Corrections to "Intro. to Homological Algebra" by C. Weibel

Cambridge University Press, 1994

These are corrections made from the hardback version to the 1995 paperback version. Please consult the list of corrections for that version for additional corrections!

p.6, line 7 of Def.1.2.1: "non-abelian" should be "non-additive"

p.13 line -1: $Z_{n-1}(b)$ should be $Z_{n-1}(B)$

p.27 line 7: the contour integral should be $\frac{1}{2\pi i}\oint f'(z)dz/f(z)$.

p.44: Exercise 2.4.1 "Show that $L_0F(f) = F(f)$ under ..."

p.47, display after "must equal": F'_{n-1} should be F''_{n-1}

p.47 line -9: $L_{m+1}(A)$ should be $L_{m+1}F(A)$.

p.47 line -6: the m^{th} *syzygy*

p.66 line 9: $pB = 0$ should be $pb = 0$.

p.85: In Exercise 3.5.3, add "for torsionfree B."

p.89 (last line): H^* should be H^n.

p.90 (line 2): add ", which is a torsionfree abelian group."

p.120 line 4: "in a concentration camp" should be "as a prisoner of war"

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p.135 **Exercises to add to end of section 5.4**

Exercise 5.4.3 (Shifting or Décalage) Given a filtration F on a chain complex C, define two new filtrations $\tilde{F}$ and Dec$F$ on C by $\tilde{F}_pC_n = F_{p-n}C_n$ and $(\text{Dec}F)_pC_n = \{x \in F_{p+n}C_n : dx \in F_{p+n-1}C_{n-1}\}$. Show that the spectral sequences for these three filtrations are isomorphic after reindexing: $E^r_{pq}(F) \cong E^{r+1}_{p+n,q-n}(\tilde{F})$ for $r \geq 0$, and $E^r_{pq}(F) \cong E^{r-1}_{p-n,q+n}(\text{Dec}F)$ for $r \geq 2$.

Exercise 5.4.4 (Eilenberg-Moore) Let $f : B \to C$ be a map of filtered chain complexes. For each $r \geq 0$, define a filtration on the mapping cone cone(f) 1.5.1 by

$$F_p\text{cone}(f)_n = F_{p-r}B_{n-1} \oplus F_pC_n.$$

Show that $E^r_p(\text{cone}f)$ is the mapping cone of $f^r : E^r_p(B) \to E^r_p(C)$. By 1.5.2 this gives a long exact sequence

$$\cdots E^r_{p+r}(\text{cone}f) \to E^r_p(B) \to E^r_p(C) \to E^r_p(\text{cone}f) \cdots .$$

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p.136 expand paragraph before 5.5.4 to read:

"When the filtration is not bounded below, convergence is more delicate. Of course we have to work within an abelian category such as R-mod, because we need axiom $(AB4^*)$ in order to even talk about E^∞ (see 5.2.8). But there are more basic problems. For example, the filtration on $H_*(C)$ need not be Hausdorff. ..."

p.140 **Theorem to add to end of 5.5**

The following result generalizes the Comparison Theorem 5.2.12 to non-convergent spectral sequences.

Eilenberg-Moore Comparison Theorem 5.5.11 Let $f : B \to C$ be a map of filtered complexes of modules, where both B and C are complete and exhaustive. Fix $r \geq 0$. Suppose that $f^r : E^r_{pq}(B) \cong E^r_{pq}(C)$ is an isomorphism for all p and q. Then $f : H_*(B) \to H_*(C)$ is an isomorphism.

Proof Consider the filtration on the mapping cone complex given by the formula $F_p\text{cone}(f) = F_{p-r}B[-1] \oplus F_pC$. This filtration is complete and exhaustive. Since f^r is an isomorphism, the long exact sequence of Exercise 5.4.4 shows that $E^r_{pq}(\text{cone}f) = 0$ for all p and q. By 5.5.10, this spectral sequence converges to $H_*\text{cone}(f)$. Hence cone(f) is an exact complex, and 1.5.4 applies. $\diamond$

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p.146 line 5 (Ex. 5.7.1): Show that ... quasi-isomorphism in $\mathcal{A}$; when A isn't bounded below, you will need to assume axiom (AB4) holds.

