Catalan numbers, Eulerian Circuits, Hamiltonian Cycles: Math 454 Lecture 13 (7/19/2017)

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Keller and Trotter Graph Theory chapter:

http://www.rellek.net/book/ch_graphs.html.

More is in Brualdi 11.2 (cycles, circuits) and 8.1 (Catalan numbers). If you can find a copy of *Modern graph theory* by Bollobas it is also a very good resource.

Contents

1 Counting Dyck Words: More Catalan

Definition 1.1. A sequence of n U's and n D's (really doesn't have to be U and D, could be anything) such that every prefix has at least as many U's as it has D's is called a *Dyck word* (pronounced *deek*). Note that this definition switches $U's$ and $D's$.

The following theorem summarizes what we saw last lecture:

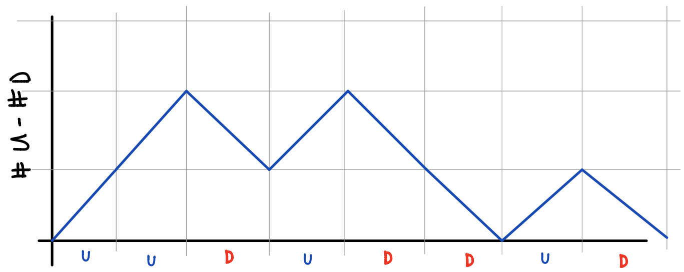


Figure 1: The graph of a Dyck word. One can view Dyck words as mountain ranges that never dip below the horizon.

Theorem 1.1. *There is a one-to-one correspondence between Dyck words of length $2n$ and ordered trees with n edges.*

Now let's count the number of Dyck words of length $2n$. There are two ways:

1.1 Counting Dyck words via a recurrence and a generating function

Let C_n be the number of Dyck words of length $2n$. Think of $C_0 = 1$ as usual. Let w be a Dyck word. Think of the graph of w as in Figure 1. Let $2 \leq 2k \leq 2n$ be the *first* point at which w's graph hits the x axis (we are justified in calling it $2k$ because we know it is even). The number of words that first hit the horizon at k is $C_{k-1}C_{n-k}$, because the possible words between 1 and $2k-1$ are the Dyck words of length $2k-2$ and the possible words between $2k$ and $2n$ are the Dyck words of length $2n-2k$. Further, for different k, these sets of words are disjoint. This means that

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}, \text{ for } n \geq 1$$

or

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}, \text{ for } n \geq 1$$

or

Theorem 1.2. *The number of Dyck words of length n, C_n, satisfies the recurrence*

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \textit{ for } n \geq 0.$$

None of the usual techniques we know work on this recurrence, but it is amenable to generating functions!

Let $h(x) = \sum_{n \geq 0} C_n$ be the generating function for the sequence $(C_n)_{n \geq 0}$. The sum

$$\sum_{i=0}^{n} C_i C_{n-i}$$

should look familiar from our formula for multiplying generating functions. If $f(x) = \sum_{i \geq 0} a_i$, and $g(x) = \sum_{j \geq 0} b_j$, then

$$f(x)g(x) = \sum_{n \geq 0} \left(\sum_{i=0}^{n} a_i b_{n-i} \right) x^n.$$

This means if we multiply $h(x)$ by itself, we get

$$h(x)h(x) = \sum_{n \geq 0} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) x^n = \sum_{n \geq 0} C_{n+1} x^n = \frac{1}{x} \left(\sum_{n \geq 1} C_n x^n \right).$$

Since $C_0 = 1$,

$$\frac{1}{x} \left(\sum_{n \geq 1} C_n x^n \right) = \frac{h(x) - 1}{x},$$

so

$$h(x) = 1 + xh(x)^2.$$

This says $h(x)$ is a solution of the equation $xh(x)^2 - h(x) + 1$, or $h(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$. By our definition, we should have $h(0) = 1$, but $\frac{1+\sqrt{1-4x}}{2x}$ tends to infinity as $x \to 0$, let's try the other root $\frac{1+\sqrt{1-4x}}{2x}$. Since this has a power series expansion, its coefficients must satisfy the same recurrence as $h(x)$, and it has first coefficient 1 in its power series, so the coefficients must actually match x. We can apply Newton's binomial theorem to obtain

$$h(x) = \frac{1}{2x} - \frac{1}{2x}(1-4x)^{1/2} = \frac{1}{2x} - \frac{1}{2x} \sum_{k \geq 0} \binom{1/2}{k} (-4)^k x^k = -\frac{1}{2} \sum_{k \geq 1} \binom{1/2}{k} (-4)^k x^{k-1}.$$

