i), i))$. Composing with $K(X) \to \mathbb{H}_{\mathrm{zar}}(X, K) \to \mathbb{H}_{\mathrm{zar}}(X, K(\mathcal{O}_X))$ and taking homotopy groups yields the desired Chern classes on X. $\square$

EXAMPLE 11.9. Let $\mathcal{K}_i$ be the Zariski sheaf on a scheme X associated to the presheaf $U \mapsto K_i(U)$. (This sheaf was discussed in 9.8 and 10.12.) Gillet showed in his 1978 thesis (see [Gillet, §8]) that there is a theory of Chern classes with coefficients $\oplus\mathcal{K}_i$, defined on the category of (smooth) varieties of finite type over a field. By 11.7 and 11.8, this yields Chern classes for each algebra, $c_{i,n} : K_n(R) \to H^{i-n}(\mathrm{Spec}(R), \mathcal{K}_i)$, and more generally Chern classes $K_n(X) \to H^{i-n}(X, \mathcal{K}_i)$. Via the isomorphism $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$ of I.5.10, the Chern class $c_{1,0} : K_0(X) \to \mathrm{Pic}(X)$ is the determinant II.2.6. The Splitting Principle II.8.8 implies that the Chern classes $c_{i,0} : K_0(X) \to H^i(X, \mathcal{K}_i) \cong CH^i(X)$ are the same as the Chern classes of II.8.9, discovered by Grothendieck (where the last isomorphism is Bloch's formula 9.8.1).

If x is a unit of R, the map $c_{1,0} : R^\times \to H^0(R, \mathcal{K}_1) = R^\times$ is the natural identification, so by the product rule the map $c_{n,n} : K_n(R) \to H^0(R, \mathcal{K}_n)$ satisfies

$$c_{n,n}(\{x_1, \dots, x_n\}) = (-1)^{n-1}(n-1)!\{x_1, \dots, x_n\}.$$

Étale Chern classes

EXAMPLE 11.10. In [GDix, p. 245], Grothendieck constructed a theory of étale Chern classes $c_i(\rho) \in H^{2i}_{\mathrm{et}}(X, G, \mu_m^{\otimes i})$ for every X over $\mathrm{Spec}(\mathbb{Z}[1/m])$. This is the case of Definition 11.5 in which $\mathcal{A}(i) = R\pi_*\mu_m^{\otimes i}[i]$ (using the shift trick of 11.2.2), where $R\pi_*$ is the direct image functor from étale sheaves to Zariski sheaves: $H^{i-n}(X, \mathcal{A}(i)) = H^{2i-n}_{\mathrm{et}}(X, \mu_m^{\otimes i})$. This yields a theory of étale Chern classes and hence (by 11.8) Chern class maps

$$c_{i,n} : K_n(X; \mathbb{Z}/m) \to H^{2i-n}(X, R\pi_*\mu_m^{\otimes i}) = H^{2i-n}_{\mathrm{et}}(X, \mu_m^{\otimes i}).$$

Soulé introduced the étale Chern classes $K_n(R;\mathbb{Z}/m) \xrightarrow{c_{i,n}} H^{2i-n}_{\mathrm{et}}(\mathrm{Spec}(R), \mu_m^{\otimes i})$ in [Sou], using them to detect many of the elements of $K_n(\mathbb{Z})$ described in Chapter VI. Suslin used the étale class $c_{2,2}$ to describe $K_3(F)$ in [Su91]; see VI.5.19 below.

By construction, $c_1(\rho)$ is the boundary of the element in $H^1_{\mathrm{et}}(X, G, \mathbb{G}_m)$ classifying the G-line bundle of $\det(\rho)$. Letting G be $R^\times$ and $\rho : G \to \mathrm{Aut}(R)$ the canonical isomorphism, we see that $c_{1,1} : R^\times \to H^1_{\mathrm{et}}(\mathrm{Spec}(R), \mu_m)$ is the Kummer map $R^\times/R^{\times n} \subset H^1_{\mathrm{et}}(\mathrm{Spec}(R), \mu_m)$. Therefore the Chern class $c_{1,1} : K_1(R) \to H^1_{\mathrm{et}}(\mathrm{Spec}(R), \mu_m)$ is the determinant $K_1(R) \to R^\times$ followed by the Kummer map.

