CHAPTER IV

DEFINITIONS OF HIGHER K-THEORY

The higher algebraic K-groups of a ring R are defined to be the homotopy groups $K_n(R) = \pi_n K(R)$ of a certain topological space $K(R)$, which we shall construct in this chapter. Of course, the space $K(R)$ is rigged so that if $n = 0, 1, 2$ then $\pi_n K(R)$ agrees with the groups $K_n(R)$ constructed in chapters II and III.

We shall also define the higher K-theory of a category $\mathcal{A}$ in each of the three settings where $K_0(\mathcal{A})$ was defined in chapter II: when $\mathcal{A}$ is a symmetric monoidal category (§4), an exact category (§6) and a Waldhausen category (§8). In each case we build a "K-theory space" $K\mathcal{A}$ and define the group $K_n\mathcal{A}$ to be its homotopy groups: $K_n\mathcal{A} = \pi_n K\mathcal{A}$. Of course the group $\pi_0 K\mathcal{A}$ will agree with the corresponding group $K_0\mathcal{A}$ defined in chapter II.

We will show these definitions of $K_n\mathcal{A}$ coincide whenever they coincide for K_0. For example, the group $K_0(R)$ of a ring R was defined in §II.2 as K_0 of the category $\mathbf{P}(R)$ of finitely generated projective R-modules, but to define $K_0\mathbf{P}(R)$ we could also regard the category $\mathbf{P}(R)$ as being either a symmetric monoidal category (II.5.2), an exact category (II.7.1) or a Waldhausen category (II.9.1.3). We will show that the various constructions give homotopy equivalent spaces $K\mathbf{P}(R)$, and hence the same homotopy groups. Thus the groups $K_n(R) = \pi_n K\mathbf{P}(R)$ will be independent of the construction used.

Many readers will not be interested in the topological details, so we have designed this chapter to allow "surfing." Since the most non-technical way to construct $K(R)$ is to use the "+"-construction, we will do this in §1 below. The short section 2 defines K-theory with finite coefficients, as the homotopy groups of $K(R)$ with finite coefficients. These have proved to be remarkably useful in describing the structure of the groups $K_n(R)$, especially as related to étale cohomology. This is illustrated in chapter VI.

In §3, we summarize the basic facts about the geometric realization BC of a category C, and the basic connection between category theory and homotopy theory needed for the rest of the constructions. Indeed, the K-theory space $K\mathcal{A}$ is constructed in each setting using the geometric realization BC of some category C, concocted out of $\mathcal{A}$. For this, we assume only that the reader has a slight familiarity with cell complexes, or *CW complexes*, which are spaces obtained by successive attachment of cells, with the weak topology.

Sections 4–9 give the construction of the K-theory spaces. Thus in §4 we have group completion constructions for a symmetric monoidal category S, such as the $S^{-1}S$ construction and the connection with the +-construction). It is used in §5 to construct λ-operations on $K(R)$. Quillen's Q-construction for abelian and

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exact categories is given in §6; in §7 we prove the "$+ = Q$" theorem, that the Q-construction and group completion constructions agree for split exact categories (II.7.1.2). The $wS.$ construction for Waldhausen categories is in §8, along with its connection to the Q-construction. In §9 we give an alternative construction for exact categories, due to Gillet and Grayson.

Section 10 gives a construction of the non-connective spectrum for algebraic K-theory of a ring, whose negative homotopy groups are the negative K-groups of Bass developed in chapter III.4. Sections 11 and 12 are devoted to Karoubi-Villamayor K-theory and the homotopy-invariant version KH of K-theory. We will return to this topic in chapter V.

§1. The BGL^+ definition for Rings

Let R be an associative ring with unit. Recall from chapter III that the *infinite general linear group* $GL(R)$ is the union of the groups $GL_n(R)$, and that its commutator subgroup is the perfect group $E(R)$ generated by the elementary matrices $e_{ij}(r)$. Moreover the group $K_1(R)$ is defined to be the quotient $GL(R)/E(R)$.

In 1969, Quillen proposed defining the higher K-theory of a ring R to be the homotopy groups of a certain topological space, which he called "$BGL(R)^+$." Before describing the elementary properties of Quillen's construction, and the related subject of acyclic maps, we present Quillen's description of $BGL(R)^+$ and define the groups $K_n(R)$ for $n \geq 1$.

For any group G, we can naturally construct a connected topological space BG whose fundamental group is G, but whose higher homotopy groups are zero. Details of this construction are in §3 below (see 3.1.3). Moreover, the homology of the topological space BG (with coefficients in a G-module M) coincides with the algebraic homology of the group G (with coefficients in M); the homology of a space X with coefficients in a $\pi_1(X)$-module is defined in [Wh, VI.1–4]. For $G = GL(R)$ we obtain the space $BGL(R)$, which is central to the following definition.

Definition 1.1. The notation $BGL(R)^+$ will denote any CW complex X which has a distinguished map $BGL(R) \to BGL(R)^+$ such that

(1) $\pi_1 BGL(R)^+ \cong K_1(R)$, and the natural map from $GL(R) = \pi_1 BGL(R)$ to $\pi_1 BGL(R)^+$ is onto with kernel $E(R)$;

(2) $H_*(BGL(R); M) \xrightarrow{\cong} H_*(BGL(R)^+; M)$ for every $K_1(R)$-module M.

We will sometimes say that X is a *model* for $BGL(R)^+$.

For $n \geq 1$, $K_n(R)$ is defined to be the homotopy group $\pi_n BGL(R)^+$.

By Theorem 1.5 below, any two models are homotopy equivalent, *i.e.,* the space $BGL(R)^+$ is uniquely defined up to homotopy. Hence the homotopy groups $K_n(R)$ of $BGL(R)^+$ are well-defined up to a canonical isomorphism.

By construction, $K_1(R)$ agrees with the group $K_1(R) = GL(R)/E(R)$ defined in chapter III. We will see in 1.7.1 below that $K_2(R) = \pi_2 BGL^+(R)$ agrees with the group $K_2(R)$ defined in chapter III.

Several distinct models for $BGL(R)^+$ are described in 1.9 below. We will construct even more models for $BGL(R)^+$ in the rest of this chapter: the space $\mathbf{P}^{-1}\mathbf{P}(R)$ of §3, the space $\Omega BQ\mathbf{P}(R)$ of §5 and the space $\Omega(\mathrm{iso}\, S.S)$ arising from the Waldhausen construction in §8.

DEFINITION 1.1.1. Write $K(R)$ for the product $K_0(R) \times BGL(R)^+$. That is, $K(R)$ is the disjoint union of copies of the connected space $BGL(R)^+$, one for each element of $K_0(R)$. By construction, $K_0(R) = \pi_0 K(R)$. Moreover, it is clear that $\pi_n K(R) = \pi_n BGL(R)^+ = K_n(R)$ for $n \geq 1$.

FUNCTORIALITY 1.1.2. Each K_n is a functor from rings to abelian groups, while the topological spaces $BGL(R)^+$ and $K(R)$ are functors from rings to the homotopy category of topological spaces. However, without more information about the models used, the topological maps $BGL(R)^+ \to BGL(R')^+$ are only well-defined up to homotopy.

To see this, note that any ring map $R \to R'$ induces a natural group map $GL(R) \to GL(R')$, and hence a natural map $BGL(R) \to BGL(R')$. This induces a map $BGL(R)^+ \to BGL(R')^+$, unique up to homotopy, by Theorem 1.5 below. Thus the group maps $K_n(R) \to K_n(R')$ are well defined. Since the identity of R induces the identity on $BGL(R)^+$, only composition remains to be considered. Given a second map $R' \to R''$, the composition $BGL(R) \to BGL(R') \to BGL(R'')$ is induced by $R \to R''$ because BGL is natural. By uniqueness in Theorem 1.5, the composition $BGL(R)^+ \to BGL(R')^+ \to BGL(R'')^+$ must be homotopy equivalent to any a priori map $BGL(R)^+ \to BGL(R'')^+$.

It is possible to modify the components of $K(R) = K_0(R) \times BGL(R)^+$ up to homotopy equivalence in order to form a homotopy-commutative H-space in a functorial way, using other constructions (see 4.11.1). Because the map $K_1(R/I) \to K_0(R, I)$ is nontrivial (see III.2.3), $K(R)$ is *not* the product of the H-space $BGL(R)^+$ and the discrete group $K_0(R)$ in a natural way.

TRANSFER MAPS 1.1.3. If $R \to S$ is a ring map such that $S \cong R^d$ as an R-module, the isomorphisms $S^m \cong R^{md}$ induce a group map $GL(S) \to GL(R)$ and hence a map $BGL(S)^+ \to BGL(R)^+$, again unique up to homotopy. On homotopy groups, the maps $K_n(S) \to K_n(R)$ are called *transfer maps*. We will see another construction of these maps in 6.3.2 below.

We shall be interested in the homotopy fiber of the map $BGL(R) \to BGL(R)^+$.

HOMOTOPY FIBER 1.2. The maps $\pi_* E \to \pi_* B$ induced by a continuous map $E \xrightarrow{f} B$ can always be made to fit into a long exact sequence, in a natural way. The *homotopy fiber* $F(f)$ of a f, relative to a basepoint $*_B$ of B, is the space of pairs (e, γ), where $e \in E$ and $\gamma \colon [0,1] \to B$ is a path in B starting at the basepoint $\gamma(0) = *_B$, and ending at $\gamma(1) = f(e)$. A sequence of based spaces $F \to E \xrightarrow{f} B$ with $F \to B$ constant is called a *homotopy fibration sequence* if the evident map $F \to F(f)$ (using $\gamma(t) = *_B$) is a homotopy equivalence.

The key property of the homotopy fiber is that (given a basepoint $*_E$ with $f(*_E) = *_B$) there is a long exact sequence of homotopy groups/pointed sets

$$\begin{aligned}\cdots \pi_{n+1} B \xrightarrow{\partial} \pi_n F(f) \to \pi_n E \to \pi_n B \xrightarrow{\partial} \pi_{n-1} F(f) \to \cdots \\ \cdots \xrightarrow{\partial} \pi_1 F(f) \to \pi_1 E \to \pi_1 B \xrightarrow{\partial} \pi_0 F(f) \to \pi_0 E \to \pi_0 B.\end{aligned}$$

When $E \to B$ is an H-map of H-spaces, $F(f)$ is also an H-space, and the maps ending the sequence are product-preserving.

Acyclic Spaces and Acyclic Maps

The definition of $BGL(R)^+$ fits into the general framework of acyclic maps, which we now discuss. Our discussion of acyclicity is taken from [HH] and [Berrick].

Definition 1.3 (Acyclic spaces). We call a topological space F *acyclic* if it has the homology of a point, that is, if $\widetilde{H}_*(F;\mathbb{Z}) = 0$.

Lemma 1.3.1. *Let F be an acyclic space. Then F is connected, its fundamental group $G = \pi_1(F)$ is a perfect group, and $H_2(G;\mathbb{Z}) = 0$ as well.*

Proof. The acyclic space F must be connected, as $H_0(F) = \mathbb{Z}$. Because $G/[G,G] = H_1(F;\mathbb{Z}) = 0$, we have $G = [G,G]$, *i.e.,* G is a perfect group. To calculate $H_2(G)$, observe that the universal covering space $\tilde{F}$ has $H_1(\tilde{F};\mathbb{Z}) = 0$. Moreover, the homotopy fiber (1.2) of the canonical map $F \to BG$ is homotopy equivalent to $\tilde{F}$; to see this, consider the long exact sequence of homotopy groups 1.2. The Serre Spectral Sequence for this homotopy fibration is $E^2_{pq} = H_p(G; H_q(\tilde{F};\mathbb{Z})) \Rightarrow H_{p+q}(F;\mathbb{Z})$ and the conclusion that $H_2(G;\mathbb{Z}) = 0$ follows from the associated exact sequence of low degree terms:

$$H_2(F;\mathbb{Z}) \to H_2(G;\mathbb{Z}) \xrightarrow{d^2} H_1(\tilde{F};\mathbb{Z})^G \to H_1(F;\mathbb{Z}) \to H_1(G;\mathbb{Z}). \qquad \square$$

Example 1.3.2 (Volodin Spaces). The Volodin space $X(R)$ is an acyclic subspace of $BGL(R)$, constructed as follows. For each n, let $T_n(R)$ denote the subgroup of $GL_n(R)$ consisting of upper triangular matrices with 1's on the diagonal. As n varies, the union of these groups forms a subgroup $T(R)$ of $GL(R)$. Similarly we may regard the permutation groups Σ_n as subgroups of $GL_n(R)$ by their representation as permutation matrices, and their union (the infinite permutation group Σ_∞) is a subgroup of $GL(R)$. For each $\sigma \in \Sigma_n$, let $T_n^\sigma(R)$ denote the subgroup of $GL_n(R)$ obtained by conjugating $T_n(R)$ by σ. For example, if $\sigma = (n\dots 1)$ then $T_n^\sigma(R)$ is the subgroup of lower triangular matrices.

Since the classifying spaces $BT_n(R)$ and $BT_n(R)^\sigma$ are subspaces of $BGL_n(R)$, and hence of $BGL(R)$, we may form their union over all n and σ: $X(R) = \bigcup_{n,\sigma} BT_n(R)^\sigma$. The space $X(R)$ is acyclic (see [Su81]). Since $X(R)$ was first described by Volodin in 1971, it is usually called the *Volodin space* of R.

The image of the map $\pi_1 X(R) \to \pi_1 BGL(R) = GL(R)$ is the group $E(R)$. To see this, note that $\pi_1(X)$ is generated by the images of the $\pi_1 BT_n(R)^\sigma$, the image of the composition $\pi_1 BT_n^\sigma(R) \to \pi_1(X) \to \pi_1 BGL(R) = GL(R)$ is the subgroup $T_n^\sigma(R)$ of $E(R)$, and every generator $e_{ij}(r)$ of $E(R)$ is contained in some $T_n^\sigma(R)$.

Definition 1.4 (Acyclic maps). Let X and Y be based connected CW complexes. A map $f\colon X \to Y$ is called *acyclic* if the homotopy fiber $F(f)$ of f is acyclic (has the homology of a point). This implies that $F(f)$ is connected and $\pi_1 F(f)$ is a perfect group.

From the exact sequence $\pi_1 F(f) \to \pi_1(X) \to \pi_1(Y) \to \pi_0 F(f)$ of homotopy groups/pointed sets, we see that if $X \to Y$ is acyclic then the map $\pi_1(X) \to \pi_1(Y)$ is onto, and its kernel P is a perfect normal subgroup of $\pi_1(X)$.

DEFINITION 1.4.1. Let P be a perfect normal subgroup of $\pi_1(X)$, where X is a based connected CW complex. An acyclic map $f: X \to Y$ is called a *+–construction* on X (relative to P) if P is the kernel of $\pi_1(X) \to \pi_1(Y)$.

EXAMPLE 1.4.2. If X is acyclic, the map $X \to$ point is acyclic. By Ex. 1.2, it is a +-construction.

When Quillen introduced the notion of acyclic maps in 1969, he observed that both Y and the map f are determined up to homotopy by the subgroup P. This is the content of the following theorem; its proof uses topological obstruction theory. Part (1) is proven in Ex. 1.4; an explicit proof may be found in §5 of [Berrick].

THEOREM 1.5 (QUILLEN). *Let P be a perfect normal subgroup of $\pi_1(X)$. Then*

(1) *There is a +–construction $f: X \to Y$ relative to P*
(2) *Let $f: X \to Y$ be a +–construction relative to P, and $g: X \to Z$ a map such that P vanishes in $\pi_1(Z)$. Then there is a map $h: Y \to Z$, unique up to homotopy, such that $g = hf$.*
(3) *In particular, if g is another +–construction relative to P, then the map h in (2) is a homotopy equivalence: $h: Y \xrightarrow{\sim} Z$.*

REMARK 1.5.1. Every group G has a unique largest perfect subgroup P, called the *perfect radical* of G, and it is a normal subgroup of G; see Ex. 1.5. If no mention is made to the contrary, the notation X^+ will always denote the +–construction relative to the perfect radical of $\pi_1(X)$.

The first construction along these lines was announced by Quillen in 1969, so we have adopted Quillen's term "+–construction" as well as his notation. A good description of his approach may be found in [HH] or [Berrick].

LEMMA 1.6. *Let X and Y be connected CW complexes. A map $f: X \to Y$ is acyclic if and only if $H_*(X, M) \cong H_*(Y, M)$ for every $\pi_1(Y)$-module M.*

PROOF. Suppose first that f is acyclic, with homotopy fiber $F(f)$. Since the map $\pi_1 F(f) \to \pi_1 Y$ is trivial, $\pi_1 F(f)$ acts trivially upon M. By the Universal Coefficient Theorem, $H_q(F(f); M) = 0$ for $q \neq 0$ and $H_0(F(f); M) = M$. Therefore $E^2_{pq} = 0$ for $q \neq 0$ in the Serre Spectral Sequence for f:

$$E^2_{pq} = H_p(Y; H_q(F(f); M)) \Rightarrow H_{p+q}(X; M).$$

Hence the spectral sequence collapses to yield $H_p(X; M) \xrightarrow{\cong} H_p(Y; M)$ for all p.

Conversely, we suppose first that $\pi_1 Y = 0$ and $H_*(X; \mathbb{Z}) \cong H_*(Y; \mathbb{Z})$. By the Comparison Theorem for the Serre Spectral Sequences for $F(f) \to X \xrightarrow{f} Y$ and $* \to Y \xrightarrow{=} Y$, we have $\tilde{H}_*(F(f); \mathbb{Z}) = 0$. Hence $F(f)$ and f are acyclic.

The general case reduces to this by the following trick. Let $\tilde{Y}$ denote the universal covering space of Y, and $\tilde{X} = X \times_Y \tilde{Y}$ the corresponding covering space of X. Then there are natural isomorphisms $H_*(\tilde{Y}; \mathbb{Z}) \cong H_*(Y; M)$ and $H_*(\tilde{X}; \mathbb{Z}) \cong H_*(X; M)$, where $M = \mathbb{Z}[\pi_1(Y)]$. The assumption that $H_*(X; M) \cong H_*(Y; M)$ implies that the map $\tilde{f}: \tilde{X} \to \tilde{Y}$ induces isomorphisms on integral homology. But $\pi_1(\tilde{Y}) = 0$, so by the special case above the homotopy fiber $F(\tilde{f})$ of $\tilde{f}$ is an acyclic space. But by path lifting we have $F(\tilde{f}) \cong F(f)$, so $F(f)$ is acyclic. Thus f is an acyclic map. □

Recall from III.5.4 that every perfect group P has a universal central extension $E \to P$, and that the kernel of this extension is the abelian group $H_2(P;\mathbb{Z})$.

Proposition 1.7. *Let P be a perfect normal subgroup of a group G, with corresponding $+$–construction $f\colon BG \to BG^+$. If $F(f)$ is the homotopy fiber of f then $\pi_1 F(f)$ is the universal central extension of P, and $\pi_2(BG^+) \cong H_2(P;\mathbb{Z})$.*

Proof. We have an exact sequence $\pi_2(BG) \to \pi_2(BG^+) \to \pi_1 F(f) \to G \to G/P \to 1$. But $\pi_2(BG) = 0$, and $\pi_2(BG^+)$ is in the center of $\pi_1 F(f)$ by [Wh, IV.3.5]. Thus $\pi_1 F(f)$ is a central extension of P with kernel $\pi_2(BG^+)$. But $F(f)$ is acyclic, so $\pi_1 F(f)$ is perfect and $H_2(F;\mathbb{Z}) = 0$ by 1.3.1. By the Recognition Theorem III.5.4, $\pi_1 F(f)$ is the universal central extension of P. $\square$

Recall from Theorem III.5.5 that the Steinberg group $St(R)$ is the universal central extension of the perfect group $E(R)$. Thus we have:

Corollary 1.7.1. *The group $K_2(R) = \pi_2 BGL(R)^+$ is isomorphic to the group $K_2(R) \cong H_2(E(R);\mathbb{Z})$ of chapter III.*

In fact, we will see in Ex. 1.8 and 1.9 that $K_n(R) \cong \pi_n(BE(R)^+)$ for all $n \geq 2$, and $K_n(R) \cong \pi_n(BSt(R)^+)$ for all $n \geq 3$, with $K_3(R) \cong H_3(St(R);\mathbb{Z})$.

Corollary 1.7.2. *The fundamental group $\pi_1 X(R)$ of the Volodin space (1.3.2) is the Steinberg group $St(R)$.*

Construction Techniques

One problem with the +construction approach is the fact that $BGL(R)^+$ is not a uniquely defined space. It is not hard to see that $BGL(R)^+$ is an H-space (see Ex. 1.11). Quillen proved that that it is also an infinite loop space, and extends to an Ω-spectrum $\mathbf{K}(R)$. We omit the proof here, because it will follow from the $+ = Q$ theorem in section 7.

Here is one of the most useful recognition criteria, due to Quillen. The proof is is an application of obstruction theory, which we omit (but see [Ger72, 1.5].)

Theorem 1.8. *The map $i\colon BGL(R) \to BGL(R)^+$ is universal for maps into H-spaces. That is, for each map $f\colon BGL(R) \to H$, where H is an H-space, there is a map $g\colon BGL(R)^+ \to H$ so that $f = gi$, and such that the induced map $\pi_i(BGL(R)^+) \to \pi_i(H)$ is independent of g.*

Remark 1.8.1. If $f_* : H_*(BGL(R),\mathbb{Z}) \cong H_*(H,\mathbb{Z})$ is an isomorphism, then f is acyclic and g is a homotopy equivalence: $BGL(R)^+ \simeq H$. This gives another characterization of $BGL(R)^+$. The proof is indicated in Exercise 1.3.

Constructions 1.9. Here are some ways that $BGL(R)^+$ may be constructed:

(i) Using point-set topology, *e.g.*, by attaching 2-cells and 3-cells to $BGL(R)$. If we perform this construction over $\mathbb{Z}$ and let $BGL(R)^+$ be the pushout of $BGL(\mathbb{Z})^+$ and $BGL(R)$ along $BGL(\mathbb{Z})$, this gives a construction which is functorial in R. This method is described Ex. 1.4, and in the books [Berrick] and [Rosenberg].

(ii) By the Bousfield-Kan integral completion functor $\mathbb{Z}_\infty$: we set $BGL(R)^+ = \mathbb{Z}_\infty BGL(R)$. This approach, which is also functorial in R, is used in [Dror] and [Ger72].

(iii) "Group completing" the H-space $\coprod_{n=0}^{\infty} BGL_n(R)$ yields an infinite loop space whose basepoint component is $BGL(R)^+$. This method will be discussed more in section 3, and is due to G. Segal [Segal].

(iv) By taking BGL of a free simplicial ring F_* with an augmentation $F_0 \to R$ such that $F_* \to R$ is a homotopy equivalence, as in [Swan]. Swan showed that the simplicial group $GL(F_*)$ and the simplicial space $BGL(F_*)$ are independent (up to simplicial homotopy) of the choice of resolution $F_* \to R$, and that $\pi_1 BGL(F_*) = \pi_0 GL(F_*) = E(R)$. The *Swan K-theory space* $\Omega K^{Sw}(R)$ is defined to be the homotopy fiber of $BGL(F_*) \to BGL(R)$, and we set $K_i^{Sw}(R) = \pi_{i-1}\Omega K^{Sw}(R)$ for $i \geq 1$ so that $K_1^{Sw}(R) = K_1(R)$ by construction. The space $\Omega K^{Sw}(R)$ is a model for the loop space $\Omega BGL(R)^+$.

As an application, if F is a free ring (without unit), we may take F_* to be the constant simplicial ring, so $\Omega K^{Sw}(F)$ is contractible, and $K_i^{Sw}(F) = 0$ for all i. Gersten proved in [Ger74] (see V.6.5) that $BGL(F)^+$ is contractible; this was used by Don Anderson [And72] to prove that the canonical map from $GL(R) = \Omega BGL(R)$ to $\Omega K^{Sw}(R)$ induces a homotopy equivalence $\Omega BGL(R)^+ \to \Omega K^{Sw}(R)$.

(v) *Volodin's construction.* Let $X(R)$ denote the acyclic Volodin space of Example 1.3.2. By Ex. 1.6, the quotient group $BGL(R)/X(R)$ is a model for $BGL(R)^+$.

An excellent survey of these constructions may be found in [Ger72], except for details on Volodin's construction, which are in [Su81].

Products

If A and B are rings, any natural isomorphism $\varphi_{pq} : A^p \otimes B^q \cong (A \otimes B)^{pq}$ of $A \otimes B$-modules allows us to define a "tensor product" homomorphism $GL_p(A) \times GL_q(B) \to GL_{pq}(A \otimes B)$. This in turn induces continuous maps $\varphi_{p,q} : BGL_p(A)^+ \times BGL_q(B)^+ \to BGL_{pq}(A \otimes B)^+ \to BGL(A \otimes B)^+$. A different choice of φ yields a tensor product homomorphism conjugate to the original, and a new map $\varphi_{p,q}$ freely homotopic to the original. It follows that $\varphi_{p,q}$ is compatible up to homotopy with stabilization in p and q.

Since the target is an H-space (Ex. 1.11), we can define new maps $\gamma_{p,q}(a,b) = \varphi_{p,q}(a,b) - \varphi_{p,q}(a,*) - \varphi_{p,q}(*,b)$, where $*$ denotes the basepoint. Since $\gamma_{p,q}(a,*) = \gamma_{p,q}(*,b) = *$, and $\gamma_{p,q}$ is compatible with stabilization in p,q, it induces a map, well defined up to weak homotopy equivalence

$$\gamma : BGL(A)^+ \wedge BGL(B)^+ \to BGL(A \otimes B)^+.$$

Combining γ with the reduced join $\pi_p(X) \otimes \pi_q(Y) \to \pi_{p+q}(X \wedge Y)$ [Wh, p. 480] allows us to define a product map :

$$K_p(A) \otimes K_q(B) \to K_{p+q}(A \otimes B).$$

Loday proved the following result in [Lo76].

THEOREM 1.10. *(Loday) The product map is natural in A and B, bilinear and associative, and if A is commutative, the induced product*

$$K_p(A) \otimes K_q(A) \to K_{p+q}(A \otimes A) \to K_{p+q}(A)$$

is graded-commutative. Moreover, the special case $K_1(A) \otimes K_1(B) \to K_2(A \otimes B)$ coincides with the product defined in III.5.12.

EXAMPLE 1.10.1. *(Steinberg symbols)* If $r_1, ..., r_n$ are units of a commutative ring R, the product of the $r_i \in K_1(R)$ is an element $\{r_1, ..., r_n\}$ of $K_n(R)$. These elements are called Steinberg symbols, since the products $\{r_1, r_2\} \in K_2(R)$ agree with the Steinberg symbols of III.5.10. If F is a field, the universal property (III.7.1) of Milnor K-theory implies that there is a ring homomorphism $K_*^M(F) \to K_*(F)$. We will see in Ex. 1.12 that it need not be an injection.

EXAMPLE 1.10.2. Associated to the unit x of $\Lambda = \mathbb{Z}[x, x^{-1}]$ we choose a map $S^1 \to BGL(\Lambda)^+$, representing $[x] \in \pi_1 BGL(\Lambda)^+$. The pairing γ induces a map $BGL(R)^+ \wedge S^1$ to $BGL(R[x, x^{-1}])^+$. By adjunction, this yields a map $BGL(R)^+ \to \Omega BGL(R[x, x^{-1}])^+$. A spectrum version of this map is given in Ex. 4.14.

EXAMPLE 1.10.3. *(The K-theory Assembly Map)* If G is any group, the inclusion $G \subset \mathbb{Z}[G]^\times = GL(\mathbb{Z}[G])$ induces a map $BG \to BGL(\mathbb{Z}[G])^+$. If R is any ring, the product map $BGL(R)^+ \wedge BGL(\mathbb{Z}[G])^+ \to BGL(R[G])^+$ induces a map from $BGL(R)^+ \wedge (BG_+)$ to $BGL(R[G])^+$, where BG_+ denotes the disjoint union of BG and a basepoint. By Ex. 1.14, there is also a map from $K(R) \wedge (BG_+)$ to $K(R[G])$.

Now for any infinite loop space (or spectrum) $\mathbf{E}$, and any pointed space X, the homotopy groups of the space $\mathbf{E} \wedge X$ give the generalized homology of X with coefficients in $\mathbf{E}$, $H_n(BG; \mathbf{E})$. For $\mathbf{E} = K(R)$, $H_n(BG; K(R))$ is the generalized homology of BG with coefficients in $K(R)$.

The map $H_n(BG; K(R)) = \pi_n \left(BGL(R)^+ \wedge BG_+ \right) \to K(R_n[G])$ which we have just constructed is called the K-theory *Assembly Map*, and it plays a critical role in the K-theory of group rings. It is due to Quinn and Loday [Lo76], who observed that for $n = 0$ it is just the map $K_0(R) \to K_0(R[G])$, while for $n = 1$ it is the map $K_1(R) \oplus G/[G, G] \to K_1(R[G])$. The *higher Whitehead Group* $Wh_n(G)$ is defined to be π_{n-1} of the homotopy fiber of the map $K(\mathbb{Z}) \wedge (BG_+) \to K(\mathbb{Z}[G])$. The above calculations show that $Wh_0(G)$ is Wall's finiteness obstruction (II.2.4), and the classical Whitehead group $Wh_1(G) = K_1(\mathbb{Z}[G])/\{\pm G\}$ of III.1.9.

If G is a torsionfree group, the *Isomorphism Conjecture* for G states that the assembly map $H_n(BG; K(R)) \to K_n(R[G])$ should be an isomorphism for any regular ring R. There is a more general Isomorphism Conjecture for infinite groups with torsion, due to Farrell-Jones [FJ]; it replaces $H_n(BG; K(R))$ by the equivariant homology of $E_{vc}G$, an equivariant version of the universal covering space EG of BG relative to the class of virtually cyclic subgroups of G.

Relative K-groups

RELATIVE GROUPS 1.11.1. Given a ring homomorphism $f\colon R \to R'$, let $K(f)$ be the homotopy fiber of $K(R) \to K(R')$, and set $K_n(f) = \pi_n K(f)$. This construction is designed so that these relative groups fit into a long exact sequence:

$$\cdots K_{n+1}(R') \xrightarrow{\partial} K_n(f) \to K_n(R) \to K_n(R') \xrightarrow{\partial} \cdots$$
$$\cdots K_1(f) \to K_1(R) \to K_1(R') \xrightarrow{\partial} K_0(f) \to K_0(R) \to K_0(R').$$

Using the functorial homotopy-commutative H-space structure on $K(R)$ (see 1.1.2), it follows that each $K_n(f)$, including $K_0(f)$, is an abelian group.

When $R' = R/I$ for some ideal I, we write $K(R,I)$ for $K(R \to R/I)$. It is easy to see (Ex. 1.15) that $K_0(R,I)$ and $K_1(R,I)$ agree with the relative groups defined in Ex. II.2.3 and III.2.2, and that the ending of this sequence is the exact sequence of III.2.3 and III.5.7. Keune and Loday have shown that $K_2(R,I)$ agrees with the relative group defined in III.5.7.

Absolute Excision 1.11.2. A non-unital ring I is said to *satisfy absolute excision* for K_n if $K_n(\mathbb{Z}\oplus I, I) \xrightarrow{\cong} K_n(R,I)$ is an isomorphism for every unital ring R containing I as an ideal; $\mathbb{Z}\oplus I$ is the canonical augmented ring (see Ex. I.1.10.) By II, Ex. 2.3, every I satisfies absolute excision for K_0. By III, Remark 2.2.1, I satisfies absolute excision for K_1 if and only if $I = I^2$.

Suslin proved in [Su95] that I satisfies absolute excision for K_n if and only if the groups $\mathrm{Tor}_i^{\mathbb{Z}\oplus I}(\mathbb{Z},\mathbb{Z})$ vanish for $i = 1,\dots,n$. (Since $\mathrm{Tor}_1(\mathbb{Z},\mathbb{Z}) = I/I^2$, this recovers the result for K_1.) In homological algebra, a non-unital ring I is called *H-unital* if every $\mathrm{Tor}_i(\mathbb{Z},\mathbb{Z})$ vanishes; Suslin's result says that I satisfies absolute excision for all K_n if and only if I is H-unital.

Together with a result of Suslin and Wodzicki [SuW], this implies that I satisfies absolute excision for $K_n \otimes \mathbb{Q}$ if and only if $I \otimes \mathbb{Q}$ satisfies absolute excision for K_n.

Suppose now that $I = I^2$. In this case the commutator subgroup of $GL(I)$ is perfect (III, Ex. 2.10). By Theorem 1.5 there is a +-construction $BGL(I)^+$ and a map from $BGL(I)^+$ to the basepoint component of $K(R,I)$. When I is H-unital, this is a homotopy equivalence; $\pi_n BGL(I)^+ \cong K_n(R,I)$ for all $n \geq 1$. This concrete version of absolute excision was proven by Suslin and Wodzicki in [SuW, 1.7].

Suspension Rings 1.11.3. Let $C(R)$ be the *cone ring* of row-and-column finite matrices over a fixed ring R (Ex. I.1.8); by II.2.1.3, $C(R)$ is flasque, so $K(C(R))$ is contractible by Ex. 1.17. The *suspension ring* $S(R)$ of III, Ex. 1.15 is $C(R)/M(R)$, where $M(R)$ is the ideal of finite matrices over R. Since $M(R) \cong M(M(R))$ and $GL(R) = GL_1(M(R))$, we have $GL(R) \cong GL(M(R))$ and hence $BGL(R)^+ \cong BGL(M(R))^+$. Since $M(R)$ is H-unital (Ex. 1.20), it satisfies absolute excision and we have a fibration sequence

$$K_0(R) \times BGL(R)^+ \to BGL(C(R))^+ \to BGL(S(R))^+.$$

Since the middle term is contractible, this proves that $K_0(R) \times BGL(R)^+ \simeq \Omega BGL(S(R))^+$ so that $K_{n+1}S(R) \cong K_n(R)$ for all $n \geq 1$. ($K_0 S(R) \cong K_{-1}(R)$ by III, Ex. 4.10.) This result was first proven by Gersten and Wagoner.

K-theory of finite fields

Next, we describe Quillen's construction for the K-theory of finite fields, arising from his work on the Adams Conjecture [Q70]. Adams had shown that the Adams operations ψ^k on topological K-theory (II.4.4) are represented by maps $\psi^k\colon BU \to BU$ in the sense that for each X the Adams operations on $\widetilde{KU}(X)$ are the maps:

$$\widetilde{KU}(X) = [X, BU] \xrightarrow{[X,\psi^k]} [X, BU] = \widetilde{KU}(X).$$

Fix a finite field $\mathbb{F}_q$ with $q = p^\nu$ elements. For each n, the Brauer lifting of the trivial and standard n-dimensional representations of $GL_n(\mathbb{F}_q)$ are n-dimensional complex representations, given by homomorphisms $1_n, \mathrm{id}_n\colon GL_n(\mathbb{F}_q) \to U$. Since BU is an H-space, we can form the difference $\rho_n = B(\mathrm{id}_n) - B(1_n)$ as a map $BGL_n(\mathbb{F}_q) \to BU$. Quillen observed that ρ_n and ρ_{n+1} are compatible up to homotopy with the inclusion of $BGL_n(\mathbb{F}_q)$ in $BGL_{n+1}(\mathbb{F}_q)$. (See 5.3.1 below.) Hence there is a map $\rho\colon BGL(\mathbb{F}_q) \to BU$, well defined up to homotopy. By Theorem 1.8, ρ induces a map from $BGL(\mathbb{F}_q)^+$ to BU, and hence maps $\rho_* : K_n(\mathbb{F}_q) \to \pi_n(BU) = \widetilde{KU}(S^n)$.

We will define operations λ^k and ψ^k on $K_*(\mathbb{F}_q)$ in 5.3.1 and Ex. 5.2 below, and show (5.5.2) that ψ^p is induced by the Frobenius on $\mathbb{F}_q$, so that ψ^q is the identity map on $K_n(\mathbb{F}_q)$. We will also see in 5.7 and 5.8 below that ρ_* commutes with the operations λ^k and ψ^k on $K_n(\mathbb{F}_q)$ and $\widetilde{KU}(S^n)$.

Theorem 1.12. *(Quillen) The map $BGL(\mathbb{F}_q)^+ \to BU$ identifies $BGL(\mathbb{F}_q)^+$ with the homotopy fiber of $\psi^q - 1$. That is, the following is a homotopy fibration.*

$$BGL(\mathbb{F}_q)^+ \xrightarrow{\rho} BU \xrightarrow{\psi^q - 1} BU$$

On homotopy groups, II.4.4.1 shows that ψ^q is multiplication by q^i on $\pi_{2i}BU = \widetilde{KU}(S^{2i})$. Using the homotopy sequence 1.2 and 1.12, we immediately deduce:

Corollary 1.13. *For every finite field $\mathbb{F}_q$, and $n \geq 1$, we have*

$$K_n(\mathbb{F}_q) = \pi_n BGL(\mathbb{F}_q)^+ \cong \begin{cases} \mathbb{Z}/(q^i - 1) & n = 2i - 1, \\ 0 & n \text{ even.} \end{cases}$$

Moreover, if $\mathbb{F}_q \subset \mathbb{F}_{q'}$ then $K_n(\mathbb{F}_q) \to K_n(\mathbb{F}_{q'})$ is an injection, identifying $K_n(\mathbb{F}_q)$ with $K_n(\mathbb{F}_{q'})^G$, where $G = \mathrm{Gal}(\mathbb{F}_{q'}/\mathbb{F}_q)$; the transfer map $K_n(\mathbb{F}_{q'}) \to K_n(\mathbb{F}_q)$ is onto (see 1.1.3).

Remark 1.13.1. Clearly all products in the ring $K_*(\mathbb{F}_q)$ are trivial. We will see in section 2 that it is also possible to put a ring structure on the homotopy groups with mod-ℓ coefficients, $K_n(\mathbb{F}_q; \mathbb{Z}/\ell) = \pi_n(BGL(\mathbb{F}_q); \mathbb{Z}/\ell)$.

If $\ell \mid (q-1)$, the long exact sequence for homotopy with mod-ℓ coefficients (2.2) shows that $K_n(\mathbb{F}_q; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$ for all $n \geq 0$. The choice of a primitive unit $\zeta \in \mathbb{F}_q^\times$ and a primitive ℓth root of unity ω gives generators ζ for $K_1(\mathbb{F}_q; \mathbb{Z}/\ell)$ and the *Bott element* β for $K_2(\mathbb{F}_q; \mathbb{Z}/\ell)$, respectively. (The Bockstein sends β to $\omega \in K_1(\mathbb{F}_q)$.) Browder has shown [Br] that $K_*(\mathbb{F}_q; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta, \zeta]/(\zeta^2)$ as a graded ring, and that the natural isomorphism from the even part $\oplus_n K_{2n}(\mathbb{F}_q; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta]$ to $\oplus \pi_{2n}(BU; \mathbb{Z}/\ell)$ is a ring isomorphism.

If $p \neq \ell$, the algebraic closure $\bar{\mathbb{F}}_p$ is the union of the $\mathbb{F}_q$ where $q = p^\nu$ and $\ell \mid (q-1)$. Hence the ring $K_*(\bar{\mathbb{F}}_p; \mathbb{Z}/\ell)$ is the direct limit of the $K_*(\mathbb{F}_q; \mathbb{Z}/\ell)$. As each ζ vanishes and the Bott elements map to each other, we have:

$$K_*(\bar{\mathbb{F}}_p; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta] \cong \pi_*(BU; \mathbb{Z}/\ell).$$

REMARK 1.13.2. Browder has also shown in [Br, 2.4] that the Bott element β in $K_2(\mathbb{C};\mathbb{Z}/m)$ maps to a generator of $\pi_2(BU;\mathbb{Z}/m)=\mathbb{Z}/m$ under the change-of-topology map. Hence the map $K_*(\mathbb{C};\mathbb{Z}/m)\to\pi_*(BU;\mathbb{Z}/m)$ is also onto. We will see in VI.1.4 that it is an isomorphism.

Homological Stability

Homological stability, the study of how the homology of a group like $GL_n(R)$ depends upon n, plays an important role in algebraic K-theory. The following theorem was proven by Suslin in [Su82], using Volodin's construction of $K(R)$. Recall from Ex. I.1.5 that the stable range of R, $sr(R)$, is defined in terms of unimodular rows; if R is commutative and noetherian it is at most $\dim(R)+1$.

THEOREM 1.14. *Let R be a ring with stable range $sr(R)$. For $r\geq\max\{2n+1,n+sr(R)\}$ the maps $\pi_nBGL_r(R)^+\to\pi_nBGL_{r+1}(R)^+$ are isomorphisms.*

Now assume that $r>sr(R)+1$, so that $E_r(R)$ is a perfect normal subgroup of $GL_r(R)$ by Ex. III.1.3. The universal covering space of $BGL_r(R)^+$ is then homotopy equivalent to $BE_r(R)^+$ for (by Ex. 1.8). Applying the Hurewicz theorem (and the Comparison Theorem) to these spaces implies:

COROLLARY 1.14.1. *In the range $r\geq\max\{2n+1,n+sr(R)\}$, the following maps are isomorphisms:*

$$H_n(BGL_r(R))\to H_n(BGL_{r+1}(R))\to H_n(BGL(R)^+)$$
$$H_n(BE_r(R))\to H_n(BE_{r+1}(R))\to H_n(BE(R)^+)$$

For example, suppose that R is an Artinian ring, so that $sr(R)=1$ by Ex. I.1.5. Then $\pi_nBGL_r(R)^+\cong K_n(R)$ and $H_n(BGL_r(R))\cong H_n(BGL(R)^+)$ for all $r>2n$. The following result, due to Suslin [Su-KM], improves this bound for fields.

PROPOSITION 1.15. *(Suslin) If F is an infinite field, $H_n(GL_r(F))\to H_n(GL(F))$ is an isomorphism for all $r\geq n$. In addition, there is a canonical isomorphism $H_n(GL_n(F))/im\,H_n(GL_{n-1}(F))\cong K_n^M(F)$.*

PROPOSITION 1.16. *(Kuku) If R is a finite ring, then $K_n(R)$ is a finite abelian group for all $n>0$.*

PROOF. The case $n=1$ follows from III.1.2.5 (or Ex. III.1.2): $K_1(R)$ is a quotient of $R^\times$. Since $K_n(R)=\pi_nBE(R)^+$ for $n>1$ by Ex. 1.8, it suffices to show that the homology groups $H_n(E(R);\mathbb{Z})$ are finite for $n>0$. But each $E_r(R)$ is a finite group, so the groups $H_n(BE_r(R);\mathbb{Z})$ are indeed finite for $n>0$. □

Rank of K_n over number fields

It is a well known theorem of Cartan and Serre that the "rational" homotopy groups $\pi_n(X)\otimes\mathbb{Q}$ of an H-space X inject into the rational homology groups $H_n(X;\mathbb{Q})$, and that $\pi_*(X)\otimes\mathbb{Q}$ forms the primitive elements in the coalgebra structure on $H_*(X;\mathbb{Q})$. (See [MM, p. 163].) For $X=BGL(R)^+$, which is an H-space by Ex. 1.11, this means that the groups $K_n(R)\otimes\mathbb{Q}=\pi_n(BGL(R)^+)\otimes\mathbb{Q}$ inject into the groups $H_*(GL(R);\mathbb{Q})=H_*(BGL(R);\mathbb{Q})=H_*(BGL(R)^+;\mathbb{Q})$ as the primitive elements. For $X = BSL(R)^+$, this means that the groups $K_n(R) \otimes \mathbb{Q}$ inject into $H_*(SL(R);\mathbb{Q})$ as the primitive elements for $n \geq 2$.

Now suppose that A is a finite dimensional semisimple algebra over $\mathbb{Q}$, such as a number field, and that R is a subring of A which is finitely generated over $\mathbb{Z}$ and has $R\otimes\mathbb{Q} = A$ (R is an *order*). In this case, Borel determined the ring $H^*(SL_m(R);\mathbb{Q})$ and hence the dual coalgebra $H_*(SL_m(R);\mathbb{Q})$ and hence its primitive part, $K_*(R)\otimes\mathbb{Q}$. (See the review MR0387496 of Borel's paper [Bor] by Garland.) The answer only depends upon the semisimple $\mathbb{R}$-algebra $A \otimes_{\mathbb{Q}} \mathbb{R}$.

More concretely, let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras (over $\mathbb{Q}$) of $SL_m(A \otimes \mathbb{R})$ and one of its maximal compact subgroups K. Borel first proved in [Bor, Thm. 1] that

$$H^q(SL_m(R);\mathbb{R}) \cong H^q(SL_m(A);\mathbb{R}) \cong H^q(\mathfrak{g},\mathfrak{k};\mathbb{R}) \quad \text{for } m \gg q.$$

By the above remarks, this proves:

Theorem 1.17 (Borel). *Let A be a finite dimensional semisimple $\mathbb{Q}$-algebra. Then for every order R in A we have $K_n(R) \otimes \mathbb{Q} \cong K_n(A) \otimes \mathbb{Q}$ for all $n \geq 2$.*

Borel also calculated the ranks of these groups. Since A is a finite product of simple algebras A_i, and $K_n(A)$ is the product of the $K_n(A_i)$ by Ex. 1.7, we may assume that A is simple, *i.e.,* a matrix algebra over a division algebra. The center of A is then a number field F. It is traditional to write r_1 and r_2 for the number of factors of $\mathbb{R}$ and $\mathbb{C}$ in the $\mathbb{R}$-algebra $F \otimes_{\mathbb{Q}} \mathbb{R}$, so that $F \otimes_{\mathbb{Q}} \mathbb{R} \cong (\mathbb{R})^{r_1} \times (\mathbb{C})^{r_2}$. Borel proved in [Bor, thm. 2] [Bor1, 12.2] that $H^*(SL(A),\mathbb{R})$ is a tensor product of r_1 exterior algebras having generators x_i in degrees $4i + 1$ ($i \geq 1$) and r_2 exterior algebras having generators x_j in degrees $2j + 1$ ($j \geq 1$). Taking primitive parts, this proves the following result:

Theorem 1.18 (Borel). *Let F be a number field, and A a central simple F-algebra. Then for $n \geq 2$ we have $K_n(A) \otimes \mathbb{Q} \cong K_n(F)$ and*

$$\operatorname{rank} K_n(A) \otimes \mathbb{Q} = \begin{cases} r_2, & n \equiv 3 \pmod 4 \\ r_1 + r_2, & n \equiv 1 \pmod 4 \\ 0, & \text{else.} \end{cases}$$

By Theorem 1.17, this also gives the rank of $K_n(R)$ for every order R.

In particular, these groups are torsion for every even $n \geq 2$.

Regulator Maps 1.18.1. Borel's construction provides a specific map from $K_n(R)$ to the real vector space P_n of primitives in $H_n(SL(R),\mathbb{R})$; Borel observed that the image is a lattice Λ. There is another canonical lattice Λ' in P_n: the image of $\pi_n(X)$ for the symmetric space X contained in $K\backslash GL_m(\mathbb{C})$. The *higher regulator* of R is defined to be the positive real number R_n such that the volume of P_n/Λ is R_n times the volume of P_n/Λ'. Borel also proved that R_{2i+1} was a positive rational number times $\sqrt{D}\ \pi^{-m(i+1)}\zeta_F(i + 1)$, where D is the discriminant of $F/\mathbb{Q}$ and ζ_F is the zeta function of F.

GROUP RINGS 1.18.2. The group ring $\mathbb{Z}[G]$ of a finite group G is an order in the semisimple algebra $\mathbb{Q}[G]$. Therefore Theorem 1.18 gives the rank of the groups $K_n(\mathbb{Z}[G])$ for $n \geq 2$. The rank of $K_1(\mathbb{Z}[G])$ was given in III.1.8, and does not follow this pattern. For example, if C_p is a cyclic group of prime order $p \geq 3$ then $r_1 + r_2 = (p+1)/2$ yet $K_1(\mathbb{Z}[C_p])$ has rank $(p-3)/2$.

$K_3(R)$ and $H_3(E(R), \mathbb{Z})$

The following material is due to Suslin [Su91]. Given an element α of $\pi_n(X)$ and an element β of $\pi_m(S^n)$, the composition product $\alpha \circ \beta$ is the element of $\pi_m(X)$ represented by $S^m \xrightarrow{\beta} S^n \xrightarrow{\alpha} X$. We will apply this to the Hopf element $\eta \in \pi_3(S^2)$, using the following observation.

If Y_n is the wedge of n copies of S^2, the Hilton-Milnor Theorem [Wh, XI(8.1)] says that

$$\Omega\Sigma Y_n \simeq \prod_{i=1}^{n} \Omega S^3 \times \prod_{i\neq j} \Omega S^5 \times Y_n',$$

where Y_n' is 5-connected and Σ is suspension. Note that $\pi_3(\Omega\Sigma S^2) = \pi_4(S^3) = \mathbb{Z}/2$, on the image of $\eta \in \pi_3(S^2)$. Hence $\pi_3(\Omega\Sigma Y_n) \cong (\mathbb{Z}/2)^n$. If Y_I is a wedge of copies of S^2 indexed by an infinite set I then (taking the filtered colimit over finite subsets of I) it follows that $\pi_3(\Omega\Sigma Y_I) \cong \oplus_I \mathbb{Z}/2$, generated by the factors $S^2 \to \Omega S^2 \to Y_I$.