The reason for the last equality is that $\binom{1/2}{0}$ is just 1, so the $1/2x$ terms cancel out. For

$k \geq 1$, we can rewrite

$$\begin{aligned}
\binom{1/2}{k} &= \frac{(1/2)(1/2-1)\dots(1/2-k+1)}{k!} \\
&= \frac{(-1)^{k-1}}{2^k}\frac{(1)(1)(3)\dots(2k-3)}{k!} \\
&= \frac{(-1)^{k-1}}{2^k}\frac{(2k-2)!}{(2k-2)(2k-4)\dots 4\cdot 2\cdot k!} \\
&= \frac{(-1)^{k-1}}{2^{2k-1}}\frac{(2k-2)!}{(k-1)!k!} \\
&= \frac{(-1)^{k-1}}{k\cdot 2^{2k-1}}\binom{2k-2}{k-1}
\end{aligned}$$

For us, everything miraculously cancels:

$$\begin{aligned}
h(x) &= -\frac{1}{2}\sum_{k\geq 1}\binom{1/2}{k}(-4)^k x^{k-1} \\
&= -\frac{1}{2}\sum_{k\geq 1}\frac{(-1)^{k-1}}{k\cdot 2^{2k-1}}\binom{2k-2}{k-1}(-4)^k x^{k-1} \\
&= \sum_{k\geq 1}\frac{(-1)^{k}}{k\cdot 2^{2k}}\binom{2k-2}{k-1}(-4)^k x^{k-1} \\
&= \sum_{k\geq 1}\frac{1}{k}\binom{2k-2}{k-1}x^{k-1}, \\
&= \sum_{k\geq 0}\frac{1}{k+1}\binom{2k}{k}x^{k},
\end{aligned}$$

So

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

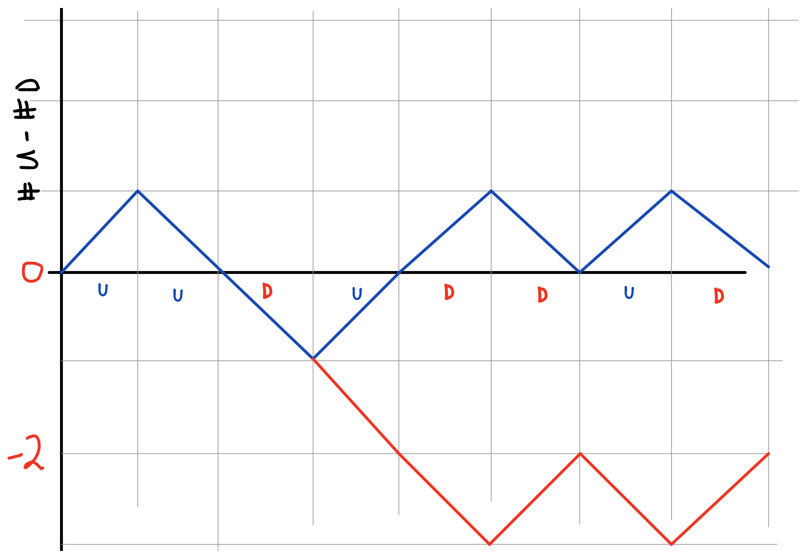
These are called the *Catalan numbers*, and they count lots of other quantities in combinatorics.

1.2 Counting Dyck words awesomely

We want to count the number of sequences of n $U's$ and $D's$ whose graph does not cross the x axis. To do this, we can also count

the total number of sequences - those that *do* cross the x axis.

The first term is, of course, $\binom{2n}{n}$. The second is a little tricker. We count these using a one-to-one and onto function. Let S be the set of sequences of n U's and n D's whose graph crosses the x-axis. If $s \in S$, let $f(s)$ be the sequence where $U's$ and $D's$ are switched *after the first time* the graph of s crosses the x-axis.



Since the graph of s just crossed the x axis, there must be $k+1$ U and k D's in s after the point at which it crossed the axis (we have to get back to the x axis by the end of s). Then $f(s)$ must have $n-1$ U's and $n+1$ D's, because swapping U's and D's after crossing increases the number of $U's$ by one and decreases the number of D's by one. Thus $f(s)$ is a sequence of $n-1$ $U's$ and $n+1$ D's.