LEMMA 11.10.1. *If R contains a primitive m^{th} root of unity ζ, with corresponding Bott element $\beta \in K_2(R;\mathbb{Z}/m)$, then $c_{1,2} : K_2(R;\mathbb{Z}/m) \to H^0_{\mathrm{et}}(R, \mu_m)$ sends $K_2(R)/m$ to 0 and satisfies $c_{1,2}(\beta) = \zeta$.*

PROOF. Recall that $H^1_{\mathrm{et}}(\mathrm{Spec}(R), \mathbb{G}_m) \cong R^\times$. We claim that the left square in the following diagram is commutative. (The right square clearly commutes.)

$$\begin{array}{ccccccc}
K_2(R) & \longrightarrow & K_2(R;\mathbb{Z}/m) & \xrightarrow{\text{Bockstein}} & K_1(R) & \xrightarrow{m} & K_1(R) \\
 & & \Big\downarrow{\scriptstyle c_{1,2}} & & \Big\downarrow{\scriptstyle \det} & & \Big\downarrow{\scriptstyle \det} \\
0 & \to & H^0_{\mathrm{et}}(R, \mu_m) & \longrightarrow & H^0_{\mathrm{et}}(R, \mathbb{G}_m) & \xrightarrow{m} & H^0_{\mathrm{et}}(R, \mathbb{G}_m).
\end{array}$$

Since the Bockstein sends β to ζ, and $K_2(R)$ to 0, the lemma will follow.

To see the claim, we fix injective resolutions $\mathbb{G}_m \to \mathcal{J}$ and $\mu_m \to \mathcal{I}$; we have a distinguished triangle

$$\mathcal{I} \to \mathcal{J} \xrightarrow{m} \mathcal{J} \xrightarrow{\delta} \mathcal{I}[1].$$

If C is the standard chain complex for G introduced before Lemma 11.3, then δ induces a map $\mathrm{Hom}(C, \mathbb{Z}) \otimes \mathcal{J}(X) \to \mathrm{Hom}(C, \mathbb{Z}) \otimes \mathcal{I}(X)[1]$. By construction, the boundary $H^1_{\mathrm{et}}(X, G, \mathbb{G}_m) \to H^2_{\mathrm{et}}(X, G, \mu_m)$ is the cohomology of this map. By Lemma 11.6, this induces a map

$$\mathrm{Hom}(H_1(G, \mathbb{Z}), H^0_{\mathrm{et}}(X, \mathbb{G}_m)) \to \mathrm{Hom}(H_2(G, \mathbb{Z}/m), H^0_{\mathrm{et}}(X, \mu_m)).$$

For $G = GL_n(R)$, $c_{1,2}$ is the image of det, and the result follows. $\square$

Motivic Chern classes

Let $\mathbb{Z}(i)$, $i \geq 0$ denote the motivic complexes (of Zariski sheaves), defined on the category $\mathcal{V}$ of smooth quasi-projective varieties over a fixed field k, as in [MVW]. The *motivic cohomology* of X, $H^{n,i}(X, \mathbb{Z})$, is defined to be $H^n(X, \mathbb{Z}(i))$. Here is our main result, due to S. Bloch [Bl86]:

THEOREM 11.11. *There is a theory of Chern classes with coefficients in motivic cohomology. The associated Chern classes are $c_{i,n} : K_n(X) \to H^{2i-n,i}(X)$.*

REMARK. The Chern classes $c_{i,0} : K_0(X) \to H^{2i,i}(X) \cong CH^i(X)$ are the same as the Chern classes of II.8.9, discovered by Grothendieck in 1957. The class $c_{1,1}$: $K_1(R) \to H^{1,1}(\mathrm{Spec}(R)) \cong R^\times$ is the same as the determinant map of III.1.1.1.

