NONCOMMUTATIVE CLUSTERS

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To the memory of Andrei Zelevinsky, 1953–2013
Светлой памяти Андрея Владленовича Зелевинского посвящается

ABSTRACT. In this paper we introduce noncommutative clusters and their mutations, which can be viewed as vast generalizations of both "classical" and quantum cluster structures. Each noncommutative cluster $\mathbf{X}$ is built on a torsion-free group G and a certain collection of its automorphisms. We assign to $\mathbf{X}$ a noncommutative algebra $\mathcal{A}(\mathbf{X})$ related to the group algebra of G, which is an analogue of the cluster algebra and establish a noncommutative Laurent Phenomenon for many algebras $\mathcal{A}(\mathbf{X})$.

Our main examples of "cluster groups" G include principal noncommutative tori which we define for any initial exchange matrix B and noncommutative triangulated groups whixh we define for all oriented surfaces. The mutations of the latter groups turn out to be noncommutative analogues of classical Ptolemy relations, which, in particular, implies the Noncommutative Laurent Phenomenon for surfaces. As a surprising byproduct, we obtain new topological invariants of closed oriented surfaces with punctures. Another application is the proof of Laurentness and positivity of a noncommutative recursion recently defined by M. Kontsevich.

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1. Introduction

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The project started about six years ago when we discovered the *noncommutative polygon* $\mathcal{A}_n$, which is an algebra generated by x_{ij} and x_{ij}^{-1}, $1\le i,j\le n$, $i\ne j$ subject to the relations:

(i) (Triangle relations) For any distinct indices i,j,k:

$$x_{ij}x_{kj}^{-1}x_{ki}=x_{ik}x_{jk}^{-1}x_{ji}\ . \tag{1.1}$$

(ii) (Exchange relations) For any cyclically listed (i,j,k,ℓ):

$$x_{j\ell}=x_{jk}x_{ik}^{-1}x_{i\ell}+x_{ji}x_{ki}^{-1}x_{k\ell}\ . \tag{1.2}$$

In the commutative case, when $x_{ij}=x_{ji}$, the triangle relations are redundant and the exchange relations lead to the famous Ptolemy-Plücker relations.

Algebra $\mathcal{A}_n$ exhibits a Laurent phenomenon: given a triangulation Δ of the n-gon, each x_{ij} belongs to a subalgebra $\mathcal{A}_\Delta$ of $\mathcal{A}_n$ generated by all x_{i_0,j_0}, $(i_0,j_0)\in\Delta$ (See Section ?? for details).

Shortly after, we attached a similar algebra $\mathcal{A}_\Sigma$ to each surface Σ with punctures (and marked boundary points if Σ has boundary) and obtained the Noncommutative Laurent Polynomial as well (Section ??)

However, we delayed publishing these results because we did not have any other examples, as well as a general definition of noncommutative cluster algebras. Fortunately, about three years ago, we realized that the Kontsevich conjecture on totally noncommutative recursions can provide both the example and the structural theory of noncommutative clusters. After proving the conjecture in [5], we started the new pursuit that led to the present paper.

First, we noticed that each noncommutative cluster can be built on a specific noncommutative group. In the case of noncommutative polygons or surfaces, this group is the *triangular* group $\mathbb{T}_\Delta$ attached to each triangulation Δ of the polygon (or surface) which is generated by all t_{ij}, $(i,j)\in\Delta$ subject to the above triangle relations:

$$t_{ij}t_{kj}^{-1}t_{ki}=t_{ik}t_{jk}^{-1}t_{ji}\ . \tag{1.3}$$

In the case of noncommutative recursions this was the free group $\mathcal{F}_2$ of rank 2.

Second, we found that each noncommutative cluster mutation, e.g., the "noncommutative Ptolemy-Plücker relations" (1.2) or Kontsevich recursion, are governed by a certain automorphism of that group.

Finally, in order to define mutations of such a cluster group, i.e, a group G with a family of certain automorphisms $\theta_1,\ldots,\theta_n$), we needed some kind of exchange matrix which normally controls mutations of both commutative or quantum clusters. Quite surprisingly, this "noncommutative exchange matrix" is just the family $\Gamma=(\Gamma_1,\ldots,\Gamma_n$ of conjugacy classes of G in respective "fixed point groups" G^{θ_1},..., G^{θ_n} where $G^\theta=\{g\in G|\theta(g)=g\}$ (See section ??? for precise definitions).

In particular, within this framework, the ordinary "commutative cluster" (of geometric type) consists of the group $G=\mathbb{Z}^m$, we view each element $\mathbf{a}=(a_1,\ldots,a_m)$ of $\mathbb{Z}^m$ as a Laurent monomial $x^{\mathbf{a}}=x_1^{a_1}\cdots x_m^{a_m}$), and each automorphism θ_i, $i=1,\ldots,n$ is given by: $\theta_i(x_j)=\begin{cases}x_j & \text{if } i\ne j\\ x_ix^{\mathbf{b}_i} & \text{if } i=j\end{cases}$ where $\mathbf{b}_i$ is the i-th column of the corresponding extended exchange matrix $\tilde{B}$ (In fact, $x^{\mathbf{b}_i}$ is fixed under θ_i because matrix $\tilde{B}$ has zero diagonal).

In this paper we introduce a notion of noncommutative principal cluster (Section ??) and one of our main results , Theorem ?? asserts that for any initial $n\times n$ exchange matrix B each commutative or quantum *principal* cluster is "covered"' by the noncommutative principal one. It is natural to expect (Conjecture ??) that the noncommutative Laurent Phenomenon holds for such principal noncommutative clusters as well.

Our approach brought a very unexpected "byproduct" for geometry/topology of surfaces. Since the cluster groups do not change under mutations, we obtained a natural isomorphism between triangular groups $\mathbb{T}_\Delta$ and $\mathbb{T}_{\Delta'}$ for any triangulations of a given surface Σ. This, in particular, helps us to prove that $\mathbb{T}_\Delta$ is free if and only if either Σ has a boundary or is a sphere with three punctures (Theorem ??). If Σ is closed with $p \geq 4$ punctures, then each T_Δ is a one-relator group, which is, in fact an invariant of Σ depending on p (Theorem ???) [1] .

Based on this, it is natural to expect that new examples of noncommutative clusters may come from the fundamental groups of 3-manifolds.

As another application of our Noncommutative Laurent Phenomenon for surfaces, we prove Laurentness of the following noncommutative recursion from a paper [21]. Namely, let noncommutative variables U_n, $n \in \mathbb{Z}$ satisfy the system

$$\begin{cases} U_{n-k}U_n = 1 + U_{n-1}U_{n-k+1} & \text{if } n \text{ is even} \\ U_nU_{n-k} = 1 + U_{n-k+1}U_{n-1} & \text{if } n \text{ is odd} \end{cases} \tag{1.4}$$

where $k \geq 3$ is a fixed odd natural number. Maxim Kontsevich conjectured in [21] that each U_n is a noncommutative Laurent polynomial in $U_0, \ldots, U_k$ with positive coefficients. We prove this conjecture (Theorem ??) by taking Σ to be a cylinder with no punctures, one marked point on the lower boundary and k marked points on the upper boundary.

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2. Definitions and main results

2.1. **Noncommutative seeds and their mutations.** For a group G, an automorphism θ of G, and a conjugacy class Γ in the fixed point subgroup $G^\theta = \{g \in G : \theta(g) = g\}$ define the "right eigenspace"

$$\mathbf{X}_{\theta,\Gamma} = \{x \in G : \theta(x) \in x\Gamma\}$$

Note that $\mathbf{X}_{\theta,\Gamma}$ is also the "left eigenspace," i.e., $\mathbf{X}_{\theta,\Gamma} = \{x \in G : \theta(x) \in \Gamma x\}$ hence $\mathbf{X}_{\theta,\Gamma} = G^\theta \cdot \mathbf{X}_{\theta,\Gamma} \cdot G^\theta$. Note also that $(\mathbf{X}_{\theta,\Gamma})^{-1} = \mathbf{X}_{\theta^{-1},\Gamma} = \mathbf{X}_{\theta,\Gamma^{-1}}$.

We say that a pair (θ, Γ) is *compatible* if:

- $\mathbf{X}_{\theta,\Gamma}$ and G^θ generate G.
- under the abelianization homomorphism $G \mapsto G^{ab} = G/[G,G]$, the corresponding automorphism θ^{ab} of G^{ab} is torsion-free.

We say that (θ, Γ) is *strongly compatible* if it is compatible and the assignment $x \mapsto x + \theta(x)$ for $x \in \mathbf{X}_{\theta,\Gamma}$ and $g \mapsto g$ for $g \in G^\theta$ defines a homomorphism of multiplicative monoids:

$$G^+ \to \mathbb{Z}G \ ,$$

where G^+ is a submonoid of G generated by $\mathbf{X}_{\theta,\Gamma}$ and G^θ.

