Solutions for Hatcher, Ch.2B #1, Ch.3.2 #3(a), 7.

2B.1 Compute $H_i(S^n - X)$ when X is homeomorphic to either $S^k \vee S^l$ or $S^k \sqcup S^l$.

Solution: To start, remember Proposition 2B.1 which states that

(a) for an embedding $h : D^k \to S^n$, $\tilde{H}_i(S^n - h(D^k)) = 0$ for all i.

(b) for an embedding $h : S^k \to S^n$, $\tilde{H}_i(S^n - h(S^k))$ is $\mathbb{Z}$ for $i = n - k - 1$ and 0 otherwise when $k < n$.

I shall abuse notation and write simply $S^n - D^k$ or $S^n - S^k$.

(a) Suppose $X \cong S^k \vee S^l$ and let $A = S^n - S^k$ and $B = S^n - S^l$. Then if $\{\star\}$ denotes the common point,

$$A \cup B = S^n - \{\star\} \cong \mathbb{R}^n, \text{ and } A \cap B = S^n - X. \tag{1}$$

By using the Mayer-Vietoris sequence,

$$\cdots \to \tilde{H}_{i+1}(A \cup B) \to \tilde{H}_i(A \cap B) \to \tilde{H}_i(A) \oplus \tilde{H}_i(B) \to \tilde{H}_i(A \cup B) \to \cdots \tag{2}$$

and the fact that

$$\tilde{H}_i(A \cup B) \cong \tilde{H}_i(\mathbb{R}^n) = 0, \ \forall i$$

we obtain simply

$$0 \to \tilde{H}_i(S^n - X) \to \tilde{H}_i(S^n - h(S^k)) \oplus \tilde{H}_i(S^n - h(S^l)) \to 0,$$

which gives, for all i,

$$\tilde{H}_i(S^n - X) \cong \tilde{H}_i(S^n - h(S^k)) \oplus \tilde{H}_i(S^n - h(S^l))$$

(b) Suppose $X \cong S^k \sqcup S^l$. Take A' as the connected component of $(S^k)^C$ which contains S^l and B' the same but with reversed roles. Let $A = A' \setminus S^l$ and $B = B' \setminus S^k$. See Figure 1.

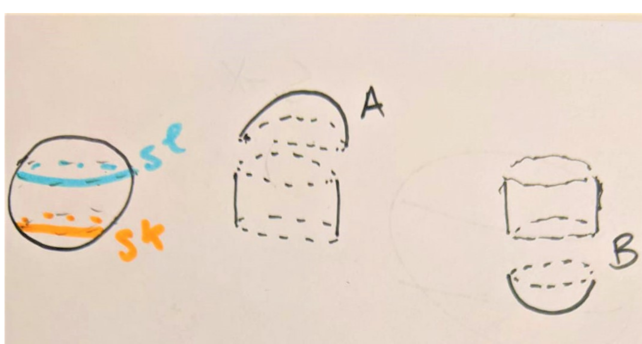

Figure 1: A poor drawing trying to explain.

Then

$$A \approx D^n - S^l,\ B \approx D^n - S^k,\ A \cup B = S^n - X,\ \text{and } A \cap B \approx S^{n-1}.$$

By using the Mayer-Vietoris sequence (2), we obtain

$$\begin{array}{ccccccc}
\tilde{H}_{i+1}(A) \oplus \tilde{H}_{i+1}(B) & \longrightarrow & \tilde{H}_{i+1}(A \cup B) & \longrightarrow & \tilde{H}_i(A \cap B) & \longrightarrow & \tilde{H}_i(A) \oplus \tilde{H}_i(B) \\
\downarrow = & & \downarrow = & & \downarrow = & & \downarrow = \\
\tilde{H}_{i+1}(D^n - S^l) \oplus \tilde{H}_{i+1}(D^n - S^k) & \longrightarrow & \tilde{H}_{i+1}(S^n - X) & \longrightarrow & \tilde{H}_i(S^{n-1}) & \longrightarrow & \tilde{H}_i(D^n - S^l) \oplus \tilde{H}_i(D^n - S^k) \\
\downarrow = & & \downarrow = & & \downarrow = & & \downarrow = \\
0 & \longrightarrow & \tilde{H}_{i+1}(S^n - X) & \longrightarrow & \tilde{H}_i(S^{n-1}) & \longrightarrow & 0
\end{array}$$

which yields

$$\tilde{H}_{i+1}(S^n - X) \cong \tilde{H}_i(S^{n-1}).$$

3.2.3 (a) Show that there is no map $\mathbb{RP}^n \to \mathbb{RP}^m$ inducing a nontrivial map $H^1(\mathbb{RP}^m; \mathbb{Z}_2) \to H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ if $n > m$. What is the corresponding result for maps $\mathbb{CP}^n \to \mathbb{CP}^m$?

Solution: Suppose for the sake of contradiction that $\varphi : \mathbb{RP}^n \to \mathbb{RP}^m$ induces a nontrivial map $\varphi^* : H^1(\mathbb{RP}^m; \mathbb{Z}_2) \to H^1(\mathbb{RP}^n; \mathbb{Z}_2)$. Let α generate $H^1(\mathbb{RP}^m; \mathbb{Z}_2)$ and β generate $H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ so that $\varphi^*(\alpha) = \beta$.

By Theorem 3.19,

$$H^1(\mathbb{RP}^m, \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{m+1}), \quad H^1(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}_2[\beta]/(\beta^{n+1}),$$

but then

$$0 = \varphi^*(\alpha^{m+1}) = (\varphi^*\alpha)^{m+1} = \beta^{m+1}$$

which contradicts $\beta^{m+1} \neq 0$ since $m + 1 < n + 1$.

The same can be done for

$$H^2(\mathbb{CP}^m; \mathbb{Z}) \to H^2(\mathbb{CP}^n; \mathbb{Z})$$

by replacing $k = 1$ for $2k$ and using Theorem 3.19 which gives

$$H^2(\mathbb{CP}^n; \mathbb{Z}) = \frac{\mathbb{Z}[\alpha]}{(\alpha^{n+1})}.$$

3.2.7 Show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$.

Solution: The cohomology of $\mathbb{RP}^3$ are

$$H^*(\mathbb{RP}^3; \mathbb{Z}_2) = \frac{\mathbb{Z}_2[\alpha]}{(\alpha^4)}$$

while for $\mathbb{R}\mathrm{P}^2 \vee S^3$ we have

$$H^*(\mathbb{R}\mathrm{P}^2 \vee S^3; \mathbb{Z}_2) = \frac{\mathbb{Z}_2[\beta]}{(\beta^3)} \times \frac{\mathbb{Z}_2[\gamma]}{(\gamma^2)},$$

which don't have the same structure.