Now our goal is to show that

$$f : \{\text{sequences that } do \text{ cross the x axis}\} \to \{\text{permutations of } \{(n-1)\cdot U, (n+1)\cdot D\}\}$$

is one-to-one and onto. Once we do that, we are done, because the right hand side has size $\binom{2n}{n+1}$.

f is one-to-one because this process is reversible - in particular there is a function

$$g : \{\text{permutations of } \{(n-1)\cdot U, (n+1)\cdot D\}\} \to \{\text{sequences that } do \text{ cross the x axis}\}$$

such that $f(g(r)) = r$ for any permutations r of $\{(n-1)\cdot U, (n+1)\cdot D\}$.

g simply takes a sequence r of $(n-1)$ U's and $(n+1)$ D's to the sequence of n U's and n D's obtained by swapping U and D after the first time r's graph crosses the x-axis - which it must, because there are $(n-1)$ U's and $(n+1)$ D's. This equalizes the number of $U's$ and $D's$ in $g(r)$, and you can see that if you apply f to $g(r)$ you will get r back.

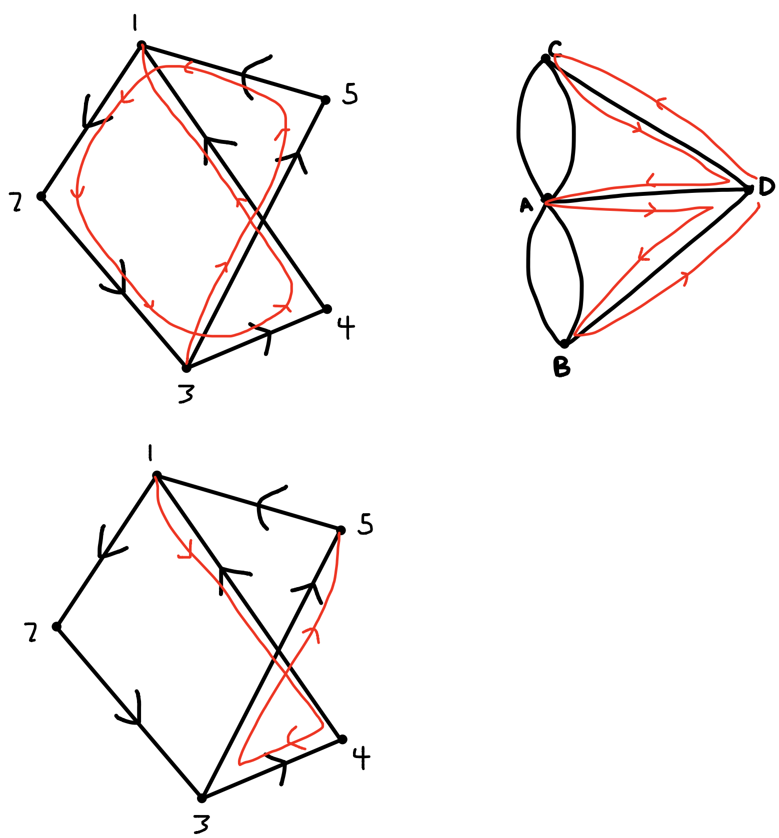
Finally, we can say $C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$.

2 Walks, Cycles, Circuits

We will be talking about looped, directed multigraphs here.

Definition 2.1. A *walk* of length l in a directed, looped, multigraph G a sequence of $l+1$ vertices $(x_0, \dots x_l)$ such that $x_{i-1}x_i$ is an edge (arc in directed graphs) of G for $i \in [l]$. A *closed walk* of length l is a walk of length l $(x_0, \dots x_l)$ where $x_0 = x_l$.

Example 1. Careful about following directions.



Top left is the walk $(3, 5, 1, 2, 3, 4, 1)$. Top right is the walk (D, C, D, A, D, B, D). Bottom left is not a walk, because it violates the direction of the edge $(1, 4)$.

3 Eulerian Circuits

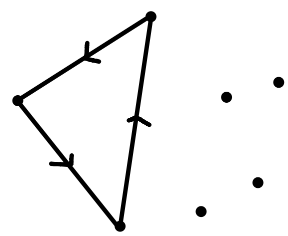
Definition 3.1. A *trail* of length l in a graph G is a walk of length l that does not repeat edges, and a *circuit* is a closed walk that does not repeat edges.

Example 2. Example in directed and in undirected.

Definition 3.2. A *Eulerian trail* in a graph G is a trail that uses every edge, and a *Eulerian circuit* is a circuit that uses every edge.

