

Bilinear Forms over a field F

Let V be a vector space. A *bilinear form* on V is a set map $B : V \times V \longrightarrow F$ which is linear in each slot. This means that for $\lambda \in F$ and $x, x', y, y' \in V$ we have

$$\begin{aligned}\lambda B(x, y) &= B(\lambda x, y) = B(x, \lambda y), \\ B(x + x', y) &= B(x, y) + B(x', y) \\ B(x, y + y') &= B(x, y) + B(x, y').\end{aligned}$$

We call B *symmetric* if $B(x, y) = B(y, x)$ and *alternating* if $B(x, y) = -B(y, x)$.

The most famous symmetric bilinear form is the dot product $B(x, y) = x \cdot y$ on F^n . Another symmetric bilinear form is the ‘‘Lorentz metric’’ on \mathbb{R}^4 : $B(x, y) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$. The most famous alternating bilinear form is the cross product $B(x, y) = x \times y = x_1y_2 - x_2y_1$ on F^2 . Every $n \times n$ matrix $A = (a_{ij})$ gives rise to a bilinear form on the vector space F^n of column vectors by the formula $B(x, y) = x^t A y = x \cdot A y$.

A variation of this definition is often used for vector spaces over \mathbb{C} . Let λ^* denote the complex conjugate of $\lambda \in \mathbb{C}$. We call a set map $B : V \times V \longrightarrow F$ *sesquilinear* if it is linear in the second slot and anti-linear in the first; that is we replace the scalar condition for a bilinear form by the condition

$$\lambda B(x, y) = B(\lambda^* x, y) = B(x, \lambda y)$$

We call B *hermitian* if $B(x, y) = B(y, x)^*$. The most famous hermitian form on \mathbb{C}^n is given by $x^* \cdot y = \sum x_i^* y_i$.

Any square matrix A gives rise to a sesquilinear form: $B(x, y) = x^* \cdot A y$. If A is a *hermitian* matrix (a matrix with $A^t = A^*$) then this is a hermitian form. Our discussion will concentrate on bilinear forms, because the sesquilinear/hermitian cases are all proven the same way (with conjugation thrown in where needed).

Proposition. *If $\dim(V) = n$, there is a 1-1 correspondence between bilinear forms and $n \times n$ matrices. The symmetric and alternating bilinear forms correspond to symmetric and alternating matrices.*

There is also a 1-1 correspondence between sesquilinear forms and $n \times n$ matrices, in which the hermitian forms correspond to hermitian matrices.

To make this correspondence, choose a basis e_1, \dots, e_n for V . The matrix $A = (a_{ij})$ associated to a bilinear form B has $a_{ij} = B(e_i, e_j)$. The formula $B(x, y) = x^t A y$ follows from bilinearity of B .

Change of basis. A change of basis for V is carried out by an invertible matrix P . Writing $x = P x_0, y = P y_0$ we see that $x^t A y = x_0^t (P^t A P) y_0$. Thus the change of basis replaces the matrix A by the matrix $P^t A P$.

Warning: the use of A to describe a linear transformation and a bilinear form result in two distinct equivalence relations on matrices: A is *similar* to $P^{-1} A P$ as a linear transformation, and is *equivalent* to $P^t A P$ as a bilinear form (or to $P^{*t} A P$ as a sesquilinear form.)

We call a form B *non-degenerate* if its corresponding matrix A has a nonzero determinant. Note that $\det(A)$ is only well-defined up to a square, since $\det(P^t A P) = \det(A) \det(P)^2$; $\det(A)$ is called the *discriminant* of B . If B is nondegenerate, the discriminant is well-defined in F^*/F^{*2} .

Lemma. *The following are equivalent for a bilinear (or even sesquilinear) form B .*

- (a) B is a non-degenerate form on V
- (b) For $x \neq 0$, the linear map $f : V \rightarrow F$ defined by $f(y) = B(x, y)$ is nonzero.
- (c) Every linear map $f : V \rightarrow F$ is $f(y) = B(x, y)$ for some unique $x \in V$.

Quadratic Forms. A set map $q : V \rightarrow F$ is called a *quadratic form* if $q(\lambda x) = \lambda^2 q(x)$ for $\lambda \in F, x \in V$ and if the function

$$B(x, y) = q(x + y) - q(x) - q(y)$$

is a symmetric bilinear form. B is called the *associated bilinear form* of q .

Every symmetric bilinear form determines a quadratic form, namely $q(x) = B(x, x)$, whose associated bilinear form is $2B$. Thus if $\text{char}(F) \neq 2$ there is a 1-1 correspondence between quadratic forms and symmetric bilinear forms.

