Math 454 Lecture 5: 7/03/2017

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Contents

1 More probability

Example 1. Flipping n fair coins. If we flip 100 coins, the probability of getting exactly 500 heads is of course

$$\frac{100!}{50!50!}2^{-100},$$

but this doesn't really give us a good idea of the size. We can use Stirling's approximation to get some estimates.

Theorem 1.1. *(Stirling's approximation:)*

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

The notation $\approx$ here means as $n \to \infty$, the ratio of the left -hand side and right-hand side tends to 1. However, Stirling's approximation does an amazing job of approximating the factorial even for small values of n.

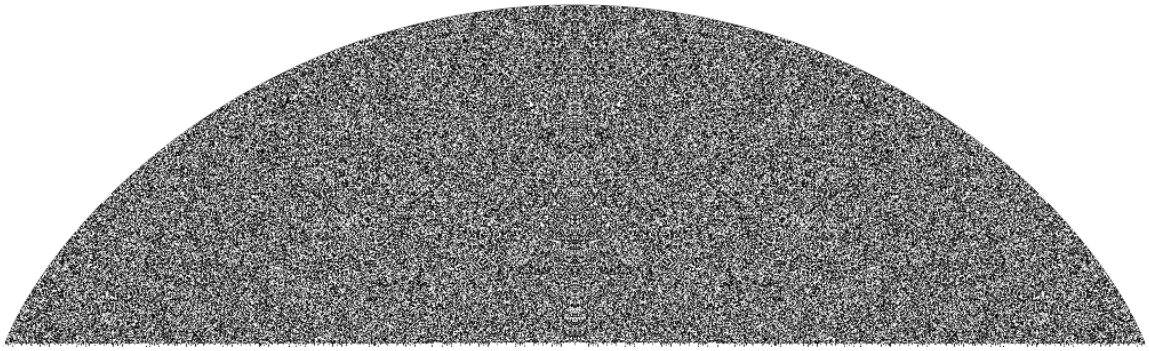


Figure 1: The decimal representations of the 1000th row of Pascal's triangle, flipped sideways. That means that the upper curve of this is roughly the graph of $\log_{10}\binom{1000}{x}$.

Using this, we can say

$$\frac{100!}{50!50!}2^{-100} \approx 2^{-100}\frac{\sqrt{2\pi 100}(100/e)^{100}}{2\pi 50(50/e)^{100}} = \frac{1}{5\sqrt{2\pi}} \approx .0798.$$

If you actually compute $\frac{100!}{50!50!}2^{-100}$ it comes out to about .0795.

In general, we get the formula

Theorem 1.2.

$$\binom{n}{n/2} \approx 2^n\sqrt{\frac{2}{\pi n}}.$$

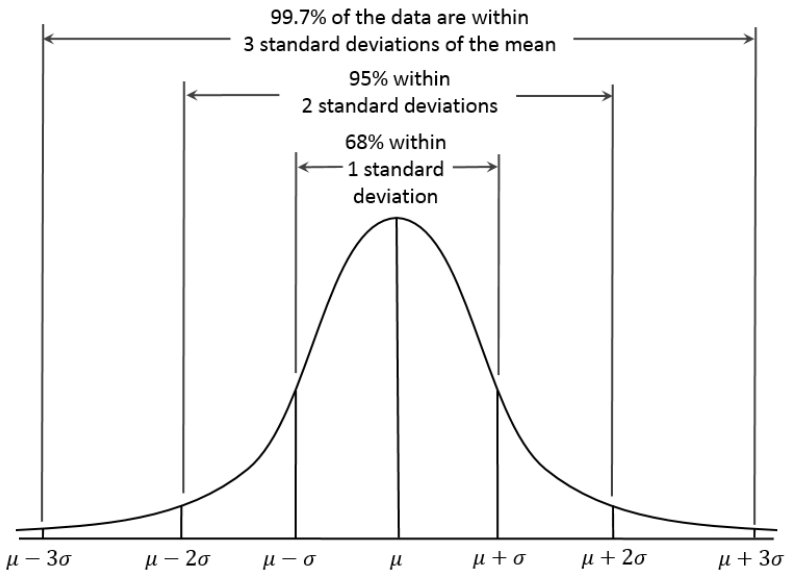
Say we flip 1000 coins instead. If you look at Figure 1, you can see that the top curve achieves a maximum at $k = 500$. This means near $k = 500$, we can approximate $\log\binom{1000}{x}$ by $\alpha - \beta(500-x)^2$ for some numbers α and β, or

$$\binom{1000}{x} \text{ is about } e^{\alpha}e^{-\beta(500-x)^2}.$$

From the theorem, we already know e^{α} is about $\frac{2^{1000}}{\sqrt{500\pi}}$. It turns out (you can also get this from Stirling's approximation) that β should be $1/500$, so

$$\binom{1000}{x} \text{ is about } \frac{2^{1000}}{\sqrt{500\pi}}e^{-\frac{1}{500}(500-x)^2}.$$