3.1 Connected even-degree graphs always have Eulerian circuits; different from class!

Remark 3.1. Warning: we need to assume the graphs have no isolated vertices for the rest of the section to make sense. Frustratingly, the following circuit in a 7 vertex graph satisfies the definition of an Eulerian circuit:



This means the theorem in class was slightly off. For the rest of this section we assume our graphs don't have isolated vertices; that is, vertices with no neighbors.

Suppose G is an undirected multigraph with no isolated vertices. What conditions do we need for there to be an Eulerian circuit?

- First of all, we need to be able to get from each edge to every other edge in G, because the circuit visits every edge. This means G must be connected.
- Secondly, the circuit will enter and leave each vertex the same number of times, so the degrees must be even.

What if G is a directed multigraph? Then we will still need to be able to get from each arc to every other arc in G, because the circuit visits every arc. However, this time we need to be able to do visit all arcs while following the directions of the arcs. For this we need

[2]By Cmglee - Own work, https://commons.wikimedia.org/w/index.php?curid=60330862

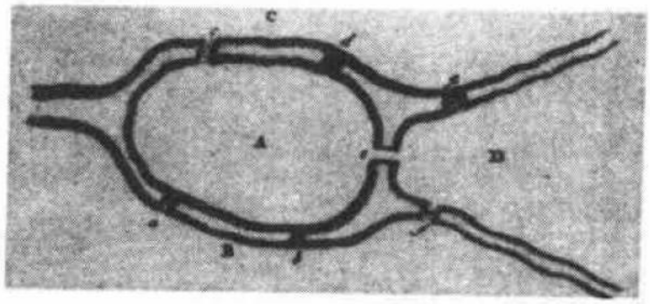


FIGURE I.14. The seven bridges on the Pregel in Königsberg.

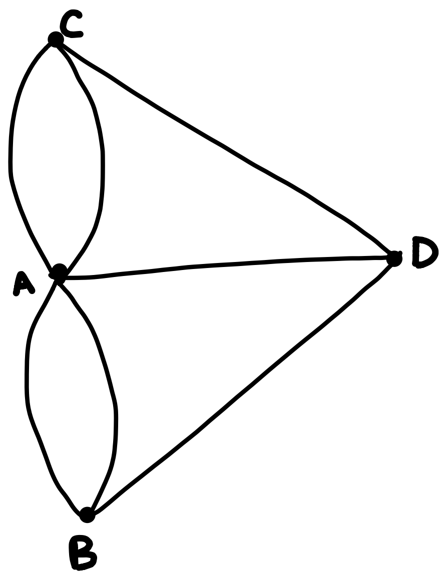


Figure 2: Top is the very old 7 bridges of Königsberg puzzle. The residents of Konigsberg asked if there was a walk that could take you across each bridge exactly once. Bottom is a graph in which the blobs of land have been treated as vertices. An Eulerian trail in this graph would solve the Königsberg puzzle in the affirmative.

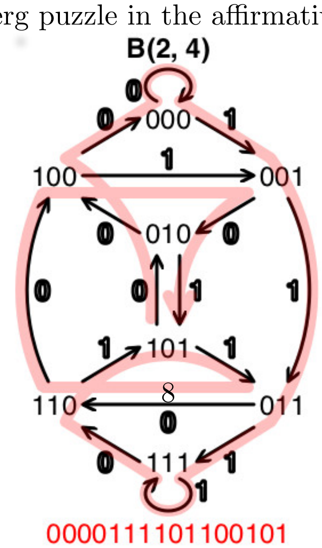


Figure 3: An Eulerian circuit in the directed graph DB_4, and the corresponding de Bruijn sequence.[2]

a condition called *strongly connected*: a directed multigraph G is strongly connected if for any two vertices x and y there is a walk from x to y in G (respecting directions of arcs). It turns out, however, that we don't need this!

Definition 3.3 (Unofficial definition:)**.** If G is a directed graph, the *undirected version of* G is the graph obtained by replacing each arc (ordered pair) by the corresponding edge (ordered pair).

We just need that the undirected version of G is connected; strong connectivity comes for free from the degree condition. Hence, we just need the following conditions.

- The undirected version of G is connected.
- Secondly, the circuit will enter and leave each vertex the same number of times, so the number of arcs leaving every vertex v, called the *in-degree* $d_-(v)$ of v, must be the same as the number of arcs entering every vertex, called the *out-degree* $d_+(v)$.