LEMMA 1.19. *If X is a simply connected loop space, the composition product with η and the Hurewicz map $h : \pi_3(X) \to H_3(X, \mathbb{Z})$ fit into an exact sequence*

$$\pi_2(X) \xrightarrow{\circ\eta} \pi_3(X) \xrightarrow{h} H_3(X, \mathbb{Z}) \to 0.$$

PROOF. Let I be a set of generators of $\pi_2(X)$; the maps $f(i) : S^2 \to X$ induce a map $f : Y \to X$, where $Y = \vee_I S^2$. The map f factors as $Y \to \Omega\Sigma Y \xrightarrow{\Omega f^*} X$, where $X = \Omega X'$ and $f^* : \Sigma Y \to X'$ is the adjoint of f. Since $\pi_2(Y) \to \pi_2(X)$ is onto, the sequence $\pi_3(\Omega\Sigma Y) \to \pi_3(X) \to H_3(X) \to 0$ is exact by Exercise 1.25.

As above, $\pi_3(\Omega\Sigma Y) \cong \oplus_I \pi_3(\Omega\Sigma S^2)$, and the i^{th} factor is the image of $\pi_3(S^2)$, generated by η. The map $\pi_3(\Omega\Sigma Y) \to \pi_3(X)$ sends the generator of the i^{th} factor to the composition product $f(i) \circ \eta : S^3 \xrightarrow{\eta} S^2 \to \Omega\Sigma S^2 \to X$. Since $\pi_2(X)$ is generated by the $f(i)$, the result follows. $\square$

REMARK 1.19.1. (Suslin) Lemma 1.19 holds for any simply connected H-space X. To see this, note that the Hilton-Milnor Theorem for $Y_n = \vee S^2$ states that the space ΩY_n is homotopy equivalent to $\prod_i \Omega S^2 \times \prod_{i\neq j} \Omega S^3 \times Y''$ where Y'' is 3-connected. Thus $\pi_3(Y_n) = \pi_2(\Omega Y_n)$ is the sum of $\mathbb{Z}^n = \oplus \pi_3(S^2)$ and $\oplus_{i\neq j} \pi_3(S^3)$, where the second summand is generated by the Whitehead products $[\iota_i, \iota_j]$ of the generators of $\pi_2(Y)$. These Whitehead products map to $[f(i), f(j)]$, which vanish in $\pi_3(X)$ when X is any H-space by [Wh, X(7.8)]. With this modification, the proof of Lemma 1.19 goes through.

COROLLARY 1.20. *For any ring R the product with $[-1] \in K_1(\mathbb{Z})$ fits into an exact sequence*

$$K_2(R) \xrightarrow{[-1]} K_3(R) \xrightarrow{h} H_3(E(R), \mathbb{Z}) \to 0.$$

PROOF. By Ex. 1.12(a), the map $\pi_3(S^2) \to K_1(\mathbb{Z})$ sends η to $[-1]$. Since $X = BGL(R^+)$ is an H-space, the composition product $\pi_2(X) \xrightarrow{\eta} \pi_3(X)$ is multiplication by the image of η in $\pi_1(X) = K_1(R)$, namely $[-1]$; see Ex. 1.12(e). The result follows from Lemma 1.19 and the observation in Exercise 1.8 that $\pi_n BE(R)^+ \to K_n(R)$ is an isomorphism for $n \geq 2$. $\square$

EXERCISES

1.1 (Kervaire) Let X be a homology n-sphere, *i.e.,* a space with $H_*(X) = H_*(S^n)$. Show that there is a homotopy equivalence $S^n \to X^+$. *Hint:* Show that $\pi_1(X)$ is perfect if $n \neq 1$, so X^+ is simply connected, and use the Hurewicz theorem.

The binary icosohedral group $\Gamma = SL_2(\mathbb{F}_5)$ embeds in $O_3(\mathbb{R})$ as the symmetry group of both the dodecahedron and icosahedron. Show that the quotient $X = S^3/\Gamma$ is a homology 3-sphere, and conclude that the canonical map $S^3 \to X^+$ is a homotopy equivalence. (The fact that it is a homology sphere was discovered by Poincaré in 1904, and X is sometimes called the *Poincaré sphere.*)

1.2 a) If F is an acyclic space, show that F^+ is contractible.

b) If $X \xrightarrow{f} Y$ is acyclic and $f_* : \pi_1(X) \cong \pi_1(Y)$, show that f is a homotopy equivalence.

1.3 Prove the assertions in Remark 1.8.1 using the following standard result: Let X and Y be H-spaces having the homotopy type of a CW complex. If $f : X \to Y$ is a map which induces an isomorphism $H_*(X, \mathbb{Z}) \cong H_*(Y, \mathbb{Z})$, show that f is a homotopy equivalence. *Hint:* Since $\pi_1(Y)$ acts trivially on the homotopy fiber F by [Wh, IV.3.6], the relative Hurewicz theorem [Wh, IV.7.2] shows that $\pi_*(F) = 0$.

1.4 Here is a point-set construction of X^+ relative to a perfect normal subgroup P. Form Y by attaching one 2-cell e_p for each element of P, so that $\pi_1(Y) = \pi_1(X)/P$. Show that $H_2(Y;\mathbb{Z})$ is the direct sum of $H_2(X;\mathbb{Z})$ and the free abelian group on the set $\{[e_p] : p \in P\}$. Next, prove that each homology class $[e_p]$ is represented by a map $h_p : S^2 \to Y$, and form Z by attaching 3-cells to Y (one for each $p \in P$) using the h_p. Finally, prove that Z is a model for X^+.

1.5 *Perfect Radicals.* Show that the subgroup generated by the union of perfect subgroups of any group G is itself a perfect subgroup. Conclude that G has a largest perfect subgroup P, called the *perfect radical* of G, and that it is a normal subgroup of G.

1.6 Let cone(i) denote the mapping cone of a map $F \xrightarrow{i} X$. If F is an acyclic space, show that the map $X \to \text{cone}(i)$ is acyclic. If F is a subcomplex of X then cone(i) is homotopy equivalent to the quotient space X/F, so $X \to X/F$ is also acyclic. Conclude that if $X(R)$ is the Volodin space of Example 1.3.2 then $BGL(R)/X(R)$ is a model for $BGL(R)^+$. *Hint:* Consider long exact sequences in homology.

1.7 Show that $BGL(R_1 \times R_2)^+ \simeq BGL(R_1)^+ \times BGL(R_2)^+$ and hence $K_n(R_1 \times R_2) \cong K_n(R_1) \times K_n(R_2)$ for every pair of rings R_1, R_2 and every n. *Hint:* Use 3.1(6) below to see that $BGL(R_1 \times R_2) \cong BGL(R_1) \times BGL(R_2)$.

1.8 Let P be a perfect normal subgroup of G, and let $BG \to BG^+$ be a +-construction relative to P. Show that BP^+ is homotopy equivalent to the universal covering space of BG^+. Hence $\pi_n(BP^+) \cong \pi_n(BG^+)$ for all $n \geq 2$. *Hint:* BP is homotopy equivalent to a covering space of BG.

For $G = GL(R)$ and $P = E(R)$, this shows that $BE(R)^+$ is homotopy equivalent to the universal covering space of $BGL(R)^+$. Thus $K_n(R) \cong \pi_n BE(R)^+$ for $n \geq 2$.

(a) If R is a commutative ring, show that $SL(R) \hookrightarrow GL(R)$ induces isomorphisms $\pi_n BSL(R)^+ \cong K_n(R)$ for $n \geq 2$, and $\pi_1 BSL(R)^+ \cong SK_1(R)$. Conclude that the map $BSL(R)^+ \times B(R^\times) \to BGL(R)^+$ is a homotopy equivalence.

(b) If A is a finite semisimple algebra over a field, the subgroups $SL_n(A)$ of $GL_n(A)$ were defined in III.1.2.4. Show that $SL(A) \hookrightarrow GL(A)$ induces isomorphisms $\pi_n BSL(A)^+ \cong K_n(A)$ for $n \geq 2$, and $\pi_1 BSL(A)^+ \cong SK_1(A)$.

1.9 Suppose that $A \to S \to P$ is a universal central extension (III.5.3). In particular, S and P are perfect groups. Show that there is a homotopy fibration $BA \to BS^+ \to BP^+$. Conclude that $\pi_n(BS^+) = 0$ for $n \leq 2$, and that $\pi_n(BS^+) \cong \pi_n(BP^+) \cong \pi_n(BG^+)$ for all $n \geq 3$. In particular, $\pi_3(BP^+) \cong H_3(S;\mathbb{Z})$.

Since the Steinberg group $St(R)$ is the universal central extension of $E(R)$, this shows that $K_n(R) \cong \pi_n St(R)^+$ for all $n \geq 3$, and that $K_3(R) \cong H_3(St(R);\mathbb{Z})$.

1.10 For $n \geq 3$, let P_n denote the normal closure of the perfect group $E_n(R)$ in $GL_n(R)$, and let $BGL_n(R)^+$ denote the $+$–construction on $BGL_n(R)$ relative to P_n. Corresponding to the inclusions $GL_n \subset GL_{n+1}$ we can choose a sequence of maps $BGL_n(R)^+ \to BGL_{n+1}(R)^+$. Show that $\varinjlim BGL_n(R)^+$ is $BGL(R)^+$.

1.11 For each m and n, the group map $\Box\colon GL_m(R) \times GL_n(R) \to GL_{m+n}(R) \subset GL(R)$ induces a map $BGL_m(R) \times BGL_n(R) \to BGL(R) \to BGL(R)^+$. Show that these maps induce an H-space structure on $BGL(R)^+$.

1.12 In this exercise, we develop some properties of $B\Sigma_\infty^+$, where Σ_∞ denotes the union of the symmetric groups Σ_n. We will see in 4.9.3 that $\pi_n(B\Sigma_\infty^+)$ is the stable homotopy group π_n^s. The permutation representations $\Sigma_n \to GL_n(\mathbb{Z})$ (1.3.2) induce a map $B\Sigma_\infty^+ \to BGL(\mathbb{Z})^+$ and hence homomorphisms $\pi_n^s \to K_n(\mathbb{Z})$.

(a) Show that $\eta \in \pi_1^s \cong \mathbb{Z}/2$ maps to $[-1] \in K_1(\mathbb{Z})$.

(b) Show that the subgroups $\Sigma_m \times \Sigma_n$ of Σ_{m+n} induce an H-space structure on $B\Sigma_\infty^+$ such that $B\Sigma_\infty^+ \to BGL(\mathbb{Z})^+$ is an H-map. (See Ex. 1.11.)

(c) Modify the construction of Loday's product (1.10) to show that product representations $\Sigma_m \times \Sigma_n \to \Sigma_{mn}$ induce a map $B\Sigma_\infty^+ \wedge B\Sigma_\infty^+ \to B\Sigma_\infty^+$ compatible with the corresponding map for $BGL(\mathbb{Z})^+$. The resulting product $\pi_m^s \otimes \pi_n^s \to \pi_{m+n}^s$ makes the stable homotopy groups into a graded-commutative ring, and makes $\pi_*^s \to K_*(\mathbb{Z})$ into a ring homomorphism.

(d) Show that the Steinberg symbol $\{-1,-1,-1,-1\}$ of 1.10.1 vanishes in $K_4(\mathbb{Z})$ and $K_4(\mathbb{Q})$. Since this symbol is nonzero in $K_4^M(\mathbb{Q})$ by III.7.2(c,d), this shows that the Milnor K-groups of a field need not inject into its Quillen K-groups. *Hint:* $\eta^3 \neq 0$ in π_3^s but $\eta^4 = 0$ in π_4^s.

(e) If $\beta \in \pi_{n+t}(S^n)$ and $\alpha \in K_n(R)$, show that the composition product $\alpha \circ \beta$ in $K_{n+t}(R)$ agrees with the product of α with $[\beta] \in \pi_t^s$.

1.13 Let A_∞ denote the union of the alternating groups A_n; it is a subgroup of Σ_∞ of index 2. A_∞ is a perfect group, since the A_n are perfect for $n \geq 5$.

(a) Show that $B\Sigma_\infty^+ \simeq BA_\infty^+ \times B(\mathbb{Z}/2)$, so $\pi_n BA_\infty^+ \cong \pi_n^s$ for all $n \geq 2$.

(b) Use Lemma 1.19 and $\pi_3^s \cong \mathbb{Z}/24$ to conclude that $H_3(A_\infty,\mathbb{Z}) \cong \mathbb{Z}/12$.

(c) Use the Künneth formula and (a) to show that $H_3(\Sigma_\infty,\mathbb{Z}) \cong H_3(\mathcal{A}_\infty,\mathbb{Z}) \oplus (\mathbb{Z}/2)^2$. This calculation was first done by Nakaoka [Nak].

1.14 Extend the product map γ of Theorem 1.10 to a map $K(A) \wedge K(B) \to K(A \otimes B)$, so that the induced maps $K_0(A) \times K_n(B) \to K_n(A \otimes B)$ agree with the products defined in III.1.6.1 and Ex. III.5.4.

1.15 Let I be an ideal in R. Show that the group $\pi_0 K(R, I)$ of 1.11.1 is isomorphic to the group $K_0(I)$ of Ex. II.2.3, and that the maps $K_1(R/I) \to K_0(I) \to K_0(R)$ in *loc. cit.* agree with the maps of 1.11.1. *Hint:* $\pi_0 K(R \oplus I, 0 \oplus I)$ must be $K_0(I)$.

Use Ex. III.2.7 to show that $\pi_1 K(R \to R/I)$ is isomorphic to the group $K_1(R, I)$ of III.2.2, and that the maps $K_2(R/I) \to K_1(R, I) \to K_1(R)$ in III.5.7.1 agree with those of 1.11.1.

1.16 If $f : R \to S$ is a ring homomorphism, show that the relative group $K_0(f)$ of 1.11.1 agrees with the relative group $K_0(f)$ of II.2.10.

1.17 (Wagoner) We say $GL(R)$ is *flabby* if there is a homomorphism $\tau : GL(R) \to GL(R)$ so that for each n the restriction $\tau_n : GL_n(R) \to GL(R)$ of τ is conjugate to the map $(1, \tau_n) : g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & \tau_n(g) \end{smallmatrix}\right)$. In particular, τ_n and $(1, \tau_n)$ induce the same map $H_*(BGL_n(R)) \to H_*(BGL(R))$ by [WHomo, 6.7.8].

(a) Assuming that $GL(R)$ is flabby, show that $BGL(R)$ is acyclic. By Ex. 1.2, this implies that $BGL(R)^+$ is contractible, *i.e.,* that $K_n(R) = 0$ for $n > 0$. *Hint:* The H-space structure (Ex. 1.11) makes $H_*(BGL(R))$ into a ring.

(b) Show that $GL(R)$ is flabby for every flasque ring R (see II.2.1.3). This shows that flasque rings have $K_n(R) = 0$ for all n. *Hint:* Modify Ex. II.2.15(a).

1.18 Suppose that I is a nilpotent ideal and that $p^\nu I = 0$ for some ν. Show that $H_*(GL(R); M) \cong H_*(GL(R/I); M)$ for every uniquely p-divisible module M. Conclude that the relative groups $K_*(R, I)$ are p-groups.

1.19 Suppose that I is a nilpotent ideal in a ring R, and that I is uniquely divisible as an abelian group. Show that $H_*(GL(R); M) \cong H_*(GL(R/I); M)$ for every torsion module M. Conclude that the relative groups $K_*(R, I)$ are uniquely divisible abelian groups.

1.20 Show that every ring with unit is H-unital (see 1.11.2). Then show that a non-unital ring I is H-unital if every finite subset of I is contained in a unital subring. (This shows that the ring $M(R)$ of finite matrices over R is H-unital.)

Finally, show that I is H-unital if for every finite subset $\{a_j\}$ of I there is an $e \in I$ such that $ea_j = a_j$. (An example of such an I is the non-unital ring of functions with compact support on $\mathbb{C}^n$.)

1.21 (Morita invariance) For each $n > 0$, we saw in III.1.1.4 that $GL(R) \cong GL(M_n(R))$ via isomorphisms $M_m(R) \cong M_m(M_n(R))$. Deduce that there is a homotopy equivalence $BGL(R)^+ \simeq BGL(M_n(R))^+$ and hence isomorphisms $K_*(R) \cong K_*(M_n(R))$. (The cases $* = 0, 1, 2$ were given in II.2.7, III.1.6.4 and III.5.6.1.) We will give a more categorical proof in 6.3.5 below.

Compare this to the approach of 1.11.2, using $M(R)$.

1.22 (Loday symbols) Let $a_1, ..., a_n$ be elements of A so that $a_n a_1 = 0$ and each $a_i a_{i+1} = 0$. Show that the elementary matrices $e_{n,1}(a_n)$ and $e_{i,i+1}(a_i)$ commute and define a ring homomorphism $B = \mathbb{Z}[x_1, 1/x_1, \dots, x_n, 1/x_n] \to M_n(A)$. Using Ex. 1.21, we define the *Loday symbol* $\langle\langle a_1, ..., a_n \rangle\rangle$ in $K_n(A)$ to be the image of $\{x_1, ..., x_n\}$ under $K_n(B) \to K_n(M_n(A)) \cong K_n(A)$.

1.23 Let $F \to E \xrightarrow{p} B$ and $F' \to E' \xrightarrow{p'} B'$ be homotopy fibrations (1.2), and suppose given pairings $e : E \wedge X \to E'$, $b : B \wedge X \to B'$ so that $p'e = b(p \wedge 1)$.

$$\begin{array}{ccccccc}
\Omega B \wedge X & \xrightarrow{\partial\wedge 1} & F \wedge X & \longrightarrow & E \wedge X & \xrightarrow{p\wedge 1} & B \wedge X \\
\downarrow{\scriptstyle \Omega b} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle b} \\
\Omega B' & \xrightarrow{\partial'} & F' & \longrightarrow & E' & \xrightarrow{p'} & B'.
\end{array}$$

Show that there is a pairing $F \wedge X \xrightarrow{f} F'$ compatible with e and such that for $\beta \in \pi_*(B)$ and $\gamma \in \pi_*(X)$ the reduced join [Wh, p. 480] satisfies $\partial(\beta \wedge_b \gamma) = \partial(\beta) \wedge_f \gamma$.

1.24 Let $f : A \to B$ be a ring homomorphism and let $K(f)$ (resp., $K(f_C)$ be the relative groups (1.11.1), *i.e.,* the homotopy fiber of $K(A) \to K(B)$ (resp., $K(A \otimes C) \to K(B \otimes C)$). Use Ex. 1.23 to show that there is an induced pairing $K_*(f) \otimes K_*(C) \to K_*(f_C)$ such that for $\beta \in K_*(B)$ and $\gamma \in K_*(C)$ we have $\partial(\beta \wedge \gamma) = \partial(\beta) \wedge \gamma$ in $K_*(f_C)$.

When f is an R-algebra homomorphism, show that $K_*(f)$ is a right $K_*(R)$-module and that the maps in the relative sequence $K_{n+1}(B) \to K_n(f) \to K_n(A) \to K_n(B)$ of (1.11.1) are $K_*(R)$-module homomorphisms.

1.25 Suppose given a homotopy fibration sequence $F \to Y \to X$ with X, Y and F simply connected. Compare the long exact homotopy sequence (see 1.2) with the exact sequence of low degree terms in the Leray-Serre Spectral sequence (see [WHomo, 5.3.3]) to show that there is an exact sequence $\pi_3(Y) \to \pi_3(X) \to H_3(X) \to 0$.

1.26 The Galois group $G = \mathrm{Gal}(\mathbb{F}_{q^i}/\mathbb{F}_q)$ acts on the group μ of units of $\mathbb{F}_{q^i}$, and also on the i-fold tensor product $\mu^{\otimes i} = \mu \otimes \cdots \otimes \mu$. By functoriality 1.1.2, G acts on $K_*(\mathbb{F}_q)$. Show that $K_{2i-1}(\mathbb{F}_q)$ is isomorphic to $\mu^{\otimes i}$ as a G-module.

1.27 *Monomial matrices* Let F be a field and consider the subgroup M of $GL(F)$ consisting of matrices with only one nonzero entry in each row and column.

(a) Show that M is the wreath product $F^\times \wr \Sigma_\infty$, and contains $F^\times \wr A_\infty$.

(b) Show that $[M, M]$ is the kernel of $\det : F^\times \wr A_\infty \to F^\times$, so $H_1(M) \cong F^\times \times \Sigma_2$.

(c) Show that $[M, M]$ is perfect, and $BM^+ \simeq B[M, M]^+ \times B(F^\times) \times B\Sigma_2$.

§2. K-theory with finite Coefficients

In addition to the usual K-groups $K_i(R)$, or the K-groups $K_i(\mathcal{C})$ of a category $\mathcal{C}$, it is often useful to study its K-groups with coefficients "mod ℓ" $K_i(R;\mathbb{Z}/\ell)$ (or $K_i(\mathcal{C};\mathbb{Z}/\ell)$,) where ℓ is a positive integer. In this section we quickly recount the basic construction from mod ℓ homotopy theory. Basic properties of mod ℓ homotopy theory may be found in [N].

Recall [N] that if $m \geq 2$ the mod ℓ *Moore space* $P^m(\mathbb{Z}/\ell)$ is the space formed from the sphere S^{m-1} by attaching an m-cell via a degree ℓ map. It is characterized as having only one nonzero reduced integral homology group, namely $\tilde{H}^m(P) = \mathbb{Z}/\ell$. The suspension of $P^m(\mathbb{Z}/\ell)$ is the Moore space $P^{m+1}(\mathbb{Z}/\ell)$, and as m varies these fit together to form a suspension spectrum $P^\infty(\mathbb{Z}/\ell)$, called the *Moore spectrum.*

Definition 2.1. If $m \geq 2$, the mod ℓ homotopy "group" $\pi_m(X;\mathbb{Z}/\ell)$ of a based topological space X is defined to be the pointed set $[P^m(\mathbb{Z}/\ell), X]$ of based homotopy classes of maps from the Moore space $P^m(\mathbb{Z}/\ell)$ to X.

For a general space X, $\pi_2(X;\mathbb{Z}/\ell)$ isn't even a group, but the $\pi_m(X;\mathbb{Z}/\ell)$ are always groups for $m \geq 3$ and abelian groups for $m \geq 4$ [N]. If X is an H-space, such as a loop space, then these bounds improve by one. If $X = \Omega Y$ then we can define $\pi_1(X;\mathbb{Z}/\ell)$ as $\pi_2(Y;\mathbb{Z}/\ell)$; this is independent of the choice of Y by Ex. 2.1. More generally, if $X = \Omega^k Y_k$ for $k >> 0$ and $P^m = P^m(\mathbb{Z}/\ell)$ then the formula

$$\pi_m(X;\mathbb{Z}/\ell) = [P^m, X] = [P^m, \Omega^k Y_k] \cong [P^{m+k}, Y_k] = \pi_{m+k}(Y_k;\mathbb{Z}/\ell)$$

shows that we can ignore these restrictions on m, and that $\pi_m(X;\mathbb{Z}/\ell)$ is an abelian group for all $m \geq 0$ (or even negative m, as long as $k > 2 + |m|$).

In particular, if X is an infinite loop space then abelian groups $\pi_m(X;\mathbb{Z}/\ell)$ are defined for all $m \in \mathbb{Z}$, using the explicit sequence of deloopings of X provided by the given structure on X.

2.1.1. If $F \to E \to B$ is a Serre fibration there is a long exact sequence of groups/pointed sets (which is natural in the fibration):

$$\begin{aligned}\cdots \to \pi_{m+1}(B;\mathbb{Z}/\ell) \to & \pi_m(F;\mathbb{Z}/\ell) \to \pi_m(E;\mathbb{Z}/\ell) \to \\ & \pi_m(B;\mathbb{Z}/\ell) \to \pi_{m-1}(F;\mathbb{Z}/\ell) \to \cdots \to \pi_2(B;\mathbb{Z}/\ell).\end{aligned}$$

This is just a special case of the fact that $\cdots \to [P,F] \to [P,E] \to [P,B]$ is exact for any CW complex P; see [Wh, III.6.18*].

If $m \geq 2$, the cofibration sequence $S^{m-1} \xrightarrow{\ell} S^{m-1} \to P^m(\mathbb{Z}/\ell)$ defining $P^m(\mathbb{Z}/\ell)$ induces an exact sequence of homotopy groups/pointed sets

$$\pi_m(X) \xrightarrow{\ell} \pi_m(X) \to \pi_m(X;\mathbb{Z}/\ell) \xrightarrow{\partial} \pi_{m-1}(X) \xrightarrow{\ell} \pi_{m-1}(X).$$

If ℓ is odd, or divisible by 4, F. Peterson showed that there is even a non-canonical splitting $\pi_m(X;\mathbb{Z}/\ell) \to \pi_m(X)/\ell$ (see [Br, 1.8]).

It is convenient to adopt the notation that if A is an abelian group then ${}_\ell A$ denotes the subgroup of all elements a of A such that $\ell \cdot a = 0$. This allows us to restate the above exact sequence in a concise fashion.

Universal Coefficient Sequence 2.2. *For all $m \geq 3$ there is a natural short exact sequence*

$$0 \to (\pi_m X) \otimes \mathbb{Z}/\ell \to \pi_m(X; \mathbb{Z}/\ell) \xrightarrow{\partial} {}_\ell(\pi_{m-1} X) \to 0.$$

This sequence is split exact (but not naturally) when $\ell \not\equiv 2 \bmod 4$.

For π_2, the sequence 2.2 of pointed sets is also exact in a suitable sense; see [N, p. 3]. However this point is irrelevant for loop spaces, so we ignore it.

Example 2.2.1. When $\ell = 2$, the sequence need not split. For example, it is known that $\pi_{m+2}(S^m; \mathbb{Z}/2) = \mathbb{Z}/4$ for $m \geq 3$, and that $\pi_2(BO; \mathbb{Z}/2) = \pi_3(O; \mathbb{Z}/2) = \mathbb{Z}/4$; see [AT65].

Here is another way to define mod ℓ homotopy groups, and hence $K_*(R; \mathbb{Z}/\ell)$.

Proposition 2.3. *Suppose that X is a loop space, and let F denote the homotopy fiber of the map $X \to X$ which is multiplication by ℓ. Then $\pi_m(X; \mathbb{Z}/\ell) \cong \pi_{m-1}(F)$ for all $m \geq 2$.*

Proof. (Neisendorfer) Let Maps(A, X) be the space of pointed maps. If $S = S^k$ is the k-sphere then the homotopy groups of Maps(S^k, X) are the homotopy groups of X (reindexed by k), while if $P = P^k(\mathbb{Z}/\ell)$ is a mod ℓ Moore space, the homotopy groups of Maps(P, X) are the mod ℓ homotopy groups of X (reindexed by k).

Now applying Maps$(-, X)$ to a cofibration sequence yields a fibration sequence, and applying Maps$(A, -)$ to a fibration sequence yields a fibration sequence; this may be formally deduced from the axioms (SM0) and (SM7) for any model structure, which hold for spaces. Applying Maps$(-, X)$ to $S^k \to S^k \to P^{k+1}(\mathbb{Z}/\ell)$ shows that Maps(P, X) is the homotopy fiber of Maps$(S^k, X) \to$ Maps(S^k, X). Applying Maps$(S^k, -)$ to $F \to X \to X$ shows that Maps(S^k, F) is also the homotopy fiber, and is therefore homotopy equivalent to Maps(P, X). Taking the homotopy groups yields the result. $\square$

Spectra 2.3.1. For fixed ℓ, the Moore spectrum $P^\infty(\mathbb{Z}/\ell)$ is equivalent to the (spectrum) cofiber of multiplication by ℓ on the sphere spectrum. If $\mathbf{E}$ is a spectrum, then (by S-duality) the homotopy groups $\pi_*(\mathbf{E}; \mathbb{Z}/\ell) = \varinjlim \pi_{*+r}(\mathbf{E}_r; \mathbb{Z}/\ell)$ are the same as the homotopy groups of the spectrum $\mathbf{E} \wedge P^\infty(\mathbb{Z}/\ell)$.

Now suppose that $\mathcal{C}$ is either a symmetric monoidal category, or an exact category, or a Waldhausen category. We will construct a K-theory space $K(\mathcal{C})$ below (in 4.3, 6.3 and 8.5); in each case $K(\mathcal{C})$ is an infinite loop space.

Definition 2.4. The mod ℓ K-groups of R are defined to be the abelian group:

$$K_m(R; \mathbb{Z}/\ell) = \pi_m(K(R); \mathbb{Z}/\ell), \quad m \in \mathbb{Z}.$$

Similarly, if the K-theory space $K(\mathcal{C})$ of a category $\mathcal{C}$ is defined then the mod ℓ K-groups of $\mathcal{C}$ are defined to be $K_m(\mathcal{C}; \mathbb{Z}/\ell) = \pi_m(K(\mathcal{C}); \mathbb{Z}/\ell)$.

By 2.1.1, if $\mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}_3$ is a sequence such that $K(\mathcal{C}_1) \to K(\mathcal{C}_2) \to K(\mathcal{C}_3)$ is a fibration, then there is a long exact sequence of abelian groups

$$\cdots \to K_{n+1}(\mathcal{C}_3; \mathbb{Z}/\ell) \to K_n(\mathcal{C}_1; \mathbb{Z}/\ell) \to K_n(\mathcal{C}_2; \mathbb{Z}/\ell) \to K_n(\mathcal{C}_3; \mathbb{Z}/\ell) \cdots$$

If $m \geq 2$ this definition states that $K_m(R;\mathbb{Z}/\ell) = [P^m(\mathbb{Z}/\ell), K(R)]$. Because $K(R) \simeq \Omega Y$, we can define $K_1(R;\mathbb{Z}/\ell)$ in a way that is independent of the choice of Y (Ex. 2.1); it agrees with the definition in III.1.7.4 (see Ex. 2.2). However, the groups $K_0(R;\mathbb{Z}/\ell)$ and $K_m(R;\mathbb{Z}/\ell)$ for $m < 0$ depend not only upon the loop space $K(R)$, but also upon the choice of the deloopings of $K(R)$ in the underlying K-theory spectrum $\mathbf{K}(R)$. In fact, the literature is not consistent about $K_m(R;\mathbb{Z}/\ell)$ when $m < 2$, even for $K_1(R;\mathbb{Z}/\ell)$. Similar remarks apply to the definition of $K_m(\mathcal{C};\mathbb{Z}/\ell)$.

By Universal Coefficients 2.2, the mod ℓ K-groups are related to the usual K-groups:

Universal Coefficient Theorem 2.5. *There is a short exact sequence*

$$0 \to K_m(R) \otimes \mathbb{Z}/\ell \to K_m(R;\mathbb{Z}/\ell) \to {}_\ell K_{m-1}(R) \to 0$$

for every $m \in \mathbb{Z}$, $\mathcal{C}$, and ℓ. It is split exact unless $\ell \equiv 2 \bmod 4$. Ex. 2.3 shows that the splitting is not natural in R.

Similarly, if the K-theory of a category $\mathcal{C}$ is defined then we have an exact sequence

$$0 \to K_m(\mathcal{C}) \otimes \mathbb{Z}/\ell \to K_m(\mathcal{C};\mathbb{Z}/\ell) \to {}_\ell K_{m-1}(\mathcal{C}) \to 0$$

Example 2.5.1. ($\ell = 2$) Since the isomorphism $\Omega^\infty\Sigma^\infty \to \mathbb{Z} \times BO$ factors through $K(\mathbb{Z})$ and $K(\mathbb{R})$, the universal coefficient theorem and 2.2.1 show that

$$K_2(\mathbb{Z};\mathbb{Z}/2) \cong K_2(\mathbb{R};\mathbb{Z}/2) \cong \pi_2(BO;\mathbb{Z}/2) = \mathbb{Z}/4.$$

It turns out [AT65] that for $\ell = 2$ the sequence for $K_m(R;\mathbb{Z}/2)$ is split whenever multiplication by $[-1] \in K_1(\mathbb{Z})$ is the zero map from $K_{m-1}(R)$ to $K_m(R)$. For example, this is the case for the finite fields $\mathbb{F}_q$, an observation made in [Br].

Example 2.5.2 (Bott elements). Suppose that R contains a primitive ℓ^{th} root of unity ζ. The Universal Coefficient Theorem 2.5 provides an element $\beta \in K_2(R;\mathbb{Z}/\ell)$, mapping to $\zeta \in {}_\ell K_1(R)$. This element is called the *Bott element*, and it plays an important role in the product structure of the ring $K_*(R;\mathbb{Z}/\ell)$. For finite fields, this role was mentioned briefly in Remark 1.13.1.

Remark 2.5.3. A priori, β depends not only upon ζ but also upon the choice of the splitting in 2.5. One way to choose β is to observe that the inclusion of μ_ℓ in $GL_1(R)$ induces a map $B\mu_\ell \to BGL(R) \to BGL(R)^+$ and therefore a set function $\mu_\ell \to K_2(R;\mathbb{Z}/\ell)$. A posteriori, it turns out that this is a group homomorphism unless $\ell \equiv 2 \pmod 4$.

Example 2.6. Let k be the algebraic closure of the field $\mathbb{F}_p$. Quillen's computation of $K_*(\mathbb{F}_q)$ in 1.13 shows that $K_n(k) = 0$ for m even ($m \geq 2$), and that $K_m(k) = \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$ for m odd ($m \geq 1$). It follows that if ℓ is prime to p then:

$$K_m(k;\mathbb{Z}/\ell) = \begin{cases} \mathbb{Z}/\ell & \text{if } m \text{ is even, } m \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In fact, $K_*(k;\mathbb{Z}/\ell)$ is the polynomial ring $\mathbb{Z}/\ell[\beta]$ on the Bott element $\beta \in K_2(k;\mathbb{Z}/\ell)$, under the K-theory product of 2.8 below. See 1.13.1 (and Chapter VI, 1.3.1) for more details.

The next result shows that we may always assume that ℓ is a power of a prime.

PROPOSITION 2.7. *If $\ell = q_1 q_2$ with q_1 and q_2 relatively prime, then $\pi_m(X;\mathbb{Z}/\ell)$ is naturally isomorphic to $\pi_m(X;\mathbb{Z}/q_1) \times \pi_m(X;\mathbb{Z}/q_1)$.*

PROOF. Set $P_1 = P^m(\mathbb{Z}/q_1)$, $P_2 = P^m(\mathbb{Z}/q_2)$ and $P = P_1 \vee P_2$. Since P has only one nonzero reduced integral homology group, namely $\tilde{H}^m(P) = \mathbb{Z}/q_1 \times \mathbb{Z}/q_2 \cong \mathbb{Z}/\ell$, the natural map $P \to P^m(\mathbb{Z}/\ell)$ must be a homotopy equivalence. But then $\pi_m(X;\mathbb{Z}/\ell)$ is naturally isomorphic to

$$[P^m(\mathbb{Z}/q_1) \vee P^m(\mathbb{Z}/q_2), X] \cong [P^m(\mathbb{Z}/q_1), X] \times [P^m(\mathbb{Z}/q_2), X],$$

which is the required group $\pi_m(X;\mathbb{Z}/q_1) \times \pi_m(X;\mathbb{Z}/q_1)$. □

Products

If $\ell \geq 3$ is prime, there is a homotopy equivalence $P^m(\mathbb{Z}/\ell^\nu) \wedge P^n(\mathbb{Z}/\ell^\nu) \simeq P^{m+n}(\mathbb{Z}/\ell^\nu) \vee P^{m+n-1}(\mathbb{Z}/\ell^\nu)$. The projections onto the first factor give a spectrum "product" map $P^\infty(\mathbb{Z}/\ell^\nu) \wedge P^\infty(\mathbb{Z}/\ell^\nu) \to P^\infty(\mathbb{Z}/\ell^\nu)$ which is homotopy associative and commutative unless $\ell^\nu = 3$. (The same thing is true when $\ell = 2$, except the product map does not exist if $2^\nu = 2$, it is not homotopy associative if $2^\nu = 4$ and it is not homotopy commutative when $2^\nu = 4, 8$.) These facts are due to Araki and Toda, and follow by S-duality from [N, 8.5–6]. So from now on, we shall exclude the pathological cases $\ell^\nu = 2, 3, 4, 8$.

If $\mathbf{E}$ is a homotopy associative and commutative ring spectrum, then so is the spectrum $\mathbf{E} \wedge P^\infty(\mathbb{Z}/\ell^\nu)$, unless $\ell^\nu = 2, 3, 4, 8$. Applying this to $\mathbf{E} = \mathbf{K}(R)$ yields the following result.

THEOREM 2.8. *Let R be a commutative ring, and suppose $\ell^\nu \neq 2, 3, 4, 8$. Then $\mathbf{K}(R) \wedge P^\infty(\mathbb{Z}/\ell^\nu)$ is a homotopy associative and commutative ring spectrum. In particular, $K_*(R;\mathbb{Z}/\ell^\nu)$ is a graded-commutative ring.*

SCHOLIUM 2.8.1. Browder [Br] has observed that if $\pi_m(\mathbf{E}) = 0$ for all even $m > 0$ (and $m < 0$) then $\mathbf{E} \wedge P^\infty(\mathbb{Z}/\ell^\nu)$ is a homotopy associative and commutative ring spectrum even for $\ell^\nu = 2, 3, 4, 8$. This applies in particular to $\mathbf{E} = \mathbf{K}(\mathbb{F}_q)$, as remarked in 1.13.1 and 2.6 above.

COROLLARY 2.8.2. *If $\ell \geq 3$ and R contains a primitive ℓ^{th} root of unity ζ, and $\beta \in K_2(R;\mathbb{Z}/\ell)$ is the Bott element (2.5.2), there is a graded ring homomorphism $\mathbb{Z}/\ell[\zeta, \beta] \to K_*(R;\mathbb{Z}/\ell)$.*

If $\zeta \notin R$, there are elements $\beta' \in K_{2\ell}(R;\mathbb{Z}/\ell)$ and $\zeta' \in K_{2\ell-1}(R;\mathbb{Z}/\ell)$ whose images in $K_{2\ell}(R[\zeta];\mathbb{Z}/\ell)$ and $K_{2\ell-1}(R[\zeta];\mathbb{Z}/\ell)$ are $\beta^{\ell-1}$ and $\beta^{\ell-2}\zeta$, respectively.

PROOF. The first assertion is immediate from 2.8 and 2.5.2. For the second assertion we may assume that $R = \mathbb{Z}$. Then the Galois group G of $\mathbb{Z}[\zeta]$ over $\mathbb{Z}$ is cyclic of order $\ell - 1$, and we define β' to be the image of $-\beta^{\ell-1} \in K_{2\ell}(\mathbb{Z}[\zeta];\mathbb{Z}/\ell)$ under the transfer map i_* (1.1.3). Since $i^* i_*$ is $\sum_{g \in G} g^*$ by Ex. 6.13,

$$i^*\beta' = -\sum g^* \beta^{\ell-1} = -(\ell - 1)\beta^{\ell-1}.$$

Similarly, $\zeta' = i_*(-\zeta\beta^{\ell-2})$ has $i^*\zeta' = \zeta\beta^{\ell-2}$. □

2.9 *The ℓ-adic completion.* Fix a prime ℓ. The *ℓ-adic completion* of a spectrum $\mathbf{E}$, $\hat{\mathbf{E}}_p$, is the homotopy limit (over ν) of the spectra $\mathbf{E} \wedge P^\infty(\mathbb{Z}/\ell^\nu)$. We let $\pi_n(\mathbf{E}; \mathbb{Z}_\ell)$ denote the homotopy groups of this spectrum; if $\mathbf{E} = \mathbf{K}(R)$ we write $K_n(R; \mathbb{Z}_\ell)$ for $\pi_n(\mathbf{K}(R); \mathbb{Z}_\ell)$. There is an extension

$$0 \to \varprojlim{}^1 \pi_{n+1}(\mathbf{E}; \mathbb{Z}/\ell^\nu) \to \pi_n(\mathbf{E}; \mathbb{Z}_\ell) \to \varprojlim \pi_n(\mathbf{E}; \mathbb{Z}/\ell^\nu) \to 0.$$

If the homotopy groups $\pi_{n+1}(\mathbf{E}; \mathbb{Z}/\ell^\nu)$ are finite, the $\varprojlim{}^1$ term vanishes and, by Universal Coefficients 2.5, $\pi_n(\mathbf{E}; \mathbb{Z}_\ell)$ is an extension of the Tate module of $\pi_{n-1}(\mathbf{E})$ by the ℓ-adic completion of $\pi_n(\mathbf{E})$. (The (ℓ-primary) *Tate module* of an abelian group A is the inverse limit of the groups $\mathrm{Hom}(\mathbb{Z}/\ell^\nu, A)$.) For example, the Tate module of $K_1(\mathbb{C}) = \mathbb{C}^\times$ is $\mathbb{Z}_\ell$, so $K_1(\mathbb{C}; \mathbb{Z}_\ell) = \pi_1(\mathbf{K}(\mathbb{C}); \mathbb{Z}_\ell)$ is $\mathbb{Z}_\ell$.

If $\mathbf{E}$ is a homotopy associative and commutative ring spectrum then so is the homotopy limit $\hat{\mathbf{E}}_p$. Thus $\pi_*(\mathbf{E}; \mathbb{Z}_\ell)$, and in particular $K_*(R; \mathbb{Z}_\ell)$, is also a graded-commutative ring.

We conclude with Gabber's Rigidity Theorem [Gabber]. If I is an ideal in a commutative ring A, we say that (A, I) is a *Hensel pair* if for every finite commutative A-algebra C the map $C \to C/IC$ induces a bijection on idempotents. A *Hensel local ring* is a commutative local ring R such that $(A, \mathfrak{m})$ is a Hensel pair. These conditions imply that I is a radical ideal (Ex. I.2.1), and (A, I) is a Hensel pair whenever I is complete by Ex. I.2.2(i).

Rigidity Theorem 2.10. *If (A, I) is a Hensel pair and $1/\ell \in A$, then for all $n \geq 1$ we have $K_n(A; \mathbb{Z}/\ell) \xrightarrow{\cong} K_n(A/I; \mathbb{Z}/\ell)$, and $\widetilde{H}_*(GL(I), \mathbb{Z}/\ell) = 0$*

Gabber proves that $\widetilde{H}_*(GL(I), \mathbb{Z}/\ell) = 0$, and observes that this is equivalent to $K_n(A; \mathbb{Z}/\ell) \to K_n(A/I; \mathbb{Z}/\ell)$ being onto.

Example 2.10.1. If $1/\ell \in R$ then $K_n(R[[x]]; \mathbb{Z}/\ell) \cong K_n(R; \mathbb{Z}/\ell)$ for all $n \geq 0$.

Examples 2.10.2. A restriction like $n \geq 0$ is necessary. Les Reid [Reid] has given an example of a 2-dimensional hensel local $\mathbb{Q}$-algebra with $K_{-2}(A) = \mathbb{Z}$, and Drinfeld [Drin] has shown that $K_{-1}(I) = 0$.

EXERCISES

2.1 Suppose that X is a loop space. Show that $\pi_1(X; \mathbb{Z}/\ell)$ is independent of the choice of Y such that $X \simeq \Omega Y$. This shows that $K_1(R; \mathbb{Z}/\ell)$ and even $K_1(\mathcal{C}; \mathbb{Z}/\ell)$ are well defined.

2.2 Show that the group $K_1(R; \mathbb{Z}/\ell)$ defined in 2.4 is isomorphic to the group defined in III.1.7.4. Using the Fundamental Theorem III.3.7 (and III.4.1.2), show that $K_0(R; \mathbb{Z}/\ell)$ and even the groups $K_n(R; \mathbb{Z}/\ell)$ for $n < 0$ which are defined in 2.4 are isomorphic to the corresponding groups defined in Ex. III.4.6.

2.3 Let R be a Dedekind domain with fraction field F. Show that the kernel of the map $K_1(R; \mathbb{Z}/\ell) \to K_1(F; \mathbb{Z}/\ell)$ is $SK_1(R)/\ell$. Hence it induces a natural map

$$_\ell \,\mathrm{Pic}(R) \xrightarrow{\rho} F^\times / F^{\times \ell} R^\times.$$

Note that $F^\times/R^\times$ is a free abelian group by I.3.6, so the target is a free $\mathbb{Z}/\ell$-module for every integer ℓ. Finally, use I.3.6 and I.3.8.1 to give an elementary description of ρ.

In particular, If R is the ring of integers in a number field F, the Bass-Milnor-Serre Theorem III.2.5 shows that the extension $K_1(R;\mathbb{Z}/\ell)$ of ${}_\ell\mathrm{Pic}(R)$ by $R^\times/R^{\times\ell}$ injects into $F^\times/F^{\times\ell}$, and that ${}_\ell K_0(R)$ is not a natural summand of $K_1(R;\mathbb{Z}/\ell)$. (If $1/\ell \in R$, the étale Chern class $K_1(R;\mathbb{Z}/\ell) \to H^1_{\mathrm{et}}(\mathrm{Spec}(R),\mu_\ell)$ of V.11.10 is an isomorphism.)

2.4 If $n \geq 2$, there is a Hurewicz map $\pi_n(X;\mathbb{Z}/\ell) \to H_n(X;\mathbb{Z}/\ell)$ sending the class of a map $f : P^n \to X$ to $f_*[e]$, where $[e] \in H_n(P^n;\mathbb{Z}/\ell) \cong H_n(S^n;\mathbb{Z}/\ell)$ is the canonical generator. Its restriction to $\pi_n(X)/\ell$ is the reduction modulo ℓ of the usual Hurewicz homomorphism $\pi_n(X) \to H_n(X;\mathbb{Z})$.

(a) If $n \geq 3$, show that the Hurewicz map is a homomorphism. (If $n = 2$ and ℓ is odd, it is also a homomorphism.) *Hint:* Since P^n is a suspension, there is a comultiplication map $P^n \to P^n \vee P^n$.

If $n = 2$ and ℓ is even, the Hurewicz map h may not be a homomorphism, even if X is an infinite loop space. The precise formula is: $h(a+b) = h(a)+h(b)+(\ell/2)\{\partial a, \partial b\}$. (See [We93].)

(b) In example 2.5.1, show that the Hurewicz map from $K_2(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/4$ to $H_2(SL(\mathbb{R});\mathbb{Z}/2) \cong \mathbb{Z}/2$ is nonzero on β and $2\beta = \{-1,-1\}$, but zero on 3β.

(c) If $n = 2$, show that the Hurewicz map is compatible with the action of $\pi_2(X)$ on $\pi_2(X;\mathbb{Z}/\ell)$ and on $H_2(X;\mathbb{Z}/\ell)$.

2.5 Show that $K_*(R;\mathbb{Z}/\ell^\nu)$ is a graded module over $K_*(R)$, associated to the evident pairing $\mathbf{K}(R) \wedge \mathbf{K}(R) \wedge P^\infty(\mathbb{Z}/\ell^\nu) \to \mathbf{K}(R) \wedge P^\infty(\mathbb{Z}/\ell^\nu)$.

2.6 Fix a prime ℓ and let $\mathbb{Z}/\ell^\infty$ denote the union of the groups $\mathbb{Z}/\ell^\nu$, which is a divisible torsion group. Show that there is a space $P^m(\mathbb{Z}/\ell^\infty) = \varinjlim P^m(\mathbb{Z}/\ell^\nu)$ such that $\pi_m(X;\mathbb{Z}/\ell^\infty) = [P^m(\mathbb{Z}/\ell^\infty), X]$ is the direct limit of the $\pi_m(X;\mathbb{Z}/\ell^\nu)$. Then show that there is a universal coefficient sequence for $m \geq 3$:

$$0 \to (\pi_m X) \otimes \mathbb{Z}/\ell^\infty \to \pi_m(X;\mathbb{Z}/\ell^\infty) \xrightarrow{\partial} (\pi_{m-1}X)_{\ell\text{-tors}} \to 0.$$

§3. Geometric realization of a small category

Recall (II.6.1.3) that a "small" category is a category whose objects form a set. If C is a small category, its *geometric realization* BC is a CW complex constructed naturally out of C. By definition, BC is the geometric realization $|NC|$ of the nerve NC of C; see 3.1.4 below. However, it is characterized in a simple way.

CHARACTERIZATION 3.1. The realization BC of a small category C is the CW complex uniquely characterized up to homeomorphism by the following properties. Let $\mathbf{n}$ denote the category with n objects $\{0, 1, \cdots, n-1\}$, with exactly one morphism $i \to j$ for each $i \le j$; $\mathbf{n}$ is an ordered set, regarded as a category.

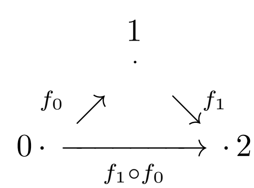
(1) (Naturality) A functor $F\colon C \to D$ induces a cellular map $BF\colon BC \to BD$, $BF \circ BG = B(FG)$ and $B(\mathrm{id}_C)$ is the identity map on BC.
(2) $B\mathbf{n}$ is the standard $(n-1)$–simplex Δ^{n-1}. The functor $\phi : \mathbf{i} \to \mathbf{n}$ induces the simplicial map $\Delta^{i-1} \to \Delta^{n-1}$ sending vertex j to vertex $\phi(j)$.
(3) BC is the colimit $\mathrm{colim}_\Phi\, B\mathbf{i}$, where Φ is the category whose objects are functors $\mathbf{n} \to C$, and whose morphisms are factorizations $\mathbf{i} \to \mathbf{n} \to C$. The corresponding map $B\mathbf{i} \to B\mathbf{n}$ is given by (2).