Associated to the dot product (a bilinear form) is the quadratic form $q(x) = x \cdot x = \|x\|^2$. This corresponds to the usual Riemannian metric on \mathbb{R}^n . Associated to the Lorentz metric on \mathbb{R}^4 is the quadratic form $q(x) = x_1^2 + x_2^2 + x_3^2 - x_4^2$.

Diagonalizability of symmetric and hermitian forms. *Every real symmetric matrix, and every complex hermitian matrix, is diagonalizable as a linear transformation. That is, V has a basis of eigenvectors for A , and all the eigenvalues of A are real numbers.*

To see that every eigenvalue λ is real, choose an eigenvector x with $\|x\| = 1$ and compute: $\lambda = x^{*t} \cdot Ax = (x^{*t} \cdot Ax)^* = \lambda^*$. If A were not diagonalizable then some eigenvalue λ would have a Jordan block of length ≥ 2 . Thus there would exist a vector x which is not an eigenvector, but such that $y = (A - \lambda)x$ is an eigenvector. But we must have $y^{*t} \cdot y = (y^{*t} \cdot Ax) - \lambda(y^{*t} \cdot x) = (\lambda^* - \lambda)(y^{*t} \cdot x) = 0$, contradiction. A non-diagonalizable symmetric matrix over \mathbb{C} is given below.

Classification of symmetric bilinear forms. *If B is a symmetric bilinear form, and $\text{char}(F) \neq 2$, then there is a basis of V , and an integer r (the rank of B) such that the matrix of B has the diagonal form*

$$\begin{pmatrix} a_1 & & & 0 \\ & \ddots & & \\ 0 & & a_r & \\ & & & 0 \end{pmatrix} \quad B(x, y) = \sum_{i=1}^r a_i x_i y_i,$$

To construct the diagonal matrix, we use the notion of *orthogonality* relative to B . Write $x \perp y$ if $B(x, y) = 0$. If $B = 0$ we are done. Otherwise choose e_1 such that $a_1 = B(e_1, e_1)$ is nonzero, and set $U = e_1^\perp = \{x \in V : x \perp e_1\}$. Then $\dim(U) = \dim(V) - 1$, so by induction on $n = \dim(V)$ we can pick e_2, \dots, e_n to diagonalize B .

The diagonal elements a_i constructed by this process are not well-defined. For example, if we replace e_i by re_i in our basis, the effect is to divide x_i, y_i by r and change a_i to a_i/r^2 . Thus over an algebraically closed field like \mathbb{C} every nondegenerate symmetric bilinear form is the dot product (up to change of basis).

Over \mathbb{R} , this only shows that we can replace each a_i by ± 1 . We will see that the number of $+1$'s and -1 's is an invariant, typically described using the *signature* of B , defined as

$$\text{signature } \sigma = (\text{number of } +1 \text{ entries}) - (\text{number of } -1 \text{ entries}).$$

Sylvester's Law of Inertia. *The rank r and signature σ of a symmetric bilinear form on $V = \mathbb{R}^n$ are well-defined. If B and B_1 are two symmetric forms with the same rank and signature, then they differ only by a change of basis matrix P : $B_1(x, y) = B(Px, Py)$. If $p = (r + \sigma)/2$, there is a basis for V such that*

$$B(x, y) = (x_1y_1 + \dots + x_py_p) - (x_{p+1}y_{p+1} + \dots + x_ry_r).$$

Indeed, if we follow the algorithm classifying symmetric forms, always choosing eigenvectors, then we see that $p = (r + \sigma)/2$ is the number of positive eigenvalues of A , and the number of negative eigenvalues of A is $(r - \sigma)/2$. As an example, the Lorentz metric on \mathbb{R}^4 has rank 4 and signature $\sigma = 2$, so it is really different than the Riemann metric.

Classification of Hermitian forms. *The rank r and signature σ of a Hermitian form is well-defined. If B and B_1 are two Hermitian forms, then they differ only by a change of basis matrix P : $B_1(x, y) = B(Px, Py)$. (The matrices A and A_1 of the forms satisfy $A_1 = P^{*t}AP$.) If $p = (r + \sigma)/2$, there is a basis for V such that*

$$B(x, y) = (x_1y_1 + \dots + x_py_p) - (x_{p+1}y_{p+1} + \dots + x_ry_r).$$

If the rank and signature are the same, the hermitian form is (up to change of basis) the usual dot product $x^{*t} \cdot y$.