Given a finite indexing set I, a *noncommutative seed* is a triple $\mathcal{B} = (G, \Theta, \Gamma)$ where G is a group, $\Theta = (\theta_i, i \in I)$ is an I-tuple of automorphisms of G and $\Gamma = (\Gamma_i, i \in I)$, where each Γ_i is a conjugacy class in G^{θ_i} such that the pair (θ_i, Γ_i) is compatible for for $i \in I$. We abbreviate $\mathbf{X}_i := \mathbf{X}_{\theta_i,\Gamma_i}$ for $i \in I$ and refer to it as an *i-th noncommutative cluster (multi)variable.*

We say that a noncommutative seed $\mathcal{B} = (G, \Theta, \Gamma)$ is *optimal* if all (θ_i, Γ_i) are strongly compatible.

To each noncommutative seed $\mathcal{B} = (G, \Theta, \Gamma)$ we assign *generalized exchange matrix* $B = |\mathcal{B}|$ which is an $I \times I$ integer matrix whose (i,j)-th entry b_{ij} is determined by:

$$\theta_i^{ab}(\gamma_j^{ab}) = \gamma_j^{ab} \cdot (\gamma_i^{ab})^{b_{ij}} \tag{2.1}$$

and γ_k^{ab} denotes the only element of $(\Gamma_k)^{ab}$ (see Section ??? for details). By definition, all $b_{ii} = 0$ (however, the matrix B need not be skew-symmetrizable).

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[1] Misha Kapovich explained to us that $\mathbb{T}_\Delta$ is related to the fundamental group of a ramified two-fold covering of Σ

Example 2.1. Let $G = \mathbb{Z}^n \times C$, where C is an abelian group (with an additional "tropical multiplication") and let $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be an $n \times n$ sign-skew-symmetric matrix, where $\mathbf{b}_j$ stands for the j-th column of B. We view each element of G as a monomial $x^{\mathbf{a}} \cdot c$, where $c \in C$ and $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. By definition, for each choice $c_1, \ldots, c_n \in C$ the triple $\{(x_1, \ldots, x_n), (c_1, \ldots, c_n), B\}$ is a Fomin-Zelevinsky seed.

Lemma 2.2. *This defines a noncommutative seed $\mathcal{B}_B = (G, \Theta, \Gamma)$, where $\theta_i : G \to G$ is an automorphism given by*

$$\theta_i(x_j) = \begin{cases} x_j & \text{if } i \neq j \\ x_i x^{\mathbf{b}_i} c_i & \text{if } i = j \end{cases},\ \Gamma_i = \{x^{\mathbf{b}_i} c_i\}$$

for $i \in I = \{1, \ldots, n\}$.

We refer to such a "noncommutative" seed $\mathcal{B}_B$ as that of *classical type.*

This example can be generalized to a purely noncommutative one as follows.

Example 2.3. Let G be a group and $X = \{x_1, \ldots, x_n\}$ be a subset of G and C be a subgroup of G such each x_i is of infinite order and that

$$G = \langle x_1 \rangle * \cdots \langle x_n \rangle * C\ ,$$

where $\langle x_i \rangle$ denotes the cyclic group generated by x_i and "$*$" denotes the free product of groups (in particular, if C is a free group of rank $m - n$ for some $m \geq n$, then G is a free group of rank m). For $i = 1, \ldots, n$ denote by G_i the subgroup of G of the form

$$G_i = \langle x_1 \rangle * \cdots * \langle x_{i-1} \rangle * \langle x_{i+1} \rangle * \cdots * \langle x_n \rangle * C\ ,$$

so that $\langle x_i \rangle \cap G_i = \{1\}$ and $G = \langle x_i \rangle * G_i$.

For $i = 1, \ldots, n$ fix a non-unit element $\gamma_i \in G_i$ and denote by Γ_i the conjugacy class of γ_i in G_i.

Lemma 2.4. *These data define a noncommutative seed $\mathbf{S} = (G, \Theta, \Gamma)$, where θ_i is an automorphism of G such that $\theta_i(x_i) = x_i \gamma_i$ and $\theta(x) = x$ for all $x \in G_i$.*

Given noncommutative seeds $\mathcal{B} = (G, \Theta, \Gamma)$ and $\mathcal{B}' = (G', \Theta', \Gamma')$ a k-th *mutation* $\mathcal{B} \to \mathcal{B}'$ is an isomorphism of groups $f : G \widetilde{\to} G'$ such that $\Theta' = (\theta'_i, i \in I)$ and $\Gamma' = (\Gamma'_i, i \in I)$ are respectively determined by:

$$f^{-1} \circ \theta'_j \circ f = \begin{cases} \theta_k \circ \theta_j \circ \theta_k^{-1} & \text{if } b_{kj} > 0 \\ \theta_i & \text{otherwise} \end{cases},\ f^{-1}(\Gamma'_j) = \begin{cases} \Gamma_j^{-1} & \text{if } j = k \\ \theta_k(\Gamma_j) & \text{if } b_{kj} > 0 \\ \Gamma_j & \text{otherwise} \end{cases} \tag{2.2}$$

.

If $G = G'$ and $f = Id_G$, we denote the k-th mutation by $\mathcal{B} \to \underline{\mu}_k \mathcal{B}$. We say that a noncommutative seed $\mathcal{B}'$ is *mutation-equivalent* to $\mathcal{B}$ if they are related by a sequence of mutations. Clearly, for each $\mathcal{B}' = (G', \Theta', \Gamma')$ mutation equivalent to $\mathcal{B} = (G, \Theta, \Gamma)$ there is an isomorphism $\tilde{f} : G \to G'$ such that $\mathcal{B}' = f \circ \underline{\mu}_{k_m} \cdots \underline{\mu}_{k_1} \mathcal{B}$ for some sequence of indices $k_1, \ldots, k_m \in I$. Even though one can always take $G = G'$ and $f = Id_G$ above, it is more convenient to view G and G' as different groups.

It is easy to see that under the k-th mutation $\mathcal{B} \to \mathcal{B}'$ the noncommutative cluster (multi)variables transform by:

$$f^{-1}(\mathbf{X}'_j) = \begin{cases} \mathbf{X}_j^{-1} & \text{if } j = k \\ \theta_k(\mathbf{X}_j) & \text{if } b_{kj} > 0 \\ \mathbf{X}_j & \text{otherwise} \end{cases} \tag{2.3}$$

It turns out that mutation of generalized exchange matrices is subject to the following generalization of Fomin-Zelevinsky rule. Define the generalized matrix mutation $B \mapsto \mu_j(B)$ of $I \times I$ integer matrices by:

$$\mu_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + b_{ik}|b_{kj}| & \text{if } b_{ki} b_{kj} \leq 0 \\ b_{ij} & \text{if } b_{ki} b_{kj} > 0 \end{cases} \tag{2.4}$$

Clearly, if B is sign-skew-symmetrizable, this coincides with Fomin-Zelevinsky matrix mutations.

Proposition 2.5. *For each noncommutative seed $\mathcal{B}$ and $j \in I$ one has*

$$|\underline{\mu}_k \mathcal{B}| = \mu_k(|\mathcal{B}|)\ .$$

Now we can define a *totally noncommutative* cluster algebra $\mathcal{A}(\mathcal{B})$ as follows. First, following Fomin-Zelevinsky, we label all mutations of $\mathcal{B}$ by the vertices of the $|I|$-valent tree $\mathbb{T}$, so that noncommutative seeds $\mathcal{B}_t = (G_t, \Theta_t, \Gamma_t)$ and $\mathcal{B}_{t'} = (G_{t'}, \Theta_{t'}, \Gamma_{t'})$ are connected by a k-th mutation for some $k \in I$ if and only if t and t' are connected by an edge colored with k. In that case, one has a fixed isomorphism $f = f_{t,t'} : G_t \to G_{t'}$ such that (2.2) holds.

We denote by $\mathcal{A}(\mathcal{B})$ the quotient of the group algebra (over $\mathbb{Z}$) of the free product of all G_t by the ideal generated by all elements of the form:

$$f_{t,t'}(x) - x - \theta_{k,t}(x)$$

for all $x \in (\mathbf{X}_{k,t})^{-1} \subset G_t$, $t \in \mathbb{T}$, $k \in I$, where t' is the only vertex connected to t by an edge colored with k.

By definition, $\mathcal{A}(\underline{\mu_k}\mathcal{B}) = \mathcal{A}(\mathcal{B})$ for $k \in I$. Also, for each $t \in \mathbb{T}$ one has a canonical homomorphism of algebras

$$\iota_t : \mathbb{Z}G_t \to \mathcal{A}(\mathcal{B}) \tag{2.5}$$

In particular (in a contrast with the commutative case) the noncommutative cluster algebra $\mathcal{A}(\mathcal{B})$ contains inverses of all cluster variables.