Summarizing,

Theorem 3.1 (Eulerian circuits for looped *undirected* multigraphs)**.** *A looped multigraph with no isolated vertices has an Eulerian circuit if and only if it is connected and all its degrees are even.*

Theorem 3.2 (Eulerian circuits for looped *directed* multigraphs)**.** *A looped directed multigraph G with no isolated vertices has an Eulerian circuit if and only if*

$$d_+(v) = d_-(v)$$

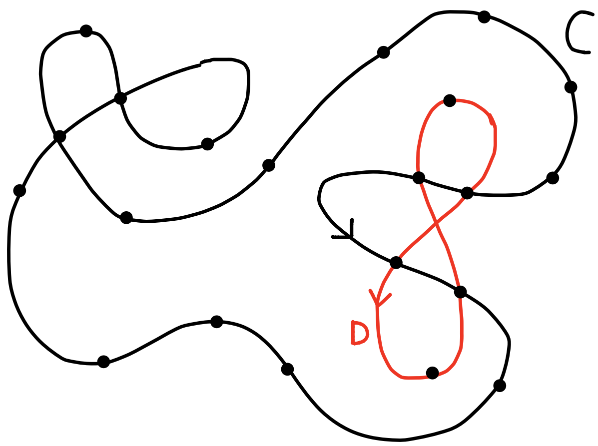
for each vertex $v \in G$, and (different from class!) the undirected version of G is connected.

This is similar to the proof from class, but less inductive and more constructive.

Proof for undirected graphs. We already saw why G must be connected with even degrees if it has an Eulerian circuit. Now suppose $G = (V, E)$ is a connected multigraph with all degrees even and no isolated vertices. Then G must have a circuit C; the endpoint x_l of the longest path $x_1 \dots x_l$ in G is of degree at least two and has no neighbors off the path (else longer path!), so must be adjacent to some vertex among $\{x_1, \dots, x_{l-1}\}$, forming a cycle (which is also a circuit).

Let this circuit C be our starting point. If C is all the edges, we are done; else, $G(V, E \setminus E(C))$ must have some connected component H with more than one vertex. Here $E(C)$ means the edge-set of C. Since G is connected, C must join the connected components of $G(V, E \setminus E(C))$, so it must visit H.

Also note that all degrees of the vertices of $G(V, E \setminus E(C))$ are even (because they decreased by an even number when we removed $E(C)$), so the degrees of H are all at least two. By the reasoning from before, this means we can find a circuit D in H through the vertex it shares with the circuit C. Let C_2 be the concatenation of C and D; that is, if $C = (x_1, \ldots, x_l)$ and $D = (y_1, \ldots y_k)$ then $C_2 = (x_1, \ldots x_l, y_1, \ldots y_k)$. Since we can assume $y_k = y_1 = x_l = x_1$ and C and D share no edges, C_2 is still a circuit. C_2 is also larger than C.



Repeat this process with C_2 and $G(V, E \setminus E(C_2))$, and so on. Eventually we will run out of edges, so at that stage we will have completed an Eulerian circuit. $\square$

Here is a sketch of how to fix the proof to work for directed multigraphs.

Proof for directed graphs: The proof is essentially the same, but after finding the circuit C we split into the components of the undirected version of $G(V, E \setminus E(C))$. Also, when we concatenate C and D we must go round D in the correct direction.

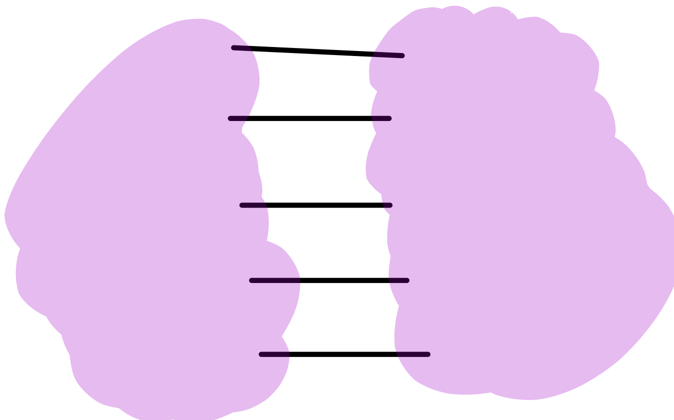
$\square$

4 Hamiltonian Cycles

Definition 4.1. A *Hamiltonian cycle* in a graph G is a cycle that visits every vertex exactly once. Recall that cycles do not repeat vertices or edges.

Lemma 4.1. *Suppose a set M of disjoint edges disconnects the graph G. Then any Hamiltonian cycle in G must pass through the edges of M an even number of times.*

Example 3.



A set of M disjoint edges connecting a graph, where the pink parts are the (at least) two connected components of $G = (V, E \setminus M)$.