In case you haven't seen it before, this is an example of a normal distribution. It has the same shape as the graph in Figure 1, though the peak will be much skinnier in the 1000 coins example.



For k very close to $n/2$, the same kind of approximation works, and so

$$\binom{n}{k} \text{ is about } 2^n \sqrt{\frac{2}{n\pi}} e^{-\frac{2}{n}\left(\frac{n}{2}-k\right)^2}.$$

Example 2. A standard deck of cards has 52 cards. There are 4 "suits"

$$\{\clubsuit, \heartsuit, \spadesuit, \diamondsuit\}$$

of 13 "ranks"

$$\{2, \ldots, 10, \text{Jack, Queen, King, Ace}\}.$$

Each card is identified by its suit and rank, hence the $52 = 4 \times 13$ cards. You don't need to remember the names of the suits and ranks to understand poker - just that there are 4 suits and 13 ranks.

Poker is a game where each player gets a hand of 5 cards, where order doesn't matter. Our experiment is to select a poker hand at random. Our sample space S has size

$$\binom{52}{5} = 2,598,960$$

Events are subsets of poker hands, and some special events are important to the game.

Two pair: "Two pair" is a hand with 2 cards of one rank and 2 cards of another, and one card of a third different rank. The probability of getting a full house is

$$\frac{\text{number of full houses}}{|S|}.$$

The number of full houses can be calculated via the multiplication principle.

1. Choose the single card (52 ways)
2. Choose the two ranks occuring in the pairs. $\binom{12}{2} = 66$ ways, because their order in the hand does not matter. Think indistinguishable bins.
3. Choose the suits of each of the two ranks occuring in the pairs. $\binom{4}{2}\binom{4}{2} = 36$ ways; ranks are distinguishable.

Thus, the probability is

$$\frac{52 \cdot 66 \cdot 36}{2,598,960} = \frac{123,552}{2,598,960} \approx .05$$

1.1 Intersections

Suppose we have two events E_1 and E_2 and we want to find out the probability that E_1 and E_2 occur. The event "E_1 and E_2" is exactly the event $E_1 \cap E_2$, the outcomes that are in the first event and the second event.

Example 3. If we flip 4 coins, what is the probability that the first and second coins are heads?

If we look at our sample space, $E_1 \cap E_2$ is the set of coin flips

$$\{HHHH, HHTH, HHHT, HHTT\}.$$

1.2 Conditional Probability

Suppose Xing has a bag with 6 red marbles, 9 blue marbles, and 5 green marbles. Suppose he draws one at random. The sample space S for this experiment is just the set of possible marbles, of which there are 20.

- What is the probability the marble is red?

  The event R that the marble in his left pocket is red has cardinality 6, since it is just the set of red marbles.

  $$\Pr(R) = 6/20.$$

Suppose he draws two at random, and without looking, he puts one in his left pocket and one in his right pocket.

The sample space S for this experiment is the set of equally likely outcomes (m_1, m_2) where m_1 is the marble that goes in his left pocket and $m_2 \neq m_1$ is the marble that goes in his right pocket.

Since there are 20 marbles, the sample space S has size $20 \cdot 19 = 380$.

- What is the probability that the marble in is left pocket is red and the marble in his right pocket is blue?

  This event has cardinality 6×9, so the probability is

  $$6 \times 9/380.$$

- Suppose Xing takes the marble out of his right pocket and sees that it is blue. Is the probability that the marble in his left pocket is red still 6/20?

Intuitively, it should be slightly higher. We can think of this as an experiment in the new (smaller) sample space S' consisting of all the equally likely events

$$\{(m_1, m_2) : m_1 \neq m_2, m_2 \text{ is blue.}\}.$$

We can calculate $|S'| = 9 \cdot 19$, since we can first choose a blue marble for the right pocket and then any marble for the left pocket.

