Math 454 Lecture 4: 6/29/2017

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1 Balls in Bins

All the problems we have done so far can be summarized by counting ways to put distinguishable or undistinguishable balls in distinguishable or undistinguishable bins. These problems behave very differently depending on distinguishability. First let's clear up what distinguishable and indistinguishable mean.

An arrangement of balls into bins is some placement of balls into bins (which we have also called parts, or subsets). One key thing is *order inside a bin never matters*, because the bins are subsets. For example, if we have bin A and bin B and balls $\{1, \dots 4\}$, then $A = \{2, 3, 1\}$ and $B = \{4\}$ is the same as $A = \{1, 2, 3\}$ and $B = \{4\}$.

1.1 Distinguishable or indistinguishable balls

Distinguishable means labels matter. For example, we can label n distinguishable balls with $\{1, \dots n\}$, or the elements of any n-element set. If the balls are distinguishable, then the arrangement

$$A = \{2, 3, 1\} \text{ and } B = \{4\}$$

is *different* from

$$A = \{2, 3, 4\} \text{ and } B = \{1\}.$$

However, if the balls are *indistinguishable*, then the labels don't matter, so

$$A = \{2, 3, 1\} \text{ and } B = \{4\}$$

is *the same as*

$$A = \{2, 3, 4\} \text{ and } B = \{1\}.$$

If labels don't matter, we might as well not have them, so we may as well just write

$$A = \{3 \cdot \bullet\} \text{ and } B = \{1 \cdot \bullet\}.$$

1.2 Distinguishable vs indistinguishable bins (corrected!)

In the previous section, our bins were distinguishable. This means their labels mattered. If our two bins are distinguishable, the arrangement

$$A = \{2, 3\} \text{ and } B = \{1, 4\}$$

is *different from*

$$B = \{2, 3\} \text{ and } A = \{1, 4\}.$$

Another way to say this is that we have a *sequence* of bins. So our first configuration is $(\{2, 3\}, \{1, 4\})$, and our second is $(\{1, 4\}, \{2, 3\})$, which are clearly different.

On the other hand, if the bins are *indistinguishable*, then their labels don't matter. For instance,

$$A = \{2, 3\} \text{ and } B = \{1, 4\}$$

is *the same as*

$$B = \{2, 3\} \text{ and } A = \{1, 4\}.$$

However,

A way to put this on sound footing is that we only care about the *set* of bins, rather than the sequence. Here our first arrangement is the set $\{\{2, 3\}, \{1, 4\}\}$, and our second is $\{\{1, 4\}, \{2, 3\}\}$, which are the same.

1.3 How to count (almost) anything (corrected!)

See Figure 1.

	K distinguishable bins: any size	K distinguishable bins: sizes $n_1, n_2 \dots n_K$, $n_1 + \dots + n_K = n$	K indistinguishable bins: any size	K indistinguishable bins: sizes $\underbrace{n_1 \dots n_1}_{K_1 \text{ times}}, \underbrace{n_2 \dots n_2}_{K_2 \text{ times}}, \dots, \underbrace{n_\ell \dots n_\ell}_{K_\ell \text{ times}}$, $\sum_{i=1}^{\ell} K_i = K$, $\sum_{i=1}^{\ell} K_i n_i = n$
n distinguishable balls	$\boxed{K^n}$ # of sequences of length n with entries from K. K = 2: # subsets of set S where $\lvert S\rvert = n$	$\binom{n}{n_1, \dots, n_K} = \frac{n!}{n_1! \dots n_K!}$ # of r-permutations of $\{n_1 \cdot 1, n_2 \cdot 2, \dots n_K \cdot K\}$ # of sequences of length n with n_i entries equal to i for $i \in [K]$. coefficient of $x_1^{n_1} \dots x_K^{n_K}$ in $(x_1 + \dots + x_K)^n$.	? You aren't expected to know this yet # partitions of an n-elt. set into ≤ K NONEMPTY unlabelled parts. $\sum_{K=0}^{n} \underbrace{\left\{ \begin{matrix} n \\ K \end{matrix} \right\}}$ Stirling # of 2^{nd} Kind = # partitions of a set into exactly K nonempty unlabelled parts.	$\frac{n!}{(n_1!)^{K_1} \dots (n_\ell)^{K_\ell} K_1! \dots K_\ell!}$ # of partitions of n-element set into K unlabelled subsets (parts) where K_1 of the subsets has size n_1, K_2 have size n_2, and so on.
n indistinguishable balls	$\binom{n+K-1}{K-1} = \binom{n+K-1}{n}$ # of integer solutions of $x_1 + \dots + x_n = n$, $x_i \geq 0$ for all i # n-combinations of $\{\infty \cdot 1, \dots \infty \cdot K\}$ # n-combinations of $\{n \cdot 1, \dots n \cdot K\}$	1	? You aren't expected to know this yet # partitions of integer n into ≤ K nonzero parts = $\underbrace{\text{\# partitions of integer } n+K \text{ into exactly } K \text{ nonzero parts}}$ called $P_K(n+K)$ → K^{th} partition # of n+K.	1

Figure 1: The definition of Stirling number I gave in class was wrong. This is the correct one. Also, note the difference between the new indistinguishable bins column and the old indistinguishable bins column.