The following useful properties are consequences of this characterization:

(4) If C is a subcategory of D, BC is a subcomplex of BD;
(5) If C is the coproduct of categories C_α, $BC = \coprod BC_\alpha$;
(6) $B(C \times D)$ is homeomorphic to $(BC) \times (BD)$, where the product is given the compactly generated topology;

Here are some useful special cases of (2) for small n:

$B\mathbf{0} = \emptyset$ is the empty set, because $\mathbf{0}$ is the empty category.
$B\mathbf{1} = \{0\}$ is a one-point space, since $\mathbf{1}$ is the one object-one morphism category.
$B\mathbf{2} = [0, 1]$ is the unit interval, whose picture is: $0 \cdot \longrightarrow \cdot 1$.
$B\mathbf{3}$ is the 2-simplex; the picture of this identification is:



The small categories form the objects of a category CAT, whose morphisms are functors. By (1), we see that geometric realization is a functor from CAT to the category of CW complexes and cellular maps.

RECIPE 3.1.1. The above characterization of the CW complex BC gives it the following explicit cellular decomposition. The 0-cells (vertices) are the objects of C. The 1-cells (edges) are the morphisms in C, excluding all identity morphisms, and they are attached to their source and target. For each pair (f, g) of composable maps in C, attach a 2-simplex, using the above picture of $B\mathbf{3}$ as the model. (Ignore pairs (f, g) where either f or g is an identity.) Inductively, given an n-tuple of composable maps in C (none an identity map), $c_0 \to c_1 \to \cdots \to c_n$, attach an n-simplex, using $B(\mathbf{n+1})$ as the model. By (3), BC is the union of these spaces, equipped with the weak topology.

Notice that this recipe implies a canonical cellular homeomorphism between BC and the realization BC^{op} of the opposite category C^{op}. In effect, the recipe doesn't notice which way the arrows run.

EXAMPLE 3.1.2. Let C_2 be the category with one object and two morphisms, 1 and σ, with $\sigma^2 = 1$. The recipe tells us that BC_2 has exactly one n-cell for each n, attached to the $(n-1)$-cell by a map of degree 2 (corresponding to the first and last faces of the n-simplex). Therefore the n-skeleton of BC_2 is the projective n-space $\mathbb{RP}^n$, and their union BC_2 is the infinite projective space $\mathbb{RP}^\infty$.

EXAMPLE 3.1.3. Any group G (or monoid) may be regarded as a category with one object. The realization BG of this category is the space studied in Section 1. The recipe 3.1.1 shows that BG has only one vertex, and one 1-cell for every nontrivial element of G.

Although the above recipe gives an explicit description of the cell decomposition of BC, it is a bit vague about the attaching maps. To be more precise, we shall assume that the reader has a slight familiarity with the basic notions in the theory of simplicial sets, as found for example in [WHomo] or [May]. A simplicial set X is a contravariant functor $\Delta \to \mathbf{Sets}$, where Δ denotes the subcategory of ordered sets on the objects $\{\mathbf{0}, \mathbf{1}, ..., \mathbf{n}, ...\}$. Alternatively, it is a sequence of sets $X_0, X_1, \dots,$ together with "face" maps $\partial_i : X_n \to X_{n-1}$ and "degeneracy maps" $\sigma_i : X_n \to X_{n+1}$ $(0 \le i \le n)$, subject to certain identities for the compositions of these maps.

We may break down the recipe for BC into two steps: we first construct a simplicial set NC, called the nerve of the category C, and then set $BC = |NC|$.

DEFINITION 3.1.4 (THE NERVE OF C). The *nerve* NC of a small category C is the simplicial set defined by the following data. Its n-simplices are functors $c\colon \mathbf{n+1} \to C$, *i.e.,* diagrams in C of the form

$$c_0 \to c_1 \to \cdots \to c_n.$$

The i^{th} face $\partial_i(c)$ of this simplex is obtained by deleting c_i in the evident way; to get the i^{th} degeneracy $\sigma_i(c)$, one replaces c_i by $c_i \xrightarrow{=} c_i$.

The geometric realization $|X.|$ of a simplicial set $X.$ is defined to be the CW complex obtained by following the recipe 3.1.1 above, attaching an n-cell for each nondegenerate n-simplex x, identifying the boundary faces of the simplex with the $(n-1)$-simplices indexed by the $\partial_i x$. See [WHomo, 8.1.6] or [May, §14] for more details.

BC is defined as the geometric realization $|NC|$ of the nerve of C. From this prescription, it is clear that BC is given by recipe 3.1.1 above.

By abuse of notation, we will say that a category is contractible, or connected, or has any other topological property if its geometric realization has that property. Similarly, we will say that a functor $F\colon C \to D$ is a homotopy equivalence if BF is a homotopy equivalence $BC \simeq BD$.

HOMOTOPY-THEORETIC PROPERTIES 3.2. A natural transformation $\eta\colon F_0 \Rightarrow F_1$ between two functors $F_i\colon C \to D$ gives a homotopy $BC \times [0,1] \to BD$ between the maps BF_0 and BF_1. This follows from (4) and (6) of 3.1, because η may be viewed as a functor from $C \times \mathbf{2}$ to D whose restriction to $C \times \{i\}$ is F_i.

As a consequence, any adjoint pair of functors $L: C \to D$, $R: D \to C$ induces a homotopy equivalence between BC and BD, because there are natural transformations $LR \Rightarrow id_D$ and $id_C \Rightarrow RL$.

EXAMPLE 3.2.1 (SMALLNESS). Any equivalence $C_0 \xrightarrow{F} C$ between small categories induces a homotopy equivalence $BC_0 \xrightarrow{\sim} BC$, because F has an adjoint.

In practice, we will often work with a category C, such as $\mathbf{P}(R)$ or $\mathbf{M}(R)$, which is not actually a small category, but which is *skeletally small* (II.6.1.3). This means that C is equivalent to a small category, say to C_0. In this case, we can use BC_0 instead of the mythical BC, because any other choice for C_0 will have a homotopy equivalent geometric realization. We shall usually overlook this fine set-theoretic point in practice, just as we did in defining K_0 in chapter II.

EXAMPLE 3.2.2 (INITIAL OBJECTS). Any category with an initial object is contractible, because then the natural functor $C \to \mathbf{1}$ has a left adjoint. Similarly, any category with a terminal object is contractible.

For example, suppose given an object d of a category C. The *comma category* C/d of *objects over* d has as its objects the morphisms $f: c \to d$ in C with target d. A morphism in the comma category from f to $f': c' \to d$ is a morphism $h: c \to c'$ so that $f = f'h$. The comma category C/d is contractible because it has a terminal object, namely the identity map $\mathrm{id}_d: d \xrightarrow{=} d$. The dual comma category $d\backslash C$ with objects $d \to c$ is similar, and left to the reader.

EXAMPLE 3.2.3 (COMMA CATEGORIES). Suppose given a functor $F: C \to D$ and an object d of D. The comma category F/d has as its objects all pairs (c, f) with c an object in C and f a morphism in D from $F(c)$ to d. By abuse of notation, we shall write such objects as $F(c) \xrightarrow{f} d$. A morphism in F/d from this object to $F(c') \xrightarrow{f'} d$ is a morphism $h: c \to c'$ in C so that the following diagram commutes in D.

$$\begin{array}{ccc} F(c) & \xrightarrow{F(h)} & F(c') \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & d & \end{array}$$

There is a canonical forgetful functor $j: F/d \to C$, $j(c, f) = c$, and there is a natural transformation $\eta_{(c,f)} = f$ from the composite $F \circ j: F/d \to D$ to the constant functor with image d. So $B(F \circ j)$ is a contractible map. It follows that there is a natural continuous map from $B(F/d)$ to the homotopy fiber of $BC \to BD$.

There is a dual comma category $d\backslash F$, whose objects are written as $d \to F(c)$, and morphisms are morphisms $h: c \to c'$ in C. It also has a forgetful functor to C, and a map from $B(d\backslash F)$ to the homotopy fiber of $BC \to BD$. In fact, $d\backslash F = (d/F^{op})^{op}$.

In the same spirit, we can define comma categories F/D (resp., $D\backslash F$); an object is just an object of F/d (resp., of $d\backslash F$) for some d in D. A morphism in F/D from $(c, F(c) \to d)$ to $(c', F(c') \to d')$ is a pair of morphisms $c \to c'$, $d \to d'$ so that the two maps $F(c) \to d'$ agree; there is an evident forgetful functor $F/D \to C \times D$. A morphism in $D\backslash F$ from $(c, d \to F(c))$ to $(c', d' \to F(c'))$ is a pair of morphisms $c \to c'$, $d' \to d$ so that the two maps $d' \to F(c')$ agree; there is an evident forgetful functor $D\backslash F \to D^{op} \times C$.

The set π_0 of components of a category

The set $\pi_0(X)$ of connected components of any CW complex X can be described as the set of vertices modulo the incidence relation of edges. For BC this takes the following form. Let obj(C) denote the set of objects of C, and write $\pi_0(C)$ for $\pi_0(BC)$.

LEMMA 3.3. *Let $\sim$ be the equivalence relation on obj(C) which is generated by the relation that $c \sim c'$ if there is a morphism in C between c and c'. Then*

$$\pi_0(C) = obj(C)/\sim .$$

TRANSLATION CATEGORIES 3.3.1. Suppose that G is a group, or even a monoid, acting on a set X. The *translation category* $G\int X$ is defined as the category whose objects are the elements of X, with $\mathrm{Hom}(x, x') = \{g \in G | g \cdot x = x'\}$. By Lemma 3.3, $\pi_0(G\int X)$ is the orbit space X/G. The components of $G\int X$ are described in Ex. 3.2.

Thinking of a G-set X as a functor $G \to CAT$, the translation category becomes a special case of the following construction, due to Grothendieck.

EXAMPLE 3.3.2. Let I be a small category. Given a functor $X: I \to \mathbf{Sets}$, let $I\int X$ denote the category of pairs (i, x) with i an object of I and $x \in X(i)$, in which a morphism $(i, x) \to (i', x')$ is a morphism $f: i \to i'$ in I with $X(f)(x) = x'$. By Lemma 3.3 we have $\pi_0(I\int X) = \mathrm{colim}_{i\in I}\ X(i)$.

More generally, given a functor $X: I \to CAT$, let $I\int X$ denote the category of pairs (i, x) with i an object of I and x an object of $X(i)$, in which a morphism $(f, \phi): (i, x) \to (i', x')$ is given by a morphism $f: i \to i'$ in I and a morphism $\phi: X(f)(x) \to x'$ in $X(i')$. Using Lemma 3.3, it is not hard to show that $\pi_0(I\int X) = \mathrm{colim}_{i\in I}\ \pi_0 X(i)$.

For example, if $F : C \to D$ is a functor then $d \mapsto F/d$ is a functor on D, and $D\int(F/-)$ is F/D, while $d \mapsto d\backslash F$ is a functor on D^{op}, and $D\int(-\backslash F)$ is $D^{op}\backslash F$.

The fundamental group π_1 of a category

Suppose that T is a set of morphisms in a category C. The *graph of* T is the 1-dimensional subcomplex of BC consisting of the edges corresponding to T and their incident vertices. We say that T is a *tree* in C if its graph is contractible (*i.e.*, a tree in the sense of graph theory). If C is connected then a tree T is maximal (a *maximal tree*) just in case every object of C is either the source or target of a morphism in T. By Zorn's Lemma, maximal trees exist when $C \neq \emptyset$.

Classically, the fundamental group $\pi_1(\Gamma)$ of the 1-skeleton Γ of BC is a free group on symbols $[f]$, one for every non-identity morphism f in C not in T. (The loop is the composite of f with the unique paths in the tree between the basepoint and the source and target of f.) The following well known formula for the fundamental group of BC is a straight-forward application of Van Kampen's Theorem.

LEMMA 3.4. *Suppose that T is a maximal tree in a small connected category C. Then the group $\pi_1(BC)$ has the following presentation: it is generated by symbols $[f]$, one for every morphism in C, modulo the relations that*

(1) $[t] = 1$ *for every $t \in T$, and $[id_c] = 1$ for the identity morphism id_c of each object c.*
(2) $[f] \cdot [g] = [f \circ g]$ *for every pair (f, g) of composable morphisms in C.*

This presentation does not depend upon the choice of the object c_0 of C chosen as the basepoint. Geometrically, the class of $f\colon c_1 \to c_2$ is represented by the unique path in T from c_0 to c_1, followed by the edge f, followed by the unique path in T from c_2 back to c_0.

Application 3.4.1 (Groups). Let G be a group, considered as a category with one object. Since BG has only one vertex, BG is connected. By Lemma 3.4 (with T empty) we see that $\pi_1(BG) = G$. In fact, $\pi_i(BG) = 0$ for all $i \geq 2$. (See Ex. 3.2.) BG is often called the *classifying space* of the group G, for reasons discussed in Examples 3.9.2 and 3.9.3 below.

Application 3.4.2 (Monoids). If M is a monoid then BM has only one vertex. This time, Lemma 3.4 shows that the group $\pi = \pi_1(BM)$ is the group completion (Ex. II.1.1) of the monoid M.

For our purposes, one important thing about BG is that its homology is the same as the ordinary Eilenberg-Mac Lane homology of the group G (see [WHomo, 6.10.5 or 8.2.3]). In fact, if M is any G-module then we may consider M as a local coefficient system on BG (see 3.5.1). The cellular chain complex used to form the homology of BG with coefficients in M is the same as the canonical chain complex used to compute the homology of G, so we have $H_*(BG; M) = H_*(G; M)$. As a special case, we have $H_1(BG; \mathbb{Z}) = H_1(G; \mathbb{Z}) = G/[G, G]$, where $[G, G]$ denotes the commutator subgroup of G, *i.e.,* the subgroup of G generated by all commutators $[g, h] = ghg^{-1}h^{-1}$ $(g, h \in G)$.

(3.5) The homology of C and BC. The i^{th} homology of a CW complex X such as BC is given by the homology of the *cellular chain complex* $C_*(X)$. By definition, $C_n(X)$ is the free abelian group on the n-cells of X. If e is an $n+1$-cell and f is an n-cell, then the coefficient of $[f]$ in the boundary of $[e]$ is the degree of the map $S^n \xrightarrow{\varepsilon} X^{(n)} \xrightarrow{f} S^n$, where ε is the attaching map of e and the second map is the projection from $X^{(n)}$ (the n-skeleton of X) onto S^n given by the n-cell f.

For example, $H_*(BC; \mathbb{Z})$ is the homology of the unreduced cellular chain complex $C_*(BC)$, which in degree n is the free abelian group on the set of all n-tuples $(f_1, ..., f_n)$ of composable morphisms in C, composable in the order $c_0 \xrightarrow{f_1} c_1 \to \cdots \xrightarrow{f_n} c_n$. The boundary map in this complex sends the generator $(f_1, ..., f_n)$ to the alternating sum obtained by succesively deleting the c_i in the evident way:

$$(f_2, ..., f_n) - (f_2 f_1, f_3, ..., f_n) + \cdots \pm (..., f_{i+1} f_i, ...) \mp \cdots \pm (..., f_n f_{n-1}) \mp (..., f_{n-1}).$$

More generally, for each functor $M\colon C \to \mathbf{Ab}$ we let $H_i(C; M)$ denote the i^{th} homology of the chain complex

$$\cdots \to \coprod_{c_0 \to \cdots \to c_n} M(c_0) \to \cdots \to \coprod_{c_0 \to c_1} M(c_0) \to \coprod_{c_0} M(c_0).$$

The final boundary map sends the copy of $M(c_0)$ indexed by $c_0 \xrightarrow{f} c_1$ to $M(c_0) \oplus M(c_1)$ by $x \mapsto (-x, fx)$. The cokernel of this map is the usual description for the colimit of the functor M, so $H_0(C; M) = \operatorname{colim}_{c \in C} M(c)$.

LOCAL COEFFICIENTS 3.5.1. A functor $C \to \mathbf{Sets}$ is said to be *morphism-inverting* if it carries all morphisms of C into isomorphisms. By Ex. 3.1, morphism-inverting functors are in 1–1 correspondence with covering spaces of BC. Therefore the morphism-inverting functors $M: C \to \mathbf{Ab}$ are in 1–1 correspondence with local coefficient systems on the topological space BC. In this case, the groups $H_i(C; M)$ are canonically isomorphic to $H_i(BC; M)$, the topologist's homology groups of BC with local coefficients M. The isomorphism is given in [Wh, VI.4.8].

Bisimplicial Sets

A *bisimplicial set* X is a contravariant functor $\Delta \times \Delta \to \mathbf{Sets}$, where Δ is the subcategory of ordered sets on the objects $\{\mathbf{0}, \mathbf{1}, ..., \mathbf{n}, ...\}$. Alternatively, it is a doubly indexed family $X_{p,q}$ of sets, together with "horizontal" face and degeneracy maps ($\partial_i^h : X_{p,q} \to X_{p-1,q}$ and $\sigma_i^h : X_{p,q} \to X_{p+1,q}$) and "vertical" face and degeneracy maps ($\partial_i^v : X_{p,q} \to X_{p,q-1}$ and $\sigma_i^v : X_{p,q} \to X_{p,q+1}$), satisfying the horizontal and vertical simplicial identities and such that horizontal maps commute with vertical maps. In particular, each $X_{p,\cdot}$ and $X_{\cdot,q}$ is a simplicial set.

DEFINITION 3.6. The *geometric realization* BX of a bisimplicial set X is obtained by taking one copy of the product $\Delta^p \times \Delta^q$ for each element of $X_{p,q}$, inductively identifying its horizontal and vertical faces with the appropriate $\Delta^{p-1} \times \Delta^q$ or $\Delta^p \times \Delta^{q-1}$, and collapsing horizontal and vertical degeneracies. This construction is sometimes described as a coend: $BX = \int_{p,q} X_{p,q} \times \Delta^p \times \Delta^q$.

There is a diagonalization functor diag from bisimplicial sets to simplicial sets ($\mathrm{diag}(X)_p = X_{p,p}$), and it is well known (see [BF,B.1]) that BX is homeomorphic to $B\,\mathrm{diag}(X)$. The following theorem is also well known; see [Wa78, p. 164–5] or [Q341, p. 98] for example.

THEOREM 3.6.1. *Let $f : X \to Y$ be a map of bisimplicial sets.*
(i) If each simplicial map $X_{p,*} \to Y_{p,*}$ is a homotopy equivalence, so is $BX \to BY$.
(ii) If Y is the nerve of a category I (constant in the second simplicial coordinate), and $f^{-1}(i) \to f^{-1}(j)$ is a homotopy equivalence for every $i \to j$ in I, then each $B(f^{-1}(i)) \to BX \to B(I)$ is a homotopy fibration sequence.

EXAMPLE 3.6.2. (Quillen) If $F : C \to D$ is a functor, the canonical functor $D\backslash F \to C$ is a homotopy equivalence, where $D\backslash F$ is the comma category of Example 3.2.3. To see this, let X denote the bisimplicial set such that $X_{p.q}$ is the set of all pairs of sequences

$$(d_q \to \cdots \to d_0 \to F(c_0),\ c_0 \to \cdots \to c_p);$$

the horizontal and vertical faces come from the nerves of C and D. Consider the projection of X onto the nerve of C. Since NC_p is the discrete set of all sequences $c_0 \to \cdots \to c_p$, the inverse image of this sequence is isomorphic to the nerve of $D\backslash F(c_0)$, and $D\backslash F(c_0)$ is contractible since it has a terminal object. Theorem 3.6.1 applies to yield $BX \xrightarrow{\sim} BC$. The simplicial set $\mathrm{diag}(X)$ is the nerve of $D\backslash F$, and the composition $B(D\backslash F) \xrightarrow{\sim} BX \xrightarrow{\sim} BC$ is the canonical map, whence the result.

Homotopy Fibers of Functors

If $F\colon C \to D$ is a functor, it is useful to study the realization map $BF\colon BC \to BD$ in terms of homotopy groups, and for this we want a category-theoretic interpretation of the homotopy fiber (1.2). The naïve approximations to the homotopy fiber are the realization of the comma categories F/d and its dual $d\backslash F$. Indeed, we saw in 3.2.3 that there are continuous maps from both $B(F/d)$ and $B(d\backslash F)$ to the homotopy fiber.

Here is the fundamental theorem used to prove that two categories are homotopy equivalent. It was proven by Quillen in [Q341]. Note that it has a dual formulation, replacing $d\backslash F$ by F/d, because $BD \simeq BD^{op}$.

3.7 Quillen's Theorem A. *Let $F\colon C \to D$ be a functor such that $d\backslash F$ is contractible for every d in D. Then $BF\colon BC \xrightarrow{\simeq} BD$ is a homotopy equivalence.*

Proof. Consider the comma category $D\backslash F$ of 3.2.3, which is equipped with functors $C \leftarrow D\backslash F \to D^{op}$ such that $BC \leftarrow B(D\backslash F)$ is a homotopy equivalence (by 3.6.2). The functor $D\backslash F \to D\backslash D$, sending $(d \to F(c), c)$ to $(d \to F(c), F(c))$, fits into a commutative diagram of categories

$$\begin{array}{ccccc} C & \xleftarrow{\ \simeq\ } & D\backslash F & \longrightarrow & D^{op} \\ {\scriptstyle F}\big\downarrow & & \big\downarrow & & \big\| \\ D & \xleftarrow{\ \simeq\ } & D\backslash D & \longrightarrow & D^{op} \end{array}$$

Therefore it suffices to show that $B(D\backslash F) \to BD^{op}$ is a homotopy equivalence. This map factors as $B(D\backslash F) \simeq BX \xrightarrow{\pi} BD^{op}$, where X is the bisimplicial set of Example 3.6.2 and π is the projection. Consider the simplicial map $\pi_{*,q}$ from $X_{*,q}$ to the qth component of the nerve of D^{op}, which is the discrete set of all sequences $d_q \to \cdots \to d_0$ in D. For each such sequence, the inverse image in $X_{*,q}$ is the nerve of $d_0\backslash F$, which is assumed to be contractible. By Theorem 3.6.1, $B(D\backslash F) \simeq BX \to BD$ is a homotopy equivalence, as required. □

Example 3.7.1. If $F\colon C \to D$ has a left adjoint L, then $d\backslash F$ is isomorphic to the comma category $L(d)\backslash C$, which is contractible by Example 3.2.2. In this case, Quillen's Theorem A recovers the observation in 3.2 that C and D are homotopy equivalent.

Example 3.7.2. Consider the inclusion of monoids $i\colon \mathbb{N} \hookrightarrow \mathbb{Z}$ as a functor between categories with one object $*$. Then $*\backslash i$ is isomorphic to the translation category $\mathbb{N}\textstyle\int\mathbb{Z}$, which is contractible (why?). Quillen's Theorem A shows that $B\mathbb{N} \simeq B\mathbb{Z} \simeq S^1$.

The *inverse image* $F^{-1}(d)$ of an object d is the subcategory of C consisting of all objects c with $F(c) = d$, and all morphisms h in C mapping to the identity of d. It is isomorphic to the full subcategory of F/d consisting of pairs $(c, F(c) \xrightarrow{=} d)$, and also to the full subcategory of pairs $(d \xrightarrow{=} F(c), c)$ of $d\backslash F$. It will usually not be homotopy equivalent to either F/d or $d\backslash F$.

One way to ensure that $F^{-1}(d)$ is homotopic to a comma category is to assume that F is either pre-fibered or pre-cofibered in the following sense.

FIBERED AND COFIBERED FUNCTORS 3.7.3. (Cf. [SGA 1, Exp. VI]) We say that a functor $F: C \to D$ is *pre-fibered* if for every d in D the inclusion $F^{-1}(d) \hookrightarrow d\backslash F$ has a right adjoint. This implies that $BF^{-1}(d) \simeq B(d\backslash F)$, and the *base change* functor $f^*: F^{-1}(d') \to F^{-1}(d)$ associated to a morphism $f: d \to d'$ in D is defined as the composite $F^{-1}(d') \hookrightarrow (d\backslash F) \to F^{-1}(d)$. F is called *fibered* if it is pre-fibered and $g^*f^* = (fg)^*$ for every pair of composable maps f, g, so that F^{-1} gives a contravariant functor from D to CAT.

Dually, we say that F is *pre-cofibered* if for every d the inclusion $F^{-1}(d) \hookrightarrow F/d$ has a left adjoint. In this case we have $BF^{-1}(d) \simeq B(F/d)$. The *cobase change* functor $f_*: F^{-1}(d) \to F^{-1}(d')$ associated to a morphism $f: d \to d'$ in D is defined as the composite $F^{-1}(d) \hookrightarrow (F/d') \to F^{-1}(d')$. F is called *cofibered* if it is pre-cofibered and $(fg)_* = f_*g_*$ for every pair of composable maps f, g, so that F^{-1} gives a covariant functor from D to CAT.

These notions allow us to state a variation on Quillen's Theorem A.

COROLLARY 3.7.4. *Suppose that $F: C \to D$ is either pre-fibered or pre-cofibered, and that $F^{-1}(d)$ is contractible for each d in D. Then BF is a homotopy equivalence $BC \simeq BD$.*

EXAMPLE 3.7.5. Cofibered functors over D are in 1–1 correspondence with functors $D \to CAT$. We have already mentioned one direction: if $F: C \to D$ is cofibered, F^{-1} is a functor from D to CAT. Conversely, for each functor $X: D \to CAT$, the category $D\int X$ of Example 3.3.2 is cofibered over D by the forgetful functor $(d, x) \mapsto d$. It is easy to check that these are inverses: C is equivalent to $D\int F^{-1}$.

Here is the fundamental theorem used to construct homotopy fibration sequences of categories. It was originally proven in[Q341]. Note that it has a dual formulation, in which $d\backslash F$ is replaced by F/d; see Ex. 3.6.

3.8 QUILLEN'S THEOREM B. *Let $F: C \to D$ be a functor such that for every morphism $d \to d'$ in D the induced functor $d'\backslash F \to d\backslash F$ is a homotopy equivalence. Then for each d in D the geometric realization of the sequence*

$$d\backslash F \xrightarrow{j} C \xrightarrow{F} D$$

is a homotopy fibration sequence. Thus there is a long exact sequence

$$\cdots \to \pi_{i+1}(BD) \xrightarrow{\partial} \pi_i B(d\backslash F) \xrightarrow{j} \pi_i(BC) \xrightarrow{F} \pi_i(BD) \xrightarrow{\partial} \cdots .$$

PROOF. We consider the projection functor $X \xrightarrow{p} ND^{op}$ of 3.6.2. Since $p^{-1}(d)$ is the nerve of $d\backslash F$, we may apply Theorem 3.6.1 to conclude that $B(d\backslash F) \to BX \to BD^{op}$ is a homotopy fibration sequence. Since $B(d\backslash F) \to B\,\mathrm{diag}(X) = B(D\backslash F) \xrightarrow{\simeq} BC$ is induced from $j : d\backslash F \to C$, the theorem follows from the diagram

$$\begin{array}{ccccc} d\backslash F & \longrightarrow & D\backslash F & \longrightarrow & D^{op} \\ \downarrow & & {\scriptstyle F}\downarrow & & \| \\ * \simeq d\backslash D & \longrightarrow & D\backslash D & \xrightarrow{\simeq} & D^{op}. \end{array} \quad \square$$

COROLLARY 3.8.1. *Suppose that F is pre-fibered, and for every $f\colon d \to d'$ in D the base change f^* is a homotopy equivalence. Then for each d in D the geometric realization of the sequence*

$$F^{-1}(d) \xrightarrow{j} C \xrightarrow{F} D$$

is a homotopy fibration sequence. Thus there is a long exact sequence

$$\cdots \to \pi_{i+1}(BD) \xrightarrow{\partial} \pi_i BF^{-1}(d) \xrightarrow{j} \pi_i(BC) \xrightarrow{F} \pi_i(BD) \xrightarrow{\partial} \cdots .$$

TOPOLOGICAL CATEGORIES 3.9. If $C = C^{\rm top}$ is a *topological* category (*i.e.,* the object and morphism sets form topological spaces), then the nerve of $C^{\rm top}$ is a simplicial topological space. Using the appropriate geometric realization of simplicial spaces, we can form the topological space $BC^{\rm top} = |NC^{\rm top}|$. It has the same underlying set as our previous realization BC^δ (the δ standing for "discrete," *i.e.,* no topology), but the topology of $BC^{\rm top}$ is more intricate. Since the identity may be viewed as a continuous functor $C^\delta \to C^{\rm top}$ between topological categories, it induces a continuous map $BC^\delta \to BC^{\rm top}$.

For example, any topological group $G = G^{\rm top}$ is a topological category, so we need to distinguish between the two connected spaces BG^δ and $BG^{\rm top}$. It is traditional to write BG for $BG^{\rm top}$, reserving the notation BG^δ for the less structured space. As noted above, BG^δ has only one nonzero homotopy group: $\pi_1(BG^\delta) = G^\delta$. In contrast, the loop space $\Omega(BG^{\rm top})$ is $G^{\rm top}$, so $\pi_i BG^{\rm top} = \pi_{i-1}G^{\rm top}$ for $i > 0$.

EXAMPLE 3.9.1. Let $G = \mathbb{R}$ be the topological group of real numbers under addition. Then $B\mathbb{R}^{\rm top}$ is contractible because $\mathbb{R}^{\rm top}$ is, but $B\mathbb{R}^\delta$ is not contractible because $\pi_1(B\mathbb{R}^\delta) = \mathbb{R}$.

EXAMPLE 3.9.2 (BU). The unitary groups U_n are topological groups, and we see from I.4.10.1 that BU_n is homotopy equivalent to the infinite complex Grassmannian manifold G_n, which classifies n-dimensional complex vector bundles by Theorem I.4.10. The unitary group U_n is a deformation retract of the complex general linear group $GL_n(\mathbb{C})^{\rm top}$. Thus BU_n and $BGL_n(\mathbb{C})^{\rm top}$ are homotopy equivalent spaces. Taking the limit as $n \to \infty$, we have a homotopy equivalence $BU \simeq BGL(\mathbb{C})^{\rm top}$.

By Theorem II.3.2, $KU(X) \cong [X, \mathbb{Z} \times BU]$ and $\widetilde{KU}(X) \cong [X, BU]$ for every compact space X. By Ex. II.3.11 we also have $KU^{-n}(X) \cong [X, \Omega^n(\mathbb{Z} \times BU)]$ for all $n \geq 0$. In particular, for the one-point space $*$ the groups $KU^{-n}(*) = \pi_n(\mathbb{Z} \times BU)$ are periodic of order 2: $\mathbb{Z}$ if n is even, 0 if not. This follows from the observation in II.3.2 that the homotopy groups of BU are periodic — except for $\pi_0(BU)$, which is zero as BU is connected.

A refinement of Bott periodicity states that $\Omega U \simeq \mathbb{Z} \times BU$. Since $\Omega(BU) \simeq U$, we have $\Omega^2(\mathbb{Z} \times BU) \simeq \Omega^2 BU \simeq \mathbb{Z} \times BU$ and $\Omega^2 U \simeq U$. This yields the periodicity formula: $KU^{-n}(X) = KU^{-n-2}(X)$.

EXAMPLE 3.9.3 (BO). The orthogonal group O_n is a deformation retract of the real general linear group $GL_n(\mathbb{R})^{\rm top}$. Thus the spaces BO_n and $BGL_n(\mathbb{R})^{\rm top}$ are homotopy equivalent, and we see from I.4.10.1 that they are also homotopy equivalent to the infinite real Grassmannian manifold G_n. In particular, they classify n-dimensional real vector bundles by Theorem I.4.10. Taking the limit as $n \to \infty$, we have a homotopy equivalence $BO \simeq BGL(\mathbb{R})^{\rm top}$.

Bott periodicity states that the homotopy groups of BO are periodic of order 8 — except for $\pi_0(BO) = 0$, and that the homotopy groups of $\mathbb{Z} \times BO$ are actually periodic of order 8. These homotopy groups are tabulated in II.3.1.1. A refinement of Bott periodicity states that $\Omega^7 O \simeq \mathbb{Z} \times BO$. Since $\Omega(BO) \simeq O$, we have $\Omega^8(\mathbb{Z} \times BO) \simeq \Omega^8(BO) \simeq \mathbb{Z} \times BO$ and $\Omega^8 O \simeq O$.

By Definition II.3.5 and Ex. II.3.11, the (real) topological K-theory of a compact space X is given by the formula $KO^{-n}(X) = [X, \Omega^n(\mathbb{Z} \times BO)], \quad n \geq 0$. This yields the periodicity formula: $KO^{-n}(X) = KO^{-n-8}(X)$.

Bicategories

One construction that has proven useful in constructing spectra is the geometric realization of a bicategory. Just as we could have regarded a small category $\mathcal{A}$ as a special type of simplicial set, via its nerve 3.1.2 ($\mathcal{A}_0$ and $\mathcal{A}_1$ are the objects and morphisms, all other $\mathcal{A}_n$ are pullbacks and $\partial_1 : \mathcal{A}_2 = \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \to \mathcal{A}_1$ defines composition), we can do the same with small bicategories.

Definition 3.10. A small *bicategory* $\mathcal{C}$ is a bisimplicial set such that every row $\mathcal{C}_{\cdot,q}$ and column $\mathcal{C}_{p,\cdot}$ is the nerve of a category. We refer to elements of $\mathcal{C}_{0,0}$, $\mathcal{C}_{1,0}$, $\mathcal{C}_{0,1}$ and $\mathcal{C}_{1,1}$ as the objects, *horizontal* and *vertical morphisms*, and *bimorphisms*. A *bifunctor* between bicategories is a morphism of the underlying bisimplicial sets.

Example 3.10.1. If $\mathcal{A}$ and $\mathcal{B}$ are categories, we can form the product bicategory $\mathcal{A} \otimes \mathcal{B}$. Its objects (resp., bimorphisms) are ordered pairs of objects (resp., morphisms) from $\mathcal{A}$ and $\mathcal{B}$. Its (p, q)-morphisms are pairs of functors $\mathbf{p+1} \to \mathcal{A}$, $\mathbf{q+1} \to \mathcal{B}$.

It is easy to see that $\text{diag}(\mathcal{A} \otimes \mathcal{B})$ is the product category $\mathcal{A} \times \mathcal{B}$, and that $B(\mathcal{A} \otimes \mathcal{B})$ is $B\mathcal{A} \times B\mathcal{B}$. In particular $B((\mathbf{p+1}) \times (\mathbf{q+1})) = \Delta^p \times \Delta^q$.

Bicategory terminology arose (in the 1960's) from the following paradigm.

Example 3.10.2. For any category $\mathcal{B}$, $\text{bi}\mathcal{B}$ is the bicategory whose degree (p, q) part consists of commutative diagrams arising from functors $\mathbf{p+1} \times \mathbf{q+1} \to \mathcal{B}$. In particular, bimorphisms are commutative squares in $\mathcal{B}$; the horizontal and vertical edges of such a square are its associated horizontal and vertical morphisms. If $\mathcal{A}$ is a subcategory, $\mathcal{A}\mathcal{B}$ is the sub-bicategory of $\text{bi}\mathcal{B}$ whose vertical maps are in $\mathcal{A}$.

We may also regard the small category $\mathcal{B}$ as a bicategory which is constant in the vertical direction ($\mathcal{B}_{p,q} = N\mathcal{B}_p$); this does not affect the homotopy type $B\mathcal{B}$ since $\text{diag}\,\mathcal{B}$ recovers the category $\mathcal{B}$. The natural inclusion into $\text{bi}\mathcal{B}$ is a homotopy equivalence by Ex. 3.13. It follows that any bifunctor $\mathcal{A} \otimes \mathcal{B} \to \text{bi}\mathcal{C}$ induces a continuous map

$$B\mathcal{A} \times B\mathcal{B} \to B\,\text{bi}\mathcal{C} \simeq B\mathcal{C}.$$

EXERCISES

3.1 *Covering spaces.* If $X : I \to \mathbf{Sets}$ is a morphism-inverting functor (3.5.1), use the recipe 3.1.1 to show that the forgetful functor $I \int X \to I$ of Example 3.3.2 makes $B(I \int X)$ into a covering space of BI with fiber $X(i)$ over each vertex i of BI.

Conversely, if $E \xrightarrow{\pi} BI$ is a covering space, show that $X(i) = \pi^{-1}(i)$ defines a morphism-inverting functor on I, where i is considered as a 0-cell of BI. Conclude that these constructions give a 1–1 correspondence between covering spaces of BI and morphism-inverting functors. (See [Q341, p. 90].)

3.2 *Translation categories.* Suppose that a group G acts on a set X, and form the translation category $G\int X$. Show that $B(G\int X)$ is homotopy equivalent to the disjoint union of the classifying spaces BG_x of the stabilizer subgroups G_x, one space for each orbit in X. For example, if X is the coset space G/H then $B(G\int X) \simeq BH$.

In particular, if $X = G$ is given the G-set structure $g \cdot g' = gg'$, this shows that $B(G\int G)$ is contractible, *i.e.,* the universal covering space of BG. Use this to calculate the homotopy groups of BG, as described in Example 3.4.1.

3.3 Let H be a subgroup of G, and $\iota: H \hookrightarrow G$ the inclusion as a subcategory.
- (a) Show that $\iota/*$ is the category $H\int G$ of Ex. 3.1. Conclude that the homotopy fiber of $BH \to BG$ is the discrete set G/H, while $B\iota^{-1}(*)$ is a point.
- (b) Use Ex. 3.2 to give another proof of (a).

3.4 If C is a filtering category [WHomo, 2.6.13], show that BC is contractible. *Hint:* It suffices to show that all homotopy groups are trivial (see [Wh, V.3.5]). Any map from a sphere into a CW complex lands in a finite subcomplex, and every finite subcomplex of BC lands in the realization BD of a finite subcategory D of C; D lies in another subcategory D' of C which has a terminal object.

3.5 *Mapping telescopes.* If $\cup\mathbf{n}$ denotes the union of the categories $\mathbf{n}$ of 3.1, then a functor $\cup\mathbf{n} \xrightarrow{C} CAT$ is just a sequence $C_0 \to C_1 \to C_2 \to \cdots$ of categories. Show that the geometric realization of the category $L = (\cup\mathbf{n})\int C$ of Example 3.3.2 is homotopy equivalent to BC, where C is the colimit of the C_n. In particular, this shows that $BL \simeq \lim_{n\to\infty} BC_n$. *Hint:* $C_n \simeq \mathbf{n}\int C$.

3.6 Suppose that $F: C \to D$ is pre-cofibered (Definition 3.7.3).
- (a) Show that $F^{op}: C^{op} \to D^{op}$ is pre-fibered. If F is cofibered, F^{op} is fibered.
- (b) Derive the dual formulation of Quillen's theorem B, using F/d and F^{op}.
- (c) If each cobase change functor f_* is a homotopy equivalence, show that the geometric realization of $F^{-1}(d) \to C \xrightarrow{F} D$ is a homotopy fibration sequence for each d in D, and there is a long exact sequence:

$$\cdots \to \pi_{i+1}(BD) \xrightarrow{\partial} \pi_i BF^{-1}(d) \to \pi_i(BC) \xrightarrow{F} \pi_i(BD) \xrightarrow{\partial} \cdots .$$

3.7 Let $F: C \to D$ be a cofibered functor (3.7.3). Construct a first quadrant double complex E^0 in which E^0_{pq} is the free abelian group on the pairs $(d_p \to \cdots \to d_0 \to F(c_0), c_0 \to \cdots \to c_q)$ of sequences of composable maps in C and D. By filtering the double complex by columns, show that the homology of the total complex Tot E^0 is $H_q(\text{Tot } E^0) \cong H_q(C; \mathbb{Z})$. Then show that the row filtration yields a spectral sequence converging to $H_*(C;\mathbb{Z})$ with $E^2_{pq} = H_p(D; H_q F^{-1})$, the homology of D with coefficients in the functor $d \mapsto H_q(F^{-1}(d); \mathbb{Z})$ described in 3.5.

3.8 A *lax functor* $\mathbf{M}: I \to CAT$ consists of functions assigning: (1) a category $\mathbf{M}(i)$ to each object i; (2) a functor $f_*: \mathbf{M}(i) \to \mathbf{M}(j)$ to every map $i \xrightarrow{f} j$ in I; (3) a natural transformation $(\mathrm{id}_i)_* \Rightarrow \mathrm{id}_{\mathbf{M}(i)}$ for each i; (4) a natural transformation $(fg)_* \Rightarrow f_* g_*$ for every pair of composable maps in I. This data is required to be "coherent" in the sense that the two transformations $(fgh)_* \Rightarrow f_* g_* h_*$ agree, and so do the various transformations $f_* \Rightarrow f_*$. For example, a functor is a lax functor in which (3) and (4) are identities.

Show that the definitions of objects and morphisms in Example 3.3.2 define a category $I\int\mathbf{M}$, where the map ϕ'' in the composition $(f'f, \phi'')$ of (f, ϕ) and (f', ϕ') is $(f'f)_*(x) \to f'_* f_*(x) \to f'_*(x') \to x''$. Show that the projection functor $\pi\colon I\int M \to I$ is pre-cofibered.

3.9 *Subdivision.* If $\mathcal{C}$ is a category, its *Segal subdivision* $Sub(\mathcal{C})$ is the category whose objects are the morphisms in $\mathcal{C}$; a morphism from $i\colon A \to B$ to $i'\colon A' \to B'$ is a pair of maps $(A' \to A, B \to B')$ so that i' is $A' \to A \xrightarrow{i} B \to B'$.

(a) Draw the Segal subdivisions of the unit interval **2** and the 2-simplex **3**.

(b) Show that the source and target functors $\mathcal{C}^{op} \leftarrow Sub(\mathcal{C}) \to \mathcal{C}$ are homotopy equivalences. *Hint:* Use Quillen's Theorem A and 3.2.2.

3.10 Given a simplicial set X, its *Segal subdivision* $Sub(X)$ is the sequence of sets $X_1, X_3, X_5, \dots$, made into a simplicial set by declaring the face maps $\partial'_i\colon X_{2n+1} \to X_{2n-1}$ to be $\partial_i\partial_{2n+1-i}$ and $\sigma'_i\colon X_{2n+1} \to X_{2n+3}$ to be $\sigma_i\sigma_{2n+1-i}$ $(0 \le i \le n)$.

If X is the nerve of a category $\mathcal{C}$, show that $Sub(X)$ is the nerve of the Segal subdivision category $Sub(\mathcal{C})$ of Ex. 3.9.

3.11 (Waldhausen) Let $f : X \to Y$ be a map of simplicial sets. For $y \in Y_n$, define the simplicial set $f/(n, y)$ to be the pullback of X and the n-simplex Δ^n along $f : X \to Y$ and the map $y : \Delta^n \to Y$. Thus an m-simplex consists of a map $\alpha : m \to n$ in Δ and an $x \in X_m$ such that $f(x) = \alpha^*(y)$. Prove that:

(A) If each $f/(n, y)$ is contractible, then f is a homotopy equivalence;

(B) If for every $m \xrightarrow{\alpha} n$ in Δ and every $y \in Y_n$ the map $f/(m, \alpha^*y) \to f/(n, y)$ is a homotopy equivalence, then each $|f/(n, y)| \to X \to Y$ is homotopy fibration sequence.

Hint: Any simplicial set X determines a category $\Delta^{op}\int X$ cofibered over Δ^{op}, by 3.3.2 and 3.7.5. Now apply Theorems A and B.

3.12 If $\mathcal{C}$ is a category, its *arrow category* $\mathcal{C}/\mathcal{C}$ has the morphisms of $\mathcal{C}$ as its objects, and a map $(a, b)\colon f \to f'$ in $\mathcal{C}/\mathcal{C}$ is a commutative diagram in $\mathcal{C}$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{f'} & B' \end{array}$$

If $f\colon A \to B$, the source $s(f) = A$ and target $t(f) = B$ of f define functors $\mathcal{C}/\mathcal{C} \to \mathcal{C}$. Show that s is a fibered functor, and that t is a cofibered functor. Then show that both s and t are homotopy equivalences.

3.13 *Swallowing Lemma.* If $\mathcal{A}$ is a subcategory of $\mathcal{B}$, show that the bicategory inclusion $\mathcal{B} \subset \mathcal{AB}$ of 3.10.2 induces a homotopy equivalence $B\mathcal{B} \simeq B(\mathcal{AB})$. When $\mathcal{A} = \mathcal{B}$ this proves that $B\mathcal{B} \simeq B(\text{bi}\mathcal{B})$. *Hint:* Show that $\mathcal{B} \simeq N_p(\mathcal{A})\mathcal{B}$ for all p.

3.14 *Diagonal Category.* (Waldhausen [Wa78]) Show that the functor from small categories to small bicategories sending $\mathcal{B}$ to bi$\mathcal{B}$ (3.10.2) has a left adjoint, sending $\mathcal{C}$ to its *diagonal category*, and that the diagonal category of the bicategory $\mathcal{A}\otimes\mathcal{B}$ is the product category $\mathcal{A}\times\mathcal{B}$. *Hint:* both the horizontal and vertical morphisms of a bicategory $\mathcal{C}$ yield morphisms, and every bimorphism yields an equivalence relation for the composition of horizontal and vertical morphisms.

§4. Symmetric Monoidal Categories

The geometric realization BS of a symmetric monoidal category is an H-space with a homotopy-commutative, homotopy-associative product. To see this, recall from definition II.5.1 that a symmetric monoidal category is a category S with a functor $\square\colon S \times S \to S$ which has a unit object "e" and is associative and is commutative, all up to coherent natural isomorphism. By 3.1(6) the geometric realization of $\square$ is the "product" map $(BS) \times (BS) \cong B(S \times S) \to BS$. The natural isomorphisms $s\square e \cong s \cong e\square s$ imply that the vertex e is an identity up to homotopy, *i.e.,* that BS is an H-space. The other axioms imply that the product on BS is homotopy commutative and homotopy associative.

In many cases e is an initial object of S, and therefore the H-space BS is contractible by Example 3.2.2. For example, any additive category $\mathcal{A}$ is a symmetric monoidal category (with $\square = \oplus$), and $e = 0$ is an initial object, so $B\mathcal{A}$ is contractible. Similarly, the category $\mathbf{Sets}_{\text{fin}}$ of finite sets is symmetric monoidal ($\square$ being disjoint union) by I.5.2, and $e = \emptyset$ is initial, so $B\mathbf{Sets}_{\text{fin}}$ is contractible.

Here is an easy way to modify S in order to get an interesting H-space.

Definition 4.1. Let $\operatorname{iso} S$ denote the subcategory of isomorphisms in S. It has the same objects as S, but its morphisms are the isomorphisms in S. Because $\operatorname{iso} S$ is also symmetric monoidal, $B(\operatorname{iso} S)$ is an H-space.

By Lemma 3.3, the abelian monoid $\pi_0(\operatorname{iso} S)$ is just the set of isomorphism classes of objects in S — the monoid S^{iso} considered in II.5. In fact, $\operatorname{iso} S$ is equivalent to the disjoint union $\coprod \operatorname{Aut}_S(s)$ of the 1–object categories $\operatorname{Aut}_S(s)$, and $B(\operatorname{iso} S)$ is homotopy equivalent to the disjoint union of the classifying spaces $B\operatorname{Aut}(s)$, $s \in S^{\text{iso}}$.

Examples 4.1.1. $B(\operatorname{iso} S)$ is often an interesting H-space.

(a) In the category $\mathbf{Sets}_{\text{fin}}$ of finite (pointed) sets, the group of automorphisms of any n-element set is isomorphic to the permutation group Σ_n. Thus the subcategory $\operatorname{iso}\mathbf{Sets}_{\text{fin}}$ is equivalent to $\coprod \Sigma_n$, the disjoint union of the one-object categories Σ_n. Thus the classifying space $B(\operatorname{iso}\mathbf{Sets}_{\text{fin}})$ is homotopy equivalent to the disjoint union of the classifying spaces $B\Sigma_n$, $n \geq 0$.

(b) The additive category $\mathbf{P}(R)$ of finitely generated projective R-modules has 0 as an initial object, so $B\mathbf{P}(R)$ is a contractible space. However, its subcategory $\mathbf{P} = \operatorname{iso}\mathbf{P}(R)$ of isomorphisms is more interesting. The topological space $B\mathbf{P}$ is equivalent to the disjoint union of the classifying spaces $B\operatorname{Aut}(P)$ as P runs over the set of isomorphism classes of finitely generated projective R-modules.

(c) Fix a ring R, and let $\mathbf{F}(R)$ be the category $\coprod GL_n(R)$ whose objects are the based free R-modules $\{0, R, R^2, \cdots, R^n, \cdots\}$ (these objects are distinct because the bases have different orders; see Section I.1). There are no maps in $\mathbf{F}(R)$ between R^m and R^n if $m \neq n$, and the self-maps of R^n form the group $GL_n(R)$. This is a symmetric monoidal category: $R^m\square R^n = R^{m+n}$ by concatenation of bases; if a and b are morphisms, $a\square b$ is the matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. The space $B\mathbf{F}(R)$ is equivalent to the disjoint union of the classifying spaces $BGL_n(R)$.

