Ramsey Theory, Probabilistic Method: Math 454 Lecture 16 (7/25/2017)

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Keller and Trotter Probabilistic Method Chapter:

http://www.rellek.net/book/ch_probmeth.html.

Ramsey Theory shows up in Chapter 3.3 of Brualdi.

Contents

1 Ramsey Theory

Ramsey Theory is a theory built upon the idea that often when one partitions objects into several parts, at least one of the parts must contain some nice substructure. An easy example of this is the pigeonhole principle, which dictates simply that if you partition a set of objects, you will get a large part.

1.1 More Pigeonhole Principle

Theorem 1.1 ("Strong" Pigeonhole Principle). *If a set of size n is partitioned into m parts, there is a part of size $\lceil n/m \rceil$.*

1.2 Ramsey numbers

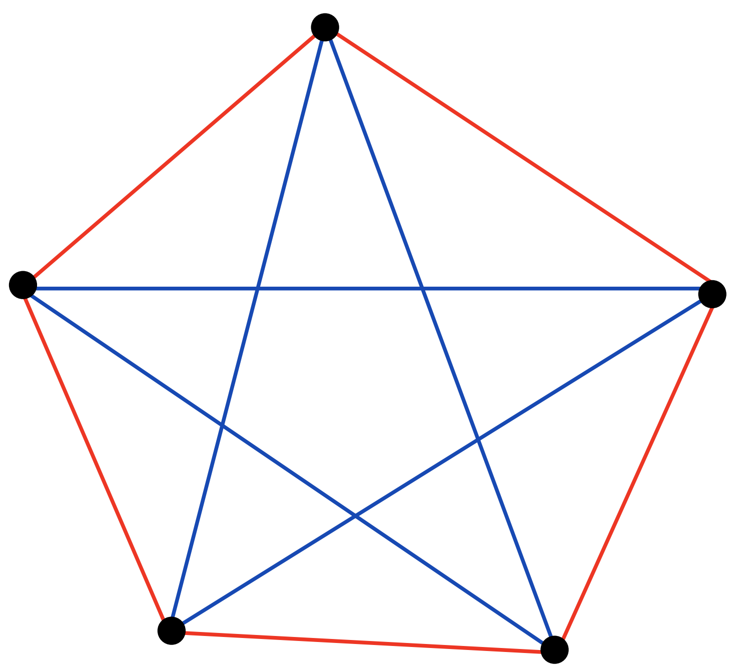
Fact 1.2.1. If the edges of K_6 are colored red and blue, then there is either a red copy of K_3 or a blue copy of K_3.

Proof. Look at a vertex v. It has five incident edges in K_6, so some 3 of them are the same color. Assume, without loss of generality, that this color is blue. Thus, v is adjacent to at least 3 vertices by blue edges. These 3 vertices have 3 edges among them. If any one of these edges is blue, then its endpoints and v form a blue triangle. If all the edges are red, then they form a red triangle. Hence, either way, K_6 contains triangle of red or blue. $\square$

We can try to come up with theorems of the same form for red K_m's and blue $K_n's$.

Definition 1.1. $R(m, n)$ is the smallest number r such that every coloring of the edges of K_r contains a red copy of K_m or a blue copy of K_n.

Example 1. Since the following contains no red or blue K_3, $R(3, 3) > 5$.



However, the fact above shows $R(3, 3) \leq 6$, so in fact $R(3, 3) = 6$.

1.3 Ramsey's Theorem for Graphs

Theorem 1.2. *For all $m \geq 1, n \geq 1$, $R(m,n)$ actually exists, and*

$$R(m,n) \leq \binom{m+n-2}{m-1}.$$

Proof. We need to show there is a number r such that if the edges of K_r are colored with red and blue, then K_r contains a red copy of K_m or a blue copy of K_n. First we show

$$R(m,n) \leq R_{m-1,n} + R_{m,n-1}.$$

Suppose $s = R_{m-1,n}$ and $t = R_{m-1,n}$. Consider a coloring of the edges of K_{s+t} with red and blue. Any vertex v has either at least s neighbors through red edges or at least t neighbors through blue edges; otherwise there are at most $1 + (s-1) + (t-1) = s+t-1$ vertices. Suppose v has s neighbors through red edges. Since $s = R(m-1,n)$, there is either a red K_{m-1} among these neighbors, in which case we can add v to get a red K_m, or a blue K_n. The argument is similar if v instead has at least t neighbors through blue edges. Either way, K_{s+t} has either a red K_m or a blue K_n.