p.152 The second display should read:

$$R\text{-}\mathbf{mod} \xrightarrow{\text{Hom}_R(S,-)} S\text{-}\mathbf{mod} \xrightarrow{\text{Hom}_S(B,-)} \mathbf{Ab}.$$

p.155 (line above 5.9.4): $\oplus H_{p+q}$

p.191 line -3: complex of

p.203 lines 1–2 When $\mathbb{F}_q$ is a finite field, and $(n,q) \neq (2,2),(2,3),(2,4),(2,9),(3,2),(3,4),(4,2)$, we know that $H_2(SL_n(\mathbb{F}_q);\mathbb{Z}) = 0$ [Suz, 2.9]. With these exceptions, it follows that

p.213 Exercise 6.11.11: ... Show that for $i \neq 0$:

$$H^i(G;\mathbb{Z}) = \begin{cases} \mathbb{Z}_{p^\infty} & i = 2 \\ 0 & \text{else.} \end{cases}$$

p.226: Line 3 of Exercise 7.3.5 should read: δ-functors (assuming that that k is a field, or that N is a projective k-module):

p.236 line 1: ... gives an extension of $\mathfrak{g}$ (it isn't central).

p.245: In Exercise 7.8.3, $\mathfrak{sl}_n$ should be $\mathfrak{sl}_2$.

p.250 line -4: formula should read $= \sum i\kappa(x_i, y_{-i})$.

p.262, correct to read: **Exercise 8.2.3** Show that $\Delta[n]$ is not fibrant if $n \neq 0$. Then show that any fibrant simplicial set X is either constant (8.1.1) or has a nondegenerate "n-cell" $x \in X_n$ for every n (8.1.6).

p.296: In Exercise 8.8.6, line 5 should read "If k is a field"

p.301 lines 4–5: the ranges should be "if $0 < i \leq n$" and "if $i = n+1$" respectively.

p.307 line -1 should read: As $\mathrm{Tor}_1^{R^e/k}(R^e, M) = 0$, the long exact relative Tor sequence (Lemma 8.7.8) yields

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p.313: The following corrects, and should replace, the text between 9.3.1 and 9.3.5. It amounts to replacing 'smooth' by 'quasi-free' in 9.3.3 and stating the smooth version correctly as 9.3.4.

Here is a variant of the Classification Theorem 9.3.1 when R is a commutative k-algebra. If a commutative algebra E is a Hochschild extension of R by an R–R bimodule M, then M must be *symmetric* in the sense that $rm = mr$ for every $m \in M$ and $r \in R$. A moment's thought shows that symmetric bimodules are the same thing as R-modules.

If we choose a k-splitting $\sigma : R \to E$ for a commutative Hochschild extension, then the corresponding factor set f must satisfy $f(r_1, r_2) = f(r_2, r_1)$, because $\sigma(r_1)$ and $\sigma(r_2)$ must commute in E. Let us call such a factor set *symmetric*. If f is a symmetric factor set, the equation $(*)$ shows that multiplication in E is commutative.

Let us write $H^2_s(R, M)$ for the submodule of $H^2(R, M)$ consisting of the equivalence classes of symmetric factor sets. With this notation, we can summarize the above discussion as follows

Commutative Extensions 9.3.1.1 *Let R be a commutative k-algebra and M an R-module. Then the equivalence classes of commutative Hochschild extensions of R by M are in 1–1 correspondence with the elements of the module $H^2_s(R, M)$.*

Remark Let k be a field. This classification, together with Exercise 8.8.4, proves that $H^2_s(R, M)$ is just the André-Quillen cohomology $D^1(R, M)$. The characteristic zero version of this was given in 8.8.9.

9.3.2 We say that a k-algebra is *quasi-free* (over k) if for every square-zero extension $0 \to M \to E \xrightarrow{\varepsilon} T \to 0$ of a k-algebra T by a T–T bimodule M and every algebra map $\nu : R \to T$, there exists a k-algebra homomorphism $u : R \to E$ lifting ν in the sense that $\varepsilon u = \nu$. For example, it is clear that every free algebra is quasi-free over k.

$$\begin{array}{ccccccccc} & & & & k & \to & R & & \\ & & & & \downarrow & \swarrow & \downarrow \nu & & \\ 0 & \to & M & \to & E & \underset{\varepsilon}{\to} & T & \to & 0 \end{array}$$

If R is quasi-free and J is a nilpotent ideal in another k-algebra E, then every algebra map $R \to E/J$ may be lifted to a map $R \to E$. In fact, we can lift it successively to $R \to E/J^2$, to $R \to E/J^3$, and so on. Since $J^m = 0$ for some m, we eventually lift it to $R \to E/J^m = E$.