The event R' that the marble in his left pocket is red is now

$$\{m_1, m_2 : m_1 \neq m_2, m_1 \text{ is red, } m_2 \text{ is blue}\},$$

and $|R'|$ is of course $6 \cdot 9$. This means

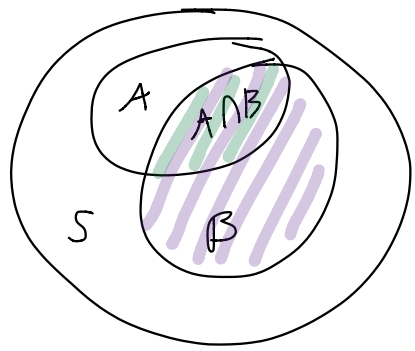
$$\Pr[R'] = \frac{|R'|}{|S'|} = \frac{6 \times 9}{9 \times 19} = 6/19.$$

Definition 1.1 (Bayes's Rule)**.** If A and B are events and B is an event with nonzero probability, then we define the probability of A given B, written $\Pr(A|B)$ to be

$$\Pr(A|B) = \Pr(A \cap B)/\Pr(B).$$

Example 4. For the marbles, we are interested in the probability of "the left marble is red (R) given that the right marble is blue (B)". We can calculate it as

$$\Pr(R|B) = \Pr(R \cap B)/\Pr(B) = \frac{(6 \cdot 9)/(20 \cdot 19)}{9/20} = 6/19.$$



$$\Pr(A|B) = \frac{\quad}{\quad}$$

We can justify this formula in this example. We can think of this as an experiment in the new (smaller) sample space B, given by

$$\{(m_1, m_2) : m_1 \neq m_2, m_2 \text{ is blue.}\}.$$

We can calculate $|B| = 9 \cdot 19$, since we can first choose a blue marble for the right pocket and then any marble for the left pocket.

The event that the marble in his left pocket is red is now $R \cap B$, or

$$\{m_1, m_2 : m_1 \neq m_2, m_1 \text{ is red, } m_2 \text{ is blue}\},$$

and so $|R \cap B|$ is of course $6 \cdot 9$. This means

$$\Pr[R \cap B \text{ in sample space } B] = \frac{|R \cap B|}{|B|} = \frac{6 \times 9}{9 \times 19} = 6/19.$$

However, we can rewrite this as

$$\frac{|R \cap B|}{|B|} = \frac{|R \cap B|}{|S|}\frac{|S|}{|B|} = \frac{\Pr(R \cap B \text{ in sample space } S)}{\Pr(B)},$$

which matches Bayes's rule.

1.3 Independence

Definition 1.2. Two events A and B are independent if $\Pr(B) = \Pr(B|A)$. Meaning, knowing whether A occurred or didn't occur has no bearing on the probability of B. Often this is intuitively clear, such as with coins.

Example 5. Suppose we flip a coin 4 times. The event E_1 that the first flip is heads and the event E_2 that the second flip is heads is independent, because

$$\Pr(E_2|E_1) = \Pr(E_2 \cap E_1)/\Pr(E_1) = \frac{1/4}{1/2} = .5.$$

2 The principle of Inclusion-Exclusion

Suppose we want to find the probability of at least one of two events happening. That is, we have events E_1 and E_2, we want to find $\Pr(E_1 \cup E_2)$, which one might say as "the probability of E_1 or E_2".)

2.1 Disjoint events

If the two events are disjoint (or mutually exclusive), meaning the events cannot happen at the same time, we can simply add the probabilities together. This could be called *the addition principle for probabilities.*

Example 6. If Xing draws a marble from his bag, what is the probability that the marble is red or it is green? The probability that it is red is 6/20 and the probability that it is green is 5/20 and these events cannot both happen, so the probability that is red or green is

$$6/20 + 5/20 = 11/20.$$

This is just the addition principle in diguise, since $R = 6$, $G = 5$ and $R \cap G = \emptyset$, so

$$\Pr(R \cup G) = |R \cup G|/|S| = |R|/|S| + |G|/|S|.$$

We also have the following theorem, which is the subtraction principle in disguise:

Theorem 2.1. *(subtraction principle for probabilities)If E is an event, then $\overline{E}$ is the event $S \setminus E$, or the event that E does not occur. Then*

$$\Pr(\overline{E}) = 1 - \Pr(E).$$

Proof. $\overline{E}$ and E are disjoint. Apply the addition principle for probabilities. □

