Solutions for Hatcher, Ch.2.1 #15,21,24,29

2.1.15 For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective. Hence for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$.

Solution: Let $\alpha, \beta, \gamma, \delta$ be the corresponding maps. By exactness,

$$\mathrm{Im}(\alpha) = \ker(\beta),\ \mathrm{Im}(\beta) = \ker(\gamma),\ \mathrm{Im}(\gamma) = \ker(\delta).$$

Note that α is surjective iff $\ker(\beta) = B$ iff $\mathrm{Im}(\beta) = 0$, and δ is injective iff $\mathrm{Im}(\gamma) = 0$ iff $\ker(\gamma) = C$. Putting both together, α is surjective and δ is injective iff $C = 0$, since $\mathrm{Im}(\beta) = \ker(\gamma)$.

For the second part, just consider the following two pieces of the long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$$

$$\cdots \to H_{n+1}(A) \to H_{n+1}(X) \to H_{n+1}(X, A) \to H_n(A) \to H_n(X) \to \cdots$$

Note that $H_*(X, A) = 0$ iff the induced map is injective (first) and surjective (second).

2.1.21 Making the preceding problem more concrete, construct explicit chain maps $s : C_n(X) \to C_{n+1}(SX)$ inducing isomorphisms $\tilde{H}_n(X) \to \tilde{H}_{n+1}(SX)$.

Solution: Let $\sigma : \Delta^n \to X$ and $C\sigma : C\Delta^n = \Delta^{n+1} \to CX$ be the cone of σ. Consider the maps

$$C_n(X, *) \to C_{n+1}(CX, X) \to C_{n+1}(SX, *),$$

where the first one is $s_1 : \sigma \mapsto C\sigma$ and the second s_2 identifies X with a single point, which yields SX. While the second map is automatically a chain map, note that we have the commutative diagram

$$\begin{array}{ccc} C_n(X,*) & \xrightarrow{\partial} & C_{n-1}(X,*) \\ \downarrow & & \downarrow \\ C_{n+1}(CX,X) & \xrightarrow{\partial} & C_n(CX,X) \end{array}$$

which implies that the composition $s = s_2 \circ s_1$ is, in fact, a chain map. Now we claim that it induces an isomorphism in homology. Consider the long exact sequence of the triple $(CX, X, *)$

$$\cdots \to H_{n+1}(X, *) \to H_{n+1}(CX, *) \to H_{n+1}(CX, X) \to H_n(X, *) \to \cdots$$

where $\delta : H_{n+1}(CX, X) \to H_n(X, *)$ is the connecting map. Note that the composition

$$H_n(X, *) \xrightarrow{(s_1)_*} H_{n+1}(CX, X) \xrightarrow{\delta} H_n(X, *)$$

is the identity map. That way, δ is an inverse of $(s_1)_*$ and since it is an isomorphism ($H_n(CX, *) = 0$), $(s_1)_*$ yields an isomorphism between $\tilde{H}_n(X)$ and $H(CX, X)$. The fact that $(s_2)_*$ is an iso follows straight from the excision theorem, since it is the collapse of X to a basepoint, so

$$\tilde{H}_{n+1}(CX, X) \cong \tilde{H}_{n+1}(CX/X) = \tilde{H}_{n+1}(SX)$$

and $(s_2 \circ s_1)_*$ is an isomorphism.

Remark: Another way would be to construct s straight from the difference between the embedded north cone and the south cone, say $s(\sigma) = S\sigma|_{north} - S\sigma|_{south}$ where S is the suspension operator.

2.1.24 Show that each n-simplex in the barycentric subdivision of Δ^n is defined by n inequalities $t_{i_0} \leq t_{i_1} \leq \cdots \leq t_{i_n}$ in its barycentric coordinates, where $(i_0, \ldots, i_n)$ is a permutation of $(0, \ldots, n)$.

Solution: Proof by induction. There is nothing to do if $n = 0$, since the barycentric subdivision of $[v_0]$ is $[v_0]$ itself. Suppose it holds for $n \geq 0$.

Consider a $(n + 1)$-simplex in the barycentric subdivision of $\Delta^{n+1} = [v_0, \ldots, v_{n+1}]$, i.e, $[b, w_0, \ldots, w_n]$ where b is the barycenter and $[w_0, \ldots, w_n]$ is an n-simplex in the barycentric subdivision of

$$[v_0, \ldots, \hat{v}_i, \ldots, v_{n+1}]$$

for some $i \in \{0, \ldots, n + 1\}$.

By induction hypothesis, the n-simplex is defined by inequalities $t_{i_0} \leq \cdots \leq t_{i_n}$ in its barycentric coordinates where $i_0, \ldots, i_n$ is a permutation of $(0, \cdots, n)$. Note that $b = 1 \cdot b$ belongs to the $(n + 1)$-simplex and so does any point in the segment connected by a point in $[w_0, \ldots, w_n]$ with b, so we obtain $t_{i_0} \leq \cdots \leq t_{i_n} \leq t_b$ where t_b is the coordinate associated to the barycenter.

Remark: If we consider without loss of generality the n-simplex given by barycentric subdivision

$$\left[v_0, \frac{v_0 + v_1}{2}, \ldots, \frac{\sum_{i=0}^n v_i}{1 + n}\right]$$

then we automatically obtain $t_0 \leq \cdots \leq t_n$. The others n-simplex are only a permutation of the indices.

2.1.29 Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Solution: Apparently, we've computed $H_*(S^1 \times S^1)$ already in Example 2.3, so

$$H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z}, & n = 2 \\ \mathbb{Z}^2, & n = 1 \\ \mathbb{Z}, & n = 0. \end{cases}$$

For the wedge sum, we have

$$\tilde{H}_n(S^1 \vee S^1 \vee S^2) = \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^2)$$

and by noting that $H_n(S^k) = \mathbb{Z}$ for $n = k$ and $n = 0$ and zero otherwise, we obtain the same homology groups.

For the second part, the universal covering space $\mathbb{R}^2$ of the torus $S^1 \times S^1$ is contractible, so $H_0(\mathbb{R}^2) = \mathbb{Z}$ while all others are zero. Thus, we only need one $n \neq 0$ such that $H_n(E)$ is non-trivial for the universal cover E of $S^1 \vee S^1 \vee S^2$.

Note that the inclusion $i : S^2 \hookrightarrow S^1 \vee S^1 \vee S^2$ factors through the universal covering space so we have the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} = H_2(S^2) & \longrightarrow & H_2(E) \\ & \searrow^{i_*} & \downarrow \\ & & H_2(S^1 \vee S^1 \vee S^2) \end{array}$$

where i_* is injetive, so $H_2(S^1 \vee S^1 \vee S^2) \neq 0$.