Now we can define *canonical* noncommutative cluster variable.

Definition 2.6. Given a noncommutative seed $\mathcal{B} = (G, \Theta, \Gamma)$, the corresponding *canonical* noncommutative cluster (multi)variable $\mathbf{X}_i^{can} = \mathbf{X}_i^{can}(\mathcal{B})$ is given by

$$\mathbf{X}_i^{can} = \mathbf{X}_i \cap \bigcap_{j \neq i} G^{\theta_j} .$$

Clearly, $\mathbf{X}_i^{can} = C\mathbf{X}_i^{can}C$ for all $i \in I$, where $C = \bigcap\limits_{i=1}^{n} G^{\theta_i}$.

Remark 2.7. Note, however, that under the k-th mutation the canonical noncommutative cluster (multi)variables $\mathbf{X}_k^{can}$ do not behave properly. For instance, it may happen that $\mathbf{X}_k^{can}(\mathcal{B}) \neq \emptyset$, but $\mathbf{X}_k^{can}(\underline{\mu}_k\mathcal{B}) = \emptyset$.

This definition is justified by the following observation.

Lemma 2.8. *????Let $\mathcal{B}_B = (G, \Theta, \Gamma)$ be a noncommutative seed of classical type as in Example 2.1. Then $\mathbf{X}_i^{can} = x_i \cdot C$ for $i = 1, \ldots, n$. , where $(X_1, \ldots, X_n)$ is the Fomin-Zelevinsky cluster and $C = \bigcap\limits_{i=1}^{n} G^{\theta_i}$ is the corresponding coefficient group.*

Definition 2.9. Let $\mathcal{B}_t = (G_t, \Theta_t, \Gamma_t)$ be a noncommutative seed. We say that $\mathcal{B}$ admits a *noncommutative Laurent Phenomenon* if for each mutation-equivalent seed $\mathcal{B}_{t'} = (G_{t'}, \Theta_{t'}, \Gamma_{t'})$ one has:

$$\emptyset \neq \mathbf{X}_{i,t'}^{can} \subset \iota_t(\mathbb{Z}G_t)$$

for all $i \in I$, where ι_t is given by (2.5).

?????OLD INTRODUCTION HOW MUCH OF IT DO WE NEED?? This is new even in the commutative or quantum case. We construct then *noncommutative cluster algebras* as group algebras of noncommutative cluster groups and define their mutations.

Our main examples of noncommutative cluster algebras come from noncommutative tori and from surfaces Σ (with marked boundary and punctures). The noncommutative clusters related to Σ, quite expectable, are parametrized by triangulations of Σ and therefore deserve the name *noncommutative triangulations.*

Since each surface can be glued out of a polygon (in many ways), the most important objects of study are noncommutative triangulations of a given polygon. In the commutative case, cluster structure (of type A) on polygons is based on the celebrated *Ptolemy relations*:

$$x_{ik}x_{j\ell} = x_{ij}x_{k\ell} + x_{i\ell}x_{jk} \tag{2.6}$$

for all quadrilaterals (i, j, i, ℓ) inscribed in a circle, so that the chords (ik) and $(j\ell)$ are diagonals of the quadrilateral, and element x_{ij}, $i \neq j$ is the Euclidean length of the chord (ij). The Ptolemy relations (2.6) can also be interpreted as Plücker identities for $2 \times n$ matrices.

In the noncommutative version we do not assume that x_{ij} equals to x_{ji} and we think of x_{ij} as a directed chord from i to j. We suggest the following noncommutative generalization of the Ptolemy identity based on the theory of noncommutative quasi-Plücker coordinates developed in [20]:

$$x_{j\ell}=x_{jk}x_{ik}^{-1}x_{i\ell}+x_{ji}x_{ki}^{-1}x_{k\ell}. \tag{2.7}$$

Note that since elements x_{ij} correspond to directed arrows, the products of the form $x_{ij}x_{k\ell}^{-1}$, $x_{\ell k}^{-1}x_{ji}$ make sense only when $\ell=j$.

The noncommutative Ptolemy relations are not enough for developing a reasonable theory of noncommutative cluster algebras, in particular, for establishing the noncommutative Laurent Phenomenon (Theorems 5.11, 5.15). However, the Phenomenon holds if we additionally impose the *triangular relations* (also suggested by properties of quasi-Plücker coordinates):

$$x_{ij}x_{kj}^{-1}x_{ki}=x_{ik}x_{jk}^{-1}x_{ik} \tag{2.8}$$

for all distinct i,j,k (of course, (2.8) is redundant in the commutative case).

These arguments extend verbatim if we replace a polygon with a surface Σ with marked points in the boundary and possibly with some punctures (see section 6 for details).

In exchange relations (2.7) both elements x_{ik} and x_{ki} are inverted. To deal with one of these elements only, we rewrite the echange relations as

$$x_{j\ell}=x_{jk}x_{ik}^{-1}x_{i\ell}+x_{jk}(x_{ik}\gamma)^{-1}x_{i\ell}$$

where $\gamma=x_{\ell k}^{-1}x_{\ell i}x_{ji}^{-1}x_{jk}$.

One can separate x_{ik} and x_{ki} and look at their mutations independently. This is why we study mutations defined by monomials $x_{jk}x_{ik}^{-1}x_{i\ell}$ and $x_{jk}(x_{ik}\gamma)^{-1}x_{i\ell}$ inside certain groups before going to group algebras. In many cases such group are free or one-relator torsion free and localization theory of their group algebras is "tame" in comparison to "wild" general theory of noncommutative realizations (see [11]).

A surprising byproduct of our approach is construction of a new topological invariant of closed surfaces Σ with $n\geq 1$ punctures. The invariant is a group $\mathbb{T}=\mathbb{T}(\Sigma)$ which is defined for any triangulation Δ of Σ with vertices in the punctures. Namely, $\mathbb{T}$ is generated by all x_{ij}, where $(i,j)\in\Delta$ subject to the triangular relations (2.8). It turns out that $\mathbb{T}_\Delta$ is a one-relator group which does not depend on the choice of Δ. The group $\mathbb{T}(\Sigma)$ looks like the fundamental group of the closure $\overline{\Sigma}$ of Σ, however it is different from $\pi_1(\overline{\Sigma})$. For instance, if Σ_n is the sphere S^2 with n punctures, then $\mathbb{T}(\Sigma_3)$ is a free group in 5 generators and $T(\Sigma_n)$ is a 1-relator torsion-free group in $4n-7$ generators if $n\geq 4$.

In any case, the association $\Sigma\mapsto\mathbb{T}(\Sigma)$ defines a functor from the (topological) category of punctured surfaces to the category of finitely generated groups (Theorem 6.4).

3. Noncommutative clusters, seeds, and mutations

3.1. **Cluster groups.** We start with the following definition.

Definition 3.1. Let G be a group, G' be a subgroup of G, and $x\in G\setminus G'$ such that x and G' generate G denote by Γ_x the set of all $\gamma\in G'$ such that the assignment

$$x\mapsto x\cdot\gamma^{-1},\ g'\mapsto g' \text{ for } g'\in G'$$

defines an automorphism $\theta_\gamma:G\to G$. Note that $\theta_{\gamma_1}\circ\theta_{\gamma_2}=\theta_{\gamma_1\gamma_2}$ for $\gamma_1,\gamma_2\in\Gamma_x$, therefore, Γ_x is a subgroup of G'.

We say that G is a *semi-free product* of the cyclic subgroup $\langle x\rangle$ and G' (over Γ_x) if:

- x and G' generate G;
- $\langle x\rangle\cap G'=\{1\}$, where $\langle x\rangle$ denotes the cyclic subgroup of G generated by x;
- $\Gamma_x\neq\{1\}$.

For the notation purposes we will sometimes denote this semi-free factorization of G by $G=\langle x\rangle *_{\Gamma_x} G'$.

The most important example of semi-free product of $\langle x\rangle$ and G' is an *HNN extension* (introduced in [16]) of G' by an injective homomorphism $\alpha:H\hookrightarrow G'$ for some subgroup H of G', the quotient of the free product $\langle x\rangle * G'$ by the relations $xhx^{-1}=\alpha(h)$ for all $h\in H$. The following result is obvious.

Lemma 3.2. *An HNN extension G of $\langle x\rangle$ and G' is a a semi-free product of $\langle x\rangle$ and G' iff the subgroup $\tilde{\Gamma}_x:=x^{-1}G'x\cap G'$ has a non-trivial centralizer in G'. In that case, $\Gamma_x=Z_{G'}(\tilde{\Gamma}_x)$.*

Furthermore, fix $n\geq 1$ and abbreviate $[n]:=\{1,\ldots,n\}$.