2.2 Two overlapping sets (not just about probability!)

What if $E_1 \cap E_2 \neq \emptyset$? We want to figure out $|E_1 \cup E_2|$. This We want to count every element in $E_1 \cup E_2$ once. We could try $|E_1| + |E_2|$, but this expression adds 1 for every element in $E_1 \setminus E_2 \cap E_1$ (those elements in E_1 but not E_2) and $E_2 \setminus E_2 \cap E_1$ (those elements in E_2 but not E_1), but it adds a 2 for those elements in $E_1 \cap E_2$. To rectify this, we can add a new term $-|E_1 \cap E_2|$. This gives us the theorem:

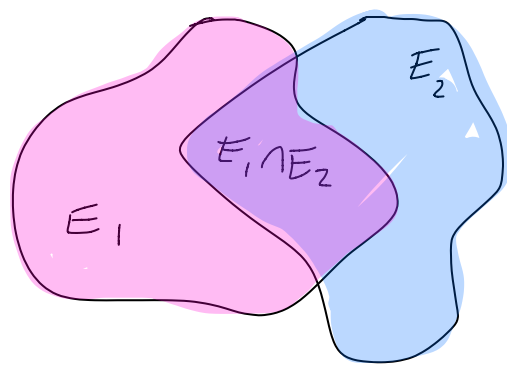


Figure 2: Venn diagram of E_1 and E_2.

Theorem 2.2 (Inclusion-Exclusion, 2 sets)**.** . *If E_1 and E_2 are finite sets,*

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|.$$

Example 7. A high school has 50 students who have taken French and 50 who have taken Spanish, and 20 who have taken both. How many have taken a language?

If the set F is the set of French takers, the set S is the set of Spanish takers, then set $S \cap F$ is those who have taken both and $S \cup F$ is those who have taken a language. From the principle of inclusion-exclusion,

$$|S \cup F| = 50 + 50 - 20 = 80$$

people have taken a language.

2.3 Three overlapping sets

How can we count $A \cup B \cup C$? Again, we could try $|A| + |B| + |C|$. This contributes 1 for elements in exactly one of the sets, but the elements who are in at least two of the sets contribute at least two to this sum, so we subtract $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$. Now elements who are in exactly two of the sets contribute 1 (as we hoped), but those who are in all three contribute $3 - 3 = 0$. So, we add $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Now we can check that this is gives the right answer for elements who are in exactly one, exactly two, or exactly three sets.

Theorem 2.3. *(Inclusion-Exclusion for 3 sets) If A B and C are finite sets, then*

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

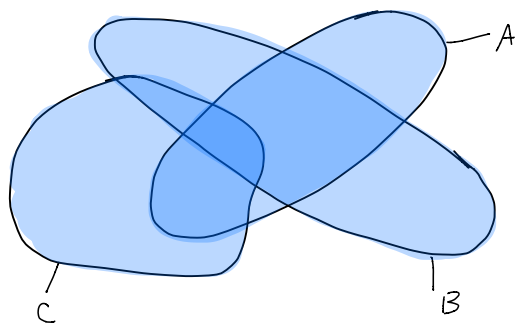


Figure 3: Venn diagram of A, B, C.

Example 8.

Since the probability of an event is exactly the size of the event divided by the sample space, both of these principles work exactly the same for probability.

Theorem 2.4. *If A, B, and C are events in a sample space, then*

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C).$$

$\Pr(A \cup B \cup C)$ *can be read "the probability of A or B or C".*

2.4 *Avoiding* Overlapping Sets

What if instead of calculating the number of objects in A or B or C (this is $|A \cup B \cup C|$), we want to count objects that are in *none* of A, B, or C? This is only meaningful if A, B, C are subsets of some larger subset S. In this case, we are interested in calculating the size of

$$(S \setminus A) \cap (S \setminus B) \cap (S \setminus C).$$

In fact, this is the same set as

$$S \setminus (A \cup B \cup C).$$

In words, the objects that are not in A, not in B, and not in C are the are the objects that are not in at least one of A or B or C. The objects that are in at least one of A, B, C is of course $A \cup B \cup C$. By the subtraction principle,

$$S \setminus (A \cup B \cup C) = |S| - |A \cup B \cup C|.$$

This gives us another way of stating the Principle of Inclusion - Exclusion:

Theorem 2.5 (Principle of Inclusion-Exclusion, subtraction version)**.**

$$\begin{aligned}&(S \setminus A) \cap (S \setminus B) \cap (S \setminus C)\\ = |S| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|.\end{aligned}$$

The sum above is equal to the number of elements of S that are in none of A, B, or C.