Here are a few examples.

distinguishable balls, distinguishable bins, fixed sizes: How many ways are there to distribute 14 numbered lottery balls to Alice, Bob, Carol, and Dale such that Alice gets 3, Bob gets 4, Carol gets 2, and Dale gets 5?

distinguishable balls, distinguishable bins, any sizes: How many ways are there to schedule 8 different classes in 3 periods? (allowing periods to be used more than once)

indistinguishable balls, distinguishable bins, any sizes: How many ways are there for a pirate to distribute 50 blue-raspberry lollipops between his first mate, his parrot, and his mother?

distinguishable balls, indistinguishable bins, fixed sizes (corrected!): How many meals with 3 plates of food (one with 4 ingredients, one with 3 ingredients, and one with 3 ingredients) can we make with the 10 ingredients of baby octopus, bok choy, oyster sauce, smoked paprika, duck breast, green onions, ginger, honey, prunes, animal crackers, and cream cheese?
Solution: There is 1 bin of size 4 and there are 2 bins of size 3. Thus, we apply the formula from the table with $n_1 = 4$, $k_1 = 1$, $n_2 = 3$, $k_2 = 2$ to obtain

$$\frac{10!}{4!(3!)^2 1!2!}$$

meals.

2 Finite Probability

Chapter 10 of Keller and Trotter has a very nice introduction to probability.

Finite probability concerns experiments with a finite number of outcomes. We will assume our outcomes are all equally likely. The set of all outcomes is called the sample space, and subsets of the sample space are called events.

Example 1. Pat spins a wheel with equally sized sections labeled $1, 2, 3, 4, 5, 6, 7, 8, 9$, as in Figure 2. The sample space for this experiment is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. One event is $A = \{1, 2, 3\}$, which in English would be called "spinning a one, two, or three."

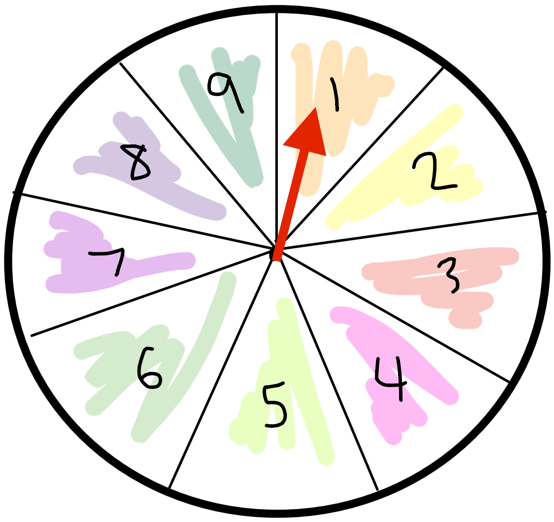

Figure 2: A wheel for Pat to spin.

Example 2. Suppose we flip a coin three times. The sample space is the set of all possible outcomes, namely the set

$$\{HHH, THH, HTH, HHT, TTH, THT, HTT, TTT\}.$$

Definition 2.1. Suppose S is a sample space and E is an event. The *probability* of E, written

$$\Pr(E),$$

is equal to

$$|E|/|S|.$$

Example 3. In the coin example, an event E_1 could be "has at least two tails". This is an informal way of saying $E_1 = \{x : x \text{ has at least two tails}\}$. There are 4 outcomes in the sample space with at least two tails. $\Pr(E_1) = 1/2$.
If E_2 is the event "the first flip and the second flip are both heads", then $|E_2| = 2$, so $\Pr(E_1) = 1/4$.

Example 4. In the wheel example, the probability of the event A is $3/7$. In English, the event would be "the spinner lands in A."

Example 5. Suppose our experiment is to flip n coins. Now the sample space S is all 2^n sequences in which each entry is H or T. Let E be the event that we get k heads. Then

$$|E| = \{\text{the number of sequences with } k \text{ heads}\},$$

which we know is

$$\binom{n}{k}.$$

Therefore,

$$\Pr(E) = \frac{\binom{n}{k}}{2^n}.$$

What if E is the event that we get an odd number of tails? By the addition principle,

$$|E| = \sum_{0 \le k \le n,\ k \text{ odd}} \binom{n}{k}.$$

On the other hand, the event that we get an even number of tails, is $S \setminus E$, and

$$|S \setminus E| = \sum_{0 \le k \le n,\ k \text{ even}} \binom{n}{k}.$$

Miraculously, $|S \setminus E| - |E| = 0$. To see why, write

$$\begin{aligned} &|S \setminus E| - |E| \\ = \sum_{0 \le k \le n,\ k \text{ even}} \binom{n}{k} - &\sum_{0 \le k \le n,\ k \text{ odd}} \binom{n}{k} \\ &= \sum_{0 \le k \le n} \binom{n}{k} (-1)^k. \end{aligned}$$

By the Binomial Theorem,

$$\begin{aligned} &\sum_{0 \le k \le n} \binom{n}{k} (-1)^k \\ = &\sum_{0 \le k \le n} \binom{n}{k} (-1)^k 1^{n-k} \\ &= ((-1) + 1)^n = 0. \end{aligned}$$

We have shown $|S \setminus E| - |E| = 0$, so $|S \setminus E| = |E|$. Since $|S \setminus E| + |E| = 2^n$ (they are disjoint, so we can use the addition principle), $|S \setminus E| = |E| = 2^n/2$. So

$$\Pr(E) = \frac{\frac{1}{2} 2^n}{2^n} = 1/2.$$