We can use the above fact to show

$$R(m,n) \leq \binom{m+n-2}{m-1}.$$

Let the above proposition be denoted $P(n,m)$. We prove it for all $n \geq 1, m \geq 1$ by double induction:
First off, note if $m \leq 2$ or $n \leq 2$, the claim is very easy. In fact,

$$R(m,2) = m = \binom{m+2-2}{m-1}$$

and

$$R(2,n) = n = \binom{n+2-2}{2-1}.$$

Thus, $P(m,2)$ for all $m \geq 0$, $P_{2,n}$ for all $n \geq 0$ are our base cases. We assume $n, m \geq 3$ to do the induction step. We use $P_{m-1,n}$ and $P_{m,n-1}$ to prove $P_{m,n}$. The idea of double induction is illustrated in Figure 2.
Thus, we assume $P_{m-1,n}$ holds and $P_{m,n-1}$ holds, that is

$$R(m-1,n) \leq \binom{m+n-3}{m-2}$$

and

$$R(m,n-1) \leq \binom{m+n-3}{m-1}.$$

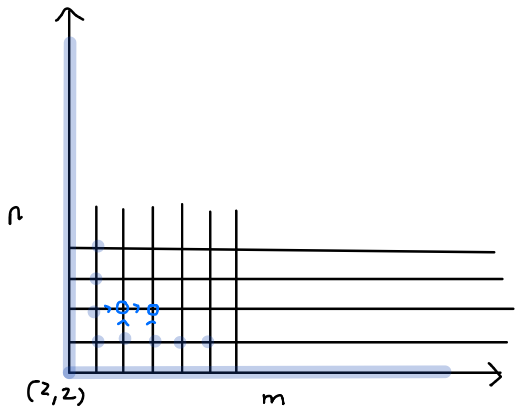


Figure 1: The idea of double induction. One could think of the axes as all the propositions $P(m, 2)$ and $P(2, n)$, highlighted in blue to indicate they are true. Then, once you know $P_{m-1,n}$ and $P_{m,n-1}$ imply $P_{m,n}$, the truth of the propositions "spreads" up and to the right via the blue arrows to fill out the whole quadrant.

We already showed

$$R(m,n) \leq R\left(m-1,n\right) + R\left(m,n-1\right).$$

However, by the induction hypothesis,

$$R\left(m-1,n\right) + R\left(m,n-1\right) \leq \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} = \binom{m+n-2}{m-1}.$$

The last equality is Pascal's Formula. □

1.4 Best known bounds for diagonal Ramsey numbers

$$\begin{aligned} R(2,2) &= 2 \\ R(3,3) &= 6 \\ R(4,4) &= 5 \\ R(5,5) &\in [43,48] \\ R(6,6) &\in [102,165] \end{aligned}$$

> Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R(5, 5) or they will destroy our planet.

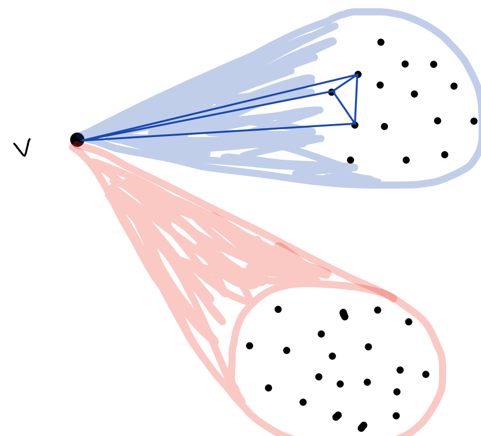


Figure 2: Obtaining a blue K_4 from a blue K_3 among the neighbors of v via blue edges.