Example 9. How many integers between 1 and 1000 (inclusive) are *not* multiples of $5, 6$, or 8?

Let $S = [1000]$. A be set of the multiples of 5 between 1 and 1000, and B be the set of multiples of 6 between 1 and 1000, and C the multiples of 8 between 1 and 1000.

We are interested in $(S \setminus A) \cap (S \setminus B) \cap (S \setminus C)$, so we can apply the subtraction version of the principle of inclusion-exclusion.

- $|A| = 1000/5 = 200$, $|B| = \lfloor 1000/6 \rfloor = 166$, $|C| = 1000/8 = 125$.
- $A \cap B$ is the number divisible by both 5 and 6, which means the numbers divisible by their least common multiple, which is 30. Therefore $|A \cap B| = \lfloor 1000/30 \rfloor = 33$. A cap C is the numbers divisible by 40, so $|A \cap C| = 25$. $|B \cap C|$ is the numbers divisible by 24, which is $\lfloor 1000/24 = 41$.
- Finally, $|A \cap B \cap C|$ is the numbers divisible by the lcm of 5, 6, 8, which is the lcm of 40 and 6, which is 120. Thus $|A \cap B \cap C| = \lfloor 1000/120 \rfloor = 8$.

Using the formula from Theorem 2.5,

$$(S \setminus A) \cap (S \setminus B) \cap (S \setminus C) = 1000 - 200 - 166 - 125 + 41 + 25 + 33 - 8 = 600.$$

2.5 Inclusion-Exclusion for Unions of Many Sets

Inclusion-Exclusion for 2 and 3 sets followed a pattern; it turns out the same pattern holds for many sets.

Theorem 2.6 (I.E. for many sets)**.** *Suppose $A_1, \dots A_n$ are finite sets. Then*

$$\begin{aligned}
|A_1 \cup A_2 \dots A_n| = & \\
& |A_1| + |A_2| + \cdots + |A_n| \\
& - \sum\nolimits_{|\{i_1,i_2\}|=2} |A_{i_1} \cap A_{i_2}| \\
& + \sum\nolimits_{|\{i_1,i_2,i_3\}|=3} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\
& \vdots \\
& +(-1)^{k-1} \sum\nolimits_{|\{i_1,\dots i_k\}|=k} |A_{i_1} \cap \cdots \cap A_{i_k}| \\
& \vdots \\
& +(-1)^{n-1} |A_1 \cap \cdots \cap A_n|.
\end{aligned}$$

In words, to compute the cardinality of $|A_1 \cup A_2 \dots A_n|$, one adds up n terms, alternating in sign, where the absolute value of the kth term is the sum of the cardinalities of all $\binom{n}{k}$ many intersections of collections of k of the sets $A_1, \dots A_n$.

Proof. Later. (same idea as for 3 or 2 sets). □

We can also use this combined with the subtraction principle to compute the number of objects avoiding all sets $A_1, \dots A_n$.

Theorem 2.7 (I.E. for the avoiding many sets)**.** *If S is a finite set, and $A_1, \dots A_n$ are subsets of S, then the number of elements of S* not in any of *$A_1, \dots A_n$, which is*

$$|(S \setminus A_1) \cap (S \setminus A_2) \cap \cdots \cap (S \setminus A_n)| = |S \setminus (A_1 \cup \cdots \cup A_n)| = |S| - |A_1 \cup \cdots \cup A_n|$$

is equal to

$$\begin{aligned}
& |S| - |A_1| - |A_2| - \cdots - |A_n| \\
& + \sum\nolimits_{|\{i_1,i_2\}|=2} |A_{i_1} \cap A_{i_2}| \\
& - \sum\nolimits_{|\{i_1,i_2,i_3\}|=3} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\
& \vdots \\
& +(-1)^{k} \sum\nolimits_{|\{i_1,\dots i_k\}|=k} |A_{i_1} \cap \cdots \cap A_{i_k}| \\
& \vdots \\
& +(-1)^{n} |A_1 \cap \cdots \cap A_n|.
\end{aligned}$$

Proof. This follows immediately from the previous theorem; subtracting just changes the signs. $\square$

Example 10. (r-combinations of multisets) Recall the pirate problem from homework 2. Let's try it for a different treasure chest:

A pirate finds a treasure chest containing 4 diamonds, 6 pearls, 5 doubloons, and 6 rubies. Unfortunately, he can only smuggle 16 objects past TSA. How many different hauls can he take (of exactly 16 items)?