If R satisfies the Invariant Basis Property (I.1.1), then $\mathbf{F}(R)$ is a full subcategory of $\operatorname{iso}\mathbf{P}(R)$. In this case, we saw in II.5.4.1 that $\mathbf{F}(R)$ is cofinal in $\operatorname{iso}\mathbf{P}(R)$.

(d) Fix a commutative ring R, and let $S = \mathbf{Pic}(R)$ be the category of invertible R-modules and their isomorphisms. This is a symmetric monoidal category in which

$\Box$ is tensor product and e is R; see II.5.2(5). In this case, $S = \text{iso}\, S$ and S^{iso} is the Picard group $\text{Pic}(R)$ discussed in section I.3. By Lemma I.3.3, $\text{Aut}(L) = R^{\times}$ for every L. Thus $\mathbf{Pic}(R)$ is equivalent to a disjoint union of copies of $R^{\times}$, and $B(\mathbf{Pic})$ is homotopy equivalent to the product $\text{Pic}(R) \times B(R^{\times})$.

(e) If F is a field, we saw in II.5.7 that the categories $\mathbf{SBil}(F)$ and $\mathbf{Quad}(F) = \mathbf{Quad}^{+}(F)$ of symmetric inner product spaces and quadratic spaces are symmetric monoidal categories. More generally, let A be any ring with involution, and $\epsilon = \pm 1$. Then the category $\mathbf{Quad}^{\epsilon}(A)$ of nonsingular ϵ-quadratic A-modules is a symmetric monoidal category with $\Box = \oplus$ and $e = 0$. See [B72, Bak] for more details.

(f) If G is a group, consider the category $G\text{-}\mathbf{Sets}_{\text{fin}}$ of free G-sets X having a finite number of orbits. This is symmetric monoidal under disjoint union (cf. II.5.2.2). If X has n orbits, then $\text{Aut}(X)$ is the wreath product $G \wr \Sigma_n$. As in (a), $B(G\text{-}\mathbf{Sets}_{\text{fin}})$ is equivalent to the disjoint union of the classifying spaces $B(G \wr \Sigma_n)$.

There is a monoidal functor $G\text{-}\mathbf{Sets}_{\text{fin}} \to \mathbf{P}(\mathbb{Z}[G])$ which sends X to the free abelian group on the set X.

The $S^{-1}S$ Construction

In [GQ], Quillen gave a construction of a category $S^{-1}S$ such that $K(S) = B(S^{-1}S)$ is a "group completion" of BS (see 4.4 below), provided that every map in S is an isomorphism and every translation $s\Box\colon \text{Aut}_S(t) \to \text{Aut}_S(s\Box t)$ is an injection. The motivation for this construction comes from the construction of the universal abelian group completion of an abelian monoid given in Chapter II, §1.

DEFINITION 4.2 ($S^{-1}S$). The objects of $S^{-1}S$ are pairs (m, n) of objects of S. A morphism in $S^{-1}S$ is an equivalence class of composites

$$(m_1, m_2) \xrightarrow{s\Box} (s\Box m_1, s\Box m_2) \xrightarrow{(f,g)} (n_1, n_2).$$

This composite is equivalent to

$$(m_1, m_2) \xrightarrow{t\Box} (t\Box m_1, t\Box m_2) \xrightarrow{(f',g')} (n_1, n_2)$$

exactly when there is an isomorphism $\alpha\colon s \xrightarrow{\simeq} t$ in S so that composition with $\alpha\Box m_i$ sends f' and g' to f and g.

A (strict) monoidal functor $S \to T$ induces a functor $S^{-1}S \to T^{-1}T$.

EXPLANATION 4.2.1. There are two basic types of morphisms in $S^{-1}S$. The first type is a pair of maps $(f_1, f_2)\colon (m_1, m_2) \to (n_1, n_2)$ with $f_i\colon m_i \to n_i$ in S, arising from the inclusion of $S \times S$ in $S^{-1}S$. The second type is a formal map $s\Box\colon (m, n) \to (s\Box m, s\Box n)$.

We shall say that *translations are faithful* in S if every translation $\text{Aut}(s) \to \text{Aut}(s\Box t)$ in S is an injection. In this case every map in $S^{-1}S$ determines s, f and g up to unique isomorphism.

REMARK 4.2.2. $S^{-1}S$ is a symmetric monoidal category, with $(m, n)\Box(m', n') = (m\Box m', n\Box n')$, and the functor $S \to S^{-1}S$ sending m to (m, e) is monoidal. Hence the natural map $BS \to B(S^{-1}S)$ is an H-space map, and $\pi_0(S) \to \pi_0(S^{-1}S)$ is a map of abelian monoids.

In fact $\pi_0(S^{-1}S)$ is an abelian group, the inverse of (m, n) being (n, m), because of the existence of a morphism η in $S^{-1}S$ from (e, e) to $(m, n)\Box(n, m) = (m\Box n, n\Box m)$. Warning: η is not a natural transformation! See Ex. 4.3.

DEFINITION 4.3. Let S be a symmetric monoidal category in which every morphism is an isomorphism. Its K-groups are the homotopy groups of $B(S^{-1}S)$:

$$K_n^{\square}(S) = \pi_n(BS^{-1}S).$$

It is sometimes convenient to write $K^{\square}(S)$ for the geometric realization $B(S^{-1}S)$, and call it the *K-theory space* of S, so that $K_n^{\square}(S) = \pi_n K^{\square}(S)$. By 4.2, a (strict) monoidal functor $S \to T$ induces a map $K^{\square}(S) \to K^{\square}(T)$ and hence homomorphisms $K_n^{\square}(S) \to K_n^{\square}(T)$.

In order to connect this definition up with the definition of $K_0^{\square}(S)$ given in section II.5, we recall from 4.2.2 that the functor $S \to S^{-1}S$ induces a map of abelian monoids from $\pi_0(S) = S^{\text{iso}}$ to $\pi_0(S^{-1}S)$.

LEMMA 4.3.1. *The abelian group $K_0^{\square}(S) = \pi_0(S^{-1}S)$ is the group completion of the abelian monoid $\pi_0(S) = S^{\text{iso}}$. Thus definition 4.3 agrees with the definition of $K_0^{\square}(S)$ given in II.5.1.2.*

PROOF. Let A denote the group completion of $\pi_0(S)$, and consider the function $\alpha(m,n) = [m] - [n]$ from the objects of $S^{-1}S$ to A. If $s \in S$ and $f_i\colon m_i \to n_i$ are morphisms in S then in A we have $\alpha(m,n) = \alpha(s\square m, s\square n)$ and $\alpha(m_1,m_2) = [m_1] - [m_2] = [n_1] - [n_2] = \alpha(n_1,n_2)$. By Lemma 3.3, α induces a set map $\pi_0(S^{-1}S) \to A$. By construction, α is an inverse to the universal homomorphism $A \to \pi_0(S^{-1}S)$. $\square$

Group Completions

Group completion constructions for K-theory were developed in the early 1970's by topologists studying infinite loop spaces. These constructions all apply to symmetric monoidal categories.

Any discussion of group completions depends upon the following well-known facts (see [Wh, III.7]). Let X be a homotopy commutative, homotopy associative H-space. Its set of components $\pi_0 X$ is an abelian monoid, and $H_0(X;\mathbb{Z})$ is the monoid ring $\mathbb{Z}[\pi_0(X)]$. Moreover, the integral homology $H_*(X;\mathbb{Z})$ is an associative graded-commutative ring with unit.

We say that a homotopy associative H-space X is *group-like* if it has a homotopy inverse; see [Wh, III.4]. Of course this implies that $\pi_0(X)$ is a group. When X is a CW complex, the converse holds: if the monoid $\pi_0(X)$ is a group, then X is group-like. (If $\pi_0(X) = 0$ this is [Wh, X.2.2]; if $\pi_0(X)$ is a group, the proof in *loc. cit.* still goes through as the shear map $\pi_0(X)^2 \to \pi_0(X)^2$ is an isomorphism.)

For example, if $S = \operatorname{iso} S$ then $\pi_0(BS)$ is the abelian monoid S^{iso} of isomorphism classes, and $H_0(BS;\mathbb{Z})$ is the monoid ring $\mathbb{Z}[S^{\text{iso}}]$. In this case, the above remarks show that BS is grouplike if and only S^{iso} is an abelian group under $\square$.

DEFINITION 4.4 (GROUP COMPLETION). Let X be a homotopy commutative, homotopy associative H-space. A *group completion* of X is an H-space Y, together with an H-space map $X \to Y$, such that $\pi_0(Y)$ is the group completion of the abelian monoid $\pi_0(X)$ (in the sense of section I.1), and the homology ring $H_*(Y;k)$ is isomorphic to the localization $\pi_0(X)^{-1}H_*(X;k)$ of $H_*(X;k)$ by the natural map, for all commutative rings k.

If X is a CW complex (such as $X = BS$), we shall assume that Y is also a CW complex. This hypothesis implies that the group completion Y is group-like.

LEMMA 4.4.1. *If X is a group-like H-space then X its own group completion, and any other group completion $f\colon X \to Y$ is a homotopy equivalence.*

PROOF. Since f is a homology isomorphism, it is an isomorphism on π_0 and π_1. Therefore the map of basepoint components is a $+$–construction relative to the subgroup 1 of $\pi_1(X)$, and Theorem 1.5 implies that $X \simeq Y$. □

EXAMPLE 4.4.2 (PICARD GROUPS). Let R be a commutative ring, and consider the symmetric monoidal category $S = \mathbf{Pic}(R)$ of Example II.5.2(5). Because $\pi_0(S)$ is already a group, S and $S^{-1}S$ are homotopy equivalent (by lemma 4.4.1). Therefore we get

$$K_0\mathbf{Pic}(R) = \mathrm{Pic}(R), \quad K_1\mathbf{Pic}(R) = U(R) \quad \text{and } K_n\mathbf{Pic}(R) = 0 \quad \text{for } n \geq 2.$$

The determinant functor from $\mathbf{P} = \mathrm{iso}\,\mathbf{P}(R)$ to $\mathbf{Pic}(R)$ constructed in section I.3 gives a map from $K(R) = K(\mathbf{P})$ to $K\mathbf{Pic}(R)$. Upon taking homotopy groups, this yields the familiar maps $\det\colon K_0(R) \to \mathrm{Pic}(R)$ of II.2.6 and $\det\colon K_1(R) \to R^\times$ of III.1.1.1.

A *phantom map* $\phi : X \to Y$ is a map such that, for every finite CW complex A, every composite $A \to X \to Y$ is null homotopic, *i.e.,* $\phi_* : [A, X] \to [A, Y]$ is the zero map. If $f : X \to Y$ is a group completion then so is $f + \phi : X \to Y$ for every phantom map ϕ. Thus the group completion is not unique up to homotopy equivalence whenever phantom maps exist.

The following result, taken from [CCMT, 1.2], shows that phantom maps are essentially the only obstruction to uniqueness of group completions. We say that two maps $X \to Y$ are *weakly homotopic* if they induce the same map on homotopy classes $[A, X] \to [A, Y]$; if Y is an H-space, this means that their difference is a phantom map.

THEOREM 4.4.3. *Let X be an H-space such that $\pi_0(X)$ is either countable or contains a countable cofinal submonoid. If $f' : X \to X'$ and $f'' : X \to X''$ are two group completions, then there is a homotopy equivalence $g : X' \to X''$, unique up to weak homotopy, such that gf' and f'' are weak homotopy equivalent. (The map g is also a weak H-map.)*

The fact that gf' and f'' are weak homotopy equivalent implies that g is a homology isomorphism, and hence is a homotopy equivalence by 4.4.1.

4.5. One can show directly that $\mathbb{Z} \times BGL(R)^+$ is a group completion of BS when $S = \coprod GL_n(R)$; see Ex. 4.9. We will see in theorem 4.8 below that the K-theory space $B(S^{-1}S)$ is another group completion of BS, and then give an explicit homotopy equivalence between $B(S^{-1}S)$ and $\mathbb{Z} \times BGL(R)^+$ in 4.9. Here are some other methods of group completion:

SEGAL'S ΩB METHOD 4.5.1.

If X is a topological *monoid*, such as $\coprod BGL_n(R)$ or $\coprod B\Sigma_n$, then we can form BX, the geometric realization of the (one-object) topological category X (see 3.9). In this case, ΩBX is an infinite loop space and the natural map $X \to \Omega BX$ is a group completion. For example, if X is the one-object monoid $\mathbb{N}$ then $B\mathbb{N} \simeq S^1$, and $\Omega B\mathbb{N} \simeq \Omega S^1 \simeq \mathbb{Z}$. That is, $\pi_0(\Omega B\mathbb{N})$ is $\mathbb{Z}$, and every component of $\Omega B\mathbb{N}$ is contractible. See [Adams] for more details.

Machine Methods 4.5.2. (See [Adams].) If X isn't quite a monoid, but the homotopy associativity of its product is nice enough, then there are constructions called "infinite loop space machines" which can construct a group completion Y of X, and give Y the structure of an infinite loop space. All machines produce the same infinite loop space Y (up to homotopy); see [MT]. Some typical machines are described in [Segal], and [May74].

The realization $X = BS$ of a symmetric monoidal category S is nice enough to be used by infinite loop space machines. These machines produce an infinite loop space $K(S)$ and a map $BS \to K(S)$ which is a group completion. Most infinite loop machines will also produce explicit deloopings of $K(S)$ in the form of an Ω-spectrum $\mathbf{K}(S)$, the *K-theory spectrum* of S, which is connective in the sense that $\pi_n\mathbf{K}(S) = 0$ for $n < 0$. The production of $\mathbf{K}(S)$ is natural enough that monoidal functors between symmetric monoidal categories induce maps of the corresponding spectra.

Pairings and Products

A *pairing* of symmetric monoidal categories is a functor $\otimes : S_1 \times S_2 \to S$ such that $s \otimes 0 = 0 \otimes s = 0$, and there is a coherent natural bi-distributivity law

$$(a + a') \otimes (b + b') \cong (a \otimes b) \square (a \otimes b') \square (a' \otimes b) \square (a' \otimes b').$$

If $S_1 = S_2 = S$, we will just call this a pairing on S. Instead of making this technical notion precise, we refer the reader to [May80, §2] and content ourselves with two examples from 4.1.1: the product of finite sets is a pairing on $\mathbf{Sets}_{\text{fin}}$, and the tensor product of based free modules is a pairing $\mathbf{F}(A) \times \mathbf{F}(B) \to \mathbf{F}(A \otimes B)$. The free module functor from $\mathbf{Sets}_{\text{fin}}$ to $\mathbf{F}(A)$ preserves these pairings. The following theorem was proven by Peter May in [May80, 1.6 and 2.1].

Theorem 4.6. *A pairing $S_1 \times S_2 \to S$ of symmetric monoidal categories determines a natural pairing $K(S_1) \wedge K(S_2) \to K(S)$ of infinite loop spaces in 4.5.2, as well as a pairing of Ω-spectra $\mathbf{K}(S_1) \wedge \mathbf{K}(S_2) \to \mathbf{K}(S)$. This in turn induces bilinear products $K_p(S_1) \otimes K_q(S_2) \to K_{p+q}(S)$. There is also a commutative diagram*

$$\begin{array}{ccccc} BS_1 \times BS_2 & \longrightarrow & BS_1 \wedge BS_2 & \xrightarrow{B\otimes} & BS \\ \downarrow & & \downarrow & & \downarrow \\ K(S_1) \times K(S_2) & \longrightarrow & K(S_1) \wedge K(S_2) & \xrightarrow{B\otimes} & K(S). \end{array}$$

From theorem 4.6 and the constructions in 1.10 and 4.9, repectively Ex. 1.12 and 4.9.3, we immediately deduce:

Corollary 4.6.1. *When S is $\mathbf{Sets}_{\text{fin}}$ or $\mathbf{F}(R)$, the product defined by Loday (in 1.10) agrees with the product in Theorem 4.6.*

Remark 4.6.2. If there is a pairing $S \times S \to S$ which is associative up to natural isomorphism, then $\mathbf{K}(S)$ can be given the structure of a ring spectrum. This is the case when S is $\mathbf{Sets}_{\text{fin}}$ or $\mathbf{F}(R)$ for commutative R.

Actions on other categories

To show that $B(S^{-1}S)$ is a group completion of BS, we need to fit the definition of $S^{-1}S$ into a more general framework.

DEFINITION 4.7. A monoidal category S is said to *act upon* a category X by a functor $\Box: S \times X \to X$ if there are natural isomorphisms $s\Box(t\Box x) \cong (s\Box t)\Box x$ and $e\Box x \cong x$ for $s, t \in S$ and $x \in X$, satisfying coherence conditions for the products $s\Box t\Box u\Box x$ and $s\Box e\Box x$ analogous to the coherence conditions defining S.

For example, S acts on itself by $\Box$. If X is a discrete category, then S acts on X exactly when the monoid $\pi_0(S)$ acts on the underlying set of objects in X.

Here is the analogue of the translation category construction (3.3.1) associated to a monoid acting on a set.

DEFINITION 4.7.1. If S acts upon X, the category $\langle S, X\rangle$ has the same objects as X. A morphism from x to y in $\langle S, X\rangle$ is an equivalence class of pairs $(s, s\Box x \xrightarrow{\phi} y)$, where $s \in S$ and ϕ is a morphism in X. Two pairs (s, ϕ) and (s', ϕ') are equivalent in case there is an isomorphism $s \cong s'$ identifying ϕ' with $s'\Box x \cong s\Box x \xrightarrow{\phi} y$.

We shall write $S^{-1}X$ for $\langle S, S \times X\rangle$, where S acts on both factors of $S \times X$. Note that when $X = S$ this definition recovers the definition of $S^{-1}S$ given in 4.2 above. If S is symmetric monoidal, then the formula $s\Box(t\Box x) = (s\Box t, x)$ defines an action of S on $S^{-1}X$.

For example, if every arrow in S is an isomorphism, then e is an initial object of $\langle S, S\rangle$ and therefore the space $S^{-1}\mathbf{1} = B\langle S, S\rangle$ is contractible.

We say that S acts *invertibly* upon X if each translation functor $s\Box: X \to X$ is a homotopy equivalence. For example, S acts invertibly on $S^{-1}X$ (if S is symmetric) by the formula $s\Box(t, x) = (s\Box t, x)$, the homotopy inverse of the translation $(t, x) \mapsto (s\Box t, x)$ being the translation $(t, x) \mapsto (t, s\Box x)$, because of the natural transformation $(t, x) \mapsto (s\Box t, s\Box x)$.

Now $\pi_0 S$ is a multiplicatively closed subset of the ring $H_0(S) = \mathbb{Z}[\pi_0 S]$, so it acts on $H_*(X)$ and acts invertibly upon $H_*(S^{-1}X)$. Thus the functor $X \to S^{-1}X$ sending x to $(0, x)$ induces a map

$$(\pi_0 S)^{-1} H_q(X) \to H_q(S^{-1}X). \tag{4.7.2}$$

THEOREM 4.8 (QUILLEN). *If every map in S is an isomorphism and translations are faithful in S, then the map (4.7.2) is an isomorphism for all X and q.*

In particular, $B(S^{-1}S)$ is a group completion of the H-space BS.

PROOF. (See [GQ, p. 221].) By Ex. 4.5, the projection functor $\rho: S^{-1}X \to \langle S, S\rangle$ is cofibered with fiber X. By Ex. 3.7 there is an associated spectral sequence $E^2_{pq} = H_p(\langle S, S\rangle; H_q(X)) \Rightarrow H_{p+q}(S^{-1}X)$. Localizing this at the multiplicatively closed subset $\pi_0 S$ of $H_0(S)$ is exact, and $\pi_0 S$ already acts invertibly on $H_*(S^{-1}X)$ by Ex. 3.7, so there is also a spectral sequence $E^2_{pq} = H_p(\langle S, S\rangle; M_q) \Rightarrow H_{p+q}(S^{-1}X)$, where $M_q = (\pi_0 S)^{-1} H_q(X)$. But the functors M_q are morphism-inverting, so by Ex. 3.1 and the contractibility of $\langle S, S\rangle$, the group $H_p(\langle S, S\rangle; M_q)$ is zero for $p \neq 0$, and equals M_q for $p = 0$. Thus the spectral sequence degenerates to the claimed isomorphism (4.7.2).

The final assertion is immediate from this and definition 4.4, given Remark 4.2.2 and Lemma 4.3.1. □

Bass gave a classical definition of $K_1(S)$ and $K_2(S)$ in [B72, p. 197]; we gave them implicitly in III.1.6.3 and III.5.6. We can now state these classical definitions, and show that they coincide with the K-groups defined in this section.

COROLLARY 4.8.1. *If $S = iso\, S$ and translations are faithful in S, then:*

$$K_1(S) = \varinjlim_{s\in S} H_1(\mathrm{Aut}(s);\mathbb{Z}),$$

$$K_2(S) = \varinjlim_{s\in S} H_2([\mathrm{Aut}(s),\mathrm{Aut}(s)];\mathbb{Z}).$$

PROOF. ([We-Az]) The localization of $H_q(BS) = \oplus_{s\in S} H_q(\mathrm{Aut}(s))$ at $\pi_0(X) = S^{\mathrm{iso}}$ is the direct limit of the groups $H_q(\mathrm{Aut}(s))$, taken over the translation category of all $s \in S$. Since $\pi_1(X) = H_1(X;\mathbb{Z})$ for every H-space X, this gives the formula for $K_1(S) = \pi_1 B(S^{-1}S)$.

For K_2 we observe that any monoidal category S is the filtered colimit of its monoidal subcategories having countably many objects. Since $K_2(S)$ and Bass' H_2 definition commute with filtered colimits, we may assume that S has countably many objects. In this case the proof is relegated to exercise 4.10. □

Relation to the +–construction

Let $S = \mathbf{F}(R) = \coprod GL_n(R)$ be the monoidal category of based free R-modules, as in example 4.1.1(c). In this section, we shall establish the following result, identifying the +–construction on $BGL(R)$ with the basepoint component of $K(S) = B(S^{-1}S)$.

THEOREM 4.9. *When S is $\coprod GL_n(R)$, $K(S) = B(S^{-1}S)$ is the group completion of $BS = \coprod BGL_n(R)$, and*

$$B(S^{-1}S) \simeq \mathbb{Z} \times BGL(R)^+.$$

As theorems 4.8 and 1.8 suggest, we first need to find an acyclic map from $BGL(R)$ to the connected basepoint component of $B(S^{-1}S)$. This is done by the following "mapping telescope" construction (illustrated in Figure 4.9.1).

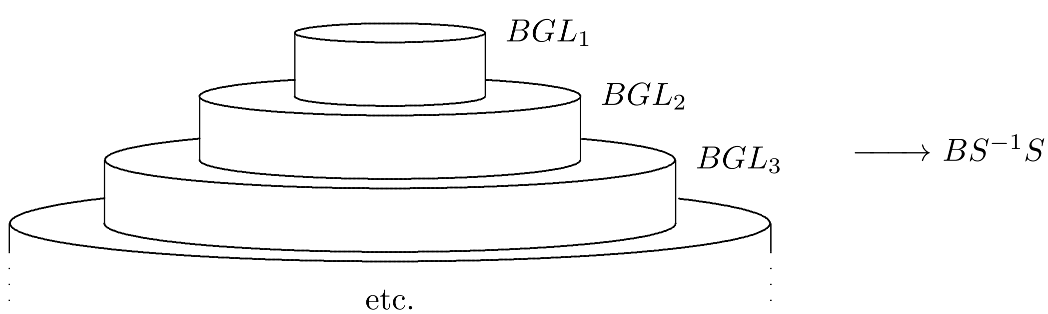


Figure 4.9.1. The mapping telescope of $BGL(R)$ and $B(S^{-1}S)$.

Any group map η from $GL_n(R)$ to $\mathrm{Aut}_{S^{-1}S}(R^n, R^n)$ gives a map from $BGL_n(R)$ to $B(S^{-1}S)$. For the specific maps $\eta = \eta_n$ defined by $\eta_n(g) = (g, 1)$, the diagram

$$\begin{array}{ccc} GL_n(R) & \xrightarrow{\eta} & \mathrm{Aut}(R^n, R^n) \\ {\scriptstyle \Box R}\downarrow & & {\scriptstyle \Box(R,R)}\downarrow \\ GL_{n+1}(R) & \xrightarrow{\eta} & \mathrm{Aut}(R^{n+1}, R^{n+1}) \end{array}$$

commutes, *i.e.,* there is a natural transformation from η to $\eta(\Box R)$. The resulting homotopy of maps $\eta \simeq \eta(\Box R) : BGL_n(R) \to B(S^{-1}S)$ gives the map from the "mapping telescope" construction of $BGL(R)$ to $B(S^{-1}S)$; see Ex. 3.5. In fact, this map lands in the connected component Y_S of the identity in $B(S^{-1}S)$. Since $B(S^{-1}S)$ is an H-space, so is the connected component Y_S of the identity.

Proof of Theorem 4.9. (Quillen) We shall show that the map $BGL(R) \to Y_S$ is an isomorphism on homology with coefficients $\mathbb{Z}$. By the remark following Theorem 1.8, this will induce a homotopy equivalence $BGL(R)^+ \to Y_S$.

Let $e \in \pi_0 BS$ be the class of R. By theorem 4.8, $H_* B(S^{-1}S)$ is the localization of the ring $H_*(BS)$ at $\pi_0(S) = \{e^n\}$. But this localization is the colimit of the maps $H_*(BS) \to H_*(BS)$ coming from the translation $\oplus R$: $S \to S$. Hence $H_* B(S^{-1}S) \cong H_*(Y_S) \otimes \mathbb{Z}[e, e^{-1}]$, where Y_S denotes the basepoint component of $B(S^{-1}S)$, and $H_*(Y_S) \cong \operatorname{colim} H_*(BGL_n(R)) = H_*(BGL(R))$. This means that the map $BGL(R) \to Y_S$ is a homology isomorphism, as required. $\Box$

Example 4.9.2. (Segal) Consider the symmetric monoidal category $S = \coprod \Sigma_n$, equivalent to the category $\mathbf{Sets}_{\text{fin}}$ of example 4.1.1(a). The infinite symmetric group Σ_∞ is the union of the symmetric groups Σ_n along the inclusions $\Box 1$ from Σ_n to Σ_{n+1}, and these inclusions assemble to give a map from the mapping telescope construction of $B\Sigma_\infty$ to $B(S^{-1}S)$, just as they did for $GL(R)$ (see Figure 4.9.1). Moreover the proof of theorem 4.9 formally goes through to prove that $B(S^{-1}S) \simeq K(\mathbf{Sets}_{\text{fin}})$ is homotopy equivalent to $\mathbb{Z} \times B\Sigma_\infty^+$. This is the equivalence of parts (a) and (b) in the following result. We refer the reader to [BP71] and [Adams, §4.2] for the equivalence of parts (b) and (c).

The Barratt-Priddy-Quillen-Segal Theorem 4.9.3. *The following three infinite loop spaces are the same:*
(a) the group completion $K(\mathbf{Sets}_{\text{fin}})$ of $B\mathbf{Sets}_{\text{fin}}$;
(b) $\mathbb{Z} \times B\Sigma_\infty^+$, where Σ_∞ is the union of the symmetric groups Σ_n; and
(c) The infinite loop space $\Omega^\infty S^\infty = \lim_{n\to\infty} \Omega^n S^n$.
Hence the groups $K_n(\mathbf{Sets}_{\text{fin}})$ are the stable homotopy groups of spheres, π_n^s.

More generally, suppose that S has a countable sequence of objects $s_1, \dots$ such that $s_{n+1} = s_n \Box a_n$ for some $a_n \in S$, and satisfying the cofinality condition that for every $s \in S$ there is an s' and an n so that $s \Box s' \cong s_n$. In this case we can form the group $\mathrm{Aut}(S) = \operatorname{colim}_{n\to\infty} \mathrm{Aut}_S(s_n)$.

Theorem 4.10. *Let $S = iso\, S$ be a symmetric monoidal category whose translations are faithful, and suppose the above condition is satisfied, so that the group*

Aut(S) exists. Then the commutator subgroup E of Aut(S) is a perfect normal subgroup, $K_1(S) = \mathrm{Aut}(S)/E$*, and the +–construction on* $B\,\mathrm{Aut}(S)$ *is the connected component of the identity in the group completion* $K(S)$*. Thus*

$$K(S) \simeq K_0(S) \times B\,\mathrm{Aut}(S)^+.$$

PROOF. ([We-Az]) The assertions about E are essentially on p. 355 of [Bass]. On the other hand, the mapping telescope construction mentioned above gives an acyclic map from $B\,\mathrm{Aut}(S)$ to the basepoint component of $B(S^{-1}S)$, and such a map is by definition a +–construction. □

EXAMPLE 4.10.1. Consider the subcategory $\coprod G \wr \Sigma_n$ of the category G-$\mathbf{Sets}_{\mathrm{fin}}$ of free G sets introduced in 4.1.1(f). The group $\mathrm{Aut}(S)$ is the (small) infinite wreath product $G \wr \Sigma_\infty = \cup G \wr \Sigma_n$, so we have $K(G\text{-}\mathbf{Sets}_{\mathrm{fin}}) \simeq \mathbb{Z} \times B(G \wr \Sigma_\infty)^+$. On the other hand, the Barratt-Priddy theorem [BP71] identifies this with the infinite loop space $\Omega^\infty S^\infty(BG_+)$ associated to the disjoint union BG_+ of BG and a point.

The monoidal functor G-$\mathbf{Sets}_{\mathrm{fin}} \to \mathbf{P}(\mathbb{Z}[G])$ of 4.1.1(f) induces a group homomorphism $G \wr \Sigma_\infty \to GL(\mathbb{Z}[G])$ and hence maps $B(G \wr \Sigma_\infty)^+ \to BGL(\mathbb{Z}[G])^+$ and $\Omega^\infty S^\infty(BG_+) \simeq K(G\text{-}\mathbf{Sets}_{\mathrm{fin}}) \to K(\mathbb{Z}[G])$.

This map is a version of the "assembly map" in the following sense. If R is any ring, there is a product map $K(R) \wedge K(\mathbb{Z}[G]) \to K(R[G])$; see 1.10 and 4.6. This yields a map from $K(R) \wedge \Omega^\infty S^\infty(BG_+)$ to $K(R[G])$. Now the space $K(R) \wedge BG$ is included as a direct factor in $K(R) \wedge \Omega^\infty S^\infty(BG_+)$ (split by the "Snaith splitting"). Since the homotopy groups of the first space give the generalized homology of BG with coefficients in $K(R)$, $H_n(BG; K(R))$, we get homomorphisms $H_n(BG; K(R)) \to K(R[G])$. It is not known if $K(R) \wedge BG$ has a complementary factor which maps trivially.

Cofinality

A monoidal functor $f\colon S \to T$ is called *cofinal* if for every t in T there is a t' and an s in S so that $t \Box t' \cong f(s)$; cf. II.5.3. For example, the functor $\mathbf{F}(R) \to \mathbf{P}(R)$ of example 4.1.1(c) is cofinal, because every projective module is a summand of a free one. For $\mathbf{Pic}(R)$, the one-object subcategory $R^\times$ is cofinal.

COFINALITY THEOREM 4.11. *Suppose that* $f\colon S \to T$ *is cofinal. Then*
(a) If T acts on X then S acts on X via f, and $S^{-1}X \simeq T^{-1}X$.
(b) If $\mathrm{Aut}_S(s) \cong \mathrm{Aut}_T(fs)$ *for all* $s \in S$ *then the basepoint components of* $K(S)$ *and* $K(T)$ *are homotopy equivalent. Thus* $K_n(S) \cong K_n(T)$ *for all* $n \geq 1$.

PROOF. By cofinality, S acts invertibly on X if and only if T acts invertibly on X. Hence Ex. 4.6 yields

$$S^{-1}X \xrightarrow{\simeq} T^{-1}(S^{-1}X) \cong S^{-1}(T^{-1}X) \xleftarrow{\simeq} T^{-1}X.$$

An alternate proof of part (a) is sketched in Ex. 4.8.

For part (b), let Y_S and Y_T denote the connected components of $B(S^{-1}S)$ and $B(T^{-1}T)$. Writing the subscript $s \in S$ to indicate a colimit over the translation category 3.3.1 of $\pi_0(S)$, and similarly for the subscript $t \in T$, theorem 4.8 yields:

$$\begin{aligned} H_*(Y_S) &= \mathrm{colim}_{s\in S}\, H_*(B\,\mathrm{Aut}(s)) = \mathrm{colim}_{s\in S}\, H_*(B\,\mathrm{Aut}(fs)) \\ &\cong \mathrm{colim}_{t\in T}\, H_*(B\,\mathrm{Aut}(t)) = H_*(Y_T). \end{aligned}$$

Hence the connected H-spaces Y_S and Y_T have the same homology, and this implies that they are homotopy equivalent. $\square$

Note that $K_0(\mathbf{F}(R)) = \mathbb{Z}$ is not the same as $K_0(\mathbf{P}(R)) = K_0(R)$ in general, although $K_n(\mathbf{F}) \cong K_n(\mathbf{P})$ for $n \geq 1$ by the Cofinality Theorem 4.11(b). By Theorem 4.9 this establishes the following important result.

COROLLARY 4.11.1. *Let $S = iso\,\mathbf{P}(R)$ be the category of finitely generated projective R-modules and their isomorphisms. Then*

$$B(S^{-1}S) \simeq K_0(R) \times BGL(R)^+.$$

REMARK 4.11.2. Consider the 0-connected cover $\mathbf{K}(R)\langle 0\rangle$ of $\mathbf{K}(R)$, the spectrum constructed by an infinite loop space machine from $\operatorname{iso}\mathbf{P}(R)$, as in 4.5.2. By 4.8 and 4.11.1, $BGL(R)^+$ is the 0^{th} space of the spectrum $\mathbf{K}(R)\langle 0\rangle$. In particular, it provides a canonical way to view $BGL(R)^+$ as an infinite loop space.

4.12. Let's conclude with a look back at the other motivating examples in 4.1.1. In each of these examples, every morphism is an isomorphism and the translations are faithful, so the classifying space of $S^{-1}S$ is a group completion of BS.

EXAMPLE 4.12.1 (STABLE HOMOTOPY). The "free R-module" on a finite set determines a functor from $\mathbf{Sets}_{\text{fin}}$ to $\mathbf{P}(R)$, or from the subcategory $\coprod \Sigma_n$ of $\mathbf{Sets}_{\text{fin}}$ to $\coprod GL_n(R)$. This functor identifies the symmetric group Σ_n with the permutation matrices in $GL_n(R)$. Applying group completions, theorem 4.9 and 4.9.3 show that this gives a map from $\Omega^\infty S^\infty$ to $K(R)$, hence maps $\pi_n^s \to K_n(R)$.

EXAMPLE 4.12.2 (L-THEORY). Let $S = \mathbf{Quad}^\epsilon(A)$ denote the category of nonsingular ϵ-quadratic A-modules, where $\epsilon = \pm 1$ and A is any ring with involution [B72, Bak]. The K-groups of this category are the L-groups ${}_\epsilon L_n(A)$ of Karoubi and others. For this category, the sequence of hyperbolic spaces H^n is cofinal (by Ex. II.5.11), and the automorphism group of H^n is the *orthogonal group* ${}_\epsilon O_n$. The infinite orthogonal group ${}_\epsilon O = {}_\epsilon O(A)$, which is the direct limit of the groups ${}_\epsilon O_n$, is the group $\operatorname{Aut}(S)$ in this case. By theorem 4.10, we have

$$K(\mathbf{Quad}^\epsilon(A)) \simeq {}_\epsilon L_0(A) \times B_\epsilon O^+.$$

When $A = \mathbb{R}$, the classical orthogonal group O is ${}_{+1}O$. When $A = \mathbb{C}$ and the involution is complex conjugation, the classical unitary group U is ${}_{+1}O(\mathbb{C})$. For more bells and whistles, and classical details, we refer the reader to [Bak].

EXAMPLE 4.12.3. When R is a topological ring (such as $\mathbb{R}$ or $\mathbb{C}$), we can think of $\mathbf{P}(R)$ as a *topological* symmetric monoidal category. Infinite loop space machines (4.5.2) also accept topological symmetric monoidal categories, and we write $K(R^{\text{top}})$ for $K(\mathbf{P}(R)^{\text{top}})$. The change-of-topology functor $\mathbf{P}(R) \to \mathbf{P}(R)^{\text{top}}$ induces natural infinite loop space maps from $K(R)$ to $K(R^{\text{top}})$. The naturality of these maps allows us to utilize infinite loop space machinery. As an example of the usefulness, we remark that

$$K(\mathbb{R}^{\text{top}}) \simeq \mathbb{Z} \times BO \qquad \text{and} \qquad K(\mathbb{C}^{\text{top}}) \simeq \mathbb{Z} \times BU.$$

EXERCISES

4.1 Let $\mathbb{N}$ be the additive monoid $\{0,1,...\}$, considered as a symmetric monoidal category with one object. Show that $\langle \mathbb{N}, \mathbb{N}\rangle$ is the union $\cup\mathbf{n}$ of the ordered categories $\mathbf{n}$, and that $\mathbb{N}^{-1}\mathbb{N}$ is a poset, each component being isomorphic to $\cup\mathbf{n}$.

4.2 Show that a sequence $X_0 \to X_1 \to \cdots$ of categories determines an action of $\mathbb{N}$ on the disjoint union $X = \coprod X_n$, and that $\langle \mathbb{N}, X\rangle$ is the mapping telescope category $\cup\mathbf{n}\int X$ of Ex. 3.5.

4.3 (Thomason) Let S be symmetric monoidal, and let $\iota\colon S^{-1}S \to S^{-1}S$ be the functor sending (m,n) to (n,m) and (f_1,f_2) to (f_2,f_1). Show that there is no natural transformation $0 \Rightarrow \mathrm{id}\Box\iota$. *Hint:* The obvious candidate is given in 4.2.2.

Thomason has shown that $B\iota$ is the homotopy inverse for the H-space structure on $B(S^{-1}S)$, but for subtle reasons.

4.4 If S is a symmetrical monoidal category, so is its opposite category S^{op}. Show that the group completions $K(S)$ and $K(S^{op})$ are homotopy equivalent.

4.5 (Quillen) Suppose that $S = \mathrm{iso}S$, and that translations in S are faithful (4.2.1). Show that the projection $S^{-1}X \xrightarrow{\rho} \langle S, S\rangle$ is cofibered, where $\rho(s,x) = s$.

4.6 Let $S = \mathrm{iso}\,S$ be a monoidal category whose translations are faithful (4.2.1). Suppose that S acts invertibly upon a category X. Show that the functors $X \to S^{-1}X$ $(x \mapsto (s,x))$ are homotopy equivalences for every s in S. If S acts upon a category Y, then S always acts invertibly upon $S^{-1}Y$, so this shows that $S^{-1}Y \simeq S^{-1}(S^{-1}Y)$. *Hint:* Use exercises 3.6 and 4.5, and the contractibility of $\langle S, S\rangle$.

4.7 Suppose that every map in X is monic, and that each translation $\mathrm{Aut}_S(s) \xrightarrow{\Box x} \mathrm{Aut}_X(s\Box x)$ is an injection. Show that the sequence $S^{-1}S \xrightarrow{\Box x} S^{-1}X \xrightarrow{\pi} \langle S, X\rangle$ is a homotopy fibration for each x in X, where π is projection onto the second factor. In particular, if $\langle S, X\rangle$ is contractible, this proves that $S^{-1}S \xrightarrow{\Box x} S^{-1}X$ is a homotopy equivalence. *Hint:* Show that π and $S^{-1}\pi\colon S^{-1}(S^{-1}X) \to \langle S, X\rangle$ are cofibered, and use the previous exercise.

4.8 Use exercises 4.5 and 4.6 to give another proof of the Cofinality Theorem 4.11(b).

4.9 Fix a ring R and set $S = \coprod GL_n(R)$. The maps $BGL_n(R) \to BGL(R) \to \{n\}\times BGL(R)^+$ assemble to give a map from BS to $\mathbb{Z}\times BGL(R)^+$. Use Ex. 1.11 to show that it is an H-space map. Then show directly that this makes $\mathbb{Z}\times BGL(R)^+$ into a group completion of BS.

4.10 Let S be a symmetric monoidal category with countably many objects, so that the group $\mathrm{Aut}(S)$ exists and its commutator subgroup E is perfect, as in 4.10. Let F denote the homotopy fiber of the H-space map $B\,\mathrm{Aut}(S)^+ \to B(K_1S)$.

(a) Show that $\pi_1(F) = 0$ and $H_2(F;\mathbb{Z}) \cong \pi_2(F) \cong K_2(S)$.

(b) ([We-Az]) Show that the natural map $BE \to F$ induces $H_*(BE) \cong H_*(F)$, so that $F = BE^+$. *Hint:* Show that K_1S acts trivially upon the homology of BE and F, and apply the comparison theorem for spectral sequences.

(c) Conclude that $K_2(S) \cong H_2(E) \cong \varinjlim_{s\in S} H_2([\mathrm{Aut}(s),\mathrm{Aut}(s)];\mathbb{Z})$.

4.11 If $f\colon X \to Y$ is a functor, we say that an action of S on X is *fiberwise* if $S \times X \xrightarrow{\Box} X \xrightarrow{f} Y$ equals the projection $S \times X \to X$ followed by f.

(a) Show that a fiberwise action on X restricts to an action of S on each fiber category $X_y = f^{-1}(y)$, and that f induces a functor $S^{-1}X \to Y$ whose fibers are the categories $S^{-1}(X_y)$.

(b) If f is a fibered functor (3.7.3), we say that a fiberwise action is *cartesian* if the base change maps commute with the action of S on the fibers. Show that in this case $S^{-1}X \to Y$ is a fibered functor.

4.12 Let G be a group, and G-$\mathbf{Sets}_{\text{fin}}$ as in 4.1.1(f).
(a) Using 4.10.1, show that $K_1(G\text{-}\mathbf{Sets}_{\text{fin}}) \cong G/[G,G] \times \{\pm 1\}$.
(b) Using Exercise II.5.9, show that the groups $K_n(\mathbb{Z}[G])$ are modules over the Burnside ring $A(G) = K_0G\text{-}\mathbf{Sets}_{\text{fin}}$.
(c) If G is abelian, show that the product of G-sets defines a pairing in the sense of Theorem 4.6. Conclude that $K_*G\text{-}\mathbf{Sets}_{\text{fin}}$ is a ring. Using the free module functor, show that $K_*G\text{-}\mathbf{Sets}_{\text{fin}} \to K_*(\mathbb{Z}[G])$ is a ring homomorphism.

4.13 (a) Show that the idempotent completion $\hat{S}$ (II.7.3) of a symmetric monoidal category S is also symmetric monoidal, and that $S \to \hat{S}$ is a cofinal monoidal functor. Conclude that the basepoint components of $K(S)$ and $K(\hat{S})$ are homotopy equivalent.
(b) Show that a pairing $S_1 \times S_2 \to S$ induces a pairing $\hat{S}_1 \times \hat{S}_2 \to \hat{S}$ and hence (by 4.6) a pairing of spectra $\mathbf{K}(\hat{S}_1) \wedge \mathbf{K}(\hat{S}_2) \to \mathbf{K}(\hat{S})$.
(c) By (b), for every pair of rings A, B there is a pairing $\mathbf{K}(A) \wedge \mathbf{K}(B) \to \mathbf{K}(A \otimes B)$. Using 4.6.1, deduce that the induced product agrees with the extension of Loday's product 1.10 described in Ex. 1.14.

4.14 Construct a morphism of spectra $S^1 \to \mathbf{K}(\mathbb{Z}[x, x^{-1}])$ which, as in 1.10.2, represents $[x] \in K_1(\mathbb{Z}[x, x^{-1}])$. Using the previous exercise, show that it induces a product map $\cup x : \mathbf{K}(R) \to \Omega\mathbf{K}(R[x, x^{-1}])$, natural in the ring R.

§5. λ-operations in higher K-theory

Let A be a commutative ring. In Chapter II.4 we introduced the operations $\lambda^k : K_0(A) \to K_0(A)$ and showed that they endow $K_0(A)$ with the structure of a special λ-ring (II.4.3.1). The purpose of this section is to extend this structure to operations $\lambda^k : K_n(A) \to K_n(A)$ for all n. Although many constructions of λ-operations have been proposed in more exotic settings, we shall restrict our attention in this section to operations defined using the $+$-construction.

We shall begin with a general construction, which produces the operations $\wedge^k$ as a special case. Fix an arbitrary group G. If $\rho : G \to \operatorname{Aut}(P)$ is any representation of G in a finitely generated projective A-module P, any isomorphism $P \oplus Q \cong A^N$ gives a map $q(\rho) : BG \to B\operatorname{Aut}(P) \to BGL_N(A) \to BGL(A)^+$. A different embedding of P in A^N will give a map which is homotopic to the first, because the two maps only differ by conjugation and $BGL(A)^+$ is an H-space. (The action of $\pi_1(H)$ on $[X, H]$ is trivial for any H-space H and any space X; see [Wh, III.4.18]). Hence the map $q(\rho)$ is well-defined up to homotopy.

Example 5.1. Recall from I.3 that the k^{th} exterior power $\wedge^k(P)$ of a finitely generated projective A-module P is also a projective module, of rank $\binom{\operatorname{rank} P}{k}$.

Because $\wedge^k$ is a functor, it determines a group map $\wedge^k_P : \mathrm{Aut}(P) \to \mathrm{Aut}(\wedge^k P)$, i.e., a representation, for each P. We write Λ^k_P for $q(\wedge^k)$. Note that $\Lambda^0_P = *$.

Now any connected H-space, such as $BGL(A)^+$, has a multiplicative inverse (up to homotopy). Given a map $f : X \to H$, this allows us to construct maps $-f$, and to take formal $\mathbb{Z}$-linear combinations of maps.

Definition 5.2. If P has rank n, we define $\lambda^k_P : B\,\mathrm{Aut}(P) \to BGL(A)^+$ to be the map

$$\lambda^k_P = \sum_{i=0}^{k-1} (-1)^i \binom{n+i-1}{i} \Lambda^{k-i}_P.$$

One can show directly that the maps λ^k_P are compatible with the inclusions of P in $P \oplus Q$, up to homotopy of course, giving the desired operations $\lambda^k : BGL(A)^+ \to BGL(A)^+$ (see Ex. 5.1). However, it is more useful to encode this bookkeeping in the Representation Ring $R_A(G)$, an approach which is due to Quillen.

Recall from II, Ex. 4.2, that the representation ring $R_A(G)$ is the Grothendieck group of the representations of G in finitely generated projective A-modules. We saw in *loc. cit.* that $R_A(G)$ is a special λ-ring.

Proposition 5.3. *If $0 \to (P', \rho') \to (P, \rho) \to (P'', \rho'') \to 0$ is a short exact sequence of representations of G, then $q(\rho) = q(\rho') + q(\rho'')$ in $[BG, BGL(A)^+]$.*

Hence there is a natural map $q : R_A(G) \to [BG, BGL(A)^+]$.

Proof. It is clear from the H-space structure on $BGL(A)^+$ that $q(\rho \oplus \rho') = q(\rho) + q(\rho')$. By the above remarks, we may suppose that P' and P'' are free modules, of ranks m and n respectively. By universality, it suffices to consider the case in which $G = G_{m,n}$ is the automorphism group of the sequence, i.e., the upper triangular group $\begin{pmatrix} \mathrm{Aut}(P') & \mathrm{Hom}(P'', P') \\ 0 & \mathrm{Aut}(P'') \end{pmatrix}$. Quillen proved in [Q76] that in the limit, the inclusions $i : \mathrm{Aut}(P') \times \mathrm{Aut}(P'') \hookrightarrow G_{m,n}$ induce a homology isomorphism

$$\varinjlim H_*(G_{m,n}) \cong H_*(GL(A) \times GL(A)).$$

It follows that for any connected H-space H we have $[\varinjlim BG_{m,n}, H] \cong [BGL(A) \times BGL(A), H]$. Taking $H = BGL(A)^+$ yields the result. $\square$

Example 5.3.1. If ρ is a representation on a rank n module P, the elements $[\rho] - n$ and $\lambda^k([\rho] - n)$ of $R_A(G)$ determine maps $BG \to BGL(A)^+$. When $G = \mathrm{Aut}(P)$ and $\rho = \mathrm{id}_P$ is the tautological representation, it follows from the formula of Ex. II.4.2 that $\lambda^k([\mathrm{id}_P] - n)$ is the map Λ^k_P of 5.1.

We can now define the operations λ^k on $[BGL(A)^+, BGL(A)^+]$. As n varies, the representations id_n of $GL_n(A)$ are related by the relation $i_n^* \mathrm{id}_{n+1} = \mathrm{id}_n \oplus 1$, where $i_n : GL_n(A) \hookrightarrow GL_{n+1}(A)$ is the inclusion. Hence the virtual characters $\rho_n = \mathrm{id}_n - n \cdot 1$ satisfy $\rho_n = i_n^* \rho_{n+1}$. Since $i^* : R_A(GL_n A) \to R_A(GL_{n+1} A)$ is a homomorphism of λ-rings, we also have $\lambda^k \rho_n = i^*(\lambda^k \rho_{n+1})$. Hence we get a compatible family of homotopy classes $\lambda^k_n \in [BGL_n(A), BGL(A)^+]$.