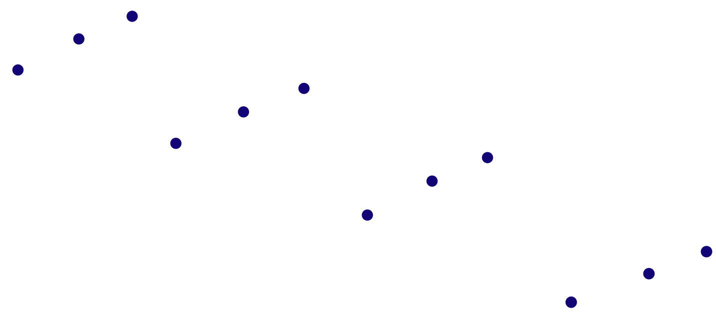
> In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for R(6, 6). In that case, he believes, we should attempt to destroy the aliens.
> - Joel Spencer

1.5 The Erdős-Szekeres Theorem

By Ramsey's Theorem, for all numbers r and s, there is a number $n(r, s)$ such that any sequence of at least $n(r, s)$ distinct real numbers has an increasing subsequence of length r or a decreasing subsequence of length s.

To see this, we use the sequence $(n_1, \ldots, n_l)$ to color K_l. Suppose $i < j$. Color ij red if $n_i < n_j$, and ij blue if $n_i > n_j$. A red K_r in this coloring is an increasing subsequence, and a blue K_r is a decreasing subsequence. Ramsey's theorem says for l large enough there must be a monochromatic K_r or K_s in the coloring. Thus, $n(r, s) \leq R(r, s)$. This is a very bad bound, however.

Example 2. Read left to right, a length $3 \cdot 4$ sequence with no increasing subsequence of length 4 and no decreasing subsequence of length 5.



We can actually get a much better bound than $R(r, s)$ for $n(r, s)$.

Theorem 1.3 (Erdős-Szekeres)**.** *A sequence of length* $(r-1)(s-1)+1$ *distinct real numbers has an increasing subsequence of length* r *or an decreasing subsequence of length* s*.*

Proof of Erdős-Szekeres. Let $(n_1, n_2, \ldots, n_{(r-1)(s-1)+1})$ be a sequence of distinct real numbers. Suppose, for contradiction, that all increasing subsequences of this sequence are of length at most $s-1$ and decreasing subsequences are of length at most $r-1$.

We will set this up as an application of the Pigeonhole Principle where two pigeons in a hole leads to a contradiction. Our pigeons will be the numbers $\{1, \ldots, (r-1)(s-1)+1\}$, and our $(r-1)(s-1)$ many holes will be pairs (a, b) of integers where $1 \leq a \leq r-1$ and $1 \leq b \leq s-1$. Assign the pigeon i to the hole (a, b) if a is the length of the longest increasing subsequence ending at i and b is the length of the longest decreasing subsequence beginning at i. By the Pigeonhole Principle, there are two numbers i and j assigned to the same pair (a, b).

Case 1. ($n_i < n_j$): The increasing subsequence of length a ending at n_i can be extended to an increasing subsequence of length $a+1$ ending at j, a contradiction (the longest increasing subsequence ending at j is a).

Case 2. ($n_i > n_j$): Do the same as Case 1, but extend the decreasing subsequence. This is also a contradiction.

□

1.6 Estimating Ramsey Numbers

So far, we have seen upper bounds for $R(n, n)$ for all n, but no lower bounds. Remember:

- To show $R(n, n) \leq r$, we need to show that all colorings of K_r contain a monochromatic K_r.

- To show $R(n,n) > r$, we need to find a coloring of K_r that *doesn't* contain a monochromatic K_r.