Before proving Theorem 11.11, we use it to generate many other applications.

Examples 11.12. A. Huber has constructed natural multiplicative transformations from motivic cohomology to other cohomology theories; see [Hu]. Applying them to the motivic classes $c_i(\rho)$ yields a theory of Chern classes, and hence Chern classes, for these other cohomology theories. Here is an enumeration.

(1) *de Rham.* If k is a field of characteristic 0, and X is smooth, the de Rham cohomology groups $H^*_{dR}(X)$ are defined using the de Rham complex Ω^*_X. Thus we have Chern classes $c_i : K_n(X) \to H^{2i-n}_{dR}(X)$.

(2) *Betti.* If X is a smooth variety over $\mathbb{C}$, there is a topological space $X(\mathbb{C})$ and its Betti (=topological) cohomology is $H^*_{\text{top}}(X(\mathbb{C}), \mathbb{Z})$. The Betti Chern classes are $c_i : K_n(X) \to H^{2i-n}_{\text{top}}(X(\mathbb{C}), \mathbb{Z})$, arising from the realization $H^{n,i}(X) \to H^n_{\text{top}}(X(\mathbb{C}), \mathbb{Z})$. Beilinson has pointed out that the image of the Betti Chern classes in $H^*_{\text{top}}(X(\mathbb{C}), \mathbb{C})$ are compatible with the de Rham Chern classes, via the Hodge structure on $H^*_{\text{top}}(X(\mathbb{C}), \mathbb{C})$.

Of course, if R is a $\mathbb{C}$-algebra the natural maps $GL(R) \to GL(R)^{\text{top}}$ (described in IV.3.9.2) induce the transformations $H_*(GL(R)) \to H^{\text{top}}_*(GL(R)^{\text{top}})$, as well as $H^{n,i}(X, GL_n(R)) \to H^n_{\text{top}}(X(\mathbb{C}), GL_n(R)^{\text{top}})$. It should come as no surprise that these Chern classes are nothing more than the maps $K_n(X) \to KU^{-n}(X(\mathbb{C}))$ followed by the topological Chern classes $c_i : KU^{-n}(X(\mathbb{C})) \to H^{2i-n}_{\text{top}}(X(\mathbb{C}))$ of II.3.7.

(3) *Deligne-Beilinson.* If X is a smooth variety defined over $\mathbb{C}$, the Deligne-Beilinson cohomology groups $H^*_{\mathcal{D}}(X, \mathbb{Z}(*))$ are defined using truncations of the augmented de Rham complex $\mathbb{Z} \to \Omega^*_X$. The map $H^{n,i}(X) \to H^n_{\mathcal{D}}(X, \mathbb{Z}(i))$ induces Chern classes $K_n(X) \to H^{2i-n}_{\mathcal{D}}(X, \mathbb{Z}(i))$. These were first described by Beilinson.

(4) *ℓ-adic.* If k is a field of characteristic $\neq \ell$, the ℓ-adic cohomology of a variety makes sense and there are maps $H^{n,i}(X) \to H^n_\ell(X, \mathbb{Z}_\ell(i))$. This gives ℓ-adic Chern classes
$$c_i : K_n(X) \to H^{2i-n}_{\text{et}}(X, \mathbb{Z}_\ell(i)).$$
The projection to $H^{2i-n}_{\text{et}}(X, \mu_{\ell^\nu}^{\otimes i})$ recovers the étale Chern classes of 11.9.

We shall now use the motivic Chern classes to prove that $K^M_*(F) \to K_*(F)$ is an injection modulo torsion for any field F. For this we use the ring map $K^M_*(F) \xrightarrow{\simeq} \oplus_i H^{i,i}(\text{Spec}(F))$, induced by the isomorphism $K^M_1(F) = F^\times \cong H^{1,1}(F)$ and the presentation of $K^M_*(F)$. It is an isomorphism, by a theorem of Totaro and Nesterenko-Suslin [NS].