Definition 3.3. Given a group G and its sub-monoid C (which will be referred to as the *coefficient monoid*) with an involutive anti-automorphism $g \mapsto \overline{g}$ such that $\overline{C} = C$. A *noncommutative cluster* on G (over C) is a subset $\mathbf{X} = \{x_1, \ldots, x_n\}$ of G such that:

- $\mathbf{X}$, $\overline{\mathbf{X}}$, and C generate G;
- For each $i \in [n]$ the group G is a semi-free product of $\langle x_i \rangle$ and $G_i := \langle C, \mathbf{X} \setminus \{x_i\}, \overline{\mathbf{X}} \setminus \{\overline{x}_i\}\rangle$ over
$$\Gamma_i = \{\gamma \in G_i \,|\, x_j \mapsto x_j \cdot \gamma^{-\delta_{ij}},\ g' \mapsto g' \text{ for } g' \in G' \text{ defines an automorphism } \theta_{i,\gamma} : G \to G\}\ .$$
- $\overline{x}_i \in G_i x_i G_i$ for $i \in [n]$;

We will refer to the pair $(G, \mathbf{X})$ as a *cluster group* (over C) and to the number n as the *rank* of $(G, \mathbf{X})$.

Given cluster groups $(G, \mathbf{X})$ and $(G', \mathbf{X}')$, a morphism $\theta : (G, \mathbf{X}) \to (G', \mathbf{X}')$ is any homomorphism of groups $\theta : G \to G'$ such that $\theta(C) = C'$ and $\theta(\mathbf{X}) = \mathbf{X}'$. We denote by **Clust** the category whose objects are cluster groups and arrows are morphisms of noncommutative seeds.

We denote by $\mathbf{Clust}_C$ the sub-category of **Clust** whose objects have a common submonoid C and morphisms are identity on C

Remark 3.4. Strictly speaking, C should be considered together with its embeddings into each involved group G, however, by a slight abuse of notation, we view C as a subgroup of all cluster groups in $\mathbf{Clust}_C$.

It would be interesting to classify noncommutative clusters on a given group G over a given monoid C. Here are some basic examples.

Example 3.5. If C is any abelian group and $G = \mathbb{Z}^n \times C$, then any generating set $\mathbf{X} = \{x_1, \ldots, x_n\}$ of $\mathbb{Z}^n$ is obviously a cluster on G with $G_i = \Gamma_i$ and $\overline{g} = g$ for any $g \in G$.

The following example slightly generalizes based quantum tori from [8].

Example 3.6. (Quantum torus) Let C_0 be an abelian group and G be generated by C_0 and $x_1, \ldots, x_m$ subject to the relations
$$x_i x_j = q_{ij} x_j x_i,\ x_i q = q x_i$$
for all $i, j \in [m]$, $q \in C_0$, where all $q_{ij} \in C_0$ are such that $q_{ji} = q_{ij}^{-1}$, $q_{ii} = 1$ for all i, j.
Let C be the sub-monoid of G generated by C_0 and $\{x_{n+1}, \ldots, x_m\}$. Then $\mathbf{X} = \{x_1, \ldots, x_n\}$ is obviously a cluster on G (i.e., (G, X) is a cluster group) over C with $\Gamma_i = Z(G_i)$ and the anti-involution $g \mapsto \overline{g}$ on G uniquely determined by $\overline{x}_j = x_j$ for $j = 1, \ldots, m$ and $\overline{c}_0 = c_0^{-1}$ for $c_0 \in C_0$.

If we assume that each $q_{ij} = v_{ij}^2$ for some $v_{ij} \in C_0$, then each element $x \in G$ uniquely factors as
$$x = cX^a$$
where $c \in C_0$, $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ and $X^a = \left(\prod_{1 \le i < j \le m} v_{ji}^{a_i a_j}\right) x_1^{a_1} \cdots x_m^{a_m}$, and the multiplication table in G becomes:
$$X^a X^b = \chi(a, b) X^{a+b}\ ,$$
where $\chi(a, b) = \prod_{1 \le i < j \le m} v_{ij}^{a_i b_j - a_j b_i}$. By construction, $\overline{X^a} = X^a$ for all $a \in \mathbb{Z}^m$ so G. Also
$$\Gamma_i = Z(G_i) = C_0 \cdot \{X^b \,|\, \chi(b, e_j) = 1, j \in [n] \setminus \{i\}\}\ ,$$
where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{Z}^n$.

The following is a "more noncommutative" version of 2-dimensional quantum tori.

Example 3.7. (Noncommutative rank 2 torus) Let $G = \mathbb{T}_{\mathbf{a}}$ be a group generated by $\mathbf{X} = \{x_1, x_2\}$ and $\mathbf{a} = \{a_{12}, a_{21}, a_{23}, a_{32}\}$, subject to the relation:
$$x_1 a_{12} x_2 a_{23} = a_{32} x_2 a_{21} x_1$$

Lemma 3.8. $(\mathbb{T}_{\mathbf{a}}, \mathbf{X})$ *is a cluster group over* $\Gamma_i = \langle \gamma_i \rangle$, $i = 1, 2$, *where* $\gamma_1 = a_{12} x_2 a_{23}$, $\gamma_2 = a_{21} x_1 a_{23}^{-1}$, *and the anti-involution* $g \mapsto \overline{g}$ *on* $\mathbb{T}_{\mathbf{a}}$ *is given by* $\overline{x}_i = x_i$, $\overline{a}_{i,i+1} = a_{i+1,i}$, $i = 1, 2$.

We will consider higher-dimensional noncommutative tori in Section 3.3.

3.2. **Noncommutative seeds and their monomial mutations.** We retain the notation of Section 3.1.

Definition 3.9. A *noncommutative seed* (over C) is a triple $\mathbf{S} = (G, \mathbf{X}, \gamma)$, where

- $(G, \mathbf{X})$ is a cluster group over C;
- $\gamma = \{\gamma_1, \ldots, \gamma_n\}$, where $\gamma_j \in \Gamma_j \setminus \{1\}$ for $i \in [n]$ such that:

$$\overline{\gamma}_j = \overline{x}_j \gamma_j \overline{x}_j^{-1} \tag{3.1}$$

for $j \in [n]$.

By Definition 3.1 of a semi-free product of $\langle x_i \rangle$ and G_i (over Γ_i), for each noncommutative seed $(G, \mathbf{X}, \gamma)$ and $i \in [n]$ the assignment

$$x_j \mapsto x_j \gamma_i^{-\delta_{ij}},\ c \mapsto c \text{ for } c \in C$$

defines an automorphism $\theta_i := \theta_{i,\gamma_i} : G \to G$. The following statement is obvious.

Lemma 3.10. *Given a noncommutative seed $(G, \mathbf{X}, \gamma)$, each automorphism θ_i of G commutes with the anti-automorphism $g \mapsto \overline{g}$ of G.*

Given noncommutative seeds $\mathbf{S} = (G, \mathbf{X}, \gamma)$ and $\mathbf{S}' = (G', \mathbf{X}', \gamma')$, a morphism $\theta : \mathbf{S} \to \mathbf{S}'$ is any homomorphism of cluster groups $\theta : (G, \mathbf{X}) \to (G', \mathbf{X}')$ such that $\theta(\gamma) = \gamma'$. We denote by **Seed** the category whose objects are noncommutative seeds and arrows are morphisms of noncommutative seeds.

Now we are ready for our main definition, *(monomial) mutation* of noncommutative seeds.

Definition 3.11. Given a noncommutative seeds $\mathbf{S} = (G, \mathbf{X}, \gamma)$ and $\mathbf{S}' = (G', \mathbf{X}', \gamma')$ over C, we say that $\mathbf{S}$ and $\mathbf{S}'$ are related by the i-th *monomial mutation*, $i \in [n]$ if there exists an isomorphism $\underline{\mu_i} : G' \to G$ such that:

- $\underline{\mu_i}(\overline{g'}) = \overline{\underline{\mu_i}(g')}$ for all $g' \in G'$;
- $\underline{\mu_i}(x_j') = x_j$ for $j \neq i$ and $\underline{\mu_i}(x_i') \in G_i x_i^{-1} G_i$;
- For $j \in [n] \setminus \{i\}$ one has $\underline{\mu_i}(x_i') \in G_j x_i^{-1} G_j \cup \theta_i^{-1}(G_j \overline{x}_i^{-1} G_j)$.
- $\underline{\mu_i}(\gamma_i') = (a_i b_i)^{-1} \overline{\gamma}_i^{-1} a_i b_i$, where $a_i, b_i \in G_i$ are such that $\underline{\mu_i}(x_i') \in G_i x_i^{-1} b_i$, $\overline{x}_i \in a_i x_i G_i$;
- For $j \in [n] \setminus \{i\}$ one has: $\underline{\mu_i}(\gamma_j') = \begin{cases} \gamma_j & \text{if } \underline{\mu_i}(x_i') \in G_j x_i^{-1} G_j \\ \theta_i^{-1}(\gamma_j) & \text{if } \underline{\mu_i}(x_i') \in \theta_i^{-1}(G_j \overline{x}_i^{-1} G_j) \end{cases}$.