Classification of alternating bilinear forms. *If B is an alternating bilinear form, and $\text{char}(F) \neq 2$, then there is a basis of V such that the matrix of B has the block form*

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In particular, if B is a nondegenerate alternating form then $\dim(V)$ is even, and

$$B\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = x_1 \cdot y_2 - x_2 \cdot y_1.$$

Exercises. 1) A real symmetric form (or a complex hermitian form) is called *positive definite* when $B(x, x) > 0$ for all nonzero x . Show that $r = \sigma$, and that there is an invertible matrix P such that $B(x, y) = (Px) \cdot (Py)$.

2) Suppose that a symmetric $n \times n$ matrix A has n distinct eigenvalues λ_i over some field F . Show that there is a basis of eigenvectors e_i for F^n such that the associated bilinear form is described by a diagonal matrix, with $a_i = \lambda_i e_i \cdot e_i$.

3) Let A be a real symmetric (or complex hermitian) matrix. Show that there is basis of orthonormal eigenvectors, i.e., eigenvectors e_i for A such that $e_i \cdot e_j = \delta_{ij}$.

4) Over \mathbb{C} , show that the symmetric matrix $A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ has Jordan form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then show that \mathbb{C}^2 cannot have a basis of orthonormal eigenvectors.

Orthogonal and Unitary matrices. If B is a symmetric bilinear form over a field F , its *orthogonal group* is the group O_B of all invertible matrices P such that $B(Px, Py) = B(x, y)$ for all x, y . When B is the dot product, we write $O_n(F)$ for this group, and can also describe it as $\{P : P^{-1} = P^t\}$. The *special orthogonal group* is the group $SO_n(F)$ of orthogonal matrices with determinant 1; O_n is the semidirect product of SO_n and the group $\{\pm 1\}$.

For example, $O_1(\mathbb{R}) = \{\pm 1\}$, and $SO_2(\mathbb{R})$ is the circle group of all rigid rotations in the plane. More generally, $SO_n(\mathbb{R})$ is the group of rigid rotations in the usual sense about the origin. The entire orthogonal group O_n includes reflections about the origin. The Lorentz group $O_{3,1}$ is the orthogonal group O_B for the Lorentz metric B , and plays a fundamental role in physics.

The *unitary group* $U_n = U_n(\mathbb{C})$ is the corresponding group for the canonical hermitian form: a quick calculation shows that $U_n = \{P : P^{-1} = P^{*t}\}$. For example, $U_1 = \{z \in \mathbb{C} : \|z\| = 1\}$. The *special unitary group* is the subgroup of all unitary matrices with determinant 1; if we write S^1 for the group of $n \times n$ matrices of the form $\{zI_n : \|z\| = 1\}$, then as a group $U_n = S^1 \times SU_n$.

Normal matrices. A matrix A over \mathbb{C} is called *normal* if $AA^{*t} = A^{*t}A$. The main theorem states that A is normal if and only if \mathbb{C}^n has a basis of orthonormal eigenvectors, i.e., eigenvectors e_i for A such that $e_i \cdot e_j = \delta_{ij}$.

Both real orthogonal and complex unitary matrices are trivially normal, the columns of these matrices forming an orthonormal basis. It is also clear that real symmetric and complex hermitian matrices are normal; the orthonormal bases were found in exercise 3 above. Real alternating matrices are also clearly normal, although they have purely imaginary eigenvalues and no real eigenvectors (why?).

If there is an orthonormal basis of eigenvectors u_i for the matrix A , the change-of-basis matrix U has the u_i as its columns. Since $u_i^{*t} \cdot u_j = \delta_{ij}$, U is a unitary matrix. Note that the matrix $U^{-1}AU = U^{*t}AU$ is the diagonal matrix D of eigenvalues of A . A quick calculation shows that A is normal.

Conversely, suppose that A is normal. If we write $A = B + iC$ with $B = \frac{1}{2}(A + A^{*t})$ and $C = \frac{1}{2i}(A - A^{*t})$ both hermitian, then $BC = CB$ and hence B preserves the eigenspaces of C : if $Cv = \lambda v$ then $w = Bv$ satisfies

$$Cw = CBv = BCv = B(\lambda v) = \lambda w.$$

Hence \mathbb{C}^n is the direct sum of subspaces V_{ij} which are eigenspaces for both B and C . By exercise 3, each V_{ij} has an orthonormal basis diagonalizing both B and C , hence A . Concatenating these yields an orthonormal basis of \mathbb{C}^n (why?), and this basis diagonalizes A , B and C .