Lemma 11.13. *Let F be a field. Then the composition of $K^M_i(F) \to K_i(F)$ and $c_{i,i} : K_i(F) \to H^{i,i}(\text{Spec}(F)) \cong K^M_i(F)$ is multiplication by $(-1)^{i-1}(i-1)!$.*

In particular, $K^M_*(F) \to K_*(F)$ is an injection modulo torsion.

Proof. Since $c_{1,1}(a) = a$, the product rule (11.3) implies that $c_{i,i} : K_i(F) \to K^M_i(F)$ takes $\{a_1, \dots, a_i\}$ to $(-1)^{i-1}(i-1)!\{a_1, \dots, a_i\}$. $\square$

The outlines of the following proof are due to Grothendieck, and use only formal properties of motivic cohomology; these properties will be axiomatized in 11.14.

Proof of Theorem 11.11. Let $B_\bullet G$ denote the simplicial group scheme
$$\text{Spec}(k) \leftleftarrows G \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, G \times G \,\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\, \cdots,$$

and similarly for $E_\bullet G$. Given a representation ρ of G in a locally free $\mathcal{O}_X$-module $\mathcal{F}$, G acts on the geometric vector bundle $\mathbb{A}$ of rank n associated to $\mathcal{F}$, and we can construct the projective bundle $\mathbb{P} = \mathbb{P}(\mathbb{A}) \times_G E_\bullet G$ over $B_\bullet G$. There is a canonical line bundle $\mathcal{L}$ on $\mathbb{P}$, and an element $\xi \in H^{2,1}(\mathbb{P}) \cong H^1(\mathbb{P}, \mathcal{O}^\times) = \mathrm{Pic}(\mathbb{P})$ associated to $\mathcal{L}$. By the Projective Bundle Theorem [MVW, 15.12], $H^{*,*}(\mathbb{P})$ is a free $H^{*,*}(B_\bullet G)$-module with basis the elements $\xi^i \in H^{2i,i}(\mathbb{P})$ $(0 \le i < n)$. That is, we have an isomorphism

$$\oplus_{i=0}^{n-1} H^{p-2i,q-i}(B_\bullet G) \xrightarrow{(1,\xi,\dots,\xi^{n-1})} H^{p,q}(\mathbb{P}).$$

The Chern classes $c_i = c_i^G(\rho) \in H^{2i,i}(B_\bullet G)$ are defined by the coefficients of $x^n \in H^{2n,n}(\mathbb{P})$ relative to this basis, by the following equation in $H^{2n,n}(\mathbb{P})$:

$$\xi^n - c_1\xi^{n-1} + \cdots + (-1)^i c_i \xi^{n-i} + \cdots + (-1)^n c_n = 0.$$

By definition, we have $c_i^G(\rho) = 0$ for $i > n$. For each n, this construction is natural in G and ρ. Moreover, if $\epsilon : G \to GL_n$ is the trivial representation, then $\mathbb{P}$ is $\mathbb{P}^{n-1} \times B_\bullet G$; since $\xi^n = 0$ in $H^{2n,n}(\mathbb{P}^{n-1})$, we have $c_i^G(\epsilon) = 0$ for all $i > 0$.

We now fix n and X. For every sheaf $\mathcal{A}$ and sheaf G of groups on X, there is a canonical map $H^*(B_\bullet G, \mathcal{A}) \to H^*(X, G, \mathcal{A})$; setting $\mathcal{A} = \mathbb{Z}(i)$, we define the classes $c_i(\rho) \in H^{2i}(X, G, \mathbb{Z}(i))$ to be the images of the $c_i \in H^{2i,i}(B_\bullet G)$ under this map.