Proposition 9.3.3 (J.H.C. Whitehead-Hochschild) *If k is a field, then a k-algebra R is quasi-free iff and only if $H^2(R,M)=0$ for all R–R bimodules M.*

Proof If R is quasi-free, ... (proof is the same until last sentence) ...
Quantifying over all such M proves that R is quasi-free. $\diamond$

Corollary 9.3.3.1 (no change; same as old 9.3.4)

Exercise 9.3.1 (no change)

9.3.1 Smooth Algebras

For the rest of this section, all the algebras we consider will be commutative.

We say that a commutative k-algebra is *smooth* (over k) if for every square-zero extension $0 \to M \to E \xrightarrow{\varepsilon} T \to 0$ of commutative k-algebras and every algebra map $\nu : R \to T$, there exists a k-algebra homomorphism $u : R \to E$ lifting ν in the sense that $\varepsilon u = \nu$. For example, it is clear that every polynomial algebra $R = k[x_1, ..., x_n]$ is smooth over k.

Proposition 9.3.4 (Whitehead-Hochschild) *Let R be an algebra over a field k. Then R is smooth if and only if $H^2_s(R,M)=0$ for all R-modules M.*

If R is smooth, then any surjection $E \to R$ of commutative k-algebras with nilpotent kernel J must be split by a k-algebra injection $\sigma : R \to E$.

Proof The proof of the Whitehead-Hochschild result 9.3.3, and the arguments in 9.3.2, go through with no changes for commutative algebras. $\diamond$

Exercise 9.3.2 (no change)

Exercise 9.3.3 (no change)

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p.315 line -1: "K–K bimodule" should read "K-module."

p.316 lines 1,2: The three occurrences of H^2 should all be H^2_s in this proof.

p.325, line 2 of 9.4.11: $\lambda_i = 2^i - 2$

p.367, 9.10.10: Feigin's name should be removed from this Theorem.

p.380: In 10.3.4(2), "Øre" should be "Ore" (for Øystein Ore). This misspelling also occurs on: p.381 (line -1); p.382 (line 2); p.383 (line 14); p.386 (line 15).

p.383: There is an error in Lemma 10.3.13. To correct it, the last two sentences of Definition 10.3.12 should be rewritten to read: "For legibility, we will write $S^{-1}\mathcal{B}$ for $(S\cap\mathcal{B})^{-1}\mathcal{B}$. $\mathcal{B}$ is called a *localizing subcategory* of $\mathcal{C}$ (for S) if the natural functor $S^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$ is fully faithful. That is, if it identifies $S^{-1}\mathcal{B}$ with the full subcategory of $S^{-1}\mathcal{C}$ on the objects of $\mathcal{B}$." Then Lemma 10.3.13 should be changed to

Lemma 10.3.13 *A full subcategory $\mathcal{B}$ of $\mathcal{C}$ is localizing for S iff (1) holds. Condition (2) implies that $\mathcal{B}$ is localizing if S is locally small on the left, and condition (3) implies that $\mathcal{B}$ is localizing if S is locally small on the right.*

1. *For each B and B' in $\mathcal{B}$, the colimit $Hom_{S\cap\mathcal{B}}(B,B')$ (taken in $\mathcal{B}$) maps bijectively to the colimit $Hom_S(B,B')$ (taken in $\mathcal{C}$).*
2. *(no change)*
3. *(no change)*

Proof The statement that $S^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$ is fully faithful means that the morphisms coincide (A.2.3), which by the Gabriel-Zisman Theorem 10.3.7 is assertion (1). Part (2) states that every left fraction $B \leftarrow C \to B''$ is equivalent to a fraction $B \leftarrow B' \to B''$, which must lie in the full subcategory $\mathcal{B}$. In particular, if two left fractions $\mathcal{B}$ are equivalent via a fraction $B \leftarrow C \to B''$ with C in $\mathcal{C}$, they are equivalent via a fraction with C in $\mathcal{B}$. Thus (2) implies (1) when S is locally small on the left. Replacing 'left' by 'right' and citing 10.3.8 proves that (3) implies (1) when S is locally small on the right. $\diamond$

p.424, line 10: show that $\mathcal{A}^I$ is a category when I is small.)

Index

p.437 should read: Connes' double complex $\mathcal{B}$, 345ff

p.439 add entry under “double chain complex”:
Connes’ — $\mathcal{B}$. *See* Connes’ double complex.
p.441 indent “∂-functor” entry.
p.443 should read: left exact functor, 27. *See* exact functor.
p.447 should read: right exact functor, 27. *See* exact functor.

References

[Mat] is out of order

[May] May, J.P. *Simplicial Objects in Algebraic Topology.* Princeton: Van Nostrand, 1967
[Spal] Spaltenstein, N. “Resolutions of unbounded complexes.” *Compositio Math.* **65** (1988): 121–154.