We say that the mutation is *positive* if:

$$\underline{\mu_i}(x_i') \in G_i^+ x_i^{-1} G_i^+,\ \theta_i(\underline{\mu_i}(x_i')) \in G_i^+ \overline{x}_i^{-1} G_i^+\ , \tag{3.2}$$

where G_i^+ is the submonoid of G_i generated by C and $\mathbf{X} \setminus \{x_i\}$, $\overline{\mathbf{X}} \setminus \{\overline{x}_i\}$;

We characterize all (positive) i-th mutations in terms of an element $\underline{\mu_i}(x_i') = y_i \in G_i x_i^{-1} G_i$ as follows.

Theorem 3.12. *Let $\mathbf{S} = (G, \mathbf{X}, \gamma)$ be a noncommutative seed over C and let $y_i \in G_i x_i^{-1} G_i$ such that*

$$y_i \in G_j x_i^{-1} G_j \cup \theta_i^{-1}(G_j \overline{x}_i^{-1} G_j) \tag{3.3}$$

for all $j \in [n] \setminus \{i\}$.

Then there exists a noncommutative seed $\mu_{y_i}(\mathbf{S}) = (G', \mathbf{X}', \gamma')$ over C such that $\mathbf{S}$ and $\mu_{y_i}(\mathbf{S})$ are related by the i-th mutation $\underline{\mu_i} : G' \to G$ given by:

$$\underline{\mu_i}(x_j') = \begin{cases} x_j & \textit{if } j \neq i \\ y_i & \textit{if } j = i \end{cases}, \underline{\mu_i}(c) = c \textit{ for } c \in C\ .$$

Moreover, the mutation is positive if and only if $y_i \in G_i^+ x_i^{-1} G_i^+$ and $\theta_i(y_i) \in (G_i^+ \overline{x}_i^{-1} G_i^+)$.

Proof. First of all, let us construct the cluster group $(G', \mathbf{X}')$ for each y_i as in the theorem. We need the following result.

Lemma 3.13. *Let $\mathbf{X}$ be a noncommutative cluster on G and let $y_i \in G_i x_i^{-1} G_i$ such that* (3.3) *holds. Then the set $\mathbf{X}' := \mathbf{X} \setminus \{x_i\} \cup \{y_i\}$ is a noncommutative cluster on G with*

$$G_j' = \begin{cases} G_i & \textit{if } j = i \\ G_j & \textit{if } j \neq i \textit{ and } y_i \in G_j x_i^{-1} G_j \\ \theta_i^{-1}(G_j) & \textit{if } j \neq i \textit{ and } y_i \in \theta_i^{-1}(G_j \overline{x}_i^{-1} G_j) \end{cases} \tag{3.4}$$

(3.5)
$$\Gamma'_j = \begin{cases} b_i^{-1}x_i\Gamma_i x_i^{-1}b_i & \textit{if } j = i \\ \Gamma_j & \textit{if } j \neq i \textit{ and } y_i \in G_j x_i^{-1}G_j \\ \theta_i^{-1}(\Gamma_j) & \textit{if } j \neq i \textit{ and } y_i \in \theta_i^{-1}(G_j\overline{x}_i^{-1}G_j) \end{cases}$$

for $j \in [n]$, where $b_i \in G_i$ is any element such that $y_i \in G_i x_i^{-1}b_i$.

Proof. For each $j \in [n]$ denote by $\tilde{G}_j$ the subgroup of G generated by $\mathbf{X}' \setminus \{x'_j\}$, $\overline{\mathbf{X}}' \setminus \{\overline{x}'_j\}$, and C and define $\tilde{\Gamma}'_j := {x'_j}^{-1}\tilde{G}_j x'_j \cap \tilde{G}_j$.

Indeed, since $y_i \in G_i x_i^{-1}G_i$, then $\tilde{G}_i = G_i$. Furthermore, the alternative (3.3) implies that for each $j \neq i$ one has:

$$\tilde{G}_j = \begin{cases} G_j & \text{if } y_i \in G_j x_i^{-1}G_j \\ \theta_i^{-1}(G_j) & \text{if } y_i \in \theta_i^{-1}(G_j\overline{x}_i^{-1}G_j) \end{cases} = G'_j,$$

for $j \neq i$.

Since θ_i is an automorphism of G, we obtain for $j \neq i$:

$$\tilde{\Gamma}'_j = \begin{cases} \Gamma_j & \text{if } y_i \in G_j x_i^{-1}G_j \\ \theta_i^{-1}(\Gamma_j) & \text{if } y_i \in \theta_i^{-1}(G_j\overline{x}_i^{-1}G_j) \end{cases} = \Gamma'_j \ .$$

If $j = i$, then, clearly, the assignments $y_i \mapsto y_i(\gamma'_i)^{-1}$, $g' \mapsto g'$ for $g' \in G_i$ define an automorphism of G for some $\gamma'_i \in G_i$ if and only if $\gamma'_i \in b_i^{-1}x_i\Gamma_i x_i^{-1}b_i$. The lemma is proved. □

Lemma 3.13 asserts that the pair

$$(G', \mathbf{X}') := (G, \mathbf{X} \setminus \{x_i\} \cup \{y_i\})$$

is a cluster group. We set $\underline{\mu_i} := Id_G$, which is, clearly, an isomorphism $G' \xrightarrow{\sim} G$. To finish the proof we have to determine the remainder of the seed $(G', \mathbf{X}', \gamma')$, the set $\gamma' = \{\gamma'_1, \ldots, \gamma'_n\}$ where each $\gamma'_j \in \Gamma'_j \setminus \{1\}$. Indeed, copying Definition 3.11, we set $\gamma'_i = b_i^{-1}x_i{\gamma_i}^{-1}x_i^{-1}b_i$, where $b_i \in G_i$ is such that $y_i \in G_i x_i^{-1}b_i$, $\overline{x}_i \in a_i x_i G_i$ and:

$$\gamma'_j = \begin{cases} \gamma_j & \text{if } y_i \in G_i x_i^{-1}G_i \\ \theta_i^{-1}(\gamma_j) & \text{if } y_i \in \theta_i^{-1}(G_j\overline{x}_i^{-1}G_j) \end{cases} .$$

Clearly, $\gamma'_j \in \Gamma'_j \setminus \{1\}$ for all $j \in [n]$. Finally, verify (3.1). Indeed, if $j \neq i$, then (3.1) holds for $(G', \mathbf{X}')$ because $\theta_i(\overline{x}_j) = \overline{\theta_i(x_j)} = \overline{x}_j$. When $j = i$, we have $\theta'_i(y_i) = y_i(\gamma'_i)^{-1} = a_i x_i^{-1}b_i(b_i^{-1}x_i{\gamma_i}^{-1}x_i^{-1}b_i)^{-1} = a_i\gamma_i x_i^{-1}b_i$ and $\theta'_i(x_j) = x_j$ if $j \neq i$, hence $\theta'_i = \theta_i$. Finally, by Lemma 3.10,

$$\overline{y}_i(\gamma'_i)^{-1} = \theta_i(\overline{y}_j) = \overline{\theta_i(y_j)} = \overline{y_i(\gamma'_i)^{-1}} = (\,\overline{\gamma}'_i\,)^{-1}\overline{y}_i \ ,$$

which proves (3.1) for $(G', \mathbf{X}')$, $j = i$.

Therefore, Theorem 3.12 is proved. □

Remark 3.14. The conditions (3.2) and (3.3) taken together are rather strong. For instance, they imply that for any noncommutative seed $\mathbf{S} = (G, X, \gamma)$, if $\gamma_i \in C$, then any y_i satisfying (3.3) must belong to $Cx_i^{-1}C$.

Example 3.15. Let $(\mathbf{X}, \tilde{B})$ be a Fomin-Zelevinsky seed of geometric type with the cluster $\mathbf{X} = \{x_1, \ldots, x_m\}$, $\tilde{B} = (b_1, \ldots, b_n)$ is the $m \times n$ extended exchange matrix, $m \geq n$. Then the triple $\mathbf{S} = \mathbf{S}(\mathbf{X}, \tilde{B}) = (G, \mathbf{X}, \gamma)$ is a noncommutative seed with:

- $G \cong \mathbb{Z}^m$, $C \cong \mathbb{Z}^{m-n}$ and C occupies last $m - n$ places in G;
- $\gamma_i = x^{b_i}$ for $i \in [n]$.