It remains to verify the axioms in Definition 11.5. The Functoriality and Triviality axioms have been already checked. For the Whitney sum formula, suppose that we are given representations ρ' and ρ'' in locally free sheaves $\mathcal{F}'$, $\mathcal{F}''$ of rank n' and n'', respectively, and a representation ρ on an extension $\mathcal{F}$. The projective space bundle $\mathbb{P}(\mathcal{F})$ contains $\mathbb{P}(\mathcal{F}')$, and $\mathbb{P}(\mathcal{F}) - \mathbb{P}(\mathcal{F}')$ is a vector bundle over $\mathbb{P}(\mathcal{F}'')$. The localization exact sequence of [MVW, (14.5.5)] becomes:

$$H^{2n-2n'',n'}(\mathbb{P}(\mathcal{F}'')) \xrightarrow{i_*} H^{2n,n}(\mathbb{P}(\mathcal{F})) \xrightarrow{j^*} H^{2n,n}(\mathbb{P}(\mathcal{F}'')).$$

Now both $i_*(1)$ and $f'' = \xi^{n''} - c_1(\rho'') + \cdots \pm c_{n''}$ generate the kernel of j^*. By the projection formula, $f' = \xi^{n'} - c_1(\rho') + \cdots \pm c_{n'}$ satisfies $0 = i_*(i^*f') = i_*(1)f'$. Hence $0 = f''f'$ in $H^{2n,n}(\mathbb{P}(\mathcal{F}))$. Since this is a monic polynomial in ξ, we must have

$$\xi^n - c_1(\rho) + \cdots + \pm c_n = (\xi^{n''} - c_1(\rho'') + \cdots \pm c_{n''})(\xi^{n'} - c_1(\rho') + \cdots \pm c_{n'}).$$

Equating coefficients give the Whitney sum formula.

For the Multiplicativity axiom, we may use the splitting principle to assume both sheaves have filtrations with line bundles as quotients. By the Whitney sum formula, we may further reduce to the case in which $\mathcal{F}_1$ and $\mathcal{F}_2$ are lines bundles. In the case the formula is just the isomorphism $\mathrm{Pic}(X) \cong H^{2,1}(X)$, $\mathcal{L} \mapsto c_1(\mathcal{L})$; see [MVW, 4.2]. □

Twisted duality theory

Although many cohomology theories (such as those listed in 11.12) come equipped with a natural map from motivic cohomology, and thus have induced Chern classes, it is possible to axiomatize the proof of Theorem 11.11 to avoid the need from such a natural map. This was done by Gillet in [Gillet], using the notion of a twisted duality theory. Fix a category $\mathcal{V}$ of schemes over a base S, and a graded sheaf $\oplus\mathcal{A}(i)$ of dg rings, as in 11.5. We assume that $\mathcal{A}$ satisfies the homotopy invariance property: $H^*(X, \mathcal{A}(*)) \cong H^*(X \times \mathbb{A}^1, \mathcal{A}(*))$.

Definition 11.14. A *twisted duality theory* on $\mathcal{V}$ with coefficients $\mathcal{A}$ consists of a covariant bigraded "homology" functor $H_n(-,\mathcal{A}(i))$ on the subcategory of proper maps in $\mathcal{V}$, equipped with contravariant maps j^* for open immersions and (for X flat over S of relative dimension d) a distinguished element η_X of $H_{2d}(X,\mathcal{A}(d))$ called the *fundamental classes* of X, subject to the following axioms.