Solution: This is the same as the number of 16-combinations of the multiset

$$\{4 \cdot \diamondsuit, 6 \cdot \circ, 5 \cdot \$, 6 \cdot r\},$$

which we know is the same as the number of integer solutions of the equation

$$\begin{aligned} & x_1 + x_2 + x_3 + x_4 = 16 \\ \text{such that } & 0 \leq x_1 \leq 4 \\ & 0 \leq x_2 \leq 6 \\ & 0 \leq x_3 \leq 5 \\ & 0 \leq x_4 \leq 6. \end{aligned}$$

We can use the principle of inclusion-exclusion for this. Let A_1 be the set of solutions in which $x_1 \geq 5$, A_2 be the set of solutions in which $x_2 \geq 7$, A_3 the set of solutions where $x_3 \geq 6$, A_4 the set where $x_4 \geq 7$. If S is the set of integer solutions to

$$\begin{aligned} & x_1 + x_2 + x_3 + x_4 = 16 \\ \text{such that } & 0 \leq x_1 \\ & 0 \leq x_2 \\ & 0 \leq x_3 \\ & 0 \leq x_4, \end{aligned}$$

then

$$S \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)$$

is our answer, because $A_1 \cup A_2 \cup A_3 \cup A_4$ is exactly the set of solutions where one of the variables is too large (the pirate took more of that commodity than actually existed). By

the principle of inclusion-exclusion for avoiding many sets,

$$\begin{aligned}
&|S| - |A_1| - |A_2| - |A_3| - |A_4| \\
&+ \sum\nolimits_{|\{i_1,i_2\}|=2} |A_{i_1} \cap A_{i_2}| \\
&- \sum\nolimits_{|\{i_1,i_2,i_3\}|=3} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\
&+(-1)^4 |A_1 \cap \cdots \cap A_4|,
\end{aligned}$$

which is

$$\begin{aligned}
&|S| - |A_1| - |A_2| - |A_3| - |A_4| \\
&+|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4| \\
&-|A_1 \cap A_2 \cap A_3| - |A_1 \cap A_2 \cap A_4| - |A_1 \cap A_3 \cap A_4| - |A_2 \cap A_3 \cap A_4| \\
&+(-1)^4 |A_1 \cap \cdots \cap A_4|.
\end{aligned}$$

Finding each and every cardinality above is an *easy* integer solution problem; let's see $|A_1|$. Well, we know A_1 is the number of integer solutions to

$$\begin{aligned}
& x_1 + x_2 + x_3 + x_4 = 16 \\
\text{such that } & 5 \leq x_1 \\
& 0 \leq x_2 \\
& 0 \leq x_3 \\
& 0 \leq x_4,
\end{aligned}$$

which is the same as the number of solutions to

$$\begin{aligned}
& x_1 + x_2 + x_3 + x_4 = 11 \\
\text{such that } & 0 \leq x_1 \\
& 0 \leq x_2 \\
& 0 \leq x_3 \\
& 0 \leq x_4,
\end{aligned}$$

which we now by our stars and bars/integer solutions theorem to be

$$|A_1| = \binom{11+4-1}{4-1} = \binom{14}{3}.$$

Next let's find $|A_1 \cap A_2|$; this one will be the number of integer solutions to

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 16 \\ \text{such that } 5 &\leq x_1 \\ 7 &\leq x_2 \\ 0 &\leq x_3 \\ 0 &\leq x_4, \end{aligned}$$

which we know as the number of integer solutions to

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 4 \\ \text{such that } 0 &\leq x_1 \\ 0 &\leq x_2 \\ 0 &\leq x_3 \\ 0 &\leq x_4, \end{aligned}$$

or

$$|A_1 \cap A_2| = \binom{7}{3}.$$

You can imagine how the rest of the terms will go; we end up with the answer

$$\begin{aligned} &\binom{19}{3} \\ &-\left(\binom{14}{3} + \binom{11}{3} + \binom{13}{3} + \binom{8}{3}\right) \\ &+\left(\binom{7}{3} + \binom{8}{3} + \binom{7}{3} + \binom{6}{3} + \binom{6}{3} + \binom{5}{3}\right) \\ &-(0+0+0+0) \\ &+0, \end{aligned}$$

which turns out to be 55.