1. If n is odd, show that the tangent bundle T to S^n has a non-trivial section.

Let $n = 2k+1$ be odd and consider S^n embedded in $\mathbb{R}^{n+1}$. Define the auxiliary vector $\upsilon : S^n \to S^n$ by

$$(x_0, x_1, \ldots, x_{2k+1}) \mapsto (x_1, -x_0, x_3, -x_2, \ldots, x_{2k+1}, -x_{2k})$$

so that $x \cdot \upsilon(x) = 0$ for all $x \in S^n$, which indicates that the following map is well defined

$$\begin{aligned} s : S^n &\to TS^n \\ x &\mapsto (x, \upsilon(x)) \end{aligned}$$

Lastly, observe that $p \circ s = \mathrm{id}_{S^n}$, so that s is a section.

2. If $E \to X$ has patching maps g_{ij}, show that the patching maps $\det(g_{ij})$ define a line bundle.

Let $g_{ij} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{R})$ be the patching maps of $E \to X$ and n the dimension of the fiber. Let $G_{ij} = \det(g_{ij})$ be the patching maps for $E' \to X$, so that whenever $U_i \cap U_j \neq \emptyset$, the map

$$\varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n$$

has the form $(x, v) \mapsto (x, G_{ij} v) = (x, \det(g_{ij}) v)$. Note that $G_{ij} : U_i \cap U_j \to GL(k, \mathbb{R})$ is just scalar multiplication, so $k = 1$.

Moreover, note that $G_{ii} = \det(g_{ii}) = \det I_n = 1$ and since

$$g_{ij} g_{jk} g_{ki} = I_n$$

it follows that $G_{ij} G_{jk} G_{ki} = 1$ whenever $U_i \cap U_j \cap U_k \neq \emptyset$, so the maps G_{ij} are indeed patching maps for a line bundle.

Remark: The bundle is given by $\wedge^n E \to X$ since $\wedge^n g_{ij} = \det g_{ij} = G_{ij}$ and $\dim(\wedge^k E) = \binom{n}{k}$ for a n-dimensional E, so $\dim \wedge^n E = 1$.

3. Show that a map $f : X \to Gr_n$ determines a vector bundle on X.

Let E be the canonical bundle over Gr_n, $\pi : E \to Gr_n$ and consider the pullback of f as in

$$\begin{array}{ccc} f^*E & \xleftarrow{\ f^*\ } & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\ f\ } & Gr_n \end{array}$$

which yields the surjectivity of p and the structure of the fiber. Moreover, for any $x \in X$, there is a local trivialization around $f(x) \in Gr_n$ given by (U, φ), i.e., an open neighborhood $U \subset Gr_n$ of $f(x)$ and an isomorphism

$$\varphi : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

such that $(\pi \circ \varphi)(x, v) = x$ for all $v \in \mathbb{R}^n$ and $v \mapsto \varphi(x, v)$ is an isomorphism between $\mathbb{R}^n$ and $\pi^{-1}(x)$.

Note that $V = f^{-1}(U)$ is an open neighborhood of x and let $\psi : V \times \mathbb{R}^n \to p^{-1}(V)$ given by

$$\begin{array}{ccccc} V \times \mathbb{R}^n = f^{-1}(U) \times \mathbb{R}^n & \xrightarrow{f \times id} & U \times \mathbb{R}^n & \xrightarrow{\varphi} & \pi^{-1}(U) \\ & \overset{\psi}{\dashrightarrow} & & & \downarrow f^* \\ & & & & p^{-1}(U) \end{array}$$

which yields a local trivialization on $x \in X$.