Because each $BGL_n(A) \to BGL_{n+1}(A)$ is a closed cofibration, it is possible to inductively construct maps $\lambda^k_n : BGL_n(A) \to BGL(A)^+$ which are strictly compatible, so that by passing to the limit they determine a continuous map $\lambda^k_\infty : BGL(A) \to BGL(A)^+$ and even

$$\lambda^k : BGL(A)^+ \to BGL(A)^+.$$

The construction in Example 5.3.1 clearly applies to any compatible family of elements in the rings $R_A(GL_n(A))$. Indeed, we have a map

$$(5.3.2)\quad \varprojlim R_A(GL_n(A)) \to \varprojlim [BGL_n(A), BGL(A)^+] = [BGL(A)^+, BGL(A)^+].$$

For example, the operations ψ^k and γ^k may be defined in this way; see Ex. 5.2.

DEFINITION 5.4. If X is any based space, and $f : X \to BGL(A)^+$ any map, we define $\lambda^k f : X \to BGL(A)^+$ to be the composition of f and λ^k. This defines operations on $[X, BGL(A)^+]$ which we also refer to as λ^k. When $X = S^n$, we get operations $\lambda^k : K_n(A) \to K_n(A)$.

EXAMPLE 5.4.1. When $n = 1$ and $a \in A^\times$ is regarded as an element of $K_1(A)$, the formulas $\lambda^k(a) = a$ and $\psi^k(a) = a^k$ are immediate from the formula 5.2 for λ^k_A.

The abelian group $[X, BGL(A)^+]$ inherits an associative multiplication from the product on $BGL(A)^+$ described in 1.10: one uses the composition $X \xrightarrow{\Delta} X \wedge X \to BGL(A)^+ \wedge BGL(A)^+$. If $X = S^n$ for $n > 0$ (or if X is any suspension), this is the zero product because then the map $X \to X \wedge X$ is homotopic to 0.

Now recall from II.4 that a λ-ring must satisfy $\lambda^0(x) = 1$, which requires an identity. In contrast, our λ^0 is zero. To fix this, we extend the operations to $K_0(A) \times [X, BGL(A)^+]$ by

$$\lambda^k(a, x) = (\lambda^k(a), \lambda^k(x) + a \cdot \lambda^{k-1}(x) + \cdots + \lambda^i(a)\lambda^{k-i}(x) + \cdots + \lambda^{k-1}(a)x.$$

Thus $\lambda^0(a, x) = (\lambda^0(a), \lambda^0(x)) = (1, 0)$, as required.

THEOREM 5.5. *For any based space X, the λ^k make $K_0(A) \times [X, BGL(A)^+]$ into a special λ-ring*

PROOF. It suffices to consider the universal case $X = BGL(A)^+$. Since $\pi_1(X) = K_1(A)$, we have a map $R_A(K_1 A) \to [X, X]$. Via the transformation q of 5.3, we are reduced to checking identities in the rings $R_A(GL_n(A))$ by (5.3.2). For example, the formula $\lambda^k(x + y) = \sum \lambda^i(x)\lambda^{k-i}(y)$ comes from the identity

$$\lambda^k \circ \oplus = \sum \lambda^i \otimes \lambda^{k-i}$$

in $R_A(GL_m(A) \times GL_n(A))$. Similarly, the formal identities for $\lambda^k(xy)$ and $\lambda^n(\lambda^k x)$, listed in II.4.3.1 and which need to hold in special λ-rings, already hold in $R_A(G)$ and so hold in our setting via the map q. □

COROLLARY 5.5.1. *If $n > 0$ then $\lambda^k : K_n(A) \to K_n(A)$ is additive, and we have $\psi^k(x) = (-1)^{k-1} k \lambda^k$.*

PROOF. Since the products are zero, this is immediate from the formulas in II.4.1 and II.4.4 for $\lambda^k(x + y)$ and $\psi^k(x)$. □

If A is an algebra over a field of characteristic p, the Frobenius endomorphism Φ of A is defined by $\Phi(a) = a^p$. We say that A is *perfect* if Φ is an automorphism, *i.e.,* if A is reduced and for every $a \in A$ there is a $b \in A$ with $a = b^p$.

COROLLARY 5.5.2. *If A is an algebra over a field of characteristic p, ψ^p is the Frobenius Φ^* on $K_n(A)$, $n>0$, and more generally on $[X, BGL(A)^+]$ for any X.*

PROOF. This follows from the fact (Ex. II.4.2) that $\psi^p = \Phi^*$ on the representation ring $R_A(G)$, together with the observation that $q(\Phi^*) : K_n(A) \to K_n(A)$ is induced by $\Phi : A \to A$ by naturality in A. □

PROPOSITION 5.6. *If A is a perfect algebra over a field of characteristic p, then $K_n(A)$ is uniquely p-divisible for all $n > 0$.*

PROOF. Since $n > 0$, we see from 5.5.1 that $\psi^p(x) = (-1)^{p-1} p\lambda^p(x)$ for $x \in K_n(A)$. Since $\psi^p = \Phi^*$ is an automorphism, so is multiplication by p. □

For any based space X there is a space FX homotopy equivalent to $B(\pi_1 X)$ and a natural map $X \to FX$ with $\pi_1(X) \xrightarrow{\cong} \pi_1(FX)$; if X is a simplicial space, FX is just the 2-coskeleton of X. Composing this map with the q of 5.3 gives a natural transformation $R_A(\pi_1 X) \to [X, BGL(A)^+]$ of functors from based spaces to groups.

PROPOSITION 5.7. *The natural transformation $R_A(\pi_1 X) \xrightarrow{q} [X, BGL(A)^+]$ is universal for maps to representable functors. That is, for any connected H-space H and any natural transformation $\eta_X : R_A(\pi_1 X) \to [X, H]$ there is a map $f : BGL(A)^+ \to H$, unique up to homotopy, such that η_X is the composite*

$$R_A(\pi_1 X) \xrightarrow{q} [X, BGL(A)^+] \xrightarrow{f} [X, H].$$

Like Theorem 1.8, this is proven by obstruction theory. Essentially, one considers the system of spaces $X = BGL_n(A)$ and the maps $BGL_n(A) \to H$ defined by $\eta_X(\mathrm{id}_n)$. See [Hiller, 2.4] for details.

EXAMPLE 5.7.1. The above construction of operations works in the topological setting, allowing us to construct λ-operations on $[X, BU]$ extending the operations in II.4.1.3. It follows that $[X, BGL(\mathbb{C})^+] \to [X, BU]$ commutes with the operations λ^k and ψ^k for every X.

EXAMPLE 5.8 (FINITE FIELDS). Let $\mathbb{F}_q$ be a finite field, and $\mathbb{F}_q^\times \to \mathbb{C}^\times$ a homomorphism. It induces a homomorphism $R_{\mathbb{F}_q}(G) \to R_{\mathbb{C}}(G)$ called the *Brauer lifting*. The composition of Brauer lifting with $R_{\mathbb{C}}(\pi_1 X) \to [X, BGL(\mathbb{C})^+]$ induces the map $BGL(\mathbb{F}_q)^+ \to BGL(\mathbb{C})^+ \to BU$ discussed in 1.12 above. Now an elementary calculation with characters (which we omit) shows that the Brauer lifting is actually a homomorphism of λ-rings. It follows from 5.7 and 5.7.1 that $[X, BGL(\mathbb{F}_q)^+] \to [X, BGL(\mathbb{C})^+] \to [X, BU]$ are homomorphisms of λ-rings. This was used in Theorem 1.12 to calculate $K_n(\mathbb{F}_q)$.

Compatibility with products

THEOREM 5.9. *The Adams operations ψ^k are compatible with the product on K-theory, in the sense that $\psi^k(x \cdot y) = \psi^k(x) \cdot \psi^k(y)$ for $x \in K_m(A)$ and $y \in K_n(A)$.*

PROOF. It suffices to show that the following diagram commutes up to weak homotopy:

$$\begin{array}{ccc} BGL(A)^+ \wedge BGL(A)^+ & \longrightarrow & BGL(A)^+ \\ {\scriptstyle \psi^k\wedge\psi^k}\downarrow & & \downarrow{\scriptstyle \psi^k} \\ BGL(A)^+ \wedge BGL(A)^+ & \longrightarrow & BGL(A)^+. \end{array}$$

Via Proposition 5.7, this follows from the fact that the $R_A(GL_m(A)\times GL_n(A))$ are λ-rings. □

EXAMPLE 5.9.1. If F is a field then $\psi^k = k^2$ on $K_2(F)$. This is because $K_2(F)$ is generated by Steinberg symbols $\{a, b\}$ (III.6.1), and Example 5.4.1 implies that $\psi^k\{a, b\} = \{a^k, b^k\} = k^2\{a, b\}$. The same argument shows that $\psi^k = k^n$ on the image of $K_n^M(F) \to K_n(F)$; see III.7.

EXAMPLE 5.9.2. For finite fields, we have $\psi^k(x) = k^i\, x$ for $x \in K_{2i-1}(\mathbb{F}_q)$. This follows from Example 5.8 and the fact (II.4.4.1) that $\psi^k = k^i$ on $\pi_{2i}BU = \widetilde{KU}(S^{2i})$.

The γ-filtration

Consider the γ-filtration (II.4.7) on $K_0(A) \times K_n(A)$; If $n > 0$ then $F^k_\gamma K_n(A)$ is generated by all $\gamma^{k'}(x)$ and $a \cdot \gamma^j(x)$ with $k' \geq k$, $a \in F^i_\gamma K_0(A)$, $x \in K_n(A)$, $i > 0$ and $i + j \geq k$. (There are other possible definitions, using the ring structure on $K_*(A)$, but they coincide up to torsion [Sou85].) For this reason, we shall ignore torsion and deal with the γ-filtration on $K_n(A) \otimes \mathbb{Q}$.

Because $x = \gamma^1(x)$ we have $K_n(A) = F^1_\gamma K_n(A)$ for $n > 0$. The next layer F^1_γ/F^2_γ of the filtration is also small.

PROPOSITION 5.10 (KRATZER). *For all commutative A, $SK_1(A) = F^2_\gamma K_1(A)$, and $F^1_\gamma K_1(A)/F^2_\gamma K_1(A) = A^\times$, and for $n \geq 2$: $K_n(A) = F^2_\gamma K_n(A)$.*

PROOF. It suffices to compute in $\pi_n BSL(A)^+$, which equals $SK_1(A)$ for $n = 1$ and $K_n(A)$ for $n > 1$ (see Ex. 1.8(a)). For $G = SL_N(A)$ the identity $\det(\mathrm{id}_N) = 1$ in $R(G)$ may be written in terms of $\rho = \mathrm{id}_N - N$ as $\gamma^1(\rho) + \gamma^2(\rho) + \cdots + \gamma^N(\rho) = 0$. Because $\gamma^i(\rho) = 0$ for $i > N$ (Exercise 5.4), this yields the identity $\sum_1^\infty \gamma^i(x) = 0$ for $x \in \pi_n BSL_N(A)^+$. Since $x = \gamma^1(x)$, this shows that $x \in F^2_\gamma \pi_n BSL_N(A)$. □

REMARK 5.10.1. Soulé has proven [Sou85, Thm. 1] that if A has stable range $sr(A) < \infty$ (I, Ex. 1.5) then γ^k vanishes on $K_n(A)$ for all $k \geq n + sr(A)$. This is a useful bound because $sr(R) \leq \dim(A) + 1$ for noetherian A. If F is a field, $\psi^k = k^n$ and $\gamma^n = (-1)^{n-1}(n-1)!$ on the image of $K_n^M(F) \to K_n(F)$, by 5.9.1, so the bound is best possible. The proof uses Volodin's construction of K-theory.

THEOREM 5.11. *For $n > 0$, the eigenvalues of ψ^k on $K_n(A)\otimes\mathbb{Q}$ are a subset of $\{1, k, k^2, \dots\}$, and the subspace $K_n^{(i)}(A)$ of eigenvectors for $\psi^k = k^i$ is independent of k. Finally, the ring $K_*(A) \otimes \mathbb{Q}$ is isomorphic to the bigraded ring $\oplus_{n,i}K_n^{(i)}(A)$.*

PROOF. Since every element of $K_n(A)$ comes from the K-theory of a finitely generated subring, we may assume that $sr(A) < \infty$. As in the proof of II.4.10, the linear operator $\prod_1^N(\psi^k - k^i)$ is trivial on each $F^i_\gamma/F^{i+1}_\gamma$ for large N, and this implies that $K_n(A)\otimes\mathbb{Q}$ is the direct sum of the eigenspaces for $\psi^k = k^i$, $1 \leq i \leq N$.

Since ψ^k and ψ^ℓ commute, it follows by downward induction on i that they have the same eigenspaces, *i.e.,* $K_n^{(i)}(A)$ is independent of k. Finally, the bigraded ring structure follows from 5.9. □

EXAMPLE 5.11.1. (Geller-Weibel) Let $A = \mathbb{C}[x_1, \dots, x_n]/(x_i x_j = 0, i \neq j)$ be the coordinate ring of the coordinate axes in $\mathbb{C}^n$. Then the Loday symbol $\langle\langle x_1, ..., x_n\rangle\rangle$ of Ex. 1.22 projects nontrivially into $K_n^{(i)}(A)$ for all i in the range $2 \leq i \leq n$. In particular, $K_n^{(i)}(A) \neq 0$ for each of these i. As $sr(A) = 2$, these are the only values of i allowed by Soulé's bound in 5.10.1.

The ring of Example 5.11.1 is not regular. In contrast, it is widely believed that the following conjecture is true for all regular rings; it may be considered to be *the* outstanding problem in algebraic K-theory.

VANISHING CONJECTURE 5.12 (BEILINSON-SOULÉ). If $i < n/2$ and A is regular then $K_n(A) = F_\gamma^i K_n(A)$. (See Ex. VI.3.6 for the connection to motivic cohomology.)

EXERCISES

5.1 Show that the composition of the cofibration $B\,\mathrm{Aut}(P) \to B\,\mathrm{Aut}(P \oplus Q)$ with $\lambda^k_{P\oplus Q}$ is homotopic to the map λ^k_P. By modifying $\lambda^k_{P\oplus Q}$, we can make the composition equal to λ^k_P. Using the free modules A^n and induction, conclude that we have maps $\lambda^k : BGL(A) \to BGL(A)^+$ and hence operations λ^k on $BGL(A)^+$, well defined up to homotopy.

One could use 4.1.1(c), 4.10 and 4.11.1 to consider the limit over $\mathrm{Aut}(P)$ for all projective modules P; the same construction will work except that there will be more bookkeeping.

5.2 Modify the construction of 5.3.2 to construct operations ψ^k and γ^k on the ring $K_0(A) \times [X, BGL(A)^+]$ for all X. (See II.4.4 and II.4.5.)

5.3 Show that the λ-operations are compatible with $K_1(A[t, 1/t]) \xrightarrow{\partial} K_0(A)$, the map in the Fundamental Theorem III.3.6, in the sense that for every $x \in K_0(A)$, $t \cdot x \in K_1(A[t, 1/t])$ satisfies $\partial\lambda^k(t \cdot x) = (-1)^{k-1}\psi^k(x)$.

5.4 (γ-dimension) Consider the γ-filtration (II.4.7) on $K_0(A) \times K_n(A)$, and show that every element of $K_n(A)$ has finite γ-dimension (II.4.5). *Hint:* Because S^n is a finite complex, each $x \in K_n(A)$ comes from some $\pi_n BGL_N(A)^+$. If $i > N$, show that γ^i kills the representation $[\mathrm{id}_N] - N$ and apply the map q.

5.5 For any commutative ring A, show that the ring structure on $R_A(G)$ induces a ring structure on $[X, K_0(A) \times BGL(A)^+]$.

5.6 Suppose that a commutative A-algebra B is finitely generated and projective as an A-module. Use 5.3 to show that the restriction of scalars map $R_B(G) \to R_A(G)$ induces a "transfer" map $BGL(B)^+ \to BGL(A)^+$. Show that it agrees on homotopy groups with the transfer maps for K_1 and K_2 in III.1.7.1 and III.5.6.3, respectively. We will encounter other constructions of the transfer in 6.3.2.

5.7 Use Ex. 5.3 to give an example of a regular ring A such that $K_3^{(2)}(A)$ and $K_3^{(3)}(A)$ are both nonzero.

§6. Quillen's Q-construction for exact categories

The higher K-theory groups of a small exact category $\mathcal{A}$ are defined to be the homotopy groups $K_n(\mathcal{A}) = \pi_{n+1}(BQ\mathcal{A})$ of the geometric realization of a certain auxiliary category $Q\mathcal{A}$, which we now define. This category has the same objects as $\mathcal{A}$, but morphisms are harder to describe. Here is the formal definition; we refer the reader to exercise 6.1 for a more intuitive interpretation of morphisms in terms of subquotients.

DEFINITION 6.1. Let $\mathcal{A}$ be an exact category. A morphism from A to B in $Q\mathcal{A}$ is an equivalence class of diagrams

$$A \overset{j}{\twoheadleftarrow} B_2 \overset{i}{\rightarrowtail} B \tag{6.1.1}$$

where j is an admissible epimorphism and i is an admissible monomorphism in $\mathcal{A}$. Two such diagrams are equivalent if there is an isomorphism between them which is the identity on A and B. The composition of the above morphism with a morphism $B \twoheadleftarrow C_2 \rightarrowtail C$ is $A \twoheadleftarrow C_1 \rightarrowtail C$, where $C_1 = B_2 \times_B C_2$.

$$\begin{array}{ccccc} & & C_1 & \rightarrowtail & C_2 & \rightarrowtail & C \\ & & \downarrow & & \downarrow & & \\ A & \twoheadleftarrow & B_2 & \rightarrowtail & B & & \end{array}$$

Two distinguished types of morphisms play a special role in $Q\mathcal{A}$: the admissible monics $A \rightarrowtail B$ (take $B_2 = A$) and the oppositely oriented admissible epis $A \twoheadleftarrow B$ (take $B_2 = B$). Both types are closed under composition, and the composition of $A \twoheadleftarrow B_2$ with $B_2 \rightarrowtail B$ is the morphism (6.1.1). In fact, every morphism in $Q\mathcal{A}$ factors as such a composition in a way that is unique up to isomorphism.

SUBOBJECTS 6.1.2. Recall from [Mac] that (in any category) a *subobject* of an object B is an equivalence class of monics $B_2 \rightarrowtail B$, two monics being equivalent if they factor through each other. In an exact category $\mathcal{A}$, we call a subobject *admissible* if any (hence every) representative $B_2 \rightarrowtail B$ is an admissible monic.

By definition, every morphism from A to B in $Q\mathcal{A}$ determines a unique admissible subobject of B in $\mathcal{A}$. If we fix a representative $B_2 \rightarrowtail B$ for each subobject in $\mathcal{A}$, then a morphism in $Q\mathcal{A}$ from A to B is a pair consisting of an admissible subobject B_2 of B and an admissible epi $B_2 \twoheadrightarrow A$.

In particular, this shows that morphisms from 0 to B in $Q\mathcal{A}$ are in 1-1 correspondence with admissible subobjects of B.

Isomorphisms in $Q\mathcal{A}$ are in 1–1 correspondence with isomorphisms in $\mathcal{A}$. To see this, note that every isomorphism $i\colon A \cong B$ in $\mathcal{A}$ gives rise to an isomorphism in $Q\mathcal{A}$, represented either by $A \overset{i}{\rightarrowtail} B$ or by $A \overset{i^{-1}}{\twoheadleftarrow} B$. Conversely, since the subobject determined by an isomorphism in $Q\mathcal{A}$ must be the maximal subobject $B \overset{=}{\rightarrowtail} B$, every isomorphism in $Q\mathcal{A}$ arises in this way.

REMARK 6.1.3. Some set-theoretic restriction is necessary for $Q\mathcal{A}$ to be a category in our universe. It suffices for $\mathcal{A}$ to be *well-powered*, *i.e.,* for each object of $\mathcal{A}$ to have a set of subobjects. We shall tacitly assume this, since we will soon need the stronger assumption that $\mathcal{A}$ is a small category.

We now consider the geometric realization $BQ\mathcal{A}$ as a based topological space, the basepoint being the vertex corresponding to the object 0. In fact, $BQ\mathcal{A}$ is a connected CW complex, because the morphisms $0 \rightarrowtail A$ in $Q\mathcal{A}$ give paths in $BQ\mathcal{A}$ from the basepoint 0 to every vertex A. (See Lemma 3.3.) The morphisms $0 \twoheadleftarrow A$ also give paths from 0 to A in $Q\mathcal{A}$.

PROPOSITION 6.2. *The geometric realization $BQ\mathcal{A}$ is a connected CW complex with $\pi_1(BQ\mathcal{A}) \cong K_0(\mathcal{A})$. The element of $\pi_1(BQ\mathcal{A})$ corresponding to $[A] \in K_0(\mathcal{A})$ is represented by the based loop composed of the two edges from 0 to A:*

$$0 \rightarrowtail A \twoheadrightarrow 0. \tag{6.2.1}$$

PROOF. Let T denote the family of all morphisms $0 \rightarrowtail A$ in $Q\mathcal{A}$. Since each nonzero vertex occurs exactly once, T is a maximal tree. By Lemma 3.4, $\pi_1(BQ\mathcal{A})$ has the following presentation: it is generated by the morphisms in $Q\mathcal{A}$, modulo the relations that $[0 \rightarrowtail A] = 1$ and $[f] \cdot [g] = [f \circ g]$ for every pair of composable arrows in $Q\mathcal{A}$. Moreover, the element of $\pi_1(BQ\mathcal{A})$ corresponding to a morphism from A to B is the based loop following the edges $0 \rightarrowtail A \to B \leftarrowtail 0$.

Since the composition $0 \rightarrowtail B_2 \rightarrowtail B$ is in T, this shows that $[B_2 \rightarrowtail B] = 1$ in $\pi_1(BQ\mathcal{A})$. Therefore $[A \twoheadleftarrow B_2 \rightarrowtail B] = [A \twoheadleftarrow B_2]$. Similarly, the composition $0 \twoheadleftarrow A \twoheadleftarrow B$ yields the relation $[A \twoheadleftarrow B][0 \twoheadleftarrow A] = [0 \twoheadleftarrow B]$. Since every morphism (6.1.1) factors, this shows that $\pi_1(BQ\mathcal{A})$ is generated by the morphisms $[0 \twoheadleftarrow A]$.

If $A \rightarrowtail B \twoheadrightarrow C$ is an exact sequence in $\mathcal{A}$, then the composition $0 \rightarrowtail C \twoheadleftarrow B$ in $Q\mathcal{A}$ is $0 \twoheadleftarrow A \rightarrowtail B$. This yields the additivity relation

$$[0 \twoheadleftarrow B] = [C \twoheadleftarrow B][0 \twoheadleftarrow C] = [0 \twoheadleftarrow A][0 \twoheadleftarrow C] \tag{6.2.2}$$

in $\pi_1(BQ\mathcal{A})$, represented by the following picture in $BQ\mathcal{A}$:

$$\begin{array}{ccccccc} 0 & \rightarrowtail & A & \rightarrowtail & B & & \\ & & \downarrow\!\!\downarrow & \nearrow & \downarrow\!\!\downarrow & \searrow\!\!\!\!\searrow & \\ & & 0 & \rightarrowtail & C & \twoheadrightarrow & 0 \end{array}$$

Since every relation $[f] \cdot [g] = [f \circ g]$ may be rewritten in terms of the additivity relation, $\pi_1(BQ\mathcal{A})$ is generated by the $[0 \twoheadleftarrow A]$ with (6.2.2) as the only relation. Therefore $K_0(\mathcal{A}) \cong \pi_1(BQ\mathcal{A})$. $\square$

EXAMPLE 6.2.3. The presentation for $\pi_1(BQ\mathcal{A})$ in the above proof yields a function from morphisms in $Q\mathcal{A}$ to $K_0(\mathcal{A})$. It sends $[A \twoheadleftarrow B_2 \rightarrowtail B]$ to $[B_1]$, where B_1 is the kernel of $B_2 \twoheadrightarrow A$.

DEFINITION 6.3. Let $\mathcal{A}$ be a small exact category. Then $K\mathcal{A}$ denotes the space $\Omega BQ\mathcal{A}$, and we set

$$K_n(\mathcal{A}) = \pi_n K\mathcal{A} = \pi_{n+1}(BQ\mathcal{A}) \qquad \text{for } \ n \geq 0.$$

Proposition 6.2 shows that this definition of $K_0(\mathcal{A})$ agrees with the one given in chapter II. Note that any exact functor $F\colon \mathcal{A} \to \mathcal{B}$ induces a functor $Q\mathcal{A} \to Q\mathcal{B}$, hence maps $BQ\mathcal{A} \to BQ\mathcal{B}$ and $K_n(\mathcal{A}) \to K_n(\mathcal{B})$. Thus the space $K\mathcal{A} = \Omega\ BQ\mathcal{A}$ and all the groups $K_n(\mathcal{A})$ are functors from exact categories and exact functors to spaces and abelian groups, respectively. Moreover, isomorphic functors induce the same map on K-groups, because they induce isomorphic functors $Q\mathcal{A} \to Q\mathcal{A}'$.

REMARK 6.3.1. If an exact category $\mathcal{A}$ is not small but has a set of isomorphism classes of objects then we define $K_n(\mathcal{A})$ to be $K_n(\mathcal{A}')$, where $\mathcal{A}'$ is a small subcategory equivalent to $\mathcal{A}$. By Ex. 6.2 this is independent of the choice of $\mathcal{A}'$. From now on, whenever we talk about the K-theory of a large exact category $\mathcal{A}$ we will use this device, assuming tacitly that we have replaced it by a small $\mathcal{A}'$. For example, this is the case in the following definitions.

DEFINITION 6.3.2. Let R be a ring with unit, and let $\mathbf{P}(R)$ denote the exact category of finitely generated projective R-modules. We set $K(R) = K\mathbf{P}(R)$ and define the K-groups of R by $K_n(R) = K_n\mathbf{P}(R)$. For $n = 0$, lemma 6.2 shows that this agrees with the definition of $K_0(R)$ in chapter II. For $n \geq 1$, agreement with the (nonfunctorial) $+$–construction definition 1.1.1 of $K(R)$ will have to wait until section 7.

Let $f : R \to S$ be a ring homomorphism such that S is finitely generated and projective as an R-module. Then there is a forgetful functor $\mathbf{P}(S) \to \mathbf{P}(R)$ and hence a "transfer" functor $f_* : K_*(S) \to K_*(R)$.

DEFINITION 6.3.3. If R is noetherian, let $\mathbf{M}(R)$ denote the category of finitely generated R-modules. Otherwise, $\mathbf{M}(R)$ is the category of pseudo-coherent modules defined in II.7.1.4. We set $G(R) = K\mathbf{M}(R)$ and define the G-groups of R by $G_n(R) = K_n\mathbf{M}(R)$. For $n = 0$, this also agrees with the definition in chapter II.

Let $f : R \to S$ be a ring map. When S is finitely generated as an R-module (and S is in $\mathbf{M}(R)$), there is a contravariant "transfer" map $f_* : G(S) \to G(R)$, induced by the forgetful functor $f_* : \mathbf{M}(S) \to \mathbf{M}(R)$, as in II.6.2.

On the other hand, if S is flat as an R-module, the exact base change functor $\otimes_R S : \mathbf{M}(R) \to \mathbf{M}(S)$ induces a covariant map $f^* : G(R) \to G(S)$ hence maps $f^* : G_n(R) \to G_n(S)$ for all n. This generalizes the base change map $G_0(R) \to G_0(S)$ of II.6.2. We will see in V.3.5 that the base change map is also defined when S has finite flat dimension over R.

DEFINITION 6.3.4. Similarly, if X is a scheme which is quasi-projective (over a commutative ring), we define $K(X) = K\mathbf{VB}(X)$ and $K_n(X) = K_n\mathbf{VB}(X)$. If X is noetherian, we define $G(X) = K\mathbf{M}(X)$ and $G_n(X) = K_n\mathbf{M}(X)$. For $n = 0$, this agrees with the definition of $K_0(X)$ and $G_0(X)$ in chapter II.

MORITA INVARIANCE 6.3.5. Recall from II.2.7 that if two rings R and S are Morita equivalent then there are equivalences $\mathbf{P}(R) \cong \mathbf{P}(S)$ and $\mathbf{M}(R) \cong \mathbf{M}(S)$. It follows that $K_n(R) \cong K_n(S)$ and $G_n(R) \cong G_n(S)$ for all n.

ELEMENTARY PROPERTIES 6.4. Here are some elementary properties of the above definition.

If $\mathcal{A}^{op}$ denotes the opposite category of $\mathcal{A}$, then $Q(\mathcal{A}^{op})$ is isomorphic to $Q\mathcal{A}$ by Ex 6.3, so we have $K_n(\mathcal{A}^{op}) = K_n(\mathcal{A})$. For example, if R is a ring then $\mathbf{P}(R^{op}) \cong \mathbf{P}(R)^{op}$ by $P \mapsto \mathrm{Hom}_R(P, R)$, so we have $K_n(R) \cong K_n(R^{op})$.

The product or direct sum $\mathcal{A} \oplus \mathcal{A}'$ of two exact categories is exact by Example II.7.1.6, and $Q(\mathcal{A} \oplus \mathcal{A}') = Q\mathcal{A} \times Q\mathcal{A}'$. Since the geometric realization preserves products by 3.1(4), we have $BQ(\mathcal{A} \oplus \mathcal{A}') = BQ\mathcal{A} \times BQ\mathcal{A}'$ and hence $K_n(\mathcal{A} \oplus \mathcal{A}') \cong K_n(\mathcal{A}) \oplus K_n(\mathcal{A}')$. For example, if R_1 and R_2 are rings then $\mathbf{P}(R_1 \times R_2) \cong \mathbf{P}(R_1) \oplus \mathbf{P}(R_2)$ and we have $K_n(R_1 \times R_2) \cong K_n(R_1) \oplus K_n(R_2)$. (Cf. Ex. 1.7.) Similarly, if a quasi-projective scheme X is the disjoint union of two components X_i, then $\mathbf{VB}(X)$ is the sum of the $\mathbf{VB}(X_i)$ and we have $K_n(X) \cong K_n(X_1) \oplus K_n(X_2)$.

The direct sum $\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is an exact functor, and its restriction to either factor is an isomorphism. It follows that $B\oplus : BQ\mathcal{A} \times BQ\mathcal{A} \to BQ\mathcal{A}$ endows $BQ\mathcal{A}$ with the structure of a homotopy-commutative H-space. (It is actually an infinite loop space; see 6.5.1).

Finally, suppose that $i \mapsto \mathcal{A}_i$ is a functor from some small filtering category I to exact categories and exact functors. Then the filtered colimit $\mathcal{A} = \varinjlim \mathcal{A}_i$ is an exact category (Ex. II.7.9), and $Q\mathcal{A} = \varinjlim Q\mathcal{A}_i$. Since geometric realization preserves filtered colimits by 3.1(3), we have $BQ\mathcal{A} = \varinjlim BQ\mathcal{A}_i$ and hence $K_n(\mathcal{A}) = \varinjlim K_n(\mathcal{A}_i)$. The K_0 version of this result was given in chapter II, as 6.2.7 and 7.1.7.

For example, if a ring R is the filtered union of subrings R_i we have $K_n(R) \cong \varinjlim K_n(R_i)$. However, $i \mapsto \mathbf{P}(R_i)$ is not a functor. One way to fix this is to replace the category $\mathbf{P}(R_i)$ by the equivalent category $\mathbf{P}'(R_i)$ whose objects are idempotent matrices over R_i; $\mathbf{P}(R)$ is equivalent to the category $\mathbf{P}'(R) = \varinjlim \mathbf{P}_i$. Alternatively one could use the Kleisli rectification, which is described in Ex. 6.5.

Cofinality 6.4.1. Let $\mathcal{B}$ be an exact subcategory of $\mathcal{A}$ which is closed under extensions in $\mathcal{A}$, and which is cofinal in the sense that for every A in $\mathcal{A}$ there is an A' in $\mathcal{A}$ so that $A \oplus A'$ is in $\mathcal{B}$. Then $BQ\mathcal{B}$ is homotopy equivalent to the covering space of $BQ\mathcal{A}$ corresponding to the subgroup $K_0(\mathcal{B})$ of $K_0(\mathcal{A}) = \pi_1(BQ\mathcal{A})$. In particular, $K_n(\mathcal{B}) \cong K_n(\mathcal{A})$ for all $n > 0$.

A special case of this is sketched in Exercise 6.6; the general case follows from this case using the version 8.9.1 of Waldhausen Cofinality below. Note that $K_0(\mathcal{B})$ is a subgroup of $K_0(\mathcal{A})$ by II.7.2.

Waldhausen constructed a delooping of $BQ\mathcal{A}$ in [Wa78, p. 194], using the "QQ" construction. This in turn provides a context for products.

Definition 6.5. When $\mathcal{A}$ is a small exact category, $QQ\mathcal{A}$ is the bicategory whose bimorphisms are equivalence classes of commutative diagrams in $\mathcal{A}$ of the form

$$\begin{array}{ccccc} \cdot & \twoheadleftarrow & \cdot & \rightarrowtail & \cdot \\ \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow \\ \cdot & \twoheadleftarrow & \cdot & \rightarrowtail & \cdot \\ \downarrowtail & & \downarrowtail & & \downarrowtail \\ \cdot & \twoheadleftarrow & \cdot & \rightarrowtail & \cdot \end{array}$$

in which the four little squares can be embedded in a 3×3 diagram with short exact rows and columns. Two such diagrams are equivalent if they are isomorphic by an isomorphism which restricts to the identity on each corner object.

Waldhausen proved that the loop space $\Omega QQ\mathcal{A}$ is homotopy equivalent to $BQ\mathcal{A}$ (see [Wa78, p. 196] and Ex. 6.8). Thus we have $K_n(\mathcal{A}) = \pi_{n+1}BQ\mathcal{A} \cong \pi_{n+2}BQQ\mathcal{A}$.

Remark 6.5.1. There are also n-fold categories $Q^n\mathcal{A}$, defined exactly as in 6.5, with $\Omega BQ^{n+1}\mathcal{A} \simeq BQ^n\mathcal{A}$. The sequence of the $BQ^n\mathcal{A}$ (using $\Omega BQ\mathcal{A}$ if $n = 0$) forms an Ω-spectrum $\mathbf{K}(\mathcal{A})$, making $K(\mathcal{A})$ into an infinite loop space.

Products

DEFINITION 6.6. If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are exact categories, a functor $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is called *biexact* if (i) each partial functor $A \otimes - : \mathcal{B} \to \mathcal{C}$ and $- \otimes B : \mathcal{A} \to \mathcal{C}$ is exact, and (ii) $A \otimes 0 = 0 \otimes B = 0$ for the distinguished zero objects of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

This is the same as the definition of biexact functor in §II.7. Note that condition (ii) can always be arranged by replacing $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ by equivalent exact categories.

Given such a biexact functor, the bicategory map $Q\mathcal{A} \otimes Q\mathcal{B} \to \mathrm{bi}(Q\mathcal{C})$ of 3.10.2 factors through the forgetful functor $QQ\mathcal{C} \to \mathrm{bi}(Q\mathcal{C})$. The functor $Q\mathcal{A} \otimes Q\mathcal{B} \to QQ\mathcal{C}$ sends a pair of morphisms $A_0 \twoheadleftarrow A_1 \rightarrowtail A_2$, $B_0 \twoheadleftarrow B_1 \rightarrowtail B_2$ to the bimorphism

$$\begin{array}{ccccc} A_0 \otimes B_0 & \twoheadleftarrow & A_1 \otimes B_0 & \rightarrowtail & A_2 \otimes B_0 \\ \uparrow\!\uparrow & & \uparrow\!\uparrow & & \uparrow\!\uparrow \\ A_0 \otimes B_1 & \twoheadleftarrow & A_1 \otimes B_1 & \rightarrowtail & A_2 \otimes B_1 \\ \downarrow & & \downarrow & & \downarrow \\ A_0 \otimes B_2 & \twoheadleftarrow & A_1 \otimes B_2 & \rightarrowtail & A_2 \otimes B_2 \end{array} \tag{6.6.1}$$

Now the geometric realization 3.6 of the bifunctor $\otimes : Q\mathcal{A} \otimes Q\mathcal{B} \to QQ\mathcal{C}$ is a map $BQ\mathcal{A} \times BQ\mathcal{B} \to BQQ\mathcal{C}$ by 3.10.1. Since $\otimes$ sends $Q\mathcal{A} \otimes 0$ and $0 \otimes Q\mathcal{B}$ to 0, by the technical condition (ii), $B\otimes$ sends $BQ\mathcal{A} \times 0$ and $0 \times BQ\mathcal{B}$ to the basepoint, and hence factors through a map

$$BQ\mathcal{A} \wedge BQ\mathcal{B} \to BQQ\mathcal{C}, \tag{6.6.2}$$

and in fact a pairing $\mathbf{K}(\mathcal{A}) \wedge \mathbf{K}(\mathcal{B}) \to \mathbf{K}(\mathcal{C})$ of spectra; see [Gillet, 7.12]. The reduced join operation [Wh, p. 480] yields bilinear maps

$$\begin{aligned} K_i(\mathcal{A}) \otimes K_j(\mathcal{B}) = \pi_{i+1}(BQ\mathcal{A}) \otimes \pi_{j+1}(BQ\mathcal{B}) \to \\ \pi_{i+j+2}(BQ\mathcal{A} \wedge BQ\mathcal{B}) \to \pi_{i+j+2}(BQ\mathcal{C}) \cong K_{i+j}(\mathcal{C}). \end{aligned} \tag{6.6.3}$$

REMARKS 6.6.4. We say that $\mathcal{A}$ *acts upon* $\mathcal{B}$ if there is a biexact $\mathcal{A} \times \mathcal{B} \to \mathcal{B}$. If there is an object A_0 of $\mathcal{A}$ so that $A_0 \otimes -$ is the identity on $\mathcal{B}$, the map $S^1 = B(0 \rightrightarrows 1) \to BQ\mathcal{A}$ given by the diagram $0 \rightarrowtail A_0 \twoheadrightarrow 0$ of 6.2 induces a map $S^1 \wedge BQ\mathcal{A} \to B\big(Q\mathcal{A} \otimes Q\mathcal{B}\big) \to BQQ\mathcal{B}$. Its adjoint $BQ\mathcal{A} \to \Omega BQQ\mathcal{A}$ is the natural map of 6.5 (see Ex. 6.8).

When there is an associative pairing $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $K_*(\mathcal{A})$ becomes a graded ring; it has a unit $[A_0] \in K_0(\mathcal{A})$ if $A_0 \otimes - = - \otimes A_0 = \mathrm{id}_{\mathcal{A}}$, by the preceeding paragraph, and $\mathbf{K}(\mathcal{A})$ is a ring spectrum. When $K(\mathcal{A})$ acts on $K(\mathcal{B})$ and the two evident functors $\mathcal{A} \times \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ agree up to natural isomorphism, the pairing makes $K_*(\mathcal{B})$ into a left $K_*(\mathcal{A})$-module.

EXAMPLE 6.6.5. These remarks apply in particular to the category $\mathcal{A} = \mathbf{P}(R)$ over a commutative ring R, and $\mathbf{VB}(X)$ over a scheme X. Tensor product makes $K_*(R) = K_*\mathbf{P}(R)$ and $K_*(X) = K_*\mathbf{VB}(X)$ into graded-commutative rings with unit. For every R-algebra A, $K_*(A)$ and $G_*(A)$ are 2-sided graded $K_*(R)$-modules, and $G_*(X)$ is a graded $K_*(X)$-module.

If $f : A \to B$ is an R-algebra map, and B is finite over A, the finite transfer $f_* : G(B) \to G(A)$ is a $K_*(R)$-module homomorphism: $f_*(x \cdot y) = f_*(x) \cdot y$ for $x \in G_*(B)$ and $y \in K_*(R)$. This fact is sometimes referred to as the *projection formula*, and holds because $f_*(x \cdot y)$ and $x \cdot f_*(y)$ arise from the isomorphic functors $M \otimes_B (B \otimes_R P) \cong M \otimes_A (A \otimes_R P)$ of functors $\mathbf{M}(B) \times \mathbf{P}(R) \to \mathbf{M}(A)$.

The $W(R)$-module $NK_*(A)$

6.7. Let k be a commutative ring. We saw in II.7.4.3 that the exact endomorphism category $\mathbf{End}(k)$ of pairs (P,α) has an associative, symmetric biexact pairing with itself, given by $\otimes_k$. This makes $K_*\mathbf{End}(k)$ into a graded-commutative ring. As in *loc. cit.*, the functors $\mathbf{P}(k)\to\mathbf{End}(k)\to\mathbf{P}(k)$ decompose this ring as a product of $K_*(k)$ and another graded-commutative ring which we call $\mathrm{End}_*(k)$.

If R is an k-algebra, $\mathbf{End}(k)$ acts associatively by $\otimes_k$ on the exact category $\mathbf{Nil}(R)$ of nilpotent endomorphisms (II.7.4.4), and on its subcategories $F_m\mathbf{Nil}(R)$ (Ex. II.7.17). As $\mathbf{Nil}(R)$ is their union, we see that $K_*\mathbf{Nil}(R)=\operatorname{colim}K_*F_m\mathbf{Nil}(R)$ is a filtered $K_*\mathbf{End}(k)$-module.

Let $\mathrm{Nil}(R)$ denote the fiber of the forgetful functor $K\mathbf{Nil}(R)\to K(R)$; since this is split, we have $K\mathbf{Nil}(R)\simeq K(R)\times\mathrm{Nil}(R)$ and $K_*\mathbf{Nil}(R)\cong K_*(R)\times\mathrm{Nil}_*(R)$, where $\mathrm{Nil}_*(R)=\pi_*\,\mathrm{Nil}(R)$ is a graded $\mathrm{End}_*(k)$-module.

By Almkvist's Theorem II.4.3, $\mathrm{End}_0(k)$ is isomorphic to the subgroup of $W(k)=(1+tk[[t]])^\times$ consisting of all quotients $f(t)/g(t)$ of polynomials in $1+tR[t]$. Stienstra observed in [St85] (cf. [St82]) that the $\mathrm{End}_0(k)$-module structure extended to a $W(k)$-module structure by the following device. There are exact functors F_m,V_m : $\mathbf{Nil}(R)\to\mathbf{Nil}(R)$ defined by $F_m(P,\nu)=(P,\nu^m)$ and $V_m(P,\nu)=(P[t]/(t^m-\nu),t)$ (see Ex. II.7.16). Stienstra proved in [St82] that $(V_m\alpha)\cdot\nu=V_m(\alpha\cdot F_m(\nu))$ for $\alpha\in\mathrm{End}_0(k)$ and $\nu\in\mathrm{Nil}_*(R)$. Since F_m is zero on $F_m\mathbf{Nil}(R)$, the elements $V_m(\alpha)$ act as zero on the image $F_m\,\mathrm{Nil}_*(R)$ of $K_*F_m\mathbf{Nil}(R)\to K_*\mathbf{Nil}(R)\to\mathrm{Nil}_*(R)$.

For example, the class of $\alpha=[(k,a)]-[(k,0)])$ in $\mathrm{End}_0(k)\subset W(R)$ is $1-at$, so $V_m(\alpha)=(1-at^m)$ acts as zero. Stienstra also proves in [St85] that if $g(t)=1+\dots$ has degree $<m$ and $f(t)$ is any polynomial then the element $1+t^m(f/g)$ of $\mathrm{End}_0(k)$ acts as zero on $F_m\,\mathrm{Nil}_*(R)$. Hence the ideal $\mathrm{End}_0(R)\cap(1+t^mR[[t]])$ is zero on $F_m\,\mathrm{Nil}_*(R)$. Writing an element of $W(k)$ as a formal factorization $f(t)=\prod_{i=1}^\infty(1-a_mt^m)$, the formula $f\cdot\nu=\sum(1-a_mt^m)\cdot\nu$ makes sense as a finite sum.

PROPOSITION 6.7.1. *If $k=\mathbb{Z}/p\mathbb{Z}$, $\mathrm{Nil}_*(R)$ is a graded p-group.*
If $k=S^{-1}\mathbb{Z}$, or if k is a $\mathbb{Q}$-algebra, $\mathrm{Nil}_*(R)$ is a graded k-module.

PROOF. If $k=S^{-1}\mathbb{Z}$, or if k is a $\mathbb{Q}$-algebra, the map $m\mapsto(1-t)^m$ defines a ring homomorphism from k into $W(k)$, so any $W(k)$-module is a k-module. If $p=0$ (or even $p^\nu=0$) in k then for each n the formal factorization of $(1-t)^{p^N}$ involves only $(1-a_mt^m)$ for $m\ge n$. It follows that p^N annihilates the image of $K_*F_n\mathbf{Nil}(R)$ in $\mathrm{Nil}_*(R)$. Since $\mathrm{Nil}_*(R)$ is the union of these images, the result follows. $\Box$

We will see in chapter V.9 that there is an isomorphism $NK_{n+1}(R)\cong\mathrm{Nil}_n(R)$, so what we have really seen is that $NK_*(R)$ is a graded $\mathrm{End}_*(k)$-module, with the properties given by 6.7.1:

COROLLARY 6.7.2. *If $k=\mathbb{Z}/p\mathbb{Z}$, each $NK_n(R)$ is a p-group.*
If $k=S^{-1}\mathbb{Z}$, or if k is a $\mathbb{Q}$-algebra, each $NK_n(R)$ is a k-module.

EXAMPLE 6.7.3. If R is an algebra over the complex numbers $\mathbb{C}$, then each $NK_n(R)$ has the structure of a $\mathbb{C}$-vector space. As an abelian group, it is either zero or else uniquely divisible and uncountable.

The endofunctor $V_m(P,\alpha)=(P[t]/t^m-\alpha,t)$ of $\mathbf{End}(R)$ (Ex. II.7.16) sends $\mathbf{Nil}(R)$ to itself, and $F_mV_m(P,\nu)=\oplus_1^m(P\nu)$. Hence V_m induces an endomorphism V_m on each $\mathrm{Nil}_n(R)$, such that F_mV_m is multiplication by m.

PROPOSITION 6.7.4 (FARRELL). *If any $NK_n(R)$ is nonzero, it cannot be finitely generated as an abelian group.*

PROOF. Since $NK_n(R) = \operatorname{colim} K_{n-1}F_n\mathbf{Nil}(R)$, every element is killed by all sufficiently large F_m. If $NK_n(R)$ were finitely generated, there would be an integer M so that the entire group is killed by F_m for all $m > M$. Pick $\beta \neq 0$ in $NK_n(R)$ and choose $m > M$ so that $m\beta \neq 0$. But $F_m(V_m\beta) = m\beta$ is nonzero, a contradiction. $\square$

Finite generation

The following conjecture is due to Bass.

BASS' FINITENESS CONJECTURE 6.8. Let R be a commutative regular ring, finitely generated as a $\mathbb{Z}$-algebra. Then the groups $K_n(R)$ are finitely generated for all n.

Quillen used a filtration of the Q-construction to prove in [Q73] that the groups $K_n(R)$ are finitely generated for any Dedekind domain R such that (1) $\operatorname{Pic}(R)$ is finite and (2) the homology groups $H_n(\operatorname{Aut}(P), st(P \otimes_R F))$ are finitely generated. He then verified (2) in [Q73] (number field case) and [GQ82] (affine curves). In other words:

THEOREM 6.9. *(Quillen) Let R be either an integrally closed subring of a number field F, finite over $\mathbb{Z}$, or else the coordinate ring of a smooth affine curve over a finite field. Then $K_n(R)$ is a finitely generated group for all n.*

EXERCISES

6.1 *Admissible subquotients.* Let B be an object in an exact category $\mathcal{A}$. An *admissible layer* in B is a pair of subobjects represented by a sequence $B_1 \rightarrowtail B_2 \rightarrowtail B$ of admissible monics, and we call the quotient B_2/B_1 an *admissible subquotient* of B. Show that a morphism $A \to B$ in $Q\mathcal{A}$ may be identified with an isomorphism $j\colon B_2/B_1 \cong A$ of A with an admissible subquotient of B, and that composition in $Q\mathcal{A}$ arises from the fact that a subquotient of a subquotient is a subquotient.

6.2 If two exact categories $\mathcal{A}$ and $\mathcal{A}'$ are equivalent (and the equivalence respects exactness), show that $Q\mathcal{A}$ and $Q\mathcal{A}'$ are equivalent. If both are small categories, conclude that $K_n(\mathcal{A}) \cong K_n(\mathcal{A}')$ for all n.

6.3 If $\mathcal{A}$ is an exact category, so is its opposite category $\mathcal{A}^{op}$ (see Example II.7.1.5). Show that $Q(\mathcal{A}^{op})$ is isomorphic to $Q\mathcal{A}$.