Lemma 3.16. *Let $(\mathbf{X}, \tilde{B})$, $(\mathbf{X}', \tilde{B}')$ be Fomin-Zelevinsky seed of geometric type. Then "noncommutative" seeds $\mathbf{S}(\mathbf{X}, \tilde{B}))$ and $\mathbf{S}(\mathbf{X}', \tilde{B}')$ are related by the i-th positive monomial mutation if and only if:*

(i) $\underline{\mu_i} : G' \to G$ is an isomorphism given by $\underline{\mu_i}(x'_j) = \begin{cases} x_j & \textit{if } j \neq i \\ x_i^{-1}x^{[-b_i]_+} & \textit{if } j = i \end{cases}$.

(ii) $\tilde{B}' = \mu_i(\tilde{B}) = (b'_1, \ldots, b'_n)$ is the Fomin-Zelevinsky i-th mutation of $\tilde{B}$, i.e.,

$$(\tilde{B}')_{kj} = \begin{cases} -b_{kj} & \textit{if } i \in \{k, j\} \\ b_{kj} + [-b_{ki}]_+ b_{ij} + b_{ki}[b_{ij}]_+ & \textit{otherwise} \end{cases}$$

for $k \in [m]$, $j \in [n]$*;*

In that case, $\gamma'_j = {x'}^{b'_j}$ *and* $\underline{\mu_i}(\gamma'_j) = x^{b_j + [b_{ij}]_+ b_i} = \begin{cases} \gamma_j & \textit{if } b_{ij} \le 0 \\ \theta_i^{-1}(\gamma_j) & b_{ij} > 0 \end{cases}$ *for all* $j \in [n]$*.*

Example 3.17. (Noncommutative triangle) Let C be a group and let $\mathbb{T}_3$ be a group generated by T_1, T_2, T_3 and C, $i, j = 1, 2, 3$, $i \ne j$ subject to the relation:

$$T_1 a_{12} T_2^{-1} a_{23} T_3 a_{31} = a_{13} T_3 a_{32} T_2^{-1} a_{21} T_1 \ ,$$

where all $a_{ij} \in C$. Assume that C admits an anti-automorphism $c \mapsto \overline{c}$ such that $\overline{a}_{ij} = a_{ji}$ for all $i \ne j$.

The following fact is obvious.

Lemma 3.18. *The set* $\mathbf{X} = \{T_1, T_2, T_3\}$ *is a noncommutative cluster on* $\mathbb{T}_3$ *with the anti-involution* $x \mapsto \overline{x}$ *extending that on* C *given by* $\overline{T}_i = T_i$, $i = 1, 2, 3$ *and semi-free factorizations (in fact, HNN-extensions)* $\mathbb{T}_3 = \langle T_i \rangle *_{\Gamma_i} (\mathbb{T}_3)_i$*, where* $\Gamma_i = \langle \gamma_i \rangle$ *and:*

$$\gamma_1 = a_{31}^{-1} T_3 a_{23}^{-1} T_2 a_{12}^{-1}, \gamma_2 = a_{12}^{-1} T_1^{-1} a_{13} T_3 a_{32}, \ \gamma_3 = a_{32} T_2^{-1} a_{21} T_1 a_{31}^{-1} \ .$$

Then the triple $\mathbf{S} = (\mathbb{T}_3, \mathbf{X}, \{\gamma_1^2, \gamma_2^2, \gamma_3^2\})$ *is a noncommutative seed which admits positive monomial mutations* $\mu_{y_i}(\mathbf{S})$*,* $i = 1, 2, 3$*, where:*

$$y_1 = T_3 a_{31} T_1^{-1} a_{13} T_3, \ y_2 = T_1 a_{12} T_2^{-1} a_{21} T_1, \ y_3 = T_2 a_{32}^{-1} T_3^{-1} a_{23}^{-1} T_2 \ .$$

Example 3.19. (Rank 2 noncommutative toric seeds) In the notation of Example 3.7 we see that for any $r_1, r_2 \ge 1$ the triple $\mathbf{S}_{r_1, r_2, \mathbf{a}} := (\mathbb{T}_\mathbf{a}, \mathbf{X}X, \{\gamma_1^{r_1}, \gamma_2^{r_2}\})$ is a noncommutative seed on $(G, \mathbf{X})$, where $\gamma_1 = a_{12} x_2 a_{23}$ and $\gamma_2 = a_{21} x_1 a_{23}^{-1}$.

The following fact easily follows from definitions.

Lemma 3.20. *Let* $r_1, r_2 \ge 1$ *and* $j \in \{1, 2\}$*. Then* $\mathbf{S}_{r_1, r_2, \mathbf{a}}$ *and* $\mathbf{S}$ *are related by an* j*-th positive mutation* μ_j *if and only if* $\mathbf{S}'$ *is isomorphic to* $\mathbf{S}_{r_1, r_2, \mathbf{a}}$*. In that case, the underlying group isomorphism* $\underline{\mu_j} : \mathbb{T}_\mathbf{a} \to \mathbb{T}_\mathbf{a}$ *is given by*

$$x_j \mapsto x_j^{-1}, x_{3-j} \mapsto x_{3-j}, \ a_{i2} \mapsto a_{4-i,2}, \ a_{2i} \mapsto a_{2,4-i}$$

for $i = 1, 3$*,* $j = 1, 2$*.*

3.3. **Principal noncommutative tori and their monomial mutations.** Let B be an $n \times n$ matrix with $b_{ii} = 0$ for all $i \in [n]$. Denote by $G(B)$ the group generated by x_i, τ_i, $i = 1, \ldots, n$ subject to the relations:

$$x_i x_j = x_j x_i, \ \tau_i x_j = x_j \tau_i, \ x_i^{b_{ij}} \tau_i \tau_j = \tau_j \tau_i x_i^{b_{ij}}$$

for all $i, j \in [n]$, $i \ne j$.

We refer to $G(B)$ as a *principal noncommutative torus.*

Remark 3.21. The quotient $\underline{G(B)}$ of $G(B)$ by the relation that all elements $q_i = x_i \tau_i x_i^{-1} \tau_i^{-1}$ are central, is a (principal) quantum torus given by

$$x_i x_j = x_j x_i, \tau_i x_j = q_i^{\delta_{ij}} x_j \tau_i, \ \tau_i \tau_j = q_i^{b_{ij}} \tau_j \tau_i$$

for all $i, j \in [n]$. In turn, this imply that B is "skew-symmetrizable": $q_i^{b_{ij}} q_j^{b_{ji}} = 1$ in $\underline{G(B)}$ for all i, j.

For any sequence $\mathbf{i} = (i_1, \ldots, i_m) \in [n]^m$ and vector $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ we define the element $\tau_\mathbf{i}^a \in G(B)$ by:

$$\tau_\mathbf{i}^a = \tau_{i_1}^{a_1} \cdots \tau_{i_m}^{a_m}$$

The following result is obvious.

Lemma 3.22. *For all* $\mathbf{i} \in [n]^m$ *and* $a \in \mathbb{Z}^m$*:*

$$\tau_\mathbf{i}^a \tau_i (\tau_\mathbf{i}^a)^{-1} = x_i^d \tau_i x_i^{-d}$$

where $d = d_\mathbf{i}(a) = \sum_{k=1}^m a_k b_{i, i_k}$*. It implies*

$$[\tau_i, \tau_j] = [\tau_j^{-1}, \tau_i] = [\tau_j, \tau_i^{-1}] \ , \tag{3.6}$$

where $[a, b]$ *denotes the group commutator* $aba^{-1}b^{-1}$*.*

Remark 3.23. Based on Lemma 3.22 one can conjecture that the subgroup $G(B)$ generated by $\tau_1, \dots, \tau_n$ is subject to the relations (3.6) and

$$[\tau_i, \tau_{\mathbf{i}}^a] = 1$$

whenever $\sum_{k=1}^m a_k b_{i,i_k} = 0$.

Denote by $G_i = G_i(B)$, $i \in [n]$ the subgroup of $G(B)$ generated by $\mathbf{X} \setminus \{x_i\}$ and $\mathbf{T} = \{t_1, \dots, t_n\}$, where $\mathbf{X} = \{x_1, \dots, x_n\}$ and

$$t_i := \tau_i \cdot \prod_{k=1}^n x_k^{-b_{ki}}$$

for $i \in [n]$;

Theorem 3.24. *Let B be any integer $n \times n$ matrix with zero diagonal. Then:*
The pair $(G(B), \mathbf{X})$ is a cluster group with:

- *the anti-involution $x \mapsto \overline{x}$ defined by $\overline{x}_i = x_i$, $\overline{\tau}_i = x_i \tau_i x_i^{-1}$ for $i \in [n]$;*
- *the submonoid $C \subset G(B)$ generated by $\mathbf{T}$ and $\overline{\mathbf{T}} = \{x_1 t_1 x_1^{-1}, \dots, x_n t_i x_n^{-1}\}$;*
- *the automorphism θ_i, $i \in [n]$ of $G(B)$ given by $x_j \mapsto x_j \cdot \tau_i^{-\delta_{ij}}$, $t_j \mapsto t_j$ for every j.*

Proof. We need the following obvious result.