(i) For every open immersion $j: U \hookrightarrow X$ and every proper map $p: Y \to X$, the following square commutes:

$$\begin{array}{ccc} H_n(Y,\mathcal{A}(i)) & \xrightarrow{j_Y^*} & H_n(U\times_X Y,\mathcal{A}(i)) \\ \big\downarrow{\scriptstyle p_!} & & \big\downarrow{\scriptstyle p_{U\,!}} \\ H_n(X,\mathcal{A}(i)) & \xrightarrow{j^*} & H_n(U,\mathcal{A}(i)). \end{array}$$

(ii) If $j: U \to X$ is an open immersion with closed complement $\iota: Z \to X$, there is a long exact sequence (natural for proper maps)

$$\cdots H_n(Z,\mathcal{A}(i)) \xrightarrow{\iota_!} H_n(X,\mathcal{A}(i)) \xrightarrow{j^*} H_n(U,\mathcal{A}(i)) \xrightarrow{\partial} H_{n-1}(Z,\mathcal{A}(i)) \cdots$$

(iii) For each (X,U,Z) as in (ii), there is a *cap product*

$$H_p(X,\mathcal{A}(r)) \otimes H_Z^q(X,\mathcal{A}(s)) \xrightarrow{\cap} H_{p-q}(Z,\mathcal{A}(r-s)),$$

which is a pairing of presheaves on each X, and such that for each proper map $p: Y \to X$ the *projection formula* holds:

$$p_!(y)\cap z = p_{Z\,!}(y\cap p^*(z)), \qquad y\in H_p(Y,\mathcal{A}(r)), z\in H_Z^q(X,\mathcal{A}(s)).$$

(iv) If X is smooth over S of relative dimension d, and $Z \hookrightarrow X$ is closed, the cap product with η_X is an isomorphism: $H_Z^{2d-n}(X,\mathcal{A}(d-s)) \xrightarrow{\simeq} H_n(Z,\mathcal{A}(s))$. (This axiom determines $H_*(Z,\mathcal{A}(*))$.) In addition:
- the isomorphism $H^0(X,\mathcal{A}(0)) \xrightarrow{\eta_X\cap} H_{2d}(X,\mathcal{A}(d))$ sends 1 to η_X;
- If Z has codimension 1, the fundamental class η_Z corresponds to an element $[Z]$ of $H_Z^2(X,\mathcal{A}(1))$, *i.e.*, $\eta_X\cap[Z]=\eta_Z$. Writing cycle(Z) for the image of $[Z]$ in $H^2(X,\mathcal{A}(1))$, we require that these cycle classes extend to a natural transformation $\mathrm{Pic}(X)\to H^2(X,\mathcal{A}(1))$.

(v) If $Z \xrightarrow{\iota} X$ is a closed immersion of smooth schemes (of codimension c) then the isomorphism $H^n(Z,\mathcal{A}(r)) \cong H_Z^{n+2c}(X,\mathcal{A}(r+c))$ in (iv) is induced by a map $\iota_!: \mathcal{A}(r) \xrightarrow{\simeq} R\iota^!\mathcal{A}(r+c)[2c]$ in the derived category of Z, where $\iota^!$ is the "sections with support" functor. The projection formula of (iii) for $Z\to X$ is represented in the derived category of X by the commutative diagram

$$\begin{array}{ccc} R\iota_!\mathcal{A}(r)\otimes^{\mathbb{L}}\mathcal{A}(s) & \xrightarrow{R\iota_!(1\otimes\iota^!)} & R\iota_!\big(\mathcal{A}(r)\otimes^{\mathbb{L}}\mathcal{A}(s)\big)\cong R\iota_!\mathcal{A}(r+s) \\ \simeq\big\downarrow{\scriptstyle \iota_!\otimes 1} & & \simeq\big\downarrow{\scriptstyle \iota_!} \\ R\iota^!\mathcal{A}(r+c)[2c]\otimes^{\mathbb{L}}\mathcal{A}(s) & \longrightarrow & R\iota^!\mathcal{A}(r+s+c)[2c]. \end{array}$$

(vi) For all $n\geq 1$ and X in $\mathcal{V}$, let $\xi\in H^2(\mathbb{P}_X^n,\mathcal{A}(1))$ be the cycle class of a hyperplane. Then the map $\pi^*: H_*(X,\mathcal{A}(*)) \to H^*(\mathbb{P}_X^n,\mathcal{A}(*))$, composed with the cap product with powers of ξ, is onto, and the cap product with powers of ξ induces an isomorphism:

$$\bigoplus_{i=0}^{n} H_{n+2i}(X, \mathcal{A}(q+i)) \xrightarrow{(1,\xi,\dots,\xi^n)\cap\pi^*} H_n(\mathbb{P}^n_X, \mathcal{A}(q)).$$

