① Show that the clutching map for the tangent bundle on S^2 has degree 2.

<u>Solution:</u> Let $f: S^1 \to GL_2(\mathbb{R})$ be the clutching function that generates TS^2, i.e.,

$$TS^2 \simeq \underbrace{D^2_+ \times \mathbb{R}^2 \sqcup D^2_- \mathbb{R}^2}_{\sim} \quad \text{where } (x,v) \sim (x, f(x)(v)) \text{ for } \begin{array}{l} (x,v) \in D^2_- \times \mathbb{R}^2 \\ (x, f(x)(v)) \in D^2_+ \times \mathbb{R}^2 \end{array}$$

Consider the vector field (x,v) defined on $D^2_+ \times \mathbb{R}^2$ given by choosing a tangent vector v_+ at the north pole and transporting it along each meridian while maintaining a constant angle with the meridian.

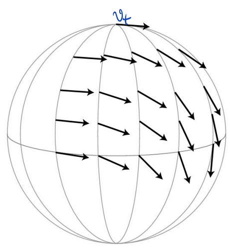

Fig. from Hatcher

Rotate v_+ counterclockwise to obtain w_+ and repeat the process. Doing the same for v_-, w_- at the south pole, obtained by reflecting across the plane of the equator, we obtain a vector field on S^2 which give us trivializations of TS^2.

By identifying and reconstructing TS^2, for each $x \in S^1$ at the equator

$$(v_-, w_-) \longmapsto f(x).(v_-, w_-) = (v_+, w_+).$$

As we go around counterclockwise, starting from a point where the trivializations agree, the angle of rotation increases from 0 to 4π. If we parametrize by $\theta \in [0, 2\pi]$, then $f(\theta)$ is a rotation by 2θ.

② Compute $KO(S^n)$ for $n = 0, 1, 2, 3$.

$n = 0$	$\widetilde{KO}(\{\pm 1\}) = \widetilde{KO}(\{1\} \sqcup \{-1\}) = \mathbb{Z}$	$\Rightarrow KO(S^0) = \mathbb{Z} \oplus \mathbb{Z}$
$n = 1$	$\widetilde{KO}(S^1) = \mathbb{Z}/2\mathbb{Z}$	$\Rightarrow KO(S^1) = \mathbb{Z} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$
$n = 2$	$\widetilde{KO}(S^2) = \mathbb{Z}/2\mathbb{Z}$	$\Rightarrow KO(S^2) = \mathbb{Z} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$
$n = 3$	$\widetilde{KO}(S^3) = 0$ since $\pi_2(O_k) = 0$ for all k	$\Rightarrow KO(S^3) = \mathbb{Z}$

③ Compute $K(X \vee Y \vee Z)$ if X, Y, Z are connected.

Since $Vect(X \vee Y \vee Z) = Vect(X) \times Vect(Y) \times Vect(Z)$, we get

$$\tilde{K}(X \vee Y \vee Z) = \tilde{K}(X) \oplus \tilde{K}(Y) \oplus \tilde{K}(Z) \quad \Rightarrow \quad K(X \vee Y \vee Z) = \mathbb{Z} \oplus K(X) \oplus K(Y) \oplus K(Z)$$