Lemma 3.25. *For each $i \in [n]$ the subgroup $\langle x_i, t_i \rangle$ generated by x_i and t_i is normal in $G(B)$.*

To prove that θ_i is an automorphism of $G(B)$, it is enough to check that θ_i is a homomorphism preserving identities including x_i and τ_i.

Note that

$$\theta_i(\tau_i) = \tau_i, \quad \theta_i(\tau_j) = \tau_j x_i^{-b_{ij}} (x_i \tau_i^{-1})^{b_{ij}}.$$

It is easy to see that $\theta_i(x_i)\theta_i(x_j) = \theta_i(x_j)\theta_i(x_i)$.

Now, let $p \neq q$. The identity $x_p \tau_q = \tau_q x_p$ can be written as $x_p t_q = t_q x_p$, the identity $x_p^{b_{pq}} \tau_p \tau_q = \tau_q \tau_p x_p^{b_{pq}}$ can be written as $x_p^{b_{pq}} \tau_p x_p^{-b_{pq}} = \tau_q \tau_p \tau_q^{-1}$ or $x_q^{b_{qp}} t_q x_q^{-b_{qp}} = t_p^{-1} t_q t_p$.

To check that $\theta_i(x_i)\theta_i(t_j) = \theta_i(t_j)\theta_i(x_i)$, it is enough to check that

$$x_i (t_i x_j^{b_{ji}})^{-1} t_j = t_j x_i (t_i x_j^{b_{ji}})^{-1}.$$

We may cancel x_i in both sides and rewrite the rest as a known identity $x_j^{b_{ji}} t_j x_j^{-b_{ji}} = t_i^{-1} t_j t_i$.

The identity $x_i^{b_{ij}} \tau_i x_i^{-b_{ij}} = \tau_j \tau_i \tau_j^{-1}$ can be rewritten as $x_j^{b_{ji}} t_j x_j^{-b_{ji}} = t_i^{-1} t_j t_i$ and remains unchanged under application of θ_i.

It remains to check that $\theta_i(x_i^{b_{ij}} \tau_i x_i^{-b_{ij}}) = \theta_i(\tau_j \tau_i \tau_j^{-1})$ or

$$(x_i \tau_i^{-1})^{b_{ij}} \tau_i (x_i \tau_i^{-1})^{-b_{ij}} = \tau_j x_i^{-b_{ij}} (x_i \tau_i^{-1})^{b_{ij}} \tau_i (x_i \tau_i^{-1})^{-b_{ij}} x_i^{b_{ij}} \tau_j^{-1}.$$

Note that

$$x_i^{b_{ij}} \tau_j^{-1} \cdot \tau_i^{-1} \cdot \tau_j x_i^{-b_{ij}} = \tau_j^{-1} (x_i^{b_{ij}} \tau_i^{-1} x_i^{-b_{ij}}) \tau_j = \tau_j^{-1} (\tau_j \tau_i^{-1} \tau_j^{-1}) \tau_j = x_i \tau_i^{-1}.$$

Since a conjugation by an invertible element is an automorphism and τ_j and x_i commute for $i \neq j$, the previous equality after conjugation by $x_i^{b_{ij}} \tau_j^{-1}$ becomes an identity. It proves that θ_i is an automorphism. □

Remark 3.26. It follows from Lemma 3.22 that $G_i(B)$ is normal in $G(B))$ iff $gcd(|b_{1i}|, \dots, |b_{ni}|) = 1$.

Remark 3.27. It is easy to see that semi-free factorizations $G(B)$ into $\langle x_i$ and $G_i(B)$ are **not** HNN-extensions. This was our primary reason for introducing semi-free products (Definition 3.1).

It follows from Theorem 3.24 that the triple $\mathbf{S}(B) := (G(B), \mathbf{X}, (\tau_1, \dots, \tau_n))$ is a noncommutative seed, which we refer to as a *principal noncommutative seed.*

In the following result we use the (generalized) matrix mutation $B \mapsto \mu_j(B)$ defined in (2.4).

Main Theorem 3.28. *Let B be any integer $n \times n$ matrix with zero diagonal and let $j \in [n]$. Then:*
(a) $\mathbf{S}(B)$ and $\mathbf{S}'$ are related by an j-th positive mutation μ_j if and only if $\mathbf{S}'$ is isomorphic to $\mathbf{S}(\mu_j(B))$.

(b) The underlying group homomorphism $\underline{\mu_j} : G(B)\widetilde{\to}G(\mu_j(B))$ *is (unique and) given by*

$$x_i \mapsto \begin{cases} x_j^{-1}\prod\limits_{k=1}^{n} x_i^{[-b_{kj}]_+} & \textit{if } i=j \\ x_i & \textit{otherwise} \end{cases}, \tau_i \mapsto \begin{cases} \overline{\tau}_j^{\,-1} & \textit{if } i=j \\ \tau_i & \textit{if } i\ne j \textit{ and } b_{ij}\ge 0 \\ \tau_i x_j^{-b_{ji}}\cdot (x_j\tau_j)^{b_{ji}} & \textit{if } i\ne j \textit{ and } b_{ij}<0 \end{cases},$$

for all $i\in[n]$.

3.4. **Noncommutative triangulated polygons, seeds, and their mutations.** For each $n\ge 3$ consider a cyclic order $i\mapsto i^+$ on $[n]=\{1,2,\ldots,n\}$ by setting

$$i^+=\begin{cases} i+1 & \text{if } i<n \\ 1 & \text{if } i=n \end{cases}$$

(and $i\mapsto i^-$ to be the inverse of $i\mapsto i^+$). In what follows we will view $[n]$ with this order as a colection of n points on a circle (or vertices of a convex n-gon) and each pair (i,j) as a chord from i to j (or as a diagonal of the n-gon).

We also say that a sequence $\mathbf{i}=(i_1,\ldots,i_\ell)$ of distinct elements in $[n]$ is *cyclic* if a cyclic permutation $\mathbf{i}\mapsto(i_k,\ldots,i_\ell,i_1,\ldots,i_{k-1})$ is strictly increasing. In particular, the sequence $(k,k+1,\ldots,n,1,\ldots,k-1)$ is cyclic for each k.

We say that a pair (i,k) *crosses* (j,ℓ) if (i,j,k,ℓ) is cyclic.

An *ordered triangulation* $\underline{\Delta}$ of $[n]$ is a maximal non-crossing subset of $[n]\times[n]$ such that if $(i,j)\in\Delta$ for some (i,j), then $(j,i)\notin\Delta$. Clearly, each ordered triangulation of $[n]$ has cardinality $2n-3$. We denote $\underline{\Delta}^{op}=\{(j,i)|(i,j)\in\underline{\Delta}\}$ the opposite ordered triangulation and the (unordered) triangulation $\Delta:=\underline{\Delta}\sqcup\underline{\Delta}^{op}$, which we refer to as the *underlying* triangulation of $\underline{\Delta}$.

For each triangulation Δ of $[n]$ define the group $\mathbb{T}_\Delta$ generated by all t_{ij}, $(i,j)\in\Delta$ subject to the triangular relations:

$$t_{ij}t_{kj}^{-1}t_{ki}=t_{ik}t_{jk}^{-1}t_{ji}$$

for all $i,j,k\in[n]$ such that $(i,j),(j,k),(k,i)\in\Delta$.

By definition, the assignment $t_{ij}\mapsto\overline{t}_{ij}=t_{ji}$ defines an anti-involution $g\mapsto\overline{g}$ on $\mathbb{T}_\Delta$.

Theorem 3.29. *For each triangulation* Δ *of* $[n]$ *the group* $\mathbb{T}_\Delta$ *is a free group in* $3n-4$ *generators.*

We prove Theorem 3.29 in Section ???. It also agrees with Theorem 6.2 for more general surfaces.

For each ordered triangulation Δ of $[n]$ define:
- The subset $X_{\underline{\Delta}}\subset\mathbb{T}_\Delta$ by $X_{\underline{\Delta}}=\{t_{ij},(i,j)\in\underline{\Delta}\setminus\{(i,i^\pm),i\in I\}\}$;
- the submonoid C of $\mathbb{T}_\Delta$ generated by all $t_{i,i^\pm}$, $i\in[n]$.
- a subgroup $G_{ij}=G_{ji}$ of $G=\mathbb{T}_\Delta$ for each $(ij)\in\Delta$ to be generated by C and $X_{\underline{\Delta}}\cup X_{\underline{\Delta}^{op}}\setminus\{t_{ij},t_{ji}\}$.