6.4 Let B be an object in an exact category $\mathcal{A}$. Show that the comma category $Q\mathcal{A}/B$ is equivalent to the poset of admissible layers of B in the sense of Ex. 6.1. If $\mathcal{P}$ is an exact subcategory of $\mathcal{A}$ and i denotes the inclusion $Q\mathcal{P} \subset Q\mathcal{A}$, show that i/B is equivalent to the poset of admissible layers of B with $B_2/B_1 \in \mathcal{P}$.

6.5 *Kleisli rectification.* Let I be a filtering category, and let $I \to CAT$ be a lax functor in the sense of Ex. 3.8. Although the family of exact categories $Q\mathcal{A}(i)$ is not filtering, the family of homotopy groups $K_n\mathcal{A}(i)$ is filtering. The following trick allows us make K-theoretic sense out of the phantom category $\mathcal{A} = \varinjlim \mathcal{A}(i)$.

Let $\mathcal{A}_i$ be the category whose objects are pairs $(A_j, j \xrightarrow{f} i)$ with A_j in $\mathcal{A}(j)$ and f a morphism in I. A morphism from $(A_j, j \xrightarrow{f} i)$ to $(A_k, k \xrightarrow{g} i)$ is a pair

$(j \xrightarrow{h} k, \theta_j)$ where $f = gh$ in I and θ_j is an isomorphism $h_*(A_j) \cong A_k$ in $\mathcal{A}(k)$. Clearly $\mathcal{A}_i$ is equivalent to $\mathcal{A}(i)$, and $i \mapsto \mathcal{A}_i$ is a functor. Thus if $\mathcal{A}$ denotes $\varinjlim \mathcal{A}_i$ we have $K_n\mathcal{A} = \varinjlim K_n\mathcal{A}(i)$.

6.6 (Gersten) Suppose given a surjective homomorphism $\phi\colon K_0(\mathcal{A}) \to G$, and let $\mathcal{B}$ denote the full subcategory of all B in $\mathcal{A}$ with $\phi[B] = 0$ in G. In this exercise we show that if $\mathcal{B}$ is cofinal in $\mathcal{A}$ then $K_n(\mathcal{B}) \cong K_n(\mathcal{A})$ for $n > 0$, and $K_0(\mathcal{B}) \subset K_0(\mathcal{A})$.

(a) Show that there is a functor $\psi\colon Q\mathcal{A} \to G$ sending the morphism (6.1.1) of $Q\mathcal{A}$ to $\phi[B_1]$, $B_1 = \ker(j)$, where G is regarded as a category with one object $*$. Using 6.2, show that the map $\pi_1(Q\mathcal{A}) \to \pi_1(G)$ is just ϕ.

(b) Show that the hypotheses of Theorem B are satisfied by ψ, so that $B(\psi/*)$ is the homotopy fiber of $BQ\mathcal{A} \to BG$.

(c) Use Theorem A to show that $Q\mathcal{B} \to \psi^{-1}(*)$ is a homotopy equivalence.

(d) Suppose in addition that $\mathcal{B}$ is cofinal in $\mathcal{A}$ (II.5.3), so that $K_0(\mathcal{B})$ is the subgroup $\ker(\phi)$ of $K_0(\mathcal{A})$ by II.7.2. Use Theorem A to show that $\psi^{-1}(*) \simeq \psi/*$. This proves that $BQ\mathcal{B} \to BQ\mathcal{A} \to BG$ is a homotopy fibration. Conclude that $K_n(\mathcal{B}) \cong K_n(\mathcal{A})$ for all $n \geq 1$.

6.7 (Waldhausen) If $\mathcal{A}$ is an exact category, let $q\mathcal{A}$ denote the bicategory (3.10) with the same objects as $\mathcal{A}$, admissible monomorphisms and epimorphisms as the horizontal and vertical morphisms, respectively; the bimorphisms in $q\mathcal{A}$ are those bicartesian squares in $\mathcal{A}$ whose horizontal edges are admissible monomorphisms, and whose vertical edges are admissible epimorphisms.

$$\begin{array}{ccc} A_{11} & \rightarrowtail & A_{10} \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ A_{01} & \rightarrowtail & A_{00} \end{array}$$

Show that the diagonal category (Ex. 3.14) of $q\mathcal{A}$ is the category $Q\mathcal{A}$.

6.8 (Waldhausen) Since the realization of the two-object category $0 \rightrightarrows 1$ is S^1, the realization of the bicategory $(0 \rightrightarrows 1) \otimes \mathcal{A}$ is $S^1 \times B\mathcal{A}$. Given a morphism $A_0 \twoheadleftarrow A_1 \rightarrowtail A_2$ show that the pair of bimorphisms in $QQ\mathcal{A}$

$$\begin{array}{ccccc} A_0 & \twoheadleftarrow & A_1 & \rightarrowtail & A_2 \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ 0 & \twoheadleftarrow & 0 & \rightarrowtail & 0, \end{array} \qquad \begin{array}{ccccc} A_0 & \twoheadleftarrow & A_1 & \rightarrowtail & A_2 \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \twoheadleftarrow & 0 & \rightarrowtail & 0 \end{array}$$

describe a map $S^1 \wedge BQ\mathcal{A} \to BQQ\mathcal{A}$. Waldhausen observed in [Wa78, p. 197] that this map is adjoint to the homotopy equivalence $BQ\mathcal{A} \simeq \Omega BQQ\mathcal{A}$.

6.9 For every biexact $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$, show that the pairing $K_0(\mathcal{A}) \otimes K_0(\mathcal{B}) \to K_0(\mathcal{C})$ of (6.6.3) agrees with the product of II.7.4.

6.10 Show that the functor $Q\mathcal{A} \otimes Q\mathcal{B} \to QQ\mathcal{C}$ of 6.6 is a map of symmetric monoidal categories (the operation on $QQ\mathcal{C}$ is slotwise direct sum). Conclude that $BQ\mathcal{A} \times BQ\mathcal{B} \to BQQ\mathcal{C}$ is an H-space map. (In fact, it is an infinite loop space map.)

6.11 Let $\mathcal{A}$ be the direct sum $\oplus_{i \in I}\mathcal{A}_i$ of exact categories. Show that $K_n(\mathcal{A}) \cong \oplus_{i \in I} K_n(\mathcal{A}_i)$.

6.12 If $f : R \to S$ is such that S is in $\mathbf{P}(R)$, show that the transfer map $f_* : K_0(S) \to K_0(R)$ of 6.3.2 agrees with the transfer functor for K_0 given in II.2.8.

6.13 If $f : R \to S$ and S is in $\mathbf{P}(R)$, show that f_*f^* is multiplication by $[S] \in K_0(R)$, and that f^*f_* is multiplication by $[S \otimes_R S] \in K_0(S)$.

If $f : k \to \ell$ is a purely inseparable field extension, show that both f^*f_* and f_*f^* are multiplication by $[\ell : k] = p^r$.

If $f : R \to S$ is a Galois extension with group G, show that $f^*f_* = \sum_{g\in G} g$.

6.14 Let $\mathcal{C}$ be a category with a distinguished zero object '0' and a coproduct $\vee$. We say that a family $\mathcal{E}$ of sequences of the form

$$0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0 \qquad (\dagger)$$

is *admissible* if the following conditions hold (cf. Ex. II.7.8): (i) Any sequence in $\mathcal{C}$ isomorphic to a sequence in $\mathcal{E}$ is in $\mathcal{E}$; (ii) If (†) is a sequence in $\mathcal{E}$ then i is a kernel for j (resp. j is a cokernel for i) in $\mathcal{C}$; (iii) the class $\mathcal{E}$ contains all of the sequences $0 \to B \to B \vee D \to D \to 0$; (iv) the class of admissible epimorphisms is closed under composition and pullback along admissible monics; (v) the class of admissible monics is closed under composition and pullback along admissible epimorphisms.

A *quasi-exact category* is a pair $(\mathcal{C}, \mathcal{E})$, where $\mathcal{E}$ is admissible in the above sense. If $\mathcal{C}$ is small, show that there is a category $Q\mathcal{C}$, defined exactly as in 6.1, and that $\pi_1(BQ\mathcal{C})$ is the group $K_0(\mathcal{C})$ defined exactly as in II.7.1: the group generated by the objects of $\mathcal{C}$ subject to the relations $[C] = [B] + [D]$ arising from the admissible exact sequences. (This formulation is due to Deitmar.)

6.15 (Waldhausen) Show that the category $\mathbf{Sets}_{\text{fin}}$ of finite pointed sets is quasi-exact, where $\mathcal{E}$ is the collection of split sequences $0 \to B \to B \vee D \to D \to 0$, and that $K_0(\mathbf{Sets}_{\text{fin}}) = \mathbb{Z}$, exacty as in II.5.2.1. The opposite category $\mathbf{Sets}_{\text{fin}}^{op}$ is not quasi-exact, because $0 \to B \to B \wedge D \to D \to 0$ is not in $\mathcal{E}^{op}$.

6.16 A monoid M with identity 1 is *pointed* if it has an element 0 with $0 \cdot m = m \cdot 0 = 0$ for all $m \in M$. A pointed M-set is a pointed set X on which M acts and $0 \cdot x = *$ for all $x \in X$. Show that the category of finitely generated pointed M-sets, and its subcategory of free pointed M-sets, are quasi-exact. Here a sequence (†) is admissible if i is an injection and j identifies D with C/B.

§7. The "$+ = Q$" Theorem

Suppose that $\mathcal{A}$ is an additive category. One way to define the K-theory of $\mathcal{A}$ is to consider the symmetric monoidal category $S = \text{iso}\,\mathcal{A}$ (where $\Box = \oplus$) and use the $S^{-1}S$ construction: $K_n^{\oplus}\mathcal{A} = \pi_n B(S^{-1}S)$ and $K^{\oplus}\mathcal{A} = K(S) = B(S^{-1}S)$.

Another way is to suppose that $\mathcal{A}$ has the structure of an exact category and form the Q-construction on $\mathcal{A}$ with the $S^{-1}S$ construction on S. Comparing the definitions of $K_0^{\oplus}\mathcal{A}$ and $K_0\mathcal{A}$ in II.5.1.2 and II.7.1, we see that the K_0 groups are not isomorphic in general, unless perhaps every exact sequence splits in $\mathcal{A}$, *i.e.,* unless $\mathcal{A}$ is a split exact category in the sense of II.7.1.2.

Here is the main theorem of this section.

Theorem 7.1 (Quillen). *If $\mathcal{A}$ is a split exact category and $S = iso\,\mathcal{A}$, then $\Omega BQ\mathcal{A} \simeq B(S^{-1}S)$. Hence $K_n(\mathcal{A}) \cong K_n(S)$ for all $n \geq 0$.*

In fact, $B(S^{-1}S)$ is the group completion of BS by theorem 4.8 and exercise 7.1. In some circumstances (see 4.9, 4.10 and 4.11.1), the $S^{-1}S$ construction is a $+$–construction. In these cases, theorem 7.1 shows that the Q–construction is also a $+$–construction. For $\mathcal{A} = \mathbf{P}(R)$, this yields the "$+ = Q$" theorem:

COROLLARY 7.2 $(+ = Q)$. *For every ring R,*

$$\Omega BQ\mathbf{P}(R) \simeq K_0(R) \times BGL(R)^+.$$

Hence $K_n(R) \cong K_n\mathbf{P}(R)$ for all $n \geq 0$.

DEFINITION 7.3. Given an exact category $\mathcal{A}$, we define the category $\mathcal{E}\mathcal{A}$ as follows. The objects of $\mathcal{E}\mathcal{A}$ are admissible exact sequences in $\mathcal{A}$. A morphism from $E' : (A' \rightarrowtail B' \twoheadrightarrow C')$ to $E : (A \rightarrowtail B \twoheadrightarrow C)$ is an equivalence class of diagrams of the following form, where the rows are exact sequences in $\mathcal{A}$:

(7.3.1)
$$\begin{array}{cccccc}
E' : & A' & \rightarrowtail & B' & \twoheadrightarrow & C' \\
 & \alpha\uparrow & & \| & & \uparrow \\
\downarrow & A & \rightarrowtail & B' & \twoheadrightarrow & C'' \\
 & \| & & \downarrow\beta & & \downarrow \\
E : & A & \rightarrowtail & B & \twoheadrightarrow & C.
\end{array}$$

Two such diagrams are equivalent if there is an isomorphism between them which is the identity at all vertices except for the C'' vertex.

Notice that the right column in (7.3.1) is just a morphism φ in $Q\mathcal{A}$ from C' to C, so the target C is a functor $t : \mathcal{E}\mathcal{A} \to Q\mathcal{A}$: $t(A \rightarrowtail B \twoheadrightarrow C) = C$. In order to improve legibility, it is useful to write $\mathcal{E}_C$ for the fiber category $t^{-1}(C)$.

FIBER CATEGORIES 7.4. If we fix φ as the identity map of $C = C'$, we see that the fiber category $\mathcal{E}_C = t^{-1}(C)$ of exact sequences with target C has for its morphisms all pairs (α, β) of isomorphisms fitting into a commutative diagram:

$$\begin{array}{ccccc}
A' & \rightarrowtail & B' & \twoheadrightarrow & C \\
\alpha\uparrow\cong & & \cong\downarrow\beta & & \| \\
A & \rightarrowtail & B & \twoheadrightarrow & C.
\end{array}$$

In particular, every morphism in $\mathcal{E}_C$ is an isomorphism.

EXAMPLE 7.4.1. The fiber category $\mathcal{E}_0 = t^{-1}(0)$ is homotopy equivalent to $S = \operatorname{iso}\mathcal{A}$. To see this, consider the functor from $\operatorname{iso}\mathcal{A}$ to $\mathcal{E}_0$ sending A to the trivial sequence $A \overset{\text{id}}{\rightarrowtail} A \twoheadrightarrow 0$. This functor is a full embedding. Moreover, every object of $\mathcal{E}_0$ is naturally isomorphic to such a trivial sequence, whence the claim.

LEMMA 7.5. *For any C in $\mathcal{A}$, $\mathcal{E}_C$ is a symmetric monoidal category, and there is a faithful monoidal functor $\eta_C : S \to \mathcal{E}_C$ sending A to $A \rightarrowtail A \oplus C \twoheadrightarrow C$.*

PROOF. Given $E_i = (A_i \rightarrowtail B_i \twoheadrightarrow C)$ in $\mathcal{E}_C$, set $E_1 * E_2$ equal to

(7.5.1)
$$A_1 \oplus A_2 \rightarrowtail (B_1 \times_C B_2) \twoheadrightarrow C.$$

This defines a symmetric product on $\mathcal{E}_C$ with identity $e \colon 0 \rightarrowtail C \twoheadrightarrow C$. It is now routine to check that $S \to \mathcal{E}_C$ is a monoidal functor, and that it is faithful. $\square$

REMARK 7.5.2. If $\mathcal{A}$ is split exact then every object of $\mathcal{E}_C$ is isomorphic to one coming from S. In particular, the category $\langle S, \mathcal{E}_C\rangle$ of 4.7.1 is connected. This fails if $\mathcal{A}$ has a non-split exact sequence.

PROPOSITION 7.6. *If $\mathcal{A}$ is split exact, each $S^{-1}S \to S^{-1}\mathcal{E}_C$ is a homotopy equivalence.*

PROOF. By Ex. 4.7 and Ex. 7.1, $S^{-1}S \to S^{-1}\mathcal{E}_C \to \langle S, \mathcal{E}_C \rangle$ is a fibration, so it suffices to prove that $L = \langle S, \mathcal{E}_C \rangle$ is contractible. First, observe that the monoidal product on $\mathcal{E}_C$ induces a monoidal product on L, so BL is an H-space (as in 4.1). We remarked in 7.5.2 that L is connected. By [Wh, X.2.2], BL is group-like, *i.e.,* has a homotopy inverse.

For every exact sequence E, there is a natural transformation $\delta_E \colon E \to E * E$ in L, where $*$ is defined by (7.5.1), given by the diagonal.

$$\begin{array}{ccccccc} E: & & A & \rightarrowtail & B & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \downarrow & & \| \\ E*E: & & A \oplus A & \rightarrowtail & B \times_C B & \twoheadrightarrow & C \end{array}$$

Now δ induces a homotopy between the identity on BL and muliplication by 2. Using the homotopy inverse to subtract the identity, this gives a homotopy between zero and the identity of BL. Hence BL is contractible. □

We also need a description of how $\mathcal{E}_C$ varies with C.

LEMMA 7.7. *For each morphism $\varphi \colon C' \to C$ in $Q\mathcal{A}$, there is a canonical functor $\varphi^* \colon \mathcal{E}_C \to \mathcal{E}_{C'}$ and a natural transformation $\eta_E \colon \varphi^*(E) \to E$ from φ^* to the inclusion of $\mathcal{E}_C$ in $\mathcal{E}\mathcal{A}$.*

In fact, $t : \mathcal{E}\mathcal{A} \to Q\mathcal{A}$ is a fibered functor with base change φ^* (Ex. 7.2). It follows (from 3.7.5) that $C \mapsto \mathcal{E}_C$ is a contravariant functor from $Q\mathcal{A}$ to CAT.

PROOF. Choose a representative $C' \twoheadleftarrow C'' \rightarrowtail C$ for φ and choose a pullback B' of B and C'' along C. This yields an exact sequence $A \rightarrowtail B' \twoheadrightarrow C''$ in $\mathcal{A}$. (Why?) The composite $B' \twoheadrightarrow C'' \twoheadrightarrow C'$ is admissible; if A' is its kernel then set

$$\varphi^*(A \rightarrowtail B \twoheadrightarrow C) = (A' \rightarrowtail B' \twoheadrightarrow C').$$

Since every morphism in $\mathcal{E}_C$ is an isomorphism, it is easy to see that φ^* is a functor, independent (up to isomorphism) of the choices made. Moreover, the construction yields a diagram (7.3.1), natural in E; the map β is an admissible monic because $A \rightarrowtail B' \xrightarrow{\beta} B'$ is. Hence (7.3.1) constitutes the natural map $\eta_E : E \to \varphi^*(E)$. □

Now the direct sum of sequences defines an operation $\oplus$ on $\mathcal{E}\mathcal{A}$, and S acts on $\mathcal{E}\mathcal{A}$ via the inclusion of S in $\mathcal{E}\mathcal{A}$ given by 7.4.1. That is, $A' \Box (A \rightarrowtail B \twoheadrightarrow C)$ is the sequence $A' \oplus A \rightarrowtail A' \oplus B \twoheadrightarrow C$. Since $t(A' \Box E) = t(E)$ we have an induced map $T = S^{-1}t \colon S^{-1}\mathcal{E}\mathcal{A} \to Q\mathcal{A}$. This is also a fibered functor (Ex. 7.2).

THEOREM 7.8. *If $\mathcal{A}$ is a split exact category and $S = iso\,\mathcal{A}$, then the sequence $S^{-1}S \to S^{-1}\mathcal{E}\mathcal{A} \xrightarrow{T} Q\mathcal{A}$ is a homotopy fibration.*

PROOF. We have to show that Quillen's Theorem B applies, *i.e.,* that the base changes φ^* of 7.7 are homotopy equivalences. It suffices to consider φ of the form $0 \rightarrowtail C$ and $0 \twoheadleftarrow C$. If φ is $0 \rightarrowtail C$, the composition of the equivalence $S^{-1}S \to S^{-1}\mathcal{E}_C$ of 7.6 with φ^* is the identity by Ex. 7.5, so φ^* is a homotopy equivalence.

Now suppose that φ is $0 \twoheadleftarrow C$. The composition of the equivalence $S^{-1}S \to S^{-1}\mathcal{E}_C$ of 7.6 with φ^* sends A to $A \oplus C$ by Ex. 7.5. Since there is a natural transformation $A \to A \oplus C$ in $S^{-1}S$, this composition is a homotopy equivalence. Hence φ^* is a homotopy equivalence. □

Proof of theorem 7.1. This will follow from theorem 7.8, once we show that $S^{-1}\mathcal{E}\mathcal{A}$ is contractible. By Ex. 7.3, $\mathcal{E}\mathcal{A}$ is contractible. Any action of S on a contractible category must be invertible (4.7.1). By Ex. 4.6 and Ex. 7.1, $\mathcal{E}\mathcal{A} \to S^{-1}\mathcal{E}\mathcal{A}$ is a homotopy equivalence, and therefore $S^{-1}\mathcal{E}\mathcal{A}$ is contractible. □

Agreement of Product Structures

Any biexact pairing $\mathcal{A}_1 \times \mathcal{A}_2 \xrightarrow{\otimes} \mathcal{A}_3$ of split exact categories (6.6) induces a pairing $S_1 \times S_2 \xrightarrow{\Box} S$ of symmetric monoidal categories, where $S_i = \operatorname{iso}\mathcal{A}_i$. We now compare the resulting pairings $K(\mathcal{A}_1) \wedge K(\mathcal{A}_2) \to K(\mathcal{A}_3)$ of 6.6 and $K(S_1) \wedge K(S_2) \to K(S_3)$ of 4.6. Waldhausen's Lemma [Wa78, 9.2.6] implies the following result; the details of the implication are given in [We81, 4.3]:

Theorem 7.9. *The homotopy equivalences $B(S_i^{-1}S_i) \to \Omega BQ\mathcal{A}_i$ of theorem 7.1 fit into a diagram which commutes up to basepoint-preserving homotopy:*

$$\begin{array}{ccc}
B(S_1^{-1}S_1) \wedge B(S_2^{-1}S_2) & \xrightarrow{\ \Box\ } & B(S_3^{-1}S_3) \\
\simeq\downarrow & & \downarrow\simeq \\
(\Omega BQ\mathcal{A}_1) \wedge (\Omega BQ\mathcal{A}_2) & \xrightarrow{\ \gamma\ } & (\Omega BQ\mathcal{A}_3) \\
\simeq\downarrow & & \downarrow\simeq \\
\Omega^2(BQ\mathcal{A}_1 \wedge BQ\mathcal{A}_2) & \xrightarrow{\ \Omega^2\otimes\ } & \Omega^2(BQQ\mathcal{A}_3).
\end{array}$$

Hence there are commutative diagrams:

$$\begin{array}{ccc}
K_p(S_1) \otimes K_q(S_2) & \xrightarrow{\ \Box\ } & K_{p+q}(S_3) \\
\cong\downarrow & & \downarrow\cong \\
K_p(\mathcal{A}_1) \otimes K_q(\mathcal{A}_2) & \xrightarrow{\ \otimes\ } & K_{p+q}(\mathcal{A}_3).
\end{array}$$

The middle map γ is induced from the H-space map $\otimes : \Omega BQ\mathcal{A}_1 \times \Omega BQ\mathcal{A}_2 \to \Omega^2 BQ\mathcal{A}_3$ of Ex. 6.10, since it sends $x \otimes 0$ and $0 \otimes y$ to 0.

EXERCISES

7.1 If $\mathcal{A}$ is an additive category, $S = \operatorname{iso}\mathcal{A}$ is equivalent to the disjoint union of one-object categories $\operatorname{Aut}(A)$, one for every isomorphism class in $\mathcal{A}$. Show that the translations $\operatorname{Aut}(A) \to \operatorname{Aut}(A \oplus B)$ are injections. Then conclude using theorem 4.8 that $B(S^{-1}S)$ is the group completion of the H-space $BS = \coprod \operatorname{Aut}(A)$.

7.2 Show that the target functor $t\colon \mathcal{E}\mathcal{A} \to Q\mathcal{A}$ is a fibered functor in the sense of Definition 3.7.3, with base change φ^* given by 7.7. Then show that the action of S on $\mathcal{E}\mathcal{A}$ is cartesian (Ex. 4.11), so that the induced functor $S^{-1}\mathcal{E}\mathcal{A} \to Q\mathcal{A}$ is also fibered, with fiber $S^{-1}S$ over 0.

7.3 Let $iQ\mathcal{A}$ denote the subcategory of $Q\mathcal{A}$ whose objects are those of $\mathcal{A}$ but whose morphisms are admissible monomorphisms. Show that the category $\mathcal{E}\mathcal{A}$ of 7.3 is equivalent to the subdivision category $Sub(iQ\mathcal{A})$ of Ex. 3.9. Conclude that the category $\mathcal{E}\mathcal{A}$ is contractible.

7.4 Show that Quillen's Theorem B *can not* apply to $\mathcal{E}\mathcal{A} \to Q\mathcal{A}$ unless $\mathcal{A} \cong 0$. *Hint:* Compare $\pi_0 S$ to $K_0\mathcal{A}$.

7.5 If φ is the map $0 \rightarrowtail C$, resp. $0 \twoheadleftarrow C$, show that $\varphi^*\colon \mathcal{E}_C \to \mathcal{E}_0 \cong S$ sends $A \rightarrowtail B \twoheadrightarrow C$ to A, resp. to B.

7.6 Describe $\mathcal{E}'\mathcal{A} = (\mathcal{E}\mathcal{A})^{op}$, which is cofibered over $(Q\mathcal{A})^{op}$ by Ex. 3.6 and 7.2. Use $\mathcal{E}'\mathcal{A}$ to prove the $+ = Q$ Theorem 7.1. *Hint:* There is a new action of S. Use pushout instead of pullback in (7.5.1) to prove the analogue of proposition 7.6.

7.7 *Finite Sets.* Let $\mathbf{Sets}_{\text{fin}}$ denote the category of finite pointed sets, and form the category $Q\mathbf{Sets}_{\text{fin}}$ by copying the Q-construction 6.1 as in Ex. 6.15.
(a) Show that there is an extension category $\mathcal{E}'\mathbf{Sets}_{\text{fin}}$, defined as in Ex. 7.6, which is cofibered over $(Q\mathbf{Sets}_{\text{fin}})^{op}$ with $S = \text{iso}\,\mathbf{Sets}_{\text{fin}}$ as the fiber over the basepoint.
(b) Modify the proof of the $+ = Q$ theorem to prove that $\Omega BQ\mathbf{Sets}_{\text{fin}} \simeq S^{-1}S$.
(c) If G is a group, let $\mathcal{F}$ be the category of finitely generated free pointed G-sets, and $Q\mathcal{F}$ as in Ex. 6.16. Using 4.10.1, show that $\Omega BQ\mathcal{F} \simeq S^{-1}S \simeq \mathbb{Z} \times \Omega^\infty S^\infty(BG_+)$.

7.8 ($\pi_1 BQ\mathcal{A}$) Given an object A in $\mathcal{A}$, lift the morphisms (6.2.1) in $Q\mathcal{A}$ to morphisms in $\mathcal{E}\mathcal{A}$, $0 \to \eta_A(0) \leftarrow \eta_0(A)$. Conclude that the isomorphism between $K_0(\mathcal{A}) = \pi_1 BQ\mathcal{A}$ and $K_0(S) = \pi_0(S^{-1}S)$ of theorem 7.1 is the canonical isomorphism of II.7.1.2, identifying $[A]$ with $[A]$.

7.9 ($\pi_2 BQ\mathcal{A}$) Given an automorphism α of an object A in $\mathcal{A}$, consider the continuous map $[0,1]^2 \to BQ\mathcal{A}$ given by the commutative diagram:

$$\begin{array}{ccccc} 0 & \rightarrowtail & A & \twoheadrightarrow & 0 \\ \| & & \alpha\downarrow & & \| \\ 0 & \rightarrowtail & A & \twoheadrightarrow & 0. \end{array}$$

Identifying the top and bottom edges to each other, the fact that the left and right edges map to the basepoint (0) means that we have a continuous function $S^2 \to BQ\mathcal{A}$, *i.e.*, an element $[\alpha]$ of $K_1(\mathcal{A}) = \pi_2(BQ\mathcal{A})$.

(a) Show that $[\alpha] + [\alpha'] = [\alpha\alpha']$ for every pair of composable automorphisms. Conclude that $\alpha \mapsto [\alpha]$ is a homomorphism $\text{Aut}(A) \to K_1(\mathcal{A})$.
(b) If $\beta \in \text{Aut}(B)$, show that the automorphism $\alpha \oplus \beta$ of $A \oplus B$ maps to $[\alpha]+[\beta]$. Using 4.8.1, this given a map from $K_1(\text{iso}\,\mathcal{A})$ to $K_1(\mathcal{A})$.
(c) Finally, lift this diagram to $\mathcal{E}\mathcal{A}$ using Ex. 7.8, representing a map $I^2 \to B\mathcal{E}\mathcal{A}$, and conclude that the isomorphism between $K_1(\mathcal{A}) = \pi_2 BQ\mathcal{A}$ and $K_1(S) = \pi_1(S^{-1}S)$ of theorem 7.1 identifies $[\alpha]$ with the class of α given by III.1.6.3 and 4.8.1.

7.10 (Canonical involution) Let R be a commutative ring. The isomorphism $\mathbf{P}(R) \to \mathbf{P}(R)^{op}$ sending P to $\text{Hom}_R(P,R)$ induces an involution on $Q\mathbf{P}(R)$ and hence $K_*(R)$ by 6.4; it is called the *canonical involution.* Show that the involution is a ring automorphism.

On the other hand, the "transpose inverse" involution of $GL(R)$ ($g \mapsto {}^tg^{-1}$) induces a homotopy involution on $BGL(R)^+$ and an involution on $K_n(R)$ for $n > 0$. Show that these two involutions agree via the '$+ = Q$' theorem 7.2.

§8. Waldhausen's $wS.$ construction

Our last construction of K-theory applies to Waldhausen categories, *i.e.,* "categories with cofibrations and weak equivalences." Unfortunately, this will occur only after a lengthy list of definitions, and we ask the reader to be forgiving.

Recall from chapter II, section 9 that a *category with cofibrations* is a category $\mathcal{C}$ with a distinguished zero object '0' and a subcategory co($\mathcal{C}$) of morphisms in $\mathcal{C}$ called "cofibrations" (indicated with feathered arrows $\rightarrowtail$). Every isomorphism in $\mathcal{C}$ is to be a cofibration, and so are the unique arrows $0 \rightarrowtail A$ for every object A in $\mathcal{C}$. In addition, the pushout $C \rightarrowtail B \cup_A C$ of any cofibration $A \rightarrowtail B$ is a cofibration. (See Definition II.9.1 for more precise statements.) These axioms imply that two constructions make sense: the coproduct $B \amalg C = B \cup_0 C$ of any two objects, and every cofibration $A \rightarrowtail B$ fits into a *cofibration sequence* $A \rightarrowtail B \twoheadrightarrow B/A$, where B/A is the cokernel of $A \rightarrowtail B$. The following is a restatement of Definition II.9.1.1:

Definition 8.1. A *Waldhausen category* $\mathcal{C}$ is a category with cofibrations, together with a family $w(\mathcal{C})$ of morphisms in $\mathcal{C}$ called "weak equivalences" (indicated with decorated arrows $\xrightarrow{\sim}$). Every isomorphism in $\mathcal{C}$ is to be a weak equivalence, and weak equivalences are to be closed under composition (so we may regard $w(\mathcal{C})$ as a subcategory of $\mathcal{C}$). In addition, the "Glueing axiom" (W3) must be satisfied, which says that the pushout of weak equivalences is a weak equivalence (see II.9.1).

A functor $f : \mathcal{A} \to \mathcal{C}$ between two Waldhausen categories is called an *exact functor* if it preserves all the relevant structure: zero, cofibrations, weak equivalences and the pushouts along a cofibration.

A *Waldhausen subcategory* $\mathcal{A}$ of a Waldhausen category $\mathcal{C}$ is a subcategory which is also a Waldhausen category in such a way that: (i) the inclusion $\mathcal{A} \subseteq \mathcal{C}$ is an exact functor, (ii) the cofibrations in $\mathcal{A}$ are the maps in $\mathcal{A}$ which are cofibrations in $\mathcal{C}$ and whose cokernel lies in $\mathcal{A}$, and (iii) the weak equivalences in A are the weak equivalences of $\mathcal{C}$ which lie in $\mathcal{A}$.

In order to describe Waldhausen's $wS.$ construction for K-theory, we need a sequence of Waldhausen categories $S_n\mathcal{C}$. $S_0\mathcal{C}$ is the zero category, and $S_1\mathcal{C}$ is the category $\mathcal{C}$, but whose objects A are thought of as the cofibrations $0 \rightarrowtail A$. The category $S_2\mathcal{C}$ is the extension category $\mathcal{E}$ of II.9.3. For convenience, we repeat its definition here.

Extension Categories 8.2. The objects of the extension category $S_2\mathcal{C}$ are the cofibration sequences $A_1 \rightarrowtail A_2 \twoheadrightarrow A_{12}$ in $\mathcal{C}$. A morphism $E \to E'$ in $S_2\mathcal{C}$ is a commutative diagram:

$$\begin{array}{cccccc} E: & A_1 & \rightarrowtail & A_2 & \twoheadrightarrow & A_{12} \\ \downarrow & \downarrow u_1 & & \downarrow u_2 & & \downarrow u_3 \\ E': & A_1' & \rightarrowtail & A_2' & \twoheadrightarrow & A_{12}' \end{array}$$

We make $S_2\mathcal{C}$ into a Waldhausen category as follows. A morphism $E \to E'$ in $S_2\mathcal{C}$ is a cofibration if $A_1 \to A_1'$, $A_{12} \to A_{12}'$ and $A_1' \cup_{A_1} A_2 \to A_2'$ are cofibrations in $\mathcal{C}$. A morphism in $S_2\mathcal{C}$ is a weak equivalence if its component maps $u_i : A_i \to A_i'$ $(i = 1, 2, 12)$ are weak equivalences in $\mathcal{C}$.

A Waldhausen category $\mathcal{C}$ is called *extensional* if it satisfies the following technically convenient axiom: weak equivalences are "closed under extensions."

EXTENSION AXIOM 8.2.1. Suppose that $f\colon E \to E'$ is a map between cofibration sequences, as in 8.2. If the source and quotient maps of f ($A \to A'$ and $C \to C'$) are weak equivalences, so is the total map of f ($B \to B'$).

DEFINITION 8.3. ($S.\mathcal{C}$) If $\mathcal{C}$ is a category with cofibrations, let $S_n\mathcal{C}$ be the category whose objects $A.$ are sequences of n cofibrations in $\mathcal{C}$:

$$A.: \quad 0 = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$$

together with a choice of every subquotient $A_{ij} = A_j/A_i$ ($0 < i < j \leq n$). These choices are to be compatible in the sense that there is a commutative diagram:

$$
\begin{array}{ccccccc}
 & & & & & & A_{n-1,n} \\
 & & & & & & \uparrow \\
 & & & & & & \cdots \\
 & & & & A_{23} & \rightarrowtail \cdots \rightarrowtail & A_{2n} \\
 & & & & \uparrow & & \uparrow \\
 & & A_{12} & \rightarrowtail & A_{13} & \rightarrowtail \cdots \rightarrowtail & A_{1n} \\
 & & \uparrow & & \uparrow & & \uparrow \\
A_1 & \rightarrowtail & A_2 & \rightarrowtail & A_3 & \rightarrowtail \cdots \rightarrowtail & A_n
\end{array}
\tag{8.3.0}
$$

The conventions $A_{0j} = A_j$ and $A_{jj} = 0$ will be convenient at times. A morphism $A. \to B.$ in $S_n\mathcal{C}$ is a natural transformation of sequences.

If we forget the choices of the subquotients A_{ij} we obtain the higher extension category $\mathcal{E}_n(\mathcal{C})$ constructed in II.9.3.2. Since we can always make such choices, it follows that the categories $S_n\mathcal{C}$ and $\mathcal{E}_n(\mathcal{C})$ are equivalent. By Ex. II.9.4, when $\mathcal{C}$ is a Waldhausen category, so is $\mathcal{E}_n(\mathcal{C})$ and hence $S_n\mathcal{C}$. Here are the relevant definitions for S_n, translated from the definitions II.9.3.2 for $\mathcal{E}_n$.

A weak equivalence in $S_n\mathcal{C}$ is a map $A. \to B.$ such that each $A_i \to B_i$ (hence, each $A_{ij} \to B_{ij}$) is a weak equivalence in $\mathcal{C}$. A map $A. \to B.$ is a cofibration when for every $0 \leq i < j < k \leq n$ the map of cofibration sequences

$$
\begin{array}{ccccc}
A_{ij} & \rightarrowtail & A_{ik} & \twoheadrightarrow & A_{jk} \\
\downarrow & & \downarrow & & \downarrow \\
B_{ij} & \rightarrowtail & B_{ik} & \twoheadrightarrow & B_{jk}
\end{array}
$$

is a cofibration in $S_2\mathcal{C}$.

The reason for including choices in the definition of the categories $S_n\mathcal{C}$ is that we can form a simplicial Waldhausen category. The maps ∂_0, ∂_1 from $\mathcal{C} = S_1\mathcal{C}$ to $0 = S_0\mathcal{C}$ are trivial; the maps $\partial_0, \partial_1, \partial_2$ from $S_2\mathcal{C}$ to $\mathcal{C}$ are q_*, t_* and s_*, respectively.

DEFINITION 8.3.1. For each $n \geq 0$, the exact functor $\partial_0\colon S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ is defined by deletion of the bottom row of (8.3.0). That is, ∂_0 is defined by the formula

$$\partial_0(A.)\colon \quad 0 = A_{11} \rightarrowtail A_{12} \rightarrowtail A_{13} \rightarrowtail \cdots \rightarrowtail A_{1n}$$

together with the choices $\partial_0(A.)_{ij} = A_{i+1,j+1}$. By Ex. 8.1, $\partial_0(A.)$ is in $S_{n-1}\mathcal{C}$.

For $0 < i \leq n$ we define the exact functors $\partial_i\colon S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ by omitting the row A_{i*} and the column containing A_i in (8.3.0), and reindexing the A_{jk} as needed. Similarly, we define the exact functors $s_i\colon S_n\mathcal{C} \to S_{n+1}\mathcal{C}$ by duplicating A_i, and reindexing with the normalization $A_{i,i+1} = 0$. (Exactness is checked in Ex. 8.2.)

By Ex. 8.2, the $S_n\mathcal{C}$ fit together to form a simplicial Waldhausen category $S.(\mathcal{C})$, and the subcategories $wS_n\mathcal{C}$ of weak equivalences fit together to form a simplicial category $wS.\mathcal{C}$. Hence their geometric realizations $B(wS_n\mathcal{C})$ fit together to form a simplicial topological space $BwS.\mathcal{C}$, and we write $|wS.\mathcal{C}|$ for the realization of $BwS.\mathcal{C}$. Since $S_0\mathcal{C}$ is trivial, $|wS.\mathcal{C}|$ is a connected space.

REMARK 8.3.2. In the realization of $BwS.\mathcal{C}$, the spaces $B(wS_n\mathcal{C})\times\Delta^n$ are glued together along the face maps. In particular, the suspension $\Sigma B(w\mathcal{C})$ is a subspace of $|wS.\mathcal{C}|$; the adjoint map is $B(w\mathcal{C})\to\Omega|wS.\mathcal{C}|$. In this way, each object of $\mathcal{C}$ yields an element of $\pi_1|wS.\mathcal{C}|$, and each weak equivalence $A\simeq A$ in $\mathcal{C}$ yields an element of $\pi_2|wS.\mathcal{C}|$.

Recall from chapter II, 9.1.2, that $K_0(\mathcal{C})$ is defined as the group generated by the set of weak equivalence classes [A] of objects of $\mathcal{C}$ with the relations that $[B]=[A]+[B/A]$ for every cofibration sequence

$$A \rightarrowtail B \twoheadrightarrow B/A.$$

PROPOSITION 8.4. *If $\mathcal{C}$ is a Waldhausen category then $\pi_1|wS.\mathcal{C}|\cong K_0(\mathcal{C})$.*

PROOF. If $X.$ is any simplicial space with X_0 a point, then $|X.|$ is connected and $\pi_1|X.|$ is the free group on $\pi_0(X_1)$ modulo the relations $\partial_1(x)=\partial_2(x)\partial_0(x)$ for every $x\in\pi_0(X_2)$. For $X.=BwS.\mathcal{C}$, $\pi_0(BwS_1\mathcal{C})$ is the set of weak equivalence classes of objects in $\mathcal{C}$, $\pi_0(BwS_2\mathcal{C})$ is the set of equivalence classes of cofibration sequences, and the maps $\partial_i\colon S_2\mathcal{C}\to S_1\mathcal{C}$ of 8.3.1 send $A\rightarrowtail B\twoheadrightarrow B/A$ to B/A, B and A, respectively. □

DEFINITION 8.5. If $\mathcal{C}$ is a small Waldhausen category, its *algebraic K-theory space* $K(\mathcal{C})=K(\mathcal{C},w)$ is the loop space

$$K(\mathcal{C})=\Omega|wS.\mathcal{C}|.$$

The K-groups of $\mathcal{C}$ are defined to be its homotopy groups:

$$K_i(\mathcal{C})=\pi_iK(\mathcal{C})=\pi_{i+1}|wS.\mathcal{C}| \qquad \text{if } i\geq 0.$$

We saw in Remark 8.3.2, there is a canonical map $B(w\mathcal{C})\to K(\mathcal{C})$.

REMARK 8.5.1. Since the subcategory $w\mathcal{C}$ is closed under coproducts in $\mathcal{C}$ by axiom (W3), the coproduct gives an H-space structure to $|wS.\mathcal{C}|$ via the map

$$|wS.\mathcal{C}|\times|wS.\mathcal{C}|\cong|wS.\mathcal{C}\times wS.\mathcal{C}|\xrightarrow{\amalg}|wS.\mathcal{C}|.$$

SIMPLICIAL MODEL 8.5.2. Suppose that $\mathcal{C}$ is a small Waldhausen category in which the isomorphisms $i\mathcal{C}$ are the weak equivalences. Let $s_n\mathcal{C}$ denote the set of objects of $S_n\mathcal{C}$; as n varies, we have a simplicial set $s.\mathcal{C}$. Waldhausen proved in [W1126, 1.4] that the inclusion $|s.\mathcal{C}|\to|iS.\mathcal{C}|$ is a homotopy equivalence. Therefore $\Omega|s.\mathcal{C}|$ is a simplicial model for the space $K(\mathcal{C})$.

RELATIVE K-THEORY SPACES 8.5.3. If $f : \mathcal{B} \to \mathcal{C}$ is an exact functor, let $S_n f$ denote the category $S_n\mathcal{B} \times_{S_n\mathcal{C}} S_{n+1}\mathcal{C}$ whose objects are pairs

$$(B_*, C_*) = (B_1 \rightarrowtail \cdots \rightarrowtail B_n, C_0 \rightarrowtail \cdots \rightarrowtail C_n)$$

such that $f(B_*)$ is $\partial_0 C_* : C_1/C_0 \rightarrowtail \cdots \rightarrowtail C_n/C_0$. Each $S_n f$ is a Waldhausen category in a natural way, containing $\mathcal{C}$ as the (Waldhausen) subcategory of all $(0, C = \cdots = C)$, and the projection $S_n f \to S_n\mathcal{B}$ is exact. We can apply the $S.$ (and $wS.$) construction degreewise to the sequence $\mathcal{C} \to S.f \to S.\mathcal{B}$ of simplicial Waldhausen categories, obtaining a sequence of bisimplicial Waldhausen categories $S.\mathcal{C} \to S.(S.f) \to S.(S.\mathcal{B})$, and a sequence $wS.\mathcal{C} \to wS.(S.f) \to wS.(S.\mathcal{B})$ of bisimplicial categories. We will see in V.1.7 (using 8.5.4) that the realization of the bisimplicial category sequence

$$wS.\mathcal{B} \to wS.\mathcal{C} \to wS.(S.f) \to wS.(S.\mathcal{B}),$$

is a homotopy fibration sequence. Thus we may regard $K(f) = \Omega^2|wS.(S.f)|$ as a relative K-theory space; the groups $K_n(f) = \pi_n K(f)$ fit into a long exact sequence involving $f_* : K_n(\mathcal{B}) \to K_n(\mathcal{C})$, ending $K_0(\mathcal{B}) \to K_0(\mathcal{C}) \to K_{-1}(f) \to 0$ (Ex. 8.11).

LEMMA 8.5.4. *If $f : \mathcal{C} \to \mathcal{C}$ is the identity, $wS.f$ is contractible.*

PROOF. In this case the simplicial category $S.f$ is just the simplicial path space construction of $S.\mathcal{C}$, and $wS.S.f$ is the simplicial path space construction of $wS.S.\mathcal{C}$ (see [WHomo, 8.3.14]). These are contractible since $S_0 f = 0$ and $wS.S_0 f$ are. □

INFINITE LOOP STRUCTURE 8.5.5. Lemma 8.5.4 implies that there are natural homotopy equivalences $|wS.\mathcal{C}| \simeq \Omega|wS.S.\mathcal{C}|$, and of course $K(\mathcal{C}) \simeq \Omega^2|wS.S.\mathcal{C}|$. In fact $K(\mathcal{C})$ is an infinite loop space.

To see this we just iterate the construction, forming the multisimplicial Waldhausen categories $S.^n\mathcal{C} = S.S. \cdots S.\mathcal{C}$ and the multisimplicial categories $wS.^n\mathcal{C}$ of their weak equivalences. By 8.5.4, we see that $|wS.^n\mathcal{C}|$ is the loop space of $|wS.^{n+1}\mathcal{C}|$, and that the sequence of spaces

$$\Omega|wS.\mathcal{C}|, |wS.\mathcal{C}|, |wS.S.\mathcal{C}|, \dots, \Omega|wS.^n\mathcal{C}|, \dots$$

forms a connective Ω-spectrum $\mathbf{K}\mathcal{C}$, called the *K-theory spectrum of $\mathcal{C}$*. Many authors think of the K-theory of $\mathcal{C}$ in terms of this spectrum. This does not affect the K-groups, because:

$$\pi_i(\mathbf{K}\mathcal{C}) = \pi_i K(\mathcal{C}) = K_i(\mathcal{C}), \qquad i \geq 0.$$

An exact functor f induces a map $f_* \colon K(\mathcal{B}) \to K(\mathcal{C})$ of spaces, and spectra, and of their homotopy groups $K_i(\mathcal{B}) \to K_i(\mathcal{C})$.

EXACT CATEGORIES 8.6. We saw in II.9.1.3 that any exact category $\mathcal{A}$ becomes a Waldhausen category in which the cofibration sequences are just the admissible exact sequences, and the weak equivalences are just the isomorphisms. We write $i(\mathcal{A})$ for the family of isomorphisms, so that we can form the K-theory space $K(\mathcal{A}) = \Omega|iS.\mathcal{A}|$. Waldhausen proved in [W1126, 1.9] that there is a homotopy equivalence between $|iS.\mathcal{A}|$ and $BQ\mathcal{A}$, so that this definition is consistent with the definition of $K(\mathcal{A})$ in definition 6.3. His proof is given in exercises 8.5 and 8.6 below.

Another important example of a Waldhausen category is $\mathcal{R}_f(X)$, introduced in II.9.1 and Ex. II.9.1. The so-called *K-theory of spaces* refers to the corresponding K-theory spaces $A(X)$, and their homotopy groups $A_n(X) = \pi_n A(X)$.

EXAMPLE 8.7 $\big(A(*)\big)$. Recall from II.9.1.4 that the category $\mathcal{R}_f = \mathcal{R}_f(*)$ of finite based CW complexes is a Waldhausen category in which the family $h\mathcal{R}_f$ of weak equivalences is the family of weak homotopy equivalences. This category is saturated (II.9.1.1) and satisfies the extension axiom 8.2.1. Following Waldhausen [W1126], we write $A(*)$ for the space $K(\mathcal{R}_f) = \Omega|hS.\mathcal{R}_f|$. We have $A_0(*) = K_0\mathcal{R}_f = \mathbb{Z}$ by II.9.1.5.

EXAMPLE 8.7.1 $\big(A(X)\big)$. More generally, let X be a CW complex. The category $\mathcal{R}(X)$ of CW complexes Y obtained from X by attaching cells, and having X as a retract, is a Waldhausen category in which cofibrations are cellular inclusions (fixing X) and weak equivalences are homotopy equivalences (see Ex. II.9.1). Consider the Waldhausen subcategory $\mathcal{R}_f(X)$ of those Y obtained by attaching only finitely many cells. Following Waldhausen [W1126], we write $A(X)$ for the space $K(\mathcal{R}_f(X)) = \Omega|hS.\mathcal{R}_f(X)|$. Thus $A_0(X) = K_0\mathcal{R}_f(X)$ is $\mathbb{Z}$ by Ex. II.9.1.

Similarly, we can form the Waldhausen subcategory $\mathcal{R}_{fd}(X)$ of those Y which are finitely dominated. We write $A^{fd}(X)$ for $K(\mathcal{R}_{fd}(X)) = \Omega|hS.\mathcal{R}_{fd}(X)|$. Note that $A_0^{fd}(X) = K_0\mathcal{R}_{fd}(X)$ is $\mathbb{Z}[\pi_1(X)]$ by Ex. II.9.1.

Cylinder Functors

When working with Waldhausen categories, it is often technically convenient to have mapping cylinders. Recall from Ex. 3.12 that the category $\mathcal{C}/\mathcal{C}$ of arrows in $\mathcal{C}$ has the morphisms of $\mathcal{C}$ as its objects, and a map $(a,b)\colon f \to f'$ in $\mathcal{C}/\mathcal{C}$ is a commutative diagram in $\mathcal{C}$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{f'} & B' \end{array} \tag{8.8.0}$$

The source $s(f) = A$ and target $t(f) = B$ of f define functors $s, t\colon \mathcal{C}/\mathcal{C} \to \mathcal{C}$.

