MATH 552 NOTES – LECTURE 9

Let I be a filtered poset[1]; this means that for every $i, j \in I$ there is a $k \in I$ such that $k < i$ and $k < j$. If $G : I \to$ **Groups** is a functor, the inverse limit $\lim_I G_i$ is the subgroup of all (g_i) in $\prod_I G_i$ such that for every $i < j$ in I, $G_i \to G_j$ sends g_i to g_j. It has natural projections $\lim_I G_i \longrightarrow G_i$ which are compatible with the maps $G_i \to G_j$.

The inverse limit has the following universal property: for every group H and every compatible system of maps $p_i : H \to G_i$, there is a unique map $H \to \lim G_i$ such that each p_i is $H \to \lim G_i \longrightarrow G_i$.

Profinite groups: A *profinite group* is a group G that is the inverse limit of a filtered poset of finite groups, i.e., $G = \lim_{i \in I} G_i$, where I is a filtered poset.

Example 1. The groups $\mathbb{Z}/p^n$ form an inverse system; the bonding maps are the natural surjections $\mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1}$. The inverse limit is the p-adic integers $\mathbb{Z}_p$. We may think of elements of $\mathbb{Z}_p$ as formal power series $r = \sum_{i \geq 0} a_i p^i$, where each $a_i \in \{0, 1, ..., p-1\}$; the projection to $\mathbb{Z}/p^n$ sends r to $\sum_{i=0}^{n-1} a_i p^i$. The natural map $\mathbb{Z} \to \mathbb{Z}_p$ induced by the projections $\mathbb{Z} \to \mathbb{Z}/p^i$ is an injection, since each integer is $r = ap^n$ for some n and $a \nmid p$.

In fact, $\mathbb{Z}_p$ is a DVR with the obvious ring structure (the maximal ideal is generated by p).

Profinite completion: If G is any group, let $\{H_i\}$ denote the family of normal subgroups H_i of finite index. Then the G/H_i form an inverse system of finite groups and the inverse limit $\widehat{G}$ is a profinite group. The map $G \to \widehat{G}$ is called the *profinite completion* of G, because if $f : G \to G_2$ is any map with G_2 profinite there is a unique map $\widehat{G} \to G_2$ such that f factors as $G \to \widehat{G} \to G_2$. That is, the inclusion of the category (!) of profinite groups into **Groups** has the profinite completion as its right adjoint: for every profinite H,

$$\mathrm{Hom}_{pro\mathbf{Groups}}(H, \widehat{G}) \cong \mathrm{Hom}_{\mathbf{Groups}}(H, G).$$

Examples 2. (i) $\mathbb{Z}_p$ is the profinite completion of $\mathbb{Z}_{(p)}$ (all rational numbers of the form a/b with b not divisible by p).
(ii) The profinite completion of $\mathbb{Z}$ is the product $\prod_p \mathbb{Z}_p$.

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[1]some authors call it a *cofiltered* poset

(iii) The profinite completion of $\mathbb{Q}/\mathbb{Z}$ is zero.
(iv) If $\bar{F}$ is the algebraic closure of F, then $\mathrm{Gal}(\bar{F}/F)$ is a profinite group. In particular, $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ is $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$.

The *separable closure* of F, F_{sep}, is the largest subfield of $\bar{F}$ which is separable over F; it equals $\bar{F}$ if $\mathrm{char}(F) = 0$ or if $\mathrm{char}(F) > 0$ and F is perfect. Note that $\mathrm{Gal}(F_{sep}/F)$ equals $\mathrm{Gal}(\bar{F}/F)$.

Topology: Forgetting the group structure, the set underlying each finite group is a compact topological space (with the discrete topology). Since the inverse limit of compact spaces is compact Hausdorff, profinite sets (and hence profinite groups) are compact Hausdorff topological spaces. They are also *totally disconnected*, meaning that every point is a connected component.

In fact, profinite sets are the same thing as totally disconnected, compact Hausdorff topological spaces. Similarly, profinite groups are the same thing as totally disconnected, compact Hausdorff topological groups.

Theorem 3 (Fundamental Theorem of Galois Theory)**.** *There is a 1–1 correspondence between the topologically closed subgroups H of $G = \mathrm{Gal}(\bar{F}/F)$ and the set of intermediate subfields K, $F \subseteq K \subseteq F_{sep}$ where $K = F_{sep}^H$ and $H = \{g \in G : (\forall x \in K) g(x) = x\}$.*

Remark 4. If H is a topologically open subgroup H of G, then H has finite index in G; these correspond to the finite field extensions of F. Finally, every open normal subgroup H is also closed, and $[G : H]$ is finite; the open normal subgroups of G correspond to the finite Galois extensions of F.