THEOREM 11.14.1. [Gillet, 2.2] *If there is a twisted duality theory on $\mathcal{V}$ with coefficients $\mathcal{A}$, then there is a theory of Chern classes on $\mathcal{V}$ with coefficients $\mathcal{A}$.*

EXERCISES

11.1 Suppose given a theory of Chern classes for a ring A with coefficients in a cochain complex C, as in 11.2, and let $R_A(G)$ denote the representation ring of G over A (see Ex. II.4.2). Show that there are well defined functions $c_n : R_A(G) \to H^i(G, C(i))$ forming Chern classes on the λ-ring $R_A(G)$ in the sense of II.4.11. If K is the space $K(C) = \prod K(C(n), n)$, show that they determine a natural transformation $R_A(G) \to [BG, K(C)]$. By the universal property of the $+$-construction (IV.5.7), this yields a unique element of $[BGL(A)^+, K(C)]$. Compare this with the construction in 11.3.

11.2 Consider the unital λ-ring $\mathbb{Z}\times(1+\prod H^n(G, A(n)))$ with multiplication $(a, f)*(b, g) = (ab, f^b g^a(f*g))$. Show that $(\mathrm{rank}, c_t) : K_*(X) \to \mathbb{Z}\times(1+\prod H^n(G, A(n)))$ is a homomorphism of unital λ-rings.

11.3 (Grothendieck) Let $H = \oplus H^n$ be a graded ring and write W for the non-unital subalgebra $1 + \prod H^n$ of $W(H)$. Show that for $x \in H^m$, $y \in H^n$ we have

$$(1-x)*(1-y) = 1 - \frac{-(m+n-1)!}{(m-1)!(n-1)!}xy + \cdots .$$

Hint: In the universal case $H = \mathbb{Z}[x, y]$, $W(H)$ embeds in $W(H\otimes\mathbb{Q})$. Now use the isomorphism $W(H\otimes\mathbb{Q}) \cong \prod H\otimes\mathbb{Q}$ and compute in the mn coordinate.

11.4 *Dennis trace as a Chern class.* Let C_* denote the standard chain complex with $H_n(C_*) = HH_n(R)$; see [WHomo, 9.1]. In the notation of 11.2.3, there is a natural chain map $B_*\otimes\mathbb{Z} \to C_*(\mathbb{Z}[G])$, representing an element of $H^0\operatorname{Hom}(B_*\otimes\mathbb{Z}, C_*(\mathbb{Z}[G]))$, which on homology is the $H_*(G,\mathbb{Z}) \to HH_*(\mathbb{Z}[G])$ of 11.1; see [WHomo, 9.7.5]. Since the $C_*(M_m(R)) \to C_*(R)$ are chain maps [WHomo, 9.5.7], any representation ρ determines a chain map and an associated element $c_1(\rho)$ of $H^0(G, C_*(R)) = H^0\operatorname{Hom}(B_*\otimes\mathbb{Z}, C_*(R))$.

Setting $A(0) = \mathbb{Z}$, $A(1) = C_*[-1]$ and $A(n) = 0$ for $n > 1$, we may regard $c_1(\rho)$ as an element of $H^1(G, A(1))$. Show that $c_1(\rho)$ determines a theory of Chern classes with $c_i = 0$ for $i \geq 2$. *Hint:* the product $c_1(\rho_1)c_1(\rho_2)$ is in $H^2(G, A(2)) = 0$, and the method of 11.2.3 applies.