In Ramsey's Theorem, we were proving upper bounds. We got

$$R(n,n) \leq \binom{2n-2}{n-1}$$

, which we can estimate using

Theorem 1.4. *(Stirling's approximation:)*

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Stirlings approximation actually gives

Theorem 1.5.

$$\binom{n}{n/2} \approx 2^n\sqrt{\frac{2}{\pi n}},$$

but we don't need such precision. For that reason, we use big-O notation:

Definition 1.2. If $f(n)$ and $g(n)$ are nonnegative functions, we write

$$f(n) = O(g(n))$$

if there is a constant $K > 0$ such that $f(n) \leq Kg(n)$ for all n.

This means we can write

$$R(n,n) = O\left(\frac{4^n}{\sqrt{n}}\right).$$

2 The Probabilistic Method

Finally we can see the brilliant idea of the probabilistic method, first employed by Szele and later perfected by Paul Erdős.

One silly way to show K_r has a coloring with no monochromatic K_n is to show that typical colorings of K_r have no monochromatic K_n. This seems much harder than just finding one such coloring, but the magic of probability says it isn't so.

2.1 Discrete Random Variables

Definition 2.1. If S is a sample space, a *random variable* X is an assignment of real numbers to the elements of S. That is, X is a function

$$X : S \to \mathbb{R}.$$

Example 3. If our sample space S were $\{a, b, c, d, e\}$ and our experiment were to draw each one with equal probability, then $X : S \to \mathbb{R}$ given by

$$X(a) = 5, X(b) = 3, X(c) = 3, X(d) = 3, X(e) = 1.$$

is a random variable.

Definition 2.2. The quantity $\sum_{s \in S} X(s) \Pr(\{s\})$ is called the *expected value* of X, and is denoted

$$\mathbb{E}[X] = \sum_{s \in S} X(s) \Pr(\{s\}).$$

Fact 2.1.1. Since we are dealing with finite sample spaces, the range of X will be a finite set. Thus, we can rearrange the definition of $\mathbb{E}[X]$ to give

$$\mathbb{E}[X] = \sum_{x} x \Pr[X = x],$$

where the sum is understood to run over the range of X.

Example 4. If X is the random variable from the previous example, then

$$\mathbb{E}[X] = 5 \cdot \frac{1}{5} + 3 \cdot 1/5 + 3 \cdot 1/5 + 3 \cdot 1/5 + 1 \cdot 1/5 = \frac{1}{5} 15 = 3.$$

Alternatively, we can write

$$\mathbb{E}[X] = 1 \cdot 1/5 + 3 \cdot 3/5 + 5 \cdot 1/5 = 3.$$

2.2 Linearity of Expectation

It is a very easy consequence of the definition of expected value that if X and Y are random variables on the same sample space, and a and b are real numbers, then

Fact 2.2.1 (Linearity of Expectation)**.**

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y].$$

We don't need any special properties of random variables to apply linearity of expectation.

Example 5. Suppose we flip 4 fair coins and let X be the number of heads. Then X is the sum of the random variables $X_i, i \in [4]$, where $X_i = 1$ if the i^{th} flip is H and 0 otherwise. Then $X = X_1 + X_2 + X_3 + X_4$, so

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4].$$

However, $\mathbb{E}[X_i] = 0 \cdot .5 + 1 \cdot .5 = .5$ for $i \in [4]$, so $\mathbb{E}[X] = 2$. If Y is the number of heads among the last two flips, then $\mathbb{E}[Y] = 1$, and $\mathbb{E}[X + Y] = 3$. Note that X and Y are not independent.

2.3 Lower bounds for Ramsey numbers

Without further ado, let us use the probabilistic method to prove something about Ramsey Theory.

Theorem 2.1. *If*

$$\binom{r}{n} 2^{1-\binom{n}{2}} < 1,$$

then $R(n, n) > r$.

Linearity of Expectation Proof: Suppose r and n satisfy the inequality above. Consider the following experiment:

For each edge of K_r, flip a fair coin to decide whether the edge is colored red or blue.