DEFINITION 8.8 (CYLINDERS). Let $\mathcal{C}$ be a Waldhausen category. A (mapping) *cylinder functor* on $\mathcal{C}$ is a functor T from the category $\mathcal{C}/\mathcal{C}$ of arrows in $\mathcal{C}$ to the category $\mathcal{C}$, together with natural transformations $j_1\colon s \Rightarrow T$, $j_2\colon t \Rightarrow T$ and $p\colon T \Rightarrow t$ so that for every $f\colon A \to B$ the diagram

$$\begin{array}{ccccc} A & \xrightarrow{j_1} & T(f) & \xleftarrow{j_2} & B \\ & f\searrow & \downarrow p & \swarrow = & \\ & & B & & \end{array}$$

commutes in $\mathcal{C}$. The following conditions must also hold:

(i) $T(0 \rightarrowtail A) = A$, with p and j_2 the identity map, for all $A \in \mathcal{C}$.
(ii) $j_1 \amalg j_2 \colon A \amalg B \rightarrowtail T(f)$ is a cofibration for all $f \colon A \to B$.
(iii) Given a map $(a, b) \colon f \to f'$ in $\mathcal{C}/\mathcal{C}$, *i.e.,* a commutative square (8.8.0), if a and b are weak equivalences in $\mathcal{C}$ then so is $T(f) \to T(f')$.
(iv) Given a map $(a, b) \colon f \to f'$ in $\mathcal{C}/\mathcal{C}$, if a and b are cofibrations in $\mathcal{C}$, then so is $T(f) \to T(f')$, and the following map, induced by condition (ii), is also a cofibration in $\mathcal{C}$.

$$A' \amalg_A T(f) \amalg_B B' \to T(f')$$

We often impose the following extra axiom on the weak equivalences of $\mathcal{C}$.

Cylinder Axiom 8.8.1. All maps $p : T(f) \to B$ are weak equivalences in $\mathcal{C}$.

Suppose $\mathcal{C}$ has a cylinder functor T. The *cone* of an object A is $\text{cone}(A) = T(A \twoheadrightarrow 0)$, and the *suspension* of A is $\Sigma A = \text{cone}(A)/A$. The cylinder axiom implies that $\text{cone}(A) \xrightarrow{\sim} 0$ is a weak equivalence. Since $A \rightarrowtail \text{cone}(A) \twoheadrightarrow \Sigma A$ is a cofibration sequence it follows from the description of $K_0(\mathcal{C})$ in II.9.1.2 that $[\Sigma A] = -[A]$ in $K_0(\mathcal{C})$. (Cf. Lemma II.9.2.1.) In fact, the Additivity Theorem (see V.1.2 below) implies that the map $\Sigma \colon K(\mathcal{C}) \to K(\mathcal{C})$ is a homotopy inverse with respect to the H-space structure on $K(\mathcal{C})$, because $\Sigma_* + 1 = \text{cone}_* = 0$.

The name 'cylinder functor' comes from the following two paradigms.

Example 8.8.2. The Waldhausen categories $\mathcal{R}_f(*)$ and $\mathcal{R}_f(X)$ of examples 8.7 and 8.7.1 have a cylinder functor: $T(f)$ is the usual (based) mapping cylinder of f. By construction, the mapping cylinder satisfies the cylinder axiom 8.8.1. Because of this paradigm, j_1 and j_2 are sometimes called the *front* and *back* inclusions.

Example 8.8.3. Let **Ch** be the Waldhausen category of chain complexes and quasi-isomorphisms constructed from an abelian (or exact) category $\mathcal{C}$; see II.9.2. The mapping cylinder of $f \colon A. \to B.$ is the usual mapping cylinder chain complex [WHomo, 1.5.5], in which

$$T(f)_n = A_n \oplus A_{n-1} \oplus B_n.$$

The suspension functor $\Sigma(A.) = A.[-1]$ here is the shift operator: $\Sigma(A.)_n = A_{n-1}$.

Example 8.8.4. Exact categories usually do not have cylinder functors. This is reflected by the fact that for some $A \in \mathcal{A}$ there may be no B such that $[A \oplus B] = 0$ in $K_0(\mathcal{A})$. However, the Waldhausen category $\mathbf{Ch}^b(\mathcal{A})$ of bounded chain complexes does have a cylinder functor, and we used it to prove that $K_0(\mathcal{A}) \cong K_0\mathbf{Ch}^b(\mathcal{A})$ in II.9.2.2. In fact, $K(\mathcal{A}) \simeq K(\mathbf{Ch}^b(\mathcal{A}))$ by the Gillet-Waldhausen theorem presented in V.2.2. Thus many results requiring mapping cylinders in Waldhausen K-theory can be translated into results for Quillen K-theory.

Cofinality

A Waldhausen subcategory $\mathcal{B}$ of $\mathcal{C}$ is said to be *cofinal* if for all C in $\mathcal{C}$ there is a C' in $\mathcal{C}$ so that $C \amalg C'$ is in $\mathcal{B}$. The K_0 version of the following theorem was proven in II.9.4. We will prove a stronger cofinality result in chapter V.

WALDHAUSEN COFINALITY 8.9. *If $\mathcal{B}$ is a cofinal Waldhausen subcategory of $\mathcal{C}$, closed under extensions, and such that $K_0(\mathcal{B}) = K_0(\mathcal{C})$. Then $wS.\mathcal{B} \to wS.\mathcal{C}$ and $K(\mathcal{B}) \to K(\mathcal{C})$ are homotopy equivalences. In particular, $K_n(\mathcal{B}) \cong K_n(\mathcal{C})$ for all n.*

REMARK 8.9.1. By Grayson's Trick (Ex. II.9.14), the assumption that $K_0(\mathcal{B}) = K_0(\mathcal{C})$ is equivalent to saying that $\mathcal{B}$ is *strictly cofinal* in $\mathcal{C}$, meaning that for every C in $\mathcal{C}$ there is a B in $\mathcal{B}$ so that $B \coprod C$ is in $\mathcal{B}$.

PROOF. By 8.5.3, it suffices to show that the "relative" bisimplicial category $wS.(S.f)$ is contractible, where $f : \mathcal{B} \to \mathcal{C}$ is the inclusion. For this it suffices to show that each $wS_n(S.f)$ is contractible. Switching simplicial directions, we can rewrite $wS_n(S_m f)$ as $wS_m(S_n f_n)$, where $f_n : S_n\mathcal{B} \to S_n\mathcal{C}$ and $S_n f_n$ is defined in 8.5.3. Since $S_n\mathcal{B}$ is equivalent to $\mathcal{E}_n(\mathcal{C})$ (see 8.3), we see from Ex. II.9.4 that $K_0(S_n\mathcal{B}) \cong K_0(S_n\mathcal{C})$. Hence the hypothesis also applies to f_n. Replacing f by f_n, we have a second reduction: it suffices to show that the simplicial category $wS.f$ is contractible.

Let $\mathcal{B}(m, w)$ denote the category of diagrams $B_0 \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} B_m$ in $\mathcal{B}$ whose maps are weak equivalences, and $f_{(m,w)}$ the inclusion of $\mathcal{B}(m, w)$ in $\mathcal{C}(m, w)$. Then the bidegree (m, n) part $w_m S_n f$ of $wS.f$ is the set $s_n f_{(m,w)}$ of objects of $S_n f_{(m,w)}$. Working with the nerve degreewise, it suffices to show that each $w_m S.f = s.f_{(m,w)}$ is contractible. Since $\mathcal{B}$ is strictly cofinal in $\mathcal{C}$ (by Grayson's trick), this implies that $f_{(m,w)}$ is also strictly cofinal by Ex. 8.12(b). The theorem now follows from Lemma 8.9.2 below. □

LEMMA 8.9.2. *If $f : \mathcal{B} \to \mathcal{C}$ is strictly cofinal then $s.f$ is contractible, where the elements of $s_n f$ are the objects of $S_n f$.*

PROOF. Strict cofinality implies that for each finite set X of objects (B^i_*, C^i_*) of $S_{n_i} f$, there is an object B' of $\mathcal{B}$ such that each $(B' \amalg B^i_*, B' \amalg C^i_*)$ is in $S_{n_i}\mathrm{id}_{\mathcal{B}}$, because each $B' \amalg C^i_j$ is in $\mathcal{B}$.

We saw in 8.5.4 that $s.\mathrm{id}_{\mathcal{B}}$ is the simplicial path space construction of $s.\mathcal{B}$, and is contractible because $s_0\mathcal{B}$ is a point. We will show that $s.f$ is contractible by showing that it is homotopy equivalent to $s.\mathrm{id}_{\mathcal{B}}$. For this we need to show that for any finite subcomplex L of $s.f$ there is a simplicial homotopy h (in the sense of [WHomo, 8.3.11]) from the inclusion $L \subset s.f$ to a map $L \to s.\mathrm{id}_{\mathcal{B}} \subset s.f$, such that each component of h sends $L \cap s.\mathrm{id}_{\mathcal{B}}$ into $s.\mathrm{id}_{\mathcal{B}}$.

If X is the set of nondegenerate elements of L, we saw above that there is a B' so that $B' \amalg X$ (and hence $B' \amalg L$) is in $s.\mathrm{id}_{\mathcal{B}}$. The desired simplicial homotopy is given by the restriction of the maps $h_i : s_n f \to s_{n+1} f$, sending (B_*, C_*) to

$$(\cdots \rightarrowtail B_j \rightarrowtail B'\amalg B_j \rightarrowtail \cdots \rightarrowtail B'\amalg B_n,\ \cdots \rightarrowtail C_j \rightarrowtail B'\amalg C_j \rightarrowtail \cdots \rightarrowtail B'\amalg C_n). \quad \square$$

QUESTION 8.9.3. If $\mathcal{B}$ is a cofinal Waldhausen subcategory of $\mathcal{C}$, but is not closed under extensions, is $K(\mathcal{B}) \simeq K(\mathcal{C})$? Using Ex. 8.12(a), the above proof shows that this is true if $\mathcal{B}$ is strictly cofinal in $\mathcal{C}$.

At the other extreme of cofinality, we have the following theorem of Thomason, which shows that by changing the weak equivalences in $\mathcal{A}$ we can force all the higher K-groups to vanish. Let $(\mathcal{A}, \mathrm{co})$ be any category with cofibrations; recall from II.9.1.3 that the group $K_0(\mathcal{A}) = K_0(\mathrm{iso}\,\mathcal{A})$ is defined in this context.

Suppose we are given a surjective homomorphism $\pi : K_0(\mathcal{A}) \to G$. Let $w(\mathcal{A})$ denote the family of morphisms $A \to A'$ in $\mathcal{A}$ such that $\pi[A] = \pi[A']$ in G. As observed in II.9.6.2, $(\mathcal{A}, w)$ is a Waldhausen category with $K_0(\mathcal{A}, w) = G$.

THEOREM 8.10. *There is a homotopy equivalence $wS.(\mathcal{A}, w) \to BG$. Hence $K(\mathcal{A})$ is homotopic to the discrete set G, and $K_n(\mathcal{A}, w) = 0$ for all $n \neq 0$.*

PROOF. (Thomason) By construction of w, the category $w\mathcal{A}$ is the disjoint union of the full subcategories $\pi^{-1}(g)$ on the objects A with $\pi[A] = g$. For each g, fix an object A_g with $\pi[A_g] = g$. For $n > 1$, consider the function $\pi : ws_n\mathcal{A} \to G^n$ sending the object $A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$ of $wS_n\mathcal{A}$ to $(\pi[A_1], \pi[A_{12}], \pi[A_{23}], \dots, \pi[A_{n-1,n}])$. By the construction of w, it induces a decomposition of $wS_n\mathcal{A}$ into the disjoint union (indexed by G^n) of the full subcategories $\pi^{-1}(g_1, ..., g_n)$ of objects mapping to $(g_1, ..., g_n)$. We will show that each of these components is contractible.

Given $\mathbf{g} = (g_1, ..., g_n)$, $\pi^{-1}(\mathbf{g})$ is not empty because it contains the object

$$A_{\mathbf{g}} : A_{g_1} \rightarrowtail (A_{g_1} \amalg A_{g_2}) \rightarrowtail (A_{g_1} \amalg A_{g_2} \amalg A_{g_3}) \rightarrowtail \cdots \rightarrowtail (\amalg_{i=1}^n A_{g_i})$$

of $wS_n\mathcal{A}$. The subcategory $\pi^{-1}(0)$ is contractible because it has initial object 0. For other $\mathbf{g}$, there is a natural transformation from the identity of $\pi^{-1}(\mathbf{g})$ to the functor $F(B) = A_{\mathbf{g}} \amalg A_{-\mathbf{g}} \amalg B$, given by the coproduct with the weak equivalence $0 \to A_{\mathbf{g}} \amalg A_{-\mathbf{g}}$. But F is null-homotopic because it factors as the composite of $F' : \pi^{-1}(\mathbf{g}) \to \pi^{-1}(0)$, $F'(B) = A_{-\mathbf{g}} \amalg B$, and $F'' : \pi^{-1}(0) \to \pi^{-1}\mathbf{g}$, $F''(C) = A_{\mathbf{g}} \amalg C$. It follows that $\pi^{-1}(\mathbf{g})$ is contractible, as desired. $\square$

Products

8.11 Our discussion in 6.6 about products in exact categories carries over to the Waldhausen setting. The following construction is taken from [Wa1126, just after 1.5.3]. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Waldhausen categories; recall from II.9.5.2 that a functor $F \colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is *biexact* if each $F(A, -)$ and $F(-, B)$ is exact, and the following condition is satisfied:

For every pair of cofibrations ($A \rightarrowtail A'$ in $\mathcal{A}$, $B \rightarrowtail B'$ in $\mathcal{B}$) the following map is a cofibration in $\mathcal{C}$:

$$F(A', B) \cup_{F(A,B)} F(A, B') \rightarrowtail F(A', B').$$

We saw in II.9.5.1 that a biexact functor induces a bilinear map $K_0(\mathcal{A}) \otimes K_0(\mathcal{B}) \to K_0(\mathcal{C})$. It also induces a morphism of bisimplicial bicategories

$$wS.\mathcal{A} \times wS.\mathcal{B} \to wwS.S.\mathcal{C}$$

which resembles (6.6.1). Upon passage to geometric realization, this factors

$$K(\mathcal{A}) \wedge K(\mathcal{B}) \to K(\mathcal{C}).$$

EXERCISES

8.1 Show that for every $0 \le i < j < k \le n$ the diagram $A_{ij} \rightarrowtail A_{ik} \twoheadrightarrow A_{jk}$ is a cofibration sequence, and that this gives an exact functor from $S_n\mathcal{C}$ to $S_2\mathcal{C}$.

8.2 Show that each functor $\partial_i \colon S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ is exact in 8.3.1. Then show that $S.\mathcal{C}$ is a simplicial category.

8.3 Let $f, f' \colon \mathcal{A} \to \mathcal{B}$ be exact functors. A natural transformation $\eta \colon f \to f'$ is called a *weak equivalence* if each $f(A) \xrightarrow{\sim} f'(A)$ is a weak equivalence in $\mathcal{B}$. Show that a weak equivalence induces a homotopy between the two maps $K(\mathcal{A}) \to K(\mathcal{B})$. *Hint:* Show that the maps $wS_n\mathcal{A} \to wS_n\mathcal{B}$ are homotopic in a compatible way.

8.4 We saw in 8.3.2 that there is a canonical map from $Bw(\mathcal{C})$ to $K(\mathcal{C})$, and hence maps $\pi_i B(w\mathcal{C}) \to K_i(\mathcal{C})$. The map $\pi_0 B(w\mathcal{C}) \to K_0(\mathcal{C})$ is described in 8.4.

(a) Every weak self-equivalence $\alpha \colon A \xrightarrow{\sim} A$ determines an element $[\alpha]$ of $K_1(\mathcal{C})$, by 3.4. If β is a weak self-equivalence of B, show that $[\alpha] + [\beta] = [\alpha \vee \beta]$. If $A = B$, show that $[\alpha] + [\beta] = [\beta\alpha]$

(b) If $\mathcal{A}$ is an exact category, considered as a Waldhausen category, show that the map $B(\mathrm{iso}\,\mathcal{A}) \to K(\mathcal{A})$ induces a map from the group $K_1^{\oplus}\mathcal{A}$ of 4.8.1 to $K_1(\mathcal{A})$.

(c) In the notation of 8.2, show that a weak equivalence in $S_2\mathcal{C}$ with $A_i = A'_i$ determines a relation $[u_1] - [u_2] + [u_{12}] = 0$ in $K_1(\mathcal{C})$.

(d) Show that every pair of cofibration sequences $A \rightarrowtail B \twoheadrightarrow C$ (with the same objects) determines an element of $K_2(\mathcal{C})$.

8.5 (Waldhausen) Let $\mathcal{A}$ be a small exact category. In this exercise we produce a map from $|iS.\mathcal{A}| \simeq |s.\mathcal{A}|$ to $BQ\mathcal{A}$, where $s.\mathcal{A}$ is defined in 8.5.2.

(a) Show that an object $A.$ of $iS_3\mathcal{A}$ determines a morphism in $Q\mathcal{A}$ from A_{12} to A_3.

(b) Show that an object $A.$ of $iS_5\mathcal{A}$ determines a sequence $A_{23} \to A_{14} \to A_5$ of row morphisms in $Q\mathcal{A}$.

(c) Recall from Ex. 3.10 that the Segal subdivision $Sub(s.\mathcal{A})$ is homotopy equivalent to $s.\mathcal{A}$. Show that (a) and (b) determine a simplicial map $Sub(s.\mathcal{A}) \to Q\mathcal{A}$. Composing with $|iS.\mathcal{A}| \simeq Sub(s.\mathcal{A})$, this yields a map from $|iS.\mathcal{A}|$ to $BQ\mathcal{A}$.

8.6 We now show that the map $|iS.\mathcal{A}| \to BQ\mathcal{A}$ constructed in the previous exercise is a homotopy equivalence. Let $iQ_n\mathcal{A}$ denote the category whose objects are the degree n elements of the nerve of $Q\mathcal{A}$, *i.e.,* sequences $A_0 \to \cdots \to A_n$ in $Q\mathcal{A}$, and whose morphisms are isomorphisms.

(a) Show that $iQ.\mathcal{A}$ is a simplicial category, and that the nerve of $Q\mathcal{A}$ is the simplicial set of objects. Waldhausen proved in [W1126, 1.6.5] that $BQ\mathcal{A} \to |iQ.\mathcal{A}|$ is a homotopy equivalence.

(b) Show that for each n there is an equivalence of categories $Sub(iS_n\mathcal{A}) \xrightarrow{\sim} iQ_n\mathcal{A}$, where $Sub(iS_n\mathcal{A})$ is the Segal subdivision category of Ex. 3.9. Then show that the equivalences form a map of simplicial categories $Sub(iS.\mathcal{A}) \to iQ.\mathcal{A}$. This map must be a homotopy equivalence, because it is a homotopy equivalence in each degree. *Hint:* The typical case $Sub(iS_3\mathcal{A}) \to iQ_3\mathcal{A}$ is illustrated in [W1126, 1.9].

(c) Show that the map of the previous exercise fits into a diagram

$$\begin{array}{ccccc} |s.\mathcal{A}| & \xleftarrow{\simeq} & |Sub(s.\mathcal{A})| & \longrightarrow & BQ\mathcal{A} \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ |iS.\mathcal{A}| & \xleftarrow{\simeq} & |Sub(iS.\mathcal{A})| & \xrightarrow{\simeq} & |iQ.\mathcal{A}|. \end{array}$$

Conclude that the map $|iS.\mathcal{A}| \to BQ\mathcal{A}$ of Ex. 8.5 is a homotopy equivalence.

8.7 Recall from 8.5.2 that $K_1(\mathcal{A}) \cong \pi_2|s.\mathcal{A}|$ for any exact category $\mathcal{A}$. Given an automorphism α of an object A in $\mathcal{A}$, show that the two 2-cells in $|s.\mathcal{A}|$ corresponding to the extensions $0 \rightarrowtail A \overset{\alpha}{\twoheadrightarrow} A$ and $A \overset{\alpha}{\rightarrowtail} A \twoheadrightarrow 0$ fit together to define an element of $\pi_2|s.\mathcal{A}|$. Then show that the map of Ex. 8.5 identifies it with the element $[\alpha]$ of $\pi_2 BQ\mathcal{A}$ described in Ex. 7.8.

8.8 *Finite Sets.* Show that the category $\mathbf{Sets}_{\rm fin}$ of finite pointed sets is a Waldhausen category, where the cofibrations are the injections and the weak equivalences are the isomorphisms. Then mimick Exercises 8.5 and 8.6 to show that the space $BQ\mathbf{Sets}_{\rm fin}$ of Ex. 6.15 is homotopy equivalent to the Waldhausen space $iS.\mathbf{Sets}_{\rm fin}$. Using Theorem 4.9.3 and Ex. 7.7, conclude that the Waldhausen K-theory space $K(\mathbf{Sets}_{\rm fin})$ is $\mathbb{Z}\times(B\Sigma_\infty)^+ \simeq \Omega^\infty S^\infty$. Thus $K_n(\mathbf{Sets}_{\rm fin}) \cong \pi_n^s$ for all n.

8.9 *G-Sets.* If G is a group, show that the category $G{-}\mathbf{Sets}_+$ of finitely generated pointed G-sets, and its subcategory $\mathcal{F}$ of free pointed G-sets, are Waldhausen categories. Then mimick Exercises 8.5 and 8.6 to show that the spaces $BQ(G{-}\mathbf{Sets}_+)$ and $BQ\mathcal{F}$ of Ex. 6.16 are homotopy equivalent to the Waldhausen spaces $iS.(G{-}\mathbf{Sets}_+)$ and $iS.\mathcal{F}$. Using Ex. 7.7, conclude that the Waldhausen K-theory space $K(\mathcal{F})$ is homotopy equivalent to $\Omega^\infty S^\infty(BG_+)$.

8.10 Given a category with cofibrations $\mathcal{C}$, let $\mathcal{E} = \mathcal{E}(\mathcal{C})$ denote the category of extensions in $\mathcal{C}$ (see II.9.3), and $s_.\mathcal{C}$ the simplicial set of 8.5.2. In this exercise we show that the source and quotient functors $s, q : \mathcal{E} \to \mathcal{C}$ induce $s_.\mathcal{E} \simeq s_.\mathcal{C} \times s_.\mathcal{C}$.
(a) Recall from Ex. 3.11 that for A in $s_n\mathcal{C}$ the simplicial set $s/(n, A)$ is the pullback of $s_.\mathcal{E}$ and Δ^n along s and $A : \Delta^n \to s_.\mathcal{C}$. Show that $s/(0, 0)$ is equivalent to $s_.\mathcal{C}$.
(b) For every vertex α of Δ^n and every A in $s_n\mathcal{C}$ that the map $s/(0, 0) \to s/(n, A)$ of Ex. 3.11 is a homotopy equivalence.
(c) Use (b) to show that $s : s_.\mathcal{E} \to s_.\mathcal{C}$ satisfies the hypothesis of Ex. 3.11(B).
(d) Use Ex. 3.11(B) to show that there is a homotopy fibration $s_.\mathcal{C} \to s_.\mathcal{E} \to s_.\mathcal{C}$ Conclude that $s \times q : s_.\mathcal{E} \to s_.\mathcal{C} \times s_.\mathcal{C}$ is a homotopy equivalence.

8.11 Given an exact functor $f : \mathcal{B} \to \mathcal{C}$, mimick the proof of 8.4 to show that the group $K_{-1}(f) = \pi_1(wS.S.f)$ of 8.5.3 is the cokernel of $K_0(\mathcal{B}) \to K_0(\mathcal{C})$.

8.12 Suppose that $\mathcal{B}$ is a strictly cofinal Waldhausen subcategory of $\mathcal{C}$.
(a) Show that $S_n\mathcal{B}$ is strictly cofinal in $S_n(\mathcal{C})$.
(b) Show that $\mathcal{B}(w, m)$ is strictly cofinal in $\mathcal{C}(w, m)$.

8.13 Any exact category $\mathcal{A}$ is cofinal in its idempotent completion $\hat{\mathcal{A}}$, by definition (see II.7.3). Let $\mathcal{A}'$ be the subcategory of $\hat{\mathcal{A}}$ consisting of all B in $\hat{\mathcal{A}}$ such that $[B]$ lies in the subgroup $K_0(\mathcal{A})$ of $K_0(\hat{\mathcal{A}})$. Show that $\mathcal{A}'$ is an exact category, closed under admissible epimorphisms in $\hat{\mathcal{A}}$, and that $\mathcal{A}$ is strictly cofinal in $\mathcal{A}'$. Hence $K(\mathcal{A}) \simeq K(\mathcal{A}')$.

8.14 Let $\mathbf{Ch}(\mathcal{C})$ be the Waldhausen category of chain complexes in an exact category $\mathcal{C}$, as in 8.8.3. Show that $\mathbf{Ch}(\mathcal{C})$ and $\mathbf{Ch}^b(\mathcal{C})$ are saturated and satisfy the Extension axiom 8.2.1, and the Cylinder Axiom 8.8.1.

8.15 If $(\mathcal{C}, \mathrm{co}, w)$ is a saturated Waldhausen category with a cylinder functor, satisfying the cylinder axiom, show that the category $\mathrm{co}\,w\mathcal{C}$ of "trivial cofibrations" (cofibrations which are weak equivalences) is homotopy equivalent to $w\mathcal{C}$. *Hint:* Use the cylinder to show that each i/C is contractible, and apply Theorem A.

§9. The Gillet-Grayson construction

Let $\mathcal{A}$ be an exact category. Following Grayson and Gillet [GG], we define a simplicial set $G. = G.\mathcal{A}$ as follows.

Definition 9.1. If $\mathcal{A}$ is a small exact category, $G.$ is the simplicial set defined as follows. The set G_0 of vertices consists of all pairs of objects (A, B) in $\mathcal{A}$. The set G_1 of edges consists of all pairs of short exact sequences with the same cokernel:

$$A_0 \rightarrowtail A_1 \twoheadrightarrow A_{01}, \quad B_0 \rightarrowtail B_1 \twoheadrightarrow A_{01}. \tag{9.1.0}$$

The degeneracy maps $G_1 \to G_0$ send (9.1.0) to (A_1, B_1) and (A_0, B_0), respectively.

The set G_n consists of all pairs of triangular commutative diagrams in $\mathcal{A}$ of the form

$$\begin{array}{ccccccc} & & & & A_{n-1,n} \\ & & & & \uparrow \\ & & & & \cdots \\ & & A_{12} & \rightarrowtail \cdots \rightarrowtail & A_{1n} \\ & & \uparrow & & \uparrow \\ & A_{01} \rightarrowtail & A_{02} & \rightarrowtail \cdots \rightarrowtail & A_{0n} \\ & \uparrow & \uparrow & & \uparrow \\ A_0 \rightarrowtail & A_1 \rightarrowtail & A_2 & \rightarrowtail \cdots \rightarrowtail & A_n \end{array} \qquad \begin{array}{ccccccc} & & & & A_{n-1,n} \\ & & & & \uparrow \\ & & & & \cdots \\ & & A_{12} & \rightarrowtail \cdots \rightarrowtail & A_{1n} \\ & & \uparrow & & \uparrow \\ & A_{01} \rightarrowtail & A_{02} & \rightarrowtail \cdots \rightarrowtail & A_{0n} \\ & \uparrow & \uparrow & & \uparrow \\ B_0 \rightarrowtail & B_1 \rightarrowtail & B_2 & \rightarrowtail \cdots \rightarrowtail & B_n \end{array} \tag{9.1.1}$$

so that each sequence $A_i \rightarrowtail A_j \twoheadrightarrow A_{ij}$ and $B_i \rightarrowtail B_j \twoheadrightarrow A_{ij}$ is exact. As in the definition of $S.\mathcal{A}$ (8.3.1), the face maps $\partial_i \colon G_n \to G_{n-1}$ are obtained by deleting the row A_{i*} and the columns containing A_i and B_i, while the degeneracy maps $\sigma_i \colon G_n \to G_{n+1}$ are obtained by duplicating A_i and B_i, and reindexing.

Suppressing the choices A_{ij} for the cokernels, we can abbreviate (9.1.1) as:

$$\frac{A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n}{B_0 \rightarrowtail B_1 \rightarrowtail B_2 \rightarrowtail \cdots \rightarrowtail B_n}\,. \tag{9.1.2}$$

Remark 9.1.3. $|G.|$ is a homotopy commutative and associative H-space. Its product $|G.| \times |G.| \to |G.|$ arises from the simplicial map $G. \times G. \to G.$ whose components $G_n \times G_n \to G_n$ are termwise $\oplus$.

Note that for each isomorphism $A \cong A'$ in $\mathcal{A}$ there is an edge in G_1 from $(0, 0)$ to (A, A'), represented by $(0 \rightarrowtail A \twoheadrightarrow A, 0 \rightarrowtail A' \twoheadrightarrow A)$. Hence (A, A') represents zero in the group $\pi_0|G.|$.

Lemma 9.2. *There is a group isomorphism* $\pi_0|G.| \cong K_0(\mathcal{A})$.

Proof. As in 3.3, $\pi_0|G.|$ is presented as the set of elements (A, B) of G_0, modulo the equivalence relation that for each edge (9.1.0) we have

$$(A_1, B_1) = (A_0, B_0).$$

It is an abelian group by 9.1.3, with operation $(A, B) \oplus (A', B') = (A \oplus A', B \oplus B')$. Since $(A \oplus B, B \oplus A)$ represents zero in $\pi_0|G.|$, it follows that (B, A) is the inverse of (A, B). From this presentation, we see that there is a map $K_0(\mathcal{A}) \to \pi_0|G.|$ sending

$[A]$ to $(A, 0)$, and a map $\pi_0|G.| \to K_0(\mathcal{A})$, sending (A, B) to $[A] - [B]$. These maps are inverses to each other. $\square$

(9.3) We now compare $G.$ with the loop space of the simplicial set $s.\mathcal{A}$ of 8.5.2. If we forget the bottom row of either of the two triangular diagrams in (9.1.1), we get a triangular commutative diagram of the form (8.3.0), *i.e.,* an element of $s_n\mathcal{A}$. The resulting set maps $G_n \to s_n\mathcal{A}$ fit together to form a simplicial map $\partial_0 \colon G. \to s.\mathcal{A}$.

Path Spaces 9.3.1. Recall from [WHomo, 8.3.14] that the *path space* $PX.$ of a simplicial set $X.$ has $PX_n = X_{n+1}$, its ith face operator is the ∂_{i+1} of $X.$, and its ith degeneracy operator is the σ_{i+1} of $X.$. The forgotten face maps $\partial_0 \colon X_{n+1} \to X_n$ form a simplicial map $PX. \to X.$, and $\pi_0(PX.) \cong X_0$. In fact, σ_0 induces a canonical simplicial homotopy equivalence from $PX.$ to the constant simplicial set X_0; see [WHomo, Ex. 9.3.7]. Thus $PX.$ is contractible exactly when X_0 is a point.

Now there are two maps $G_n \to s_{n+1}\mathcal{A}$, obtained by forgetting one of the two triangular diagrams (9.1.1) giving an element of G_n. The face and degeneracy maps of $G.$ are defined so that these yield two simplicial maps from $G.$ to the path space $P. = P(s.\mathcal{A})$. Clearly, either composition with the canonical map $P. \to s.\mathcal{A}$ yields the map $\partial_0 \colon G. \to s.\mathcal{A}$. Thus we have a commutative diagram

$$\begin{array}{ccc} G. & \longrightarrow & P. \\ \downarrow & & \downarrow \\ P. & \longrightarrow & s.\mathcal{A}. \end{array} \tag{9.3.2}$$

Since $s_0\mathcal{A}$ is a point, the path space $|P.|$ is canonically contractible. Therefore this diagram yields a canonical map $|G.| \to \Omega|s.\mathcal{A}|$. On the other hand, we saw in 8.6 that $|s.\mathcal{A}| \simeq BQ\mathcal{A}$, so $\Omega|s.\mathcal{A}| \simeq \Omega BQ\mathcal{A} = K(\mathcal{A})$.

We cite the following result from [GG, 3.1]. Its proof uses simplicial analogues of Quillen's theorems A and B.

Theorem 9.4. *(Gillet-Grayson) Let $\mathcal{A}$ be a small exact category. Then the map of (9.3) is a homotopy equivalence:*

$$|G.| \simeq \Omega|s.A| \simeq K(\mathcal{A}).$$

Hence $\pi_i|G.| = K_i(\mathcal{A})$ for all $i \geq 0$.

Example 9.5. A *double s.e.s.* in $\mathcal{A}$ is a pair ℓ of short exact sequences in $\mathcal{A}$ on the same objects:

$$\ell\colon \qquad A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C, \quad A \overset{f'}{\rightarrowtail} B \overset{g'}{\twoheadrightarrow} C.$$

Thus ℓ is an edge (in G_1) from (A, A) to (B, B). To ℓ we attach the element $[\ell]$ of $K_1(\mathcal{A}) = \pi_1|G.|$ given by the following 3-edged loop.

$$\begin{array}{ccc} (A,A) & \overset{\ell}{\longrightarrow} & (B,B) \\ e_A \nwarrow & & \nearrow e_B \\ & (0,0) & \end{array}$$

where e_A denotes the canonical double s.e.s. $(0 \rightarrowtail A \twoheadrightarrow A,\ 0 \rightarrowtail A \twoheadrightarrow A)$.

The following theorem was proven by A. Nenashev in [Nen].

NENASHEV'S THEOREM 9.6. *$K_1(\mathcal{A})$ may be described as follows.*
(a) Every element of $K_1(\mathcal{A})$ is represented by the loop $[\ell]$ of a double s.e.s.;
(b) $K_1(\mathcal{A})$ is presented as the abelian group with generators the double s.e.s. in $\mathcal{A}$, subject to two relations:

(i) If E is a short exact sequence, the loop of the double s.e.s. (E, E) is zero;

(ii) for any diagram of six double s.e.s. (9.6.1) such that the "first" diagram commutes, and the "second" diagram commutes, then

$$[r_0] - [r_1] + [r_2] = [c_0] - [c_1] + [c_2],$$

where r_i is the i^{th} row and c_i is the i^{th} column of (9.6.1).

(9.6.1)
$$\begin{array}{ccccc} A' & \rightrightarrows & A & \rightrightarrows & A'' \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ B' & \rightrightarrows & B & \rightrightarrows & B'' \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ C' & \rightrightarrows & C & \rightrightarrows & C''. \end{array}$$

EXAMPLE 9.6.2. If α is an automorphism of A, the class $[\alpha] \in K_1(\mathcal{A})$ is the class of the double s.e.s. $(0 \rightarrowtail A \overset{\alpha}{\twoheadrightarrow} A, 0 \rightarrowtail A \overset{=}{\twoheadrightarrow} A)$.

If β is another automorphism of A, the relation $[\alpha\beta] = [\alpha] + [\beta]$ comes from relation (ii) for

$$\begin{array}{ccccc} 0 & \rightrightarrows & 0 & \rightrightarrows & 0 \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ 0 & \rightrightarrows & A & \underset{1}{\overset{\alpha}{\rightrightarrows}} & A \\ \downdownarrows & & 1 \downdownarrows 1 & & \beta \downdownarrows 1 \\ 0 & \rightrightarrows & A & \underset{1}{\overset{\alpha\beta}{\rightrightarrows}} & A. \end{array}$$

EXERCISES

9.1 Verify that condition 9.6(i) holds in $\pi_1|G.|$.

9.2 Show that omitting the choice of quotients A_{ij} from the definition of $G.\mathcal{A}$ yields a homotopy equivalent simplicial set $G'.\mathcal{A}$. An element of $G'_n\mathcal{A}$ is a diagram (9.1.2) together with a compatible family of isomorphisms $A_j/A_i \cong B_j/B_i$.

9.3 Consider the involution on $G.$ which interchanges the two diagrams in (9.1.1). We saw in 9.2 that it induces multiplication by -1 on $K_0(\mathcal{A})$. Show that this involution is an additive inverse map for the H-space structure 9.1.3 on $|G.|$.

9.4 If $\alpha\colon A \cong A$ is an isomorphism, use relation (ii) in Nenashev's presentation 9.6 to show that $[\alpha^{-1}] \in K_1(\mathcal{A})$ is represented by the loop of the double s.e.s.:

$$\frac{A \overset{\alpha}{\to} A}{\overline{A \underset{=}{\to} A}}$$

9.5 If $\mathcal{A}$ is a split exact category, use Nenashev's presentation 9.6 to show that $K_1(\mathcal{A})$ is generated by automorphisms (9.6.2).

§10. Non-connective spectra for algebraic K-theory

In III.4 we introduced the negative K-groups of a ring using Bass' Fundamental Theorem for $K_0(R[t,t^{-1}])$, III.3.7. For many applications, it is useful to have a spectrum-level version of this construction, viz., a non-connective "Bass K-theory spectrum" $\mathbf{K}^B(R)$ with $\pi_n\mathbf{K}^B(R) = K_n(R)$ for all $n < 0$. In this section we constuct such a non-connective spectrum starting from any one of the functorial models of a connective K-theory spectrum $\mathbf{K}(R)$. (See 1.9(iii), 4.5.2 and 8.5.5.)

Let $\mathbf{E}$ be a functor from rings to spectra. Since the inclusions of $\mathbf{E}(R)$ in $\mathbf{E}(R[x])$ and $\mathbf{E}(R[x^{-1}])$ split, the homotopy pushout $\mathbf{E}(R[x]) \vee_{\mathbf{E}(R)} \mathbf{E}(R[x^{-1}])$ is the wedge of $\mathbf{E}(R)$ and these two complementary factors.

Definition 10.1. Write $L\mathbf{E}(R)$ for the spectrum homotopy cofiber of the map f_0 from this homotopy pushout to $\mathbf{E}(R[x,x^{-1}])$, and $\Lambda\mathbf{E}(R)$ for its desuspension $\Omega L\mathbf{E}(R)$.

Since the mapping cone is natural, $L\mathbf{E}$ and $\Lambda\mathbf{E}$ are functors and there is a cofibration sequence, natural in $\mathbf{E}$ and R:

$$\Lambda\mathbf{E}(R) \to \mathbf{E}(R[x]) \vee_{\mathbf{E}(R)} \mathbf{E}(R[x^{-1}]) \xrightarrow{f_0} \mathbf{E}(R[x,x^{-1}]) \to L\mathbf{E}(R).$$

The algebraic version of the Fundamental Theorem of higher K-theory, established in V.6.2 and V.8.2, states that there is a split exact sequence

$$0 \to K_n(R) \to K_n(R[x]) \oplus K_n(R[x^{-1}]) \to K_n(R[x,x^{-1}]) \overset{\leftarrow}{\longrightarrow} K_{n-1}(R) \to 0,$$

in which the splitting is multiplication by $x \in K_1(\mathbb{Z}[x,x^{-1}])$. Applying π_n to the case $\mathbf{E} = \mathbf{K}$ of Definition 10.1 shows that $\pi_n L\mathbf{K}(R) \cong K_{n-1}(R)$ for all $n > 0$. The Fundamental Theorems for K_1 and K_0 (III, 3.6 and 3.7) imply that $\pi_0\Lambda\mathbf{K}(R) = K_0(R)$, $\pi_{-1}\Lambda\mathbf{K}(R) = K_{-1}(R)$ and that $\pi_n\Lambda\mathbf{K}(R) = 0$ for $n < -1$.

We will need the following topological version of the Fundamental Theorem, also established in the next chapter (in V.8.4). Fix a map $S^1 \to \mathbf{K}(\mathbb{Z}[x,x^{-1}])$, represented by the element $x \in K_1(\mathbb{Z}[x,x^{-1}])$. Recall from 1.10.2 and Ex. 4.14 that this map induces a product map $\mathbf{K}(R) \xrightarrow{\cup x} \Omega\mathbf{K}(R[x,x^{-1}])$, natural in the ring R. Composing with $\Omega\mathbf{K}(R[x,x^{-1}]) \to \Omega L\mathbf{K}(R) \xleftarrow{\simeq} \Lambda\mathbf{K}(R)$ yields a map of spectra $\mathbf{K}(R) \to \Lambda\mathbf{K}(R)$.

Fundamental Theorem 10.2. *For any ring R, the map $\mathbf{K}(R) \to \Lambda\mathbf{K}(R)$ induces a homotopy equivalence between $\mathbf{K}(R)$ and the (-1)-connective cover of the spectrum $\Lambda\mathbf{K}(R)$. In particular, $K_n(R) \cong \pi_n\Lambda\mathbf{K}(R)$ for all $n \geq 0$.*

By induction on k, we have natural maps

$$\Lambda^{k-1}\mathbf{K}(R) \xrightarrow{\cup x} \Lambda^{k-1}\Omega\mathbf{K}(R[x,x^{-1}]) \to \Lambda^{k-1}\Omega L\mathbf{K}(R) \xleftarrow{\simeq} \Lambda^k\mathbf{K}(R).$$

Corollary 10.3. *For $k > 0$ the map $\Lambda^{k-1}\mathbf{K}(R) \to \Lambda^k\mathbf{K}(R)$ induces a homotopy equivalence between $\Lambda^{k-1}\mathbf{K}(R)$ and the $(-k)$-connective cover of $\Lambda^k\mathbf{K}(R)$, with $K_n(R) \cong \pi_n\Lambda^{k-1}\mathbf{K}(R) \cong \pi_n\Lambda^k\mathbf{K}(R)$ for $n > -k$, and $K_{-k}(R) \cong \pi_{-k}\Lambda^k\mathbf{K}(R)$.*

Proof. We proceed by induction on k, the case $k = 1$ being Theorem 10.2. Set $\mathbf{E} = \Lambda^{k-1}\mathbf{K}$; we have a natural isomorphism $K_n(R) \cong \pi_n\mathbf{E}(R)$ for $n > -k$, such that $\cup x : K_n(R) \to K_{n+1}(R[x, x^{-1}])$ agrees with π_n of $\mathbf{E}(R) \to \Omega\mathbf{E}(R[x, x^{-1}])$ up to isomorphism. The map $\pi_n\mathbf{E}(R[x]) \vee_{\mathbf{E}(R)} \mathbf{E}(R[x^{-1}]) \to \pi_n\mathbf{E}(R[x, x^{-1}])$ in 10.1 is an injection for all n, being either the injection from $K_n(R[x]) \oplus K_n(R[x^{-1}])/K_n(R)$ to $K_n(R[x, x^{-1}])$ of III.4.1.2 and 10.2 (for $n > -k$) or $0 \to 0$ (for $n \le -k$). It follows from III.4.1.2 that for $n > -k$ the maps $K_n(R) \cong \pi_n\Lambda^{k-1}\mathbf{K}(R) \to \pi_n\Lambda^k\mathbf{K}(R)$ are isomorphisms, and that the composite

$$K_{-k}(R) \xrightarrow{\cup x} K_{1-k}(R[x, x^{-1}]) \cong \pi_{1-k}\Lambda^{k-1}\mathbf{K}(R[x, x^{-1}]) \to \pi_{-k}\Lambda^k\mathbf{K}(R)$$

is an isomorphism. Since $\Lambda^{k-1}\mathbf{K}(R)$ is $(-k)$-connected, it is the $(-k)$-connected cover of $\Lambda^k\mathbf{K}(R)$. It is also clear from 10.1 that $\pi_n\Lambda^k\mathbf{K}(R) = 0$ for $n < -k$. $\square$

Definition 10.4. We define $\mathbf{K}^B(R)$ to be the homotopy colimit of the diagram

$$\mathbf{K}(R) \to \Omega L\mathbf{K}(R) \xleftarrow{\simeq} \Lambda\mathbf{K}(R) \to \cdots \Lambda^{k-1}\mathbf{K}(R) \to \Lambda^{k-1}\Omega L\mathbf{K}(R) \xleftarrow{\simeq} \Lambda^k\mathbf{K}(R) \to \cdots$$

(The homotopy colimit may be obtained by inductively replacing each portion $\cdot \xleftarrow{\simeq} \cdot \to \cdot$ by a pushout and then taking the direct limit of the resulting sequence of spectra.)

By 10.3, the canonical map $\mathbf{K}(R) \to \mathbf{K}^B(R)$ induces isomorphisms $K_n(R) \cong \pi_n\mathbf{K}^B(R)$ for $n \ge 0$, and $K_n(R) \cong \pi_n\mathbf{K}^B(R)$ for all $n \le 0$ as well.

Variant 10.4.1. The "suspension ring" $S(R)$ of R provides an alternative way of constructing a non-connective spectrum for K-theory. Recall from III, Ex. 1.15, that $S(R)$ is defined to be $C(R)/M(R)$. In III, Ex. 4.10, we saw that there are isomorphisms $K_n(R) \cong K_0S^{|n|}(R)$ for $n \le 0$. In fact, Gersten and Wagoner proved that $K_0(R) \times BGL(R)^+ \simeq \Omega BGL(S(R))^+$ so that $K_n(R) \cong K_{n+1}S(R)$ for all $n \ge 0$. It follows that the sequence of spaces $\mathbf{K}^{GW}(R)_i = K_0(S^i(R)) \times BGL(S^i(R))^+$ form a nonconnective spectrum with $\pi_n\mathbf{K}^{GW}(R) \cong K_n(R)$ for all n. We leave it as an exercise to show that a homotopy equivalence between the 0th space of $\mathbf{K}(R)$ and $K_0(R) \times BGL(R)^+$ induces a homotopy equivalence of spectra $\mathbf{K}^B(R) \simeq \mathbf{K}^{GW}(R)$.

We now introduce a delooping of Quillen's space $K(\mathcal{A}) = \Omega BQ\mathcal{A}$ (or spectrum) associated to an exact category $\mathcal{A}$, as $K(S\mathcal{A})$ for a different exact category $S\mathcal{A}$. Iterating this yields a non-connective spectrum with connective cover $K(\mathcal{A})$, which agrees with the construction of Definition 10.1 when $\mathcal{A} = \mathbf{P}(R)$.

Big vector bundles 10.5. Many constructions require that $K(X)$ be strictly functorial in X. For this we introduce the notion of big vector bundles, which I learned from Thomason; see Ex. 10.3. Let $\mathcal{V}$ be a small category of schemes over a fixed scheme X. By a *big vector bundle* over X we will mean the choice of a vector bundle $\mathcal{E}_Y$ on Y for each morphism $Y \to X$ in $\mathcal{V}$, equipped with an isomorphism $f^*\mathcal{E}_Y \to \mathcal{E}_Z$ for every $f : Z \to Y$ over X such that: (i) to the identity on Y we associate the identity on $\mathcal{E}_Y$, and (ii) for each composition $W \xrightarrow{g} Z \xrightarrow{f} Y$, the map $(fg)^*$ is the compoition $g^*f^*\mathcal{E}_Y \to g^*\mathcal{E}_Z \to \mathcal{E}_W$. Let $\mathbf{VB}_{\mathcal{V}}(X)$ denote the category of big vector bundles over X. The obvious forgetful functor $\mathbf{VB}_{\mathcal{V}}(X) \to \mathbf{VB}(X)$ is an equivalence of categories, and $X \mapsto \mathbf{VB}_{\mathcal{V}}(X)$ is clearly a contravariant functor from $\mathcal{V}$ to exact categories. Since K-theory is a functor on exact categories, $X \mapsto K\mathbf{VB}_{\mathcal{V}}(X)$ is a presheaf of spectra on $\mathcal{V}$.

If $\mathcal{V}$ is a small category of noetherian schemes and flat maps, a big coherent module over X for $\mathcal{V}$ is the choice of a coherent $\mathcal{O}_Y$-module $\mathcal{F}$ on Y for each $Y \to X$, equipped with a natural isomorphism $f^*\mathcal{F}_Y \to \mathcal{F}_Z$ for every (flat) $f : Z \to Y$ over X, subject to the usual conditions on identity maps and compositions. Let $\mathbf{M}_{\mathcal{V}}(X)$ denote the category of big coherent modules over X. The obvious forgetful functor $\mathbf{M}_{\mathcal{V}}(X) \to \mathbf{M}(X)$ is an equivalence of categories, and $X \mapsto \mathbf{M}_{\mathcal{V}}(X)$ is clearly a contravariant functor from $\mathcal{V}$ to exact categories. Since K-theory is a functor on exact categories, $X \mapsto K\mathbf{M}_{\mathcal{V}}(X)$ is a presheaf of spectra on $\mathcal{V}$.

Non-connective K-theory of schemes 10.6. Let $\mathcal{V}$ be a small category of quasi-projective schemes such that whenever X is in $\mathcal{V}$ then so are $X \times \mathbb{A}^1$ and $X \times \operatorname{Spec}(\mathbb{Z}[x, x^{-1}])$. Using big vector bundles on $\mathcal{V}$, we may arrange that $X \mapsto \mathbf{K}(X)$ is a functor from $\mathcal{V}$ to spectra. In this way, Construction 10.1 may be made functorial in X.