11.5 Suppose that R contains both $1/m$ and the group μ_m of primitive m^{th} roots of unity. Let $\rho : \mu_m \to GL_2(R)$ be the representation $\rho(\zeta)(x, y) = (\zeta x, \zeta^{-1}y)$. Show that the étale Chern class $c_{2,2}(\rho) : H_4(\mu_m, \mathbb{Z}/m) \to H^0_{\mathrm{et}}(R, \mu_m^{\otimes 2}) \cong \mathbb{Z}/m$ is an isomorphism. *Hint:* if $\lambda : \mu_m \to R^\times$ is the tautological representation, show that $c_{2,2} = c_{1,1}(\lambda)^2$.

11.6 Show that a theory of Chern classes with coefficients $\mathcal{A}(i)$ yields Chern classes with coefficients $\mathcal{B}(i)=\mathcal{A}(i)\otimes^{\mathbb{L}}\mathbb{Z}/m$, and that there is a commutative diagram

$$\begin{array}{ccccc} K_n(X) & \longrightarrow & K_n(X;\mathbb{Z}/m) & \longrightarrow & K_{n-1}(R) \\ \downarrow{\scriptstyle c_i} & & \downarrow{\scriptstyle c_i} & & \downarrow{\scriptstyle c_i} \\ H^{i-n}(X,\mathcal{A}(i)) & \longrightarrow & H^{i-n}(X,\mathcal{B}(i)) & \longrightarrow & H^{i+1-n}(X,\mathcal{A}(i)). \end{array}$$

Hint: There is a triangle $\mathrm{Hom}(C_*,\mathcal{A})[1]\to\mathrm{Hom}(C_*,\mathcal{A}\otimes^{\mathbb{L}}\mathbb{Z}/m)\to\mathrm{Hom}(C_*,\mathcal{A})\to$.

As an application of this, let $\mathcal{A}(i)$ be the complex for Deligne-Beilinson cohomology of a complex variety; $\mathcal{B}(i)$ computes étale cohomology, and we have a commutative diagram

$$\begin{array}{ccccc} K_n(X) & \longrightarrow & K_n(X;\mathbb{Z}/m) & \longrightarrow & K_{n-1}(R) \\ \downarrow{\scriptstyle c_i} & & \downarrow{\scriptstyle c_i} & & \downarrow{\scriptstyle c_i} \\ H^{2i-n}_{\mathcal{D}}(X,\mathbb{Z}(i)) & \longrightarrow & H^{2i-n}_{\mathrm{et}}(X,\mu_m^{\otimes i}) & \longrightarrow & H^{2i+1-n}_{\mathcal{D}}(X,\mathbb{Z}(i)). \end{array}$$

11.7 Recall from Kummer theory that $H^1_{\mathrm{et}}(X,\mu_m)\cong {}_m\mathrm{Pic}(X)\oplus U(X)/mU(X)$, where $U(X)=\mathcal{O}(X)^{\times}$. Modifying Example 11.10, show that the Chern class $c_{1,1}:K_1(X;\mathbb{Z}/m)\to H^1_{\mathrm{et}}(X,\mu_m)$ sends $K_1(X)/m$ to $U(X)/mU(X)$ by the determinant, and that the induced quotient map $c_{1,1}:{}_mK_0(X)\to{}_m\mathrm{Pic}(X)$ may be identified with the canonical map of II.8.1.

11.8 (Suslin) Suppose given a theory of Chern classes for a field F with coefficients A. Show that the Chern class $c_n:K_n(F)\to H^0A(n)$ factors through Suslin's map $K_n(F)\to K_n^M(F)$ described in IV.1.15. In particular, the K-cohomology Chern class $c_n:K_n(F)\to H^0(\mathrm{Spec}(F),\mathcal{K}_n)\cong K_n(F)$ factors through $K_n(F)\to K_n^M(F)$. *Hint:* Use $K_n^M(F)\cong H_n(GL_n(F))/H_n(GL_{n-1}(F))$ to reduce to the Triviality axiom 11.2(2) that $c_n(\mathrm{id}_{n-1})=0$.