The sample space is all $2^{\binom{r}{2}}$ colorings of the edges of K_r. Define a random variable X on this sample space by

$$X(c) = \text{ number of monochromatic } K_n \text{ in the coloring } c.$$

The critical thing is that X is a sum of other random variables: If S is a fixed n-subset of $[r]$, then define X_S to be the random variable that is 1 if the edges among S form a monochromatic K_n and 0 if not. Then

$$X = \sum_{|S|=r, S \subset [n]} X_S,$$

and by linearity of expectation,

$$\mathbb{E}[X] = \sum_{|S|=r, S \subset [n]} \mathbb{E}[X_S].$$

However, because X_S only takes values 0 or 1,

$$\mathbb{E}[X_S] = \Pr[X_S = 1],$$

which is exactly $2 \cdot 2^{-\binom{n}{2}}$. (it is the probability that the coin flips for the $\binom{n}{2}$ edges among S all come up heads or all come up tails). Thus,

$$\mathbb{E}[X] = \binom{r}{n} \cdot 2^{1-\binom{n}{2}} < 1.$$

A final, and critical step: Since $\mathbb{E}[X] < 1$, there must be some element c of the sample space with $X(c) < 1$. Otherwise, the expectation would certainly be at least 1. However, $X(c)$ can only take on integer values, so $X(c) = 0$. Thus, c has no monochromatic K_n's. $\square$

There is an alternate proof avoiding the use of expected value, which exemplifies the technique of seemingly wasteful upper bounds - a staple in probabilistic combinatorics.

Union-Bound Proof: Flip a fair coin to decide whether each edge of the complete graph on $[r]$ is colored red or blue. We want to look at the probability there is a blue K_n. If S is a fixed n-subset of $[r]$, then define A_S to be the event that the complete graph on vertex set S, a K_n, is monochromatic. We want to bound the probability at least one of the events A_S occurs. That is, we want to bound

$$\Pr\left(\bigcup_{|S|=r, S\subset[n]} A_S\right).$$

We could use inclusion-exclusion, but that would be unnecessarily complicated. Instead, we use what is known as the seemingly crude *union-bound*:

$$\Pr\left(\bigcup_{|S|=r, S\subset[n]} A_S\right) \le \sum_{|S|=r, S\subset[n]} \Pr(A_S).$$

In words, the probability of the union is at most what it would be if the events were disjoint.

However, A_S occurs if and only if the $\binom{n}{2}$ independent flips coloring the edges among S all come up heads or all come up tails, so

$$\Pr(A_S) = 2 \cdot 2^{-\binom{n}{2}}.$$

Further, there are $\binom{r}{n}$ n-subsets of $[r]$, so

$$\sum_{|S|=r, S\subset[n]} \Pr(A_S) = \binom{r}{n} 2^{1-\binom{n}{2}}.$$

If

$$\binom{r}{n} 2^{1-\binom{n}{2}} < 1$$

$\square$

2.4 Subgraphs

Sometimes we can find probabilistic proofs of statements that seem to have nothing to do with probability. The following fact is actually not hard to prove in other ways, but this proof is very clean.

Fact 2.4.1. Suppose G is a graph on n vertices and m edges. Then there is a subgraph of G with k vertices that has

$$m \cdot \frac{\binom{k}{2}}{\binom{n}{2}}$$

edges.

Proof. Take a random k-subset S of the vertices of $G = (V, E)$. Let X be the random variable counting the number of edges of G contained in S. If e is an edge of G, define X_e to be 1 if $e \subset S$ and 0 otherwise. Then

$$X = \sum_{e \in E} X_e,$$

and so

$$\mathbb{E}[X] = \sum_{e \in E} \mathbb{E}[X_e].$$

This time, $\mathbb{E}[X_e]$ is the probability that $e \subset S$, which is

$$\frac{\binom{k}{2}}{\binom{n}{2}}.$$

Thus,

$$\mathbb{E}[X] = m \frac{\binom{k}{2}}{\binom{n}{2}},$$

so there must be some set S of k vertices containing with at least (and one with at most!) $m \frac{\binom{k}{2}}{\binom{n}{2}}$ edges. The subgraph

$$H = (S, \{e : e \in S\})$$

is the desired subgraph.

$\square$