Trees: Math 454 Lecture 11 (7/17/2017)

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Some of these notes follow sections 5.1 and 5.6 of Keller and Trotter, which you can find here:

http://www.rellek.net/book/ch_graphs.html.

More is in Brualdi 11.5. We also talked a bit about isomorphism today; some of that stuff can be found in Lecture 10.

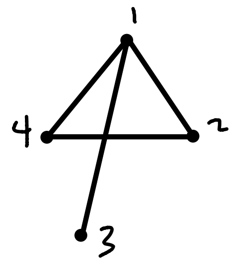
Remark 0.1. One key take-away from the definition of isomorphism is that any property of graphs which doesn't depend on the names of the vertices is preserved under isomorphism. For example, if G and H are isomorphic, then the property "contains a copy of C_3" is satisfied by G if and only if it is satisfied by H. A non-example of such a property is the property "ab is an edge of G", which clearly depends on the labels of vertices, so we needn't have ab an edge of H even if G and H are isomorphic. We'll define containment formally later in these notes.

Contents

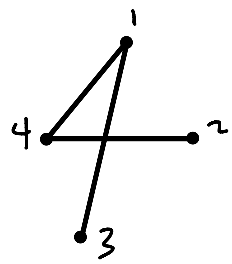
1 Subgraphs

Definition 1.1 (Subgraph)**.** If $G = (V, E)$ is a graph, then $H = (W, F)$ is a subgraph of G if $W \subset V$ and $F \subset E$. Sometimes we say G *contains* H. If G contains a subgraph isomorphic to H, we say G *contains a copy* of H.

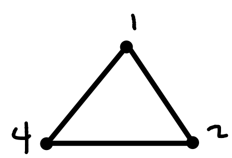
Example 1. The graph



contains the subgraph

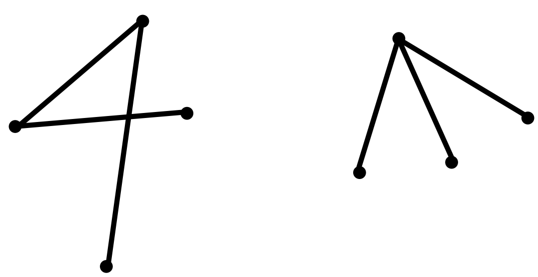


and also the subgraph



Definition 1.2 (Acyclic)**.** A graph G is *acyclic* if it contains no cycles.

Example 2. An acyclic graph:



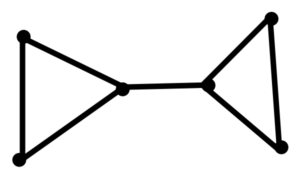
2 Connectivity

Definition 2.1 (Connected, Connected Component)**.** A graph $G = (V, E)$ is *connected* if for any pair of distinct vertices x and y, G contains a path with x and y as endpoints.

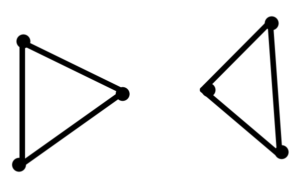
Intuitively, you can get from any place in the graph to another by walking along the edges. A graph that is not connected is said to be *disconnected.* If G is a graph, then a *connected component* C of G is a maximal connected subgraph of G. That is, you can't add any more vertices to C and still have it be a connected subgraph of G.

A connected component is everything you can get to from a particular vertex in G by walking on the edges.

A connected graph:



A disconnected graph:



Theorem 2.1. *A connected graph G with n vertices has at least $n-1$ edges.*

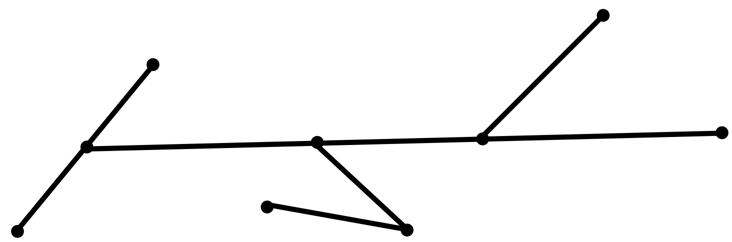
Proof. Think about this by adding the edges of G one at a time. Before we have added any, there are n components. Each time we add a new edge, we can decrease the number of components by at most one (either we join two, or we don't!). Thus, we need to add at least $n-1$ edges to get one component (i.e., a connected graph). □

3 Trees

Trees are the "smallest" connected graphs.

Definition 3.1. A connected, acyclic graph is called a *tree.*

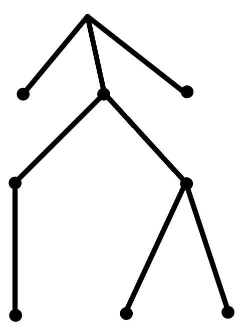
Example 3. A tree:



Definition 3.2. The *neighborhood* $N(v)$ of a vertex v is all vertices adjacent to v.

Remark 3.1. Let T be a tree. Let's try to draw T. Start at some arbitrary vertex r. Look at its neighbors; draw them below r. For each of the neighbors, look at their neighborhoods other than r. Those neighborhoods can't intersect, because otherwise there would be a cycle. Thus, we can draw their neighborhoods (other than r) below them. We can do this for the vertices we just drew; because there are no cycles, we will be able to place the new neighborhoods directly below the corresponding existing vertex. Continuing this process gives a nice (upside down) tree-like picture.

Example 4. A better way of drawing the last tree we saw is



3.1 Important properties of trees

Theorem 3.1. *(Big tree theorem) Let T be a graph on n vertices. The following are equivalent.*

1. *T is a tree.*
2. *T is connected but removing any edge from T results in a disconnected graph.*
3. *T is connected and has exactly $n-1$ edges.*

4. *T is acyclic and has exactly $n-1$ edges.*

Proof of 2 using 1: Assume T is a tree, so it is connected and acyclic. Consider an edge xy of T. If removing xy doesn't connect the graph, then there is a path between x and y that doesn't use xy. Adding xy back in gives a cycle in T, so T must contain a cycle; this is a contradiction.

Proof of 3 using 2 and 1: Let's do this by induction on n, the order of the tree. A single vertex is a tree and has 0 edges, so the base case $n = 1$ holds. Let T be a tree with $n \geq 2$ vertices, then there must be an edge xy or else the graph cannot be connected. $T' = (V, E \setminus \{xy\})$ must have two components, because by the last argument we know there is are at most two and there cannot be just one component by part 1 of the Theorem. Thus, T' has two components, each of which is a tree. Say one of the components has $1 \leq k \leq n-1$ vertices; then the number of edges in these two trees is $(k-1)+(n-k-1) = n-2$ by induction. When we include xy, we get $n-1$.

Proof of 2 using 3 : If T is connected but has $n-1$ edges, then removing any edge disconnects it by Theorem 2.1. Hence 2 holds.

Proof of 1 using 2: Suppose T satisfies 2 but T has a cycle. Then removing some edge of this cycle does not disconnect T, a contradiction. Hence T is a cyclic and connected, and so it is a tree.

Proof of 4 using 3 and 1: Easy.

Proof of 1 using 4 and the equivalence of 1 and 3: If T is acyclic, each of the k connected components of T must be trees. By the equivalence of 1 and 3 which we have already shown, if the components have $n_1, \ldots, n_k$ vertices, then they have $n_1 - 1, \ldots, n_k - 1$ edges. Thus, T has $(n_1 - 1) + \cdots + (n_1 - 1) = n - k$ edges, so $k = 1$, implying T is connected. Thus T is a tree.