There is also a Fundamental Theorem like 10.2 for the algebraic K-theory of a quasi-projective scheme X (and even for quasi-compact, quasi-separated schemes), due to Thomason and Trobaugh [TT, 6.1]. Using this and functoriality of $\Lambda^k\mathbf{K}(X)$, the proof of 10.3 goes through, and we define $\mathbf{K}^B(X)$ to be the homotopy colimit of the $\Lambda^k\mathbf{K}(X)$. If $X = \operatorname{Spec}(R)$ then $\mathbf{K}(X)$ is homotopy equivalent to $\mathbf{K}(R)$ and hence $\mathbf{K}^B(X)$ is homotopy equivalent to $\mathbf{K}^B(R)$. As for rings, the canonical map $\mathbf{K}(X) \to \mathbf{K}^B(X)$ induces isomorphisms $K_n(X) \cong \pi_n\mathbf{K}^B(X)$ for $n \geq 0$, and $K_n(X) \cong \pi_n\mathbf{K}^B(X)$ for all $n \leq 0$ as well.

EXERCISES

10.1 Let I be an ideal in a ring R, and write $\mathbf{K}^B(R, I)$ for the homotopy fiber of $\mathbf{K}^B(R) \to \mathbf{K}^B(R/I)$. Let $\mathbf{K}^{\leq 0}(R, I)$ denote the homotopy cofiber of the 0-connected cover $\mathbf{K}^B(R, I)\langle 0\rangle \to \mathbf{K}^B(R, I)$, as in 4.11.2. Thus $\pi_n\mathbf{K}^{\leq 0}(R, I) = 0$ for $n > 0$, and $\pi_0\mathbf{K}^{\leq 0}(R, I) \cong K_0(I)$ by Ex. 1.15. Use III.2.3 to show that $\pi_n\mathbf{K}^{\leq 0}(R, I) \cong K_n(I)$ for all $n < 0$.

10.2 Let $\mathcal{A}$ be the category $\mathbf{VB}(X)$. Use the method of 10.4.1 to produce a non-connective spectrum with connective cover $\mathbf{K}(X)$.

10.3 Let $\mathcal{V}$ be a small category of schemes, so that $X \mapsto \mathbf{VB}(X)$ is a contravariant lax functor on $\mathcal{V}$. Recall the Kleisli rectification of $\mathbf{VB}_X$ in Exercise 6.5, whose objects are pairs $(Y \to X, \mathcal{E}_Y)$, and whose morphisms are pairs $(Z \xrightarrow{h} Y, h^*(\mathcal{E}_Y) \cong \mathcal{E}_Z)$. Given a morphism $f : T \to X$ in $\mathcal{V}$, use the natural isomorphism $h^*f^* \cong f^*h^*$ to construct an exact functor $f^* : \mathbf{VB}_X \to \mathbf{VB}_T$. Compare this with the construction of big vector bundles in 10.5.

§11. Karoubi-Villamayor K-theory

Following Gersten, we say that a functor F from rings (or rings without unit) to sets is *homotopy invariant* if $F(R) \cong F(R[t])$ for every R. Similarly, a functor F from rings to CW complexes (spaces) is called *homotopy invariant* if for every ring R the natural map $R \to R[t]$ induces a homotopy equivalence $F(R) \simeq F(R[t])$. Note that each homotopy group $\pi_n F(R)$ also forms a homotopy invariant functor.

Of course, this notion may be restricted to functors defined on any subcategory of rings which is closed under polynomial extensions and contains the evaluations as well as the inclusion $R \subset R[t]$. For example, we saw in II, 6.5 and 7.9.3 that $G_0(R)$ is a homotopy invariant functor defined on noetherian rings (and schemes) and maps of finite flat dimension.

Conversely, recall from III.3.4 that R is called F*-regular* if $F(R) \cong F(R[t_1, ..., t_n])$ for all n. Clearly, any functor F from rings to sets becomes homotopy invariant when restricted to the subcategory of F-regular rings. For example, we see from II.7.8 that K_0 becomes homotopy invariant when restricted to regular rings. The Fundamental Theorem in chapter V, 6.3 implies that the functors K_n are also homotopy invariant when restricted to regular rings.

There is a canonical way to make F into a homotopy invariant functor.

Strict homotopization 11.1. Let F be a functor from rings to sets. Its *strict homotopization* $[F]$ is defined as the coequalizer of the evaluations at $t = 0, 1$: $F(R[t]) \rightrightarrows F(R)$. In fact, $[F]$ is a homotopy invariant functor and there is a universal transformation $F(R) \to [F](R)$; see Ex. 11.1. Moreover, if F takes values in groups then so does $[F]$; see Ex. 11.3.

Example 11.1.1. Recall that a matrix is called *unipotent* if it has the form $1 + \nu$ for some nilpotent matrix ν. Let $Unip(R)$ denote the subgroup of $GL(R)$ generated by the unipotent matrices. This is a normal subgroup of $GL(R)$, because the unipotent matrices are closed under conjugation. Since every elementary matrix $e_{ij}(r)$ is unipotent, this contains the commutator subgroup $E(R)$ of $GL(R)$.

We claim that $[E]R = [Unip]R = 1$ for every R. Indeed, if $1 + \nu$ is unipotent, $(1 + t\nu)$ is a matrix in $Unip(R[t])$ with $\partial_0(1 + t\nu) = 1$ and $\partial_1(1 + t\nu) = (1 + \nu)$. Since $Unip(R)$ is generated by these elements, $[Unip]R$ must be trivial. The same argument applies to the elementary group $E(R)$.

We now consider $GL(R)$ and its quotient $K_1(R)$. A priori, $[GL]R \to [K_1]R$ is a surjection. In fact, it is an isomorphism.

Lemma 11.2. *Both* $[GL]R$ *and* $[K_1]R$ *are isomorphic to* $GL(R)/Unip(R)$.

Definition 11.2.1. For each ring R, we define $KV_1(R)$ to be $GL(R)/Unip(R)$. Thus $KV_1(R)$ is the strict homotopization of $K_1(R) = GL(R)/E(R)$.

Proof. The composite $Unip(R) \to GL(R) \to [GL]R$ is trivial, as it factors through $[Unip]R = 1$. Hence $[GL]R$ (and $[K_1]R$) are quotients of $GL(R)/Unip(R)$. By Higman's trick III.3.5.1, if $g \in GL(R[t])$ is in the kernel of ∂_0 then $g \in Unip(R[t])$ and hence $\partial_1(g) \in Unip(R)$. Hence $\partial_1(NGL(R)) = Unip(R)$. Hence $GL(R)/Unip(R)$ is a strictly homotopy invariant functor; universality implies that the induced maps $[GL]R \to [K_1]R \to GL(R)/Unip(R)$ must be isomorphisms. □

To define the higher Karoubi-Villamayor groups, we introduce the simplicial ring $R[\Delta^\cdot]$, and use it to define the notion of homotopization. The simplicial ring $R[\Delta^\cdot]$ also plays a critical role in the construction of higher Chow groups and motivic cohomology, which is used in Chapter VI.

DEFINITION 11.3. For each ring R the coordinate rings of the standard simplices form a simplicial ring $R[\Delta^\cdot]$. It may described by the diagram

$$R \leftleftarrows R[t_1] \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} R[t_1, t_2] \overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}} \cdots R[t_1, \dots, t_n] \cdots$$

with $R[\Delta^n] = R[t_0, t_1, \cdots, t_n]/\left(\sum t_i = 1\right) \cong R[t_1, \cdots, t_n]$. The face maps ∂_i are given by: $\partial_i(t_i) = 0$; $\partial_i(t_j)$ is t_j for $j < i$ and t_{j-1} for $j > i$. Degeneracies σ_i are given by: $\sigma_i(t_i) = t_i + t_{i+1}$; $\sigma_i(t_j)$ is t_j for $j < i$ and t_{j+1} for $j > i$.

DEFINITION 11.4. Applying the functor GL to $R[\Delta^\cdot]$ gives us a simplicial group $GL. = GL(R[\Delta^\cdot])$. For $n \geq 1$, we define the Karoubi-Villamayor groups to be $KV_n(R) = \pi_{n-1}(GL.) = \pi_n(BGL.)$.

Since $\pi_0(GL.)$ is the coequalizer of $GL(R[t]) \rightrightarrows GL(R)$, we see from 11.2 that definitions 11.2.1 and 11.4 of $KV_1(R)$ agree: $KV_1(R) = GL(R)/Unip(R) \cong \pi_0(GL.)$.

The proof in Ex. 1.11 that $BGL(R)^+$ is an H-space also applies to $BGL(R[\Delta^\cdot])$ (exercise 11.9). From the universal property in theorem 1.8 we deduce the following elementary result.

LEMMA 11.4.1. *The map $BGL(R) \to BGL(R[\Delta^\cdot])$ factors through an H-map $BGL(R)^+ \to BGL(R[\Delta^\cdot])$. Thus there are canonical maps $K_n(R) \to KV_n(R)$, $n \geq 1$.*

REMARK 11.4.2. In fact, $BGL(R[\Delta^\cdot])^+$ is an infinite loop space; it is the 0^{th} space of the geometric realization $\mathbf{KV}(R)$ of the simplicial spectrum $\mathbf{K}(R[\Delta^\cdot])\langle 0\rangle$ of 4.11.2. (For any (-1)-connected simplicial spectrum $\mathbf{E}^\cdot$, the 0^{th} space of $|\mathbf{E}^\cdot|$ is the realization of the 0^{th} simplicial space.) Since $R[\Delta^0] = R$, there is a canonical morphism of spectra $\mathbf{K}(R) \to \mathbf{K}(R)$. This shows that the map $BGL(R)^+ \to BGL(R[\Delta^\cdot])$ of 11.4.1 is in fact an infinite loop space map.

It is useful to put the definition of KV_* into a more general context:

DEFINITION 11.5 (HOMOTOPIZATION). Let F be a functor from rings to CW complexes. Its *homotopization* $F^h(R)$ is the geometric realization of the simplicial space $F(R[\Delta^\cdot])$. Thus F^h is also a functor from rings to CW complexes, and there is a canonical map $F(R) \to F^h(R)$.

LEMMA 11.5.1. *Let F be a functor from rings to CW complexes. Then:*

(1) *F^h is a homotopy invariant functor;*
(2) *$\pi_0(F^h)$ is the strict homotopization $[F_0]$ of the functor $F_0(R) = \pi_0 F(R)$;*
(3) *If F is homotopy invariant then $F(R) \simeq F^h(R)$ for all R.*

COROLLARY 11.5.2. *The abelian groups $KV_n(R)$ are homotopy invariant, i.e.,*

$$KV_n(R) \cong KV_n(R[x]) \quad \textit{for every } n \geq 1.$$

PROOF OF 11.5.1. We claim that the inclusion $R[\Delta^\cdot] \subset R[x][\Delta^\cdot]$ is a simplicial homotopy equivalence, split by evaluation at $x = 0$. For this, we define ring maps

$h_i\colon R[x][\Delta^n] \to R[x][\Delta^{n+1}]$ by: $h_i(f) = \sigma_i(f)$ if $f \in R[\Delta^n]$ and $h_i(x) = x(t_{i+1} + \cdots + t_{n+1})$. These maps define a simplicial homotopy (see [WHomo]) between the identity map of $R[x][\Delta^\cdot]$ and the composite

$$R[x][\Delta^\cdot] \xrightarrow{x=0} R[\Delta^\cdot] \subset R[x][\Delta^\cdot].$$

Applying F gives a simplicial homotopy equivalence between $F^h(R[\Delta^\cdot])$ and $F^h(R[x][\Delta^\cdot])$. Geometric realization converts this into a topological homotopy equivalence between $F^h(R)$ and $F^h(R[x])$.

Part (2) follows from the fact that, for any simplicial space $X.$, the group $\pi_0(|X.|)$ is the coequalizer of $\partial_0, \partial_1\colon \pi_0(X_1) \rightrightarrows \pi_0(X_0)$. In this case $\pi_0(X_0) = \pi_0 F(R)$ and $\pi_0(X_1) = \pi_0 F(R[t])$.

Finally, if F is homotopy invariant then the map from the constant simplicial space $F(R)$ to $F(R[\Delta^\cdot])$ is a homotopy equivalence in each degree. It follows (see [Wa78]) that their realizations $F(R)$ and $F^h(R)$ are homotopy equivalent. □

It is easy to see that $F \to F^h$ is universal (up to homotopy equivalence) for natural transformations from F to homotopy invariant functors. A proof of this fact is left to Ex. 11.2.

EXAMPLE 11.6. Suppose that $G(R)$ is a group-valued functor. Then $G^h(R)$ is the realization of the simplicial group $G(R[\Delta^\cdot])$. This shows that G^h may have higher homotopy groups even if G does not.

In fact, the groups $\pi_n(G.)$ of any simplicial group $G.$ may be calculated using the formula $\pi_p(G.) = H_n(N^*G.)$, where $N^*G.$ is the *Moore complex*; see [WHomo, 11.3.6] [May, 17.3]. By definition, the Moore complex of a simplicial group $G.$ is the chain complex of groups with $N^0G. = G_0$, $N^1G. = \ker(\partial_0 : G_1 \to G_0)$ and $N^nG. = \cap_{i=0}^{n-1} \ker(\partial_i)$ for $n \geq 1$, with differential $(-1)^n\partial_n$. See Ex. 11.4.

In the case that $G.(R) = G(R[\Delta^\cdot])$, $N^1G.(R)$ is the group $NG(R)$ of III.3.3, and $N^nG.(R) \subset G(R[t_1, ..., t_n])$ is the n^{th} iterate of this functor.

A related situation arises when $F(R) = BG(R)$. Then $|G(R[\Delta^\cdot])|$ is the loop space of $F^h(R)$, which is a connected space with $\pi_{n+1}F^h(R) = H_n(N^*G(R[\Delta^\cdot]))$.

EXAMPLE 11.6.1. Suppose that $F(R) = |G.(R)|$ for some functor $G.$ from rings to simplicial groups. Then $F^h(R)$ is the geometric realization of a bisimplicial group $G_{pq} = G_q(R[\Delta^p])$. We can calculate the homotopy groups of any bisimplicial space $G..$ using the standard spectral sequence [Q66]

$$E^1_{pq} = \pi_q(G_{p\cdot}) \Rightarrow \pi_{p+q}|G..|.$$

As a special case, if $F(R) \simeq F(R[t_1, ..., t_n])$ for all n then $G_{p\bullet} \simeq G_\bullet(R)$ for all p, so the spectral sequence degenerates to yield $F(R) \simeq F^h(R)$.

THEOREM 11.7. *If $F(R)$ is any functorial model of $BGL(R)^+$ then we also have $KV_n(R) = \pi_n F^h(R)$ for all $n \geq 1$. Moreover, there is a first quadrant spectral sequence (for $p \geq 0, q \geq 1$):*

$$E^1_{pq} = K_q(R[\Delta^p]) \Rightarrow KV_{p+q}(R). \tag{11.7.1}$$

PROOF. (Anderson) We may assume (by Ex. 11.2) that $F(R) = |G.(R)|$ for a functor $G.$ from rings to simplicial groups which is equipped with a natural transformation $BGL \to G.$ such that $BGL(R) \to |G.(R)|$ identifies $|G.(R)|$ with $BGL(R)^+$. Such functors exist; see 1.9. The spectral sequence of 11.6.1 becomes (11.7.1) once we show that $KV_n(R) = \pi_n G.(R[\Delta^\cdot])$. Thus it suffices to prove that $BGL^h(R) \simeq |G.(R)|^h$. Since $BGL^h(R)$ is an H-space (Ex. 11.9), the proof is standard, and relegated to Exercise 11.10. □

THEOREM 11.8. *If R is regular, then $K_p(R) \cong KV_p(R)$ for all $p \geq 1$.*

PROOF. If R is regular, then each simplicial group $K_p(R[\Delta^\cdot])$ is constant (by the Fundamental Theorem in chapter V, 6.3). Thus the spectral sequence (11.7.1) degenerates at E^2 to yield the result. □

We now quickly develop the key points in KV-theory.

DEFINITION 11.9. We say that a ring map $f\colon R \to S$ is a *GL-fibration* if

$$GL(R[t_1, ..., t_n]) \times GL(S) \to GL(S[t_1, ..., t_n])$$

is onto for every n. Note that we do not require R and S to have a unit.

EXAMPLE 11.9.1. If I is a nilpotent ideal in R, then $R \to R/I$ is a GL-fibration. This follows from Ex. I.1.12(iv), because each $I[t_1, ..., t_n]$ is also nilpotent.

REMARK 11.9.2. Any GL-fibration must be onto. That is, $S \cong R/I$ for some ideal I of R. To see this, consider the $(1, 2)$ entry α_{12} of a preimage of the elementary matrix $e_{12}(st)$. Since $f(\alpha_{12}) = st$, evaluation at $t = 1$ gives an element of R mapping to $s \in S$. However, not every surjection is a GL-fibration; see Ex. 11.6(d).

EXAMPLE 11.9.3. Any ring map $R \to S$ is homotopic to a GL-fibration. Indeed, the inclusion of R into the graded ring $R' = R \oplus xS[x] = R \times_S S[x]$ induces a homotopy equivalence $GL(R[\Delta^\cdot]) \simeq GL(R'[\Delta^\cdot])$ by Ex. 11.5, so that $KV_*(R) \cong KV_*(R')$. Moreover, the map $R' \to S$ sending x to 1 is a GL-fibration by Ex. 11.6(a,c).

The definition of $KV_n(I)$ makes sense if I is a ring without unit using the group $GL(I)$ of III.2: $KV_n(I) = \pi_n BGL(I[\Delta^\cdot])$. Since $GL(R \oplus I)$ is the semidirect product of $GL(R)$ and $GL(I)$, we clearly have $KV_n(R \oplus I) \cong KV_n(R) \oplus KV_n(I)$. This generalizes as follows.

THEOREM 11.10 (EXCISION). *If $R \to R/I$ is a GL-fibration, there is a long exact sequence*

$$\begin{aligned} KV_{n+1}(R/I) &\to KV_n(I) \to KV_n(R) \to KV_n(R/I) \to \cdots \\ &\to KV_1(I) \to KV_1(R) \to KV_1(R/I) \to K_0(I) \to K_0(R) \to K_0(R/I). \end{aligned}$$

Theorem 11.10 is called an "Excision Theorem" because it says that (whenever $R \to R/I$ is a GL-fibration) $KV_n(R, I) \cong KV_n(I)$ for all $n \geq 1$.

PROOF. Let $G_n \subset GL(R/I[\Delta^n])$ denote the image of $GL(R[\Delta^n])$. Then there is an exact sequence of simplicial groups

$$1 \to GL(I[\Delta^\cdot]) \to GL(R[\Delta^\cdot]) \to G. \to 1. \tag{11.10.1}$$

Now any short exact sequence of simplicial groups is a fibration sequence, meaning there is a long exact sequence of homotopy groups. Moreover, the quotient $GL(R/I[\Delta^\cdot])/G.$ is a constant simplicial group, by Ex. 11.7. It is now a simple matter to splice the long exact sequences together to get the result. The splicing details are left to Ex. 11.7. □

Corollary 11.10.2. *For any ring map $\phi : R \to S$, set $I' = R \times_S xS[x] = \{(r, x\,f(x)) \in R \times xS[x] : \phi(r) + f(1) = 0\}$. Then there is a long exact sequence*

$$\cdots \to KV_{n+1}(S) \to KV_n(I') \to KV_n(R) \to KV_n(S) \to \cdots$$

ending in $KV_1(I') \to KV_1(R) \to KV_1(S) \to K_0(I')$.

Proof. Set $R' = R \oplus xS[x]$ and note that $R' \to S$ is a GL-fibration with kernel I' by 11.9.3. Since R' is graded, $KV_n(R) \cong KV_n(R')$ for all $n \geq 1$. The desired long exact sequence comes from Theorem 11.10. $\square$

Remark 11.10.3. When $R \to R/I$ is a GL-fibration, then $KV_*(I) \cong KV_*(I')$, and the long exact sequences of 11.10 and 11.10.2 coincide (with $S = R/I$). This follows from the 5-lemma, since the map ϕ factors through $R' \to R/I$ yielding a morphism of long exact sequences.

Theorem 11.10 fails if $R \to R/I$ is not a GL-fibration. Not only does the extension of Theorem 11.10 to K_0 fail (as the examples $\mathbb{Z} \to \mathbb{Z}/8$ and $\mathbb{Z} \to \mathbb{Z}/25$ show), but we need not even have $KV_*(I) \cong KV_*(I')$, as exercise 11.14 shows.

Corollary 11.11. *If I is a nilpotent ideal in a ring R, then $KV_n(I) = 0$ and $KV_n(R) \cong KV_n(R/I)$ for all $n \geq 1$.*

Proof. By 11.9.1, 11.10 and Lemma II.2.2, it suffices to show that $KV_n(I) = \pi_n GL(I[\Delta^\cdot]) = 0$. (A stronger result, that $GL(I[\Delta^\cdot])$ is simplicially contractible, is relegated to Ex. 11.11.) By exercise I.1.12(iii), $GL_m(I[\Delta^n])$ consists of the matrices $1 + x$ in $M_m(I[\Delta^n])$, so if $T = (t_0 t_1 \cdots t_{n-1})$ then the degree $n + 1$ part of the Moore complex (11.6) consists of matrices $1 + xT$, and $\partial_n(1 + xT) = 1$ exactly when $x = t_n y$ for some matrix y. Regarding y as a matrix over $I[t_0, ..., t_{n-1}]$, the element $1 + yTt_n$ in $GL(I[\Delta^{n+1}])$ belongs to the Moore complex and ∂_{n+1} maps $1 + yTt_n$ to $1 + xT$. $\square$

Theorem 11.12 (Mayer-Vietoris). *Let $\varphi : R \to S$ be a map of rings, sending an ideal I of R isomorphically onto an ideal of S. If $S \to S/I$ is a GL-fibration, then $R \to R/I$ is also a GL-fibration, and there is a long exact Mayer-Vietoris sequence*

$$\cdots \to KV_{n+1}(S/I) \to KV_n(R) \to KV_n(R/I) \oplus KV_n(S) \to KV_n(S/I) \to \cdots$$
$$\to KV_1(R/I) \oplus KV_1(S) \to KV_1(S/I) \to K_0(R) \to K_0(R/I) \oplus K_0(S).$$

It is compatible with the Mayer-Vietoris sequence for K_1 and K_0 in III.2.6.

Proof. To see that $R \to R/I$ is a GL-fibration, fix $\bar{g} \in GL(R/I[t_1, ..., t_n])$ with $\bar{g}(0) = I$. Since $S \to S/I$ is a GL-fibration, there is a $g' \in GL(S[t_1, ..., t_n])$ which is $\varphi(\bar{g})$ modulo I. Since R is the pullback of S and R/I, there is a g in $GL(R[t_1, ..., t_n])$ mapping to g' and $\bar{g}$. Hence $R \to R/I$ is a GL-fibration.

As in the proof of theorem III.5.8, there is a morphism from the (exact) chain complex of 11.10 for (R, I) to the corresponding chain complex for (S, I). Since every third term of this morphism is an isomorphism, the required Mayer-Vietoris sequence follows from a diagram chase. $\square$

Here is an application of this result. Since $R[x] \to R$ has a section, it is a GL-fibration. By homotopy invariance, it follows that $KV_n(xR[x]) = 0$ for all $n \geq 1$. (Another proof is given in Ex. 11.5.)

DEFINITION 11.13. For any ring R (with or without unit), define ΩR to be the ideal $(x^2-x)R[x]$ of $R[x]$. Iterating yields $\Omega^2 R = (x_1^2-x_1)(x_2^2-x_2)R[x_1,x_2]$, etc.

The following corollary of 11.10 shows that, for $n \geq 2$, we can also define $KV_n(R)$ as $KV_1(\Omega^{n-1}R)$, and hence in terms of K_0 of the rings $\Omega^n R$ and $\Omega^n R[x]$.

COROLLARY 11.13.1. *For all R, $KV_1(R)$ is isomorphic to the kernel of the map $K_0(\Omega R) \to K_0(xR[x])$, and $KV_n(R) \cong KV_{n-1}(\Omega R)$ for all $n \geq 2$.*

PROOF. The map $xR[x] \xrightarrow{x=1} R$ is a GL-fibration by Ex. 11.6(c), with kernel ΩR. The result now follows from Theorem 11.10. $\square$

We conclude with an axiomatic treatment, due to Karoubi and Villamayor.

DEFINITION 11.14. A *positive homotopy K-theory* (for rings) consists of a sequence of functors K_n^h, $n \geq 1$, on the category of rings without unit, together with natural connecting maps $\delta_n \colon K_{n+1}^h(R/I) \to K_n^h(I)$ and $\delta_0 \colon K_1^h(R/I) \to K_0(I)$, defined for every GL-fibration $R \to R/I$, satisfying the following axioms:

(1) The functors K_n^h are homotopy invariant;
(2) For every GL-fibration $R \to R/I$ the resulting sequence is exact:

$$K_{n+1}^h(R/I) \xrightarrow{\delta} K_n^h(I) \to K_n^h(R) \to K_n^h(R/I) \xrightarrow{\delta} K_{n-1}^h(I) \to$$
$$K_1^h(R) \to K_1^h(R/I) \xrightarrow{\delta} K_0(I) \to K_0(R) \to K_0(R/I).$$

THEOREM 11.14.1. *Up to isomorphism, there is a unique positive homotopy K-theory, namely $K_n^h = KV_n$.*

PROOF. The fact that KV_n form a positive homotopy K-theory is given by 11.4, 11.5.2 and 11.10. The axioms imply that any other positive homotopy K-theory must satisfy the conclusion of 11.13.1, and so must be isomorphic to KV-theory. $\square$

EXERCISES

11.1 Let F be a functor from rings to sets. Show that $[F]$ is a homotopy invariant functor, and that every natural transformation $F(R) \to H(R)$ to a homotopy invariant functor H factors uniquely through $F(R) \to [F](R)$.

11.2 Let F and H be functors from rings to CW complexes, and assume that H is homotopy invariant. Show that any natural transformation $F(R) \to H(R)$ factors through maps $F^h(R) \to H(R)$ such that for each ring map $R \to S$ the map $F^h(R) \to F^h(S) \to H(S)$ is homotopy equivalent to $F^h(R) \to F^h(S) \to H(S)$.

11.3 If G is a functor from rings to groups, let $NG(R)$ denote the kernel of the map $t=0 \colon G(R[t]) \to G(R)$. Show that the image $G_0(R)$ of the induced map $t = 1 \colon NG(R) \to G(R)$ is a normal subgroup of $G(R)$, and that $[G]R = G(R)/G_0(R)$. Thus $[G]R$ is a group.

11.4 (Moore) If $G.$ is a simplicial group and $N^nG. = \cap_{i=0}^{n-1}\ker(\partial_i)$ as in 11.6, show that $\partial_{n+1}(N^{n+1}G.)$ is a normal subgroup of G_n. Conclude that $\pi_n(G.)$ is also a group. *Hint:* conjugate elements of $N^{n+1}G$ by elements of s_nG_n.

11.5 Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded ring. Show that for every homotopy invariant functor F on rings we have $F(R_0) \simeq F(R)$. In particular, if F is defined on rings without unit then $F(xR[x]) \simeq F(0)$ for every R. *Hint:* Copy the proof of III.3.4.1.

11.6 *GL-fibrations.* Let $f\colon R \to S$ be a GL-fibration with kernel I. Show that:
(a) If f factors as $R \to R' \to S$, then $R' \to S$ is a GL-fibration.
(b) Both $xR[x] \to xS[x]$ and $\Omega R \to \Omega S$ are GL-fibrations.
(c) The map $xR[x] \to R$, $f(x) \mapsto f(1)$, is a GL-fibration with kernel ΩR.
(d) $\mathbb{Z} \to \mathbb{Z}/4$ is not a GL-fibration, but $GL(\mathbb{Z}) \to GL(\mathbb{Z}/4)$ is onto.
(e) If S is a regular ring (with unit), then every surjection $R \to S$ is a GL-fibration. *Hint:* $K_1(S) \cong K_1(S[x])$ by III.3.8.

11.7 Let $f\colon R \to S$ be a GL-fibration with kernel I, and define $G_.$ as in the proof of Theorem 11.10. Show that $GL(S[\Delta^\cdot])/G.$ is a constant simplicial group. Use this to show that $\pi_i(G_.) = KV_{i+1}(S)$ for all $i > 0$, but that the cokernel of $\pi_0(G_.) \to \pi_0 GL(S[\Delta^\cdot])$ is the image of $K_1(S)$ in $K_0(I)$ under the map of III.2.3. Combining this with the long exact sequence of homotopy groups for (11.10.1), finish the proof of 11.10.

11.8 Consider the unit functor U on rings. The identity $U(M_m(R)) = GL_m(R)$ implies that $U^h(M(R)) = KV(R)$. If R is commutative, use I.3.12 to show that $U^h(R) \simeq U(R_{\text{red}})$.

11.9 Show that $BGL(R[\Delta^\cdot])$ is an H-space. This is used to prove Lemma 11.4.1. *Hint:* See Ex. 1.11; the permutation matrices lie in $E(R)$, and $E(R[\Delta^\cdot])$ is path connected.

11.10 (Anderson) Use exercise 11.9 to complete the proof of theorem 11.7.

11.11 If I is nilpotent, show that the simplicial sets $GL_m(I[\Delta^\cdot])$ and $GL(I[\Delta^\cdot])$ have a simplicial contraction [WHomo, 8.4.6]. *Hint:* multiply by t_n.

11.12 If R is K_i-regular for all $i \le n$, show that $K_i(R) \cong KV_i(R)$ for all $i \le n$, and that $KV_{n+1}(R) \cong [K_n]R$.

11.13 (Strooker) Consider the ring $R = \mathbb{Z}[x]/(x^2 - 4)$. In this exercise, we use two different methods to show that the map $K_2(R) \to KV_2(R)$ is not onto, and that its cokernel is $\mathbb{Z}/2$. Note that $KV_1(R) = K_1(R) = R^\times = \{\pm 1\}$ by Ex. III.5.13.

(a) Compare the Mayer-Vietoris sequences III.5.8 and 11.12 to show that the natural map $K_2(R) \to K_2(\mathbb{Z})^2$ has cokernel $\mathbb{Z}/2$, yet $KV_2(R) \cong K_2(\mathbb{Z})^2 = (\mathbb{Z}/2)^2$.

(b) Use III.5.8 and Ex. III.5.15 to compute $K_1(R[t])$ and $K_1(R[t_1, t_2])$. Then show that the sequence $N^2K_1(R) \to NK_1(R) \to K_1(R)$ is not exact. Use the spectral sequence (11.7.1) to conclude that the map $K_2(R) \to KV_2(R)$ is not onto, and that its cokernel is $\mathbb{Z}/2$.

11.14 Let k be a field of characteristic 0, and set $S = k[x, (x+1)^{-1}]$
(a) Show that $K_1(\mathbb{Z} \oplus I, I) \cong K_1(S, I)$. *Hint:* Use the obstruction described in III, Ex. 2.6, showing that $3\psi(a\, dx \otimes x^2) = 0$.
(b) Use III, Ex. 5.14(c) to show that $K_1(S, I)$ is the cokernel of $d\ln : k^\times \to \Omega_k$, $d\ln(a) = da/a$.
(c) Show that $KV_1(I) = 0$, and conclude that the sequence $KV_1(I) \to KV_1(S) \to KV_1(S/I)$ is not exact.

§12. Homotopy K-theory

In order to define a truly homotopy invariant version of algebraic K-theory, we need to include K_0 and even the negative K-groups. This is most elegantly done at the level of spectra, and that approach begins by constructing the non-connective "Bass K-theory spectrum" $\mathbf{K}^B(R)$ out of any one of the functorial models of a connective K-theory spectrum $\mathbf{K}(R)$. (See 1.9(iii), 4.5.2 and 8.5.5.)

Let R be an associative ring with unit. By 11.3 there is a simplicial ring $R[\Delta^\cdot]$ and hence a simplicial spectrum $\mathbf{K}^B(R[\Delta^\cdot])$.

Definition 12.1. Let $KH(R)$ denote the (fibrant) geometric realization of the simplicial spectrum $\mathbf{K}^B(R[\Delta^\cdot])$. For $n \in \mathbb{Z}$, we write $KH_n(R)$ for $\pi_n KH(R)$.

It is clear from the definition that $KH(R)$ commutes with filtered colimits of rings, and that there are natural transformations $K_n(R) \to KH_n(R)$ which factor through $KV_n(R)$ when $n \geq 1$. Indeed, the spectrum map $\mathbf{K}(R)\langle 0\rangle \to \mathbf{K}^B(R) \to KH(R)$ factors through the spectrum $\mathbf{KV}(R) = \mathbf{K}(R[\Delta^\cdot])\langle 0\rangle$ of 11.4.2

Theorem 12.2. *Let R be a ring. Then:*

(1) $KH(R) \simeq KH(R[x])$, *i.e.,* $KH_n(R) \cong KH_n(R[x])$ *for all* n.
(2) $KH(R[x,x^{-1}]) \simeq KH(R) \times \Omega^{-1}KH(R)$, *i.e.,*

$$KH_n(R[x,x^{-1}]) \cong KH_n(R) \oplus KH_{n-1}(R) \quad \textit{for all } n.$$

(3) *If* $R = R_0 \oplus R_1 \oplus \cdots$ *is a graded ring then* $KH(R) \simeq KH(R_0)$.

Proof. Part (1) is a special case of 11.5.1. Part (2) follows from the Fundamental Theorem 10.2 and (1). Part (3) follows from (1) and Ex. 11.5. □

The homotopy groups of a simplicial spectrum are often calculated by means of a standard right half-plane spectral sequence. In the case at hand, *i.e.,* for $KH(R)$, the edge maps are the canonical maps $K_q(R) \to KH_q(R)$, induced by $\mathbf{K}^B(R) \to KH(R)$, and the spectral sequence specializes to yield:

Theorem 12.3. *For each ring R there is an exhaustive convergent right half-plane spectral sequence:*

$$E^1_{p,q} = N^p K_q(R) \Rightarrow KH_{p+q}(R).$$

The edge map from $E^1_{0,q} = K_q(R)$ to $KH_q(R)$ identifies $E^2_{p,0}$ with the strict homotopization $[K_p](R)$ of $K_p(R)$, defined in 11.1.

The phrase "exhaustive convergent" in 12.3 means that for each n there is a filtration $0 \subseteq F_0KH_n(R) \subseteq \cdots \subseteq F_{p-1}KH_n(R) \subseteq F_pKH_n(R) \subseteq \cdots$ with union $KH_n(R)$, zero for $p<0$, and isomorphisms $E^\infty_{p,q} \cong F_pKH_n(R)/F_{p-1}KH_n(R)$ for $q = n-p$. (A discussion of Convergence may be found in [WHomo, 5.2.11].)

As pointed out in 11.8, we will see in chapter V, 6.3 that regular rings are K_q-regular for all q, *i.e.,* that $N^pK_q(R) = 0$ for every q and every $p > 0$. For such rings, the spectral sequence 12.3 degenerates at E^1, showing that the edge maps are isomorphisms. We record this as follows:

COROLLARY 12.3.1. *If R is regular noetherian, then $\mathbf{K}(R) \simeq KH(R)$. In particular, $K_n(R) \simeq KH_n(R)$ for all n.*

For the next application, we use the fact that if R is K_i-regular for some i, then it is K_q-regular for all $q \le i$. If $i \le 0$, this was proven in III, 4.2.3. For $i = 1$ it was shown in III, Ex. 3.9. In the remaining case $i > 1$, the result will be proven in chapter V.

COROLLARY 12.3.2. *Suppose that the ring R is K_i-regular for some fixed i. Then $KH_n(R) \cong K_n(R)$ for all $n \le i$, and $KH_{i+1}(R) = [K_{i+1}]R$.*

If R is K_0-regular then $KV_n(R) \cong KH_n(R)$ for all $n \ge 1$, and $KH_n(R) \cong K_n(R)$ for all $n \le 0$. In this case the spectral sequences of (11.7.1) and 12.3 coincide.

PROOF. In this case, the spectral sequence degenerates below the line $q = i$, yielding the first assertion. If R is K_0-regular, the morphism $\mathbf{KV}(R) \to KH(R)$ induces a morphism of spectral sequences, from (11.7.1) to 12.3, which is an isomorphism on $E^1_{p,q}$ (except when $p = 0$ and $q \le 0$). The comparison theorem yields the desired isomorphism $\mathbf{KV}(R) \to KH(R)\langle 0 \rangle$. $\square$

THEOREM 12.3.3. *If $1/\ell \in R$ then $KH_n(R;\mathbb{Z}/\ell) \cong K_n(R;\mathbb{Z}/\ell)$ for all n.*

PROOF. The proof of 12.3 goes through with finite coefficients to yield a spectral sequence with $E^1_{p,q} = N^pK_q(R;\mathbb{Z}/\ell) \Rightarrow KH_{p+q}(R;\mathbb{Z}/\ell)$. When $1/\ell \in R$ and $p > 0$, the groups $N^pK_q(R)$ are $\mathbb{Z}[1/\ell]$-modules (uniquely ℓ-divisible groups) by 6.7.2. By the Universal Coefficient Sequence 2.2 we have $N^pK_q(R;\mathbb{Z}/\ell) = 0$, so the spectral sequence degenerates to yield the result. $\square$

If I is a non-unital ring, we define $KH(I)$ to be $KH(\mathbb{Z} \oplus I)/KH(\mathbb{Z})$ and set $KH_n(I) = \pi_n KH(I)$. If I is an ideal in a ring R, recall (from 1.11 or Ex. 10.1) that $\mathbf{K}^B(R, I)$ denotes the homotopy fiber of $\mathbf{K}^B(R) \to \mathbf{K}^B(R/I)$; it depends upon R. The following result, which shows that the KH-analogue does not depend upon R, is one of the most important properties of KH-theory.

THEOREM 12.4 (EXCISION). *Let I be an ideal in a ring R. Then $KH(I) \to KH(R) \to KH(R/I)$ is a homotopy fibration. Thus there is a long exact sequence*

$$\cdots \to KH_{n+1}(R/I) \to KH_n(I) \to KH_n(R) \to KH_n(R/I) \to \cdots$$

PROOF. Let $KH(R, I)$ denote the homotopy fiber of $KH(R) \to KH(R/I)$. By standard simplicial homotopy theory, $KH(R, I)$ is homotopy equivalent to $|\mathbf{K}^B(R[\Delta^\cdot], I[\Delta^\cdot])|$. It suffices to prove that $KH(I) \to KH(R, I)$ is a homotopy equivalence.

We first claim that $KH_n(I) \to KH_n(R, I)$ is an isomorphism for $n \le 0$. For each $p \ge 0$, let $\mathbf{K}^B(R[\Delta^p], I[\Delta^p])\langle 0 \rangle$ be the 0-connected cover of $\mathbf{K}^B(R[\Delta^p], I[\Delta^p])$, and define $\mathbf{K}^{\le 0}(R[\Delta^p], I[\Delta^p])$ by the termwise "Postnikov" homotopy fibration:

$$\mathbf{K}^B(R[\Delta^p], I[\Delta^p])\langle 0 \rangle \to \mathbf{K}^B(R[\Delta^p], I[\Delta^p]) \to \mathbf{K}^{\le 0}(R[\Delta^p], I[\Delta^p]).$$

Let C_R denote the geometric realization of $\mathbf{K}^{\le 0}(R[\Delta^\cdot], I[\Delta^\cdot])$. Comparing the standard spectral sequence for C_R and the spectral sequence of Theorem 12.3, we see that $KH_n(R, I) \cong \pi_n(C_R)$ for all $n \le 0$. By Exercise 10.1, the ring map

$A = \mathbb{Z} \oplus I \to R$ induces $\pi_n \mathbf{K}^{\le 0}(A[\Delta^p], I[\Delta^p]) \cong \pi_n \mathbf{K}^{\le 0}(R[\Delta^p], I[\Delta^p])$ for all n and p. Hence we have homotopy equivalences for each p and hence a homotopy equivalence on realizations, $C_A \simeq C_R$. The claim follows.

For $n > 0$, we consider the homotopy fiber sequence $\mathbf{KV}(R, I) \to KH(R, I) \to C_R$, where $\mathbf{KV}(R, I)$ is the geometric realization of $\mathbf{K}^B(R[\Delta^\cdot], I[\Delta^\cdot])\langle 0\rangle$. Comparing with the spectrum $\mathbf{KV}(R)$ defined in 11.4.2, we see that $\mathbf{KV}(R, I)$ is the 0-connected cover of the homotopy fiber of $\mathbf{KV}(R) \to \mathbf{KV}(R/I)$.

The theorem now follows when $R \to R/I$ is a GL-fibration, since in this case $KV_n(I) \cong KV_n(R, I)$ for all $n \ge 1$ by 11.10. Combining this with the above paragraph, the 5-lemma shows that in this case $KH(I) \simeq KH(R, I)$, as required.

An important GL-fibration is given by the non-unital map $xR[x] \to R$ (or the unital $\mathbb{Z} \oplus xR[x] \to \mathbb{Z} \oplus R$) with kernel ΩR; see Ex. 11.6(c). In the following diagram, the bottom two rows are homotopy fibration sequences by the previous paragraph, and the terms in the top row are defined so that the columns are homotopy fibrations:

$$\begin{array}{ccccc}
KH(\mathbb{Z} \oplus \Omega R, \Omega I) & \longrightarrow & KH(\mathbb{Z} \oplus xR[x], xI[x]) & \longrightarrow & KH(R, I) \\
\downarrow & & \downarrow & & \downarrow \\
KH(\Omega R) & \longrightarrow & KH(xR[x]) & \longrightarrow & KH(R) \\
\downarrow & & \downarrow & & \downarrow \\
KH(\Omega R/I) & \longrightarrow & KH(xR/I[x]) & \longrightarrow & KH(R/I).
\end{array}$$

Since $KH(xR[x])$ is contractible (by 12.2), the top middle term is contractible, and we have a natural homotopy equivalence $\Omega KH(R) \simeq KH(\Omega R)$. Since the top row must also be a homotopy fibration, we also obtain a natural homotopy equivalence $\Omega K(R, I) \to KH(\mathbb{Z} \oplus \Omega R, \Omega I)$. Applying π_n yields isomorphisms $KH_{n+1}(R, I) \cong KH_n(\mathbb{Z} \oplus \Omega R, \Omega I)$ for all n.

Now suppose by induction on $n \ge 0$ that, for all rings R' and ideals I', the canonical map $I' \to R'$ induces an isomorphism $KH_n(I') \cong KH_n(R', I')$. In particular, $\Omega I \to \Omega R$ induces $KH_n(\Omega I) \cong KH_n(\mathbb{Z} \oplus \Omega R, \Omega I)$. It follows that the map from $A = \mathbb{Z} \oplus I$ to R induces a commutative diagram of isomorphisms:

$$\begin{array}{ccccc}
KH_{n+1}(I) & = & KH_{n+1}(A, I) & \xrightarrow{\simeq} & KH_n(\mathbb{Z} \oplus \Omega A, \Omega I) \\
 & & \downarrow & & \downarrow \simeq \\
 & & KH_{n+1}(R, I) & \xrightarrow{\simeq} & KH_n(\mathbb{Z} \oplus \Omega R, \Omega I).
\end{array}$$

This establishes the inductive step. We have proven that for all R and I, $KH_n(I) \cong KH_n(R, I)$ for all n, and hence $KH(I) \simeq KH(R, I)$, as required. $\square$

COROLLARY 12.5. *If I is a nilpotent ideal in a ring R, then the spectrum $KH(I)$ is contractible and $KH(R) \simeq KH(R/I)$. In particular, $KH_n(I) = 0$ and $KH_n(R) \cong KH_n(R/I)$ for all integers n.*

PROOF. By Ex. II.2.5, I is K_0-regular, and $K_n(I) = 0$ for $n \le 0$ (see III, Ex. 4.3). By 12.3.2 and 11.11, we have $KH_n(I) \cong KV_n(I) = 0$ for $n > 0$, and $KV_n(I) = 0$ for $n \le 0$. Since $KH_n(I) = 0$ for all n, $KH(I)$ is contractible. The remaining assertions now follow from Excision 12.4. $\square$

EXAMPLE 12.5.1. Let R be a commutative Artinian ring, with associated reduced ring $R_{\text{red}} = R/\text{nilradical}(R)$. As R_{red} is regular, we see from 12.5 and 12.3.1 that $KH_n(R) \cong K_n(R_{\text{red}})$ for all n. In particular, $KH_0(R) = K_0(R) = H^0(R)$ and $KH_n(R) = 0$ for $n < 0$.

EXAMPLE 12.5.2. Let R be a 1-dimensional commutative noetherian ring. Then $KH_0(R) \cong H^0(R) \oplus [\text{Pic}]R$, $KH_{-1}(R)$ is torsionfree, and $KH_n(R) = 0$ for all $n \leq -2$. This follows from 12.3.2 and Ex. III.4.4, which states that R is K_{-1}-regular and computes $K_n(R)$ for $n \leq 0$. An example in which $KV_1(R) \to KH_1(R)$ is not onto is given in Ex. 12.2.

If in addition R is seminormal, then R is Pic-regular by Traverso's theorem I.3.12. In this case we also have $KH_0(R) = K_0(R) = H^0(R) \oplus \text{Pic}(R)$ and $KV_1(R) \cong KH_1(R)$.

COROLLARY 12.6 (CLOSED MAYER-VIETORIS). *Let $R \to S$ be a map of commutative rings, sending an ideal I of R isomorphically onto an ideal of S. Then there is a long exact Mayer-Vietoris sequence (for all integers n):*

$$\cdots \to KH_{n+1}(S/I) \to KH_n(R) \to KH_n(R/I) \oplus KH_n(S) \to KH_n(S/I) \to \cdots$$

Recall (10.4) that $\mathbf{K}^B(R)$ is the homotopy colimit of a diagram of spectra $L^q\mathbf{K}(R)$. Since geometric realization commutes with homotopy colimits, at least up to weak equivalence, we have $KH(R) = \text{colim}_q\, |L^q\mathbf{K}(R[\Delta^\cdot])|$.

DEFINITION 12.7. Let X be a scheme. Using the functorial nonconnective spectrum $\mathbf{K}^B$ of 10.6, let $KH(X)$ denote the (fibrant) geometric realization of the simplicial spectrum $\mathbf{K}^B(X \times \Delta^\cdot)$, where $\Delta^\cdot = \text{Spec}(R[\Delta^\cdot])$ as in 11.3. For $n \in \mathbb{Z}$, we write $KH_n(X)$ for $\pi_n KH(X)$.

LEMMA 12.8. *For any quasi-projective scheme X we have:*
1. *$KH(X) \simeq KH(X \times \mathbb{A}^1)$.*
2. *$KH(X \times \text{Spec}(\mathbb{Z}[x, x^{-1}]) \simeq KH(X) \times \Omega^{-1}KH(R)$, i.e.,*

$$KH_n(X[x, x^{-1}]) \cong KH_n(X) \oplus KH_{n-1}(X) \quad \textit{for all } n.$$

3. *If X is regular noetherian, then $\mathbf{K}(X) \simeq KH(X)$. In particular, $K_n(X) \simeq KH_n(X)$ for all n.*

PROOF. The proof of 11.5.1 goes through to show (1). From the Fundamental Theorem (see 10.6), we get (2). We will see in V.6.13.2 that if X is a regular noetherian scheme then $K(X) \simeq K(X \times \mathbb{A}^1)$ and hence $\mathbf{K}^B(X) \simeq \mathbf{K}^B(X \times \mathbb{A}^1)$. It follows that $KH(X) = \mathbf{K}^B(X \times \Delta^\cdot)$ is homotopy equivalent to the constant simplicial spectrum $\mathbf{K}^B(X)$. $\square$

EXERCISES

12.1 *Dimension shifting.* Fix a ring R, and let $\Delta^d(R)$ denote the coordinate ring $R[t_0,\dots,t_d]/(f)$, $f = t_0\cdots t_d(1-\sum t_i)$ of the d-dimensional tetrahehron over R. Show that for all n, $KH_n(\Delta^d(R)) \cong KH_n((R) \oplus KH_{d+n}(R)$. If R is regular, conclude that $KH_n(\Delta^d(R)) \cong K_n(R) \oplus K_{d+n}(R)$, and that $K_0(\Delta^d(R)) \cong K_0(R) \oplus K_d(R)$. *Hint:* Use the Mayer-Vietoris squares of III.4.3.1, where we saw that $K_j(\Delta^n(R)) \cong K_{j+1}(\Delta^{n-1}(R))$ for $j<0$. In III, Ex. 4.8 we saw that each $\Delta^n(R)$ is K_0-regular if R is.

12.2 (KV_1 need not map onto KH_1.) Let k be a field of characteristic 0, I the ideal of $S = k[x,(x+1)^{-1}]$ generated by x^2, and $R = k \oplus I$. Show that $KH_n(R) \cong KH_n(S)$ for all n, but that there is an exact sequence $0 \to KV_1(R) \to KH_1(R) \to \mathbb{Z} \to 0$. *Hint:* Use the Mayer-Vietoris sequence for $R \to S$ and apply I.3.12 to show that $K_0(R) = \mathbb{Z} \oplus k/\mathbb{Z}$, $NK_0(R) \cong tk[t]$ and $N^2K_0(R) \cong t_1t_2k[t_1,t_2]$. Alternatively, note that $KV_1(I) = 0$ by Ex. 11.14.

12.3 The seminormalization R^+ of a reduced commutative ring R was defined in I, Ex. 3.15. Show that $KH_n(R) \cong KH_n(R^+)$ for all n. *Hint:* show that KH is invariant under subintegral extensions.