Solutions for Hatcher, Ch.3.1 #8(a,c),9,12,13

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8. (a) Compute $H^i(S^n;G)$ by induction on n in two ways: using the long exact sequence of a pair, and using Mayer-Vietoris sequence.

Solution: Only the induction step.

(LES) Use the long exact sequence of the good pair $(D^{n+1}, \partial(D^{n+1}) \approx S^n)$

$$\cdots \to H^i(D^{n+1};G) \to H^i(S^n;G) \to H^{i+1}(D^{n+1},S^n;G) \to H^{i+1}(D^{n+1};G) \to \cdots$$

which is equivalent to

$$\cdots \to H^i(\star;G) \to H^i(S^n;G) \to H^{i+1}(S^{n+1};G) \to H^{i+1}(\star;G) \to \cdots$$

and yields the pieces

$$0 \to H^i(S^n;G) \to H^{i+1}(S^{n+1};G) \to 0.$$

Thus $H^{i+1}(S^{n+1};G) = H^i(S^n;G)$.

(MV) Use Mayer-Vietoris sequence of $X = S^{n+1}$, A, B the complement of the south/north pole respectively, so $X = int(A) \cup int(B)$ and $A \cap B \approx S^n$. Then the MV-sequence

$$\cdots \to H^i(A;G) \oplus H^i(B;G) \to H^i(A \cap B;G) \to H^{i+1}(X;G) \to H^{i+1}(A;G) \oplus H^{i+1}(B;G) \to \cdots$$

i.e.,

$$\cdots \to H^i(\star;G) \oplus H^i(\star;G) \to H^i(S^n;G) \to H^{i+1}(S^{n+1};G) \to H^{i+1}(\star;G) \oplus H^{i+1}(\star;G) \to \cdots$$

which yields the same sequences as above

$$0 \to H^i(S^n;G) \to H^{i+1}(S^{n+1};G) \to 0.$$

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8. (c) Show that if A is a retract of X then $H^n(X;G) \approx H^n(A;G) \oplus H^n(X,A;G)$.

Solution: Let $r : X \to A$ be a retraction so that it is a right inverse for the inclusion $i : A \hookrightarrow X$. Thus, r^* is a left-inverse for i^*, hold that thought.

Now consider the long exact sequence given by

$$\cdots \to H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\delta} H^{n+1}(X,A;G) \to \cdots$$

which gives us the following short exact sequence since i^* is injetive

$$0 \to H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \to 0$$

and by the thought we've been holding, we can use the Splitting Lemma (p.147).

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9. Show that if $f : S^n \to S^n$ has degree d then $f^* : H^n(S^n; G) \to H^n(S^n; G)$ is multiplication by d.

Solution: Observe that $\deg(f) = d$ implies that $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by d. Now, applying the Universal Coefficient Theorem for Cohomology, we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(S^n), G) & \longrightarrow & H^n(S^n; G) & \xrightarrow{h} & \mathrm{Hom}(H_n(S^n), G) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle (f_*)^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle (f_*)^*} & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(S^n), G) & \longrightarrow & H^n(S^n; G) & \xrightarrow{h} & \mathrm{Hom}(H_n(S^n), G) & \longrightarrow & 0 \end{array}$$

Since $H_{n-1}(S^n)$ is free for any n, $H^n(S^n) = G$, $H_n(S^n) = \mathbb{Z}$ and h is an isomorphism, we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & G & \xrightarrow{h} & \mathrm{Hom}(\mathbb{Z}, G) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle (\cdot d)^*} & & \\ 0 & \longrightarrow & G & \xrightarrow{h} & \mathrm{Hom}(\mathbb{Z}, G) & \longrightarrow & 0 \end{array}$$

and for any $\phi \in \mathrm{Hom}(\mathbb{Z}, G)$

$$(\cdot d)^* (\phi(1)) = \phi(d) = d \cdot \phi(1).$$

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12. Show that $H^k(X, X^n; G) = 0$ if X is a CW complex and $k \leq n$, by using the cohomology version of the second proof of the corresponding result for homology in Lemma 2.34.

Solution: Note that $H^k(X, X^n; G) \approx H^k(X/X^n; G)$. By the Universal Coefficient Theorem for cohomology we have

$$0 \to \mathrm{Ext}(H_{k-1}(X/X^n), G) \to H^k(X/X^n; G) \to \mathrm{Hom}(H_k(X/X^n), G) \to 0,$$

where $H_k(X/X^n) = 0$ for $k \leq n$. (Proof of lemma 2.34) That yields

$$0 \to 0 \to H^k(X/X^n; G) \to 0$$

for $k \leq n$ and the result follows by the initial remark.

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13. Let $\langle X, Y \rangle$ denote the set of basepoint-preserving homotopy classes of basepoint-preserving maps $X \to Y$. Using Proposition 1B.9, show that if X is a connected CW complex and G is an abelian group, then the map $\langle X, K(G,1) \rangle \to H^1(X; G)$ sending a map $f : X \to K(G,1)$ to the induced homomorphism $f_* : H_1(X) \to H_1(K(G,1)) \approx G$ is a bijection, where we identify $H^1(X; G)$ with $\mathrm{Hom}(H_1(X), G)$ via the universal coefficient theorem.

Solution: The Proposition 1B.9 states that there is a bijection between

$$\langle X, K(G,1)\rangle \leftrightarrow \mathrm{Hom}(\pi_1(X), \pi_1(K(G,1))) = \mathrm{Hom}(\pi_1(X), G).$$

If we remember that $H_1(X) = \pi_1(X)^{ab}$, the fact that G is abelian implies that every $\varphi \in \mathrm{Hom}(\pi_1(X), G)$ factors uniquely through $H_1(X)$, the abelianization of $\pi_1(X)$, i.e.,

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\varphi} & G \\ \Big\downarrow h & \overset{\exists!\psi}{\nearrow} & \\ H_1(X) & & \end{array}$$

Therefore,

$$\mathrm{Hom}(\pi_1(X), G) = \mathrm{Hom}(\pi_1(X)^{ab}, G) = \mathrm{Hom}(H_1(X), G),$$

and by the initial remark, we obtain the bijection between $\langle X, K(G,1)\rangle$ and $\mathrm{Hom}(H_1(X), G)$, which is exactly $H^1(X; G)$.