Theorem 3.30. *For each triangulation* Δ *of* $[n]$ *the pair* $(\mathbb{T}_\Delta,X_\Delta)$ *is a cluster group such that for each* $(i,j)\in\Delta$, $j\notin\{i^+,i^-\}$ *one has a semi-free factorization (in fact, an HNN-extension):*

$$\mathbb{T}_\Delta=\langle x_{ij}\rangle *_{\Gamma_{ij}} G_{ij}$$

where Γ_{ij} *is the cyclic group generated by* $\gamma_{ij}=t_{\ell j}^{-1}t_{\ell i}t_{ki}^{-1}t_{kj}$, *where* $(k\ell)\in[n]\times[n]\setminus\Delta$ *is a unique (up to the transposition* $k\leftrightarrow\ell$*) pair such that* $\Delta'=\Delta\setminus\{(ij),(ji)\}\cup\{(k\ell),(\ell k)\}$ *is a triangulation of* $[n]$.

Theorem 3.31. *For each ordered triangulation* Δ *of* $[n]$ *the triple* $\mathbf{S}_{\underline{\Delta}}:=(\mathbb{T}_\Delta,X_{\underline{\Delta}},\gamma_{\underline{\Delta}})$, *where* $\gamma_{\underline{\Delta}}=\{\gamma_{ij}|(i,j)\in\underline{\Delta}\setminus\{(i,i^\pm),i\in I\}\}$ *is a noncommutative seed on* $(\mathbb{T}_\Delta,X_{\underline{\Delta}})$.

Theorem 3.32. *??? Given an ordered triangulation* $\underline{\Delta}$ *and a noncommutative seed* $\mathbf{S}=(G,X,\gamma)$. *Then* $\mathbf{S}_{\underline{\Delta}}$ *and* $\mathbf{S}$ *are related by an* (ij)*-th positive mutation if and only if* $\mathbf{S}$ *is isomorphic to* $\mathbf{S}_{\underline{\Delta}'}$, *where* $\underline{\Delta}'$ *is an ordered triangulation of* $[n]$ *such that* $\underline{\Delta}\setminus\underline{\Delta}'=\{(ij)\}$. *In that case, the underlying isomorphism* $\underline{\mu_{ij}}:\mathbb{T}_{\Delta'}\to\mathbb{T}_\Delta$ *is (unique and) given by (here* $\{(k,\ell)\}=\underline{\Delta}'\setminus\underline{\Delta}$*):*

$$t_{k\ell}\mapsto t_{kj}t_{ij}^{-1}t_{i\ell},\ t_{\ell k}\mapsto t_{\ell j}t_{ji}^{-1}t_{jk},\ t_{k'\ell'}\mapsto t_{k'\ell'} \textit{ if } (k',\ell')\in\Delta'\setminus\{(k,\ell),(\ell,k)\}\ .$$

We prove Theorem 3.32 in Section ??.

3.5. **Binomial mutations and cluster algebras.** For each cluster group $(G, \mathbf{X})$ denote by G^+ the submonoid of G generated by $\mathbf{X}$ and C.

For each group G denote by $\mathbb{Z}G$ the group ring of G with integer coefficients and and by $\mathbb{Z}G^+$ the subalgebra of $\mathbb{Z}G$ generated by G^+. We start with a simple fact on monomial mutations.

Lemma 3.33. *Let $\mathbf{S} \in \mathbf{Seed}_C(G)$. Then for any $i \in [n]$ there exists a unique homomorphism of algebras $\mu_i : \Bbbk G'^+ \to \Bbbk G$ such that $\mu_i(c) = c$ for $c \in C$ and:*

$$\mu_i(x_j') = \begin{cases} x_j & \text{if } j \neq i \\ \underline{\mu_i}(x_i) + \theta_i \underline{\mu_i}(x_i) & \text{if } j = i \end{cases} .$$

Definition 3.34. (Cluster algebra) For each noncommutative seed $\mathbf{S}$ define its *cluster algebra* to be the $\mathbb{Z}$-algebra $\mathcal{A}(\mathbf{S})$ to be generated by the groups G' of all noncommutative seeds $\mathbf{S}' = (G', \mathbf{X}', \gamma') \in \langle S \rangle$ subject to the relations:

$$x_i'' = \mu_i(x_j')$$

for $i \in [n]$ and $\mathbf{S}'' = (G'', \mathbf{X}'', \mathbf{y}'', \gamma'')$ in $\langle S \rangle$ such that $\mathbf{S}'$ and $\mathbf{S}''$ are related by an i-th positive mutation and

$$x_j'' = x_j'$$

for $j \in [n]$ and $\mathbf{S}'$, $\mathbf{S}''$ in $\langle S \rangle$ such that $\mathbf{S}' \cong \mathbf{S}''$.

By definition $\mathcal{A}(\mathbf{S}) = \mathcal{A}(\mathbf{S}')$ for any $\mathbf{S}'$ mutation-equivalent to $\mathbf{S}$.

We say that a noncommutative seed $\mathbf{S} = (G, \mathbf{X}, \mathbf{y}, \gamma)$ is *complete* for each $i \in [n]$ there exists a unique (up to an isomorphism) positive mutation $\mu_i(\mathbf{S})$ of $\mathbf{S}$.

For each complete noncommutative seed $\mathbf{S}$ we define the *upper cluster algebra* $\mathcal{U}(\mathbf{S}) \subset \mathbb{Z}G$ of $\mathbf{S}$ by:

$$\mathcal{U}(\mathbf{S}) := \bigcap_{i=1}^{n} \mathbb{Z}\langle C, x_1^{\pm 1}, \ldots, x_{i-1}^{-1}, x_i, x_i', x_{i+1}^{\pm 1}, \ldots, x_n^{\pm 1} \rangle ,$$

where $x_i' = \underline{\mu_i}(x_i) + \theta_i \underline{\mu_i}(x_i)$ and $\mathbb{Z}\langle M \rangle$ stands for the subalgebra of $\mathbb{Z}G$ generated by a subset $M \subset \mathbb{Z}G$.

The upper cluster algebra of $\mathbf{S}$ is, clearly, a generalization of the *upper bound* for the ordinary and quantum cluster algebras. However, we do not know if $\mathcal{U}(\mathbf{S})$ is mutation-invariant in general.

3.6. **Localizations and Laurent seeds.** For each group G we denote by $\mathbb{Z}_{>0}G$ te $\mathbb{Z}_{>0}$-linear span of elements of G in $\mathbb{Z}G$. By definition, $\mathbb{Z}_{>0}G$ is a sub-semiring of $\mathbb{Z}G$ not containing 0. Denote by $\mathcal{A}_G$ the universal localization $\mathbb{Z}G[(\mathbb{Z}_{>0}G)^{-1}]$ (see Section 8 below for details) and denote by $\mathbf{j}$ the canonical homomorphism

$$\mathbb{Z}G \to \mathcal{A}_G \ . \tag{3.7}$$

For instance, if $G = \mathbb{Z}_2$ then $\mathcal{A}_G = \mathbb{Q}$ and if $G = \mathbb{Z}^n$, then $\mathbb{Z}G = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is the Laurent polynomial algebra and $\mathcal{A}_G$ is the set of all rational functions f/g in $x_1, \ldots, x_n$ such that $f \in \Bbbk[x_1, \ldots, x_n]$ and $g \in \mathbb{Q}_{>0}[x_1, \ldots, x_n]$.

This choice of "denominators" for $\mathcal{A}_G$ is more convenient than other localizations of $\mathbb{Z}G$. ???? In particular, $\mathcal{A}_G$ respects group homomorphisms (see e.g., Lemma 3.43 below). Denote by $\mathcal{A}_G^+$ the sub-semiring of $\mathcal{A}_G$ generated by $\mathbb{Z}_{>0}G$ and $(\mathbb{Z}_{>0}G)^{-1}$.

Note that if $g \in G$ is of order m, then the canonical homomorphism $\mathbf{j} : \mathbb{Z}G \to \mathcal{A}_G$ satisfies $\varphi(g-1) = 0$ because $1 + g + \cdots + g^{m-1} \in \mathbb{Z}_{>0}G$. This implies the following result.

Lemma 3.35. *If $\mathbf{j}$ is injective, then G is torsion-free.*

However, converse of the lemma is not known. The following reformulation of Lemma 8.3 addresses this issue.

Lemma 3.36. *If G is such that $\Bbbk G$ can be embedded into a skew field, then $\mathbf{j}$ is injective.*

Remark 3.37. A well-known Embedding Conjecture (see e.g., [17, Chapter IV.2.2]) asserts that that for each torsion-free group G the group algebra $\Bbbk G$ can be embedded into a skew field (in particular, $\Bbbk G$ has no zero divisors). This would imply Lemma 3.36 for all torsion-free groups G. Based on this, we can conjecture that $\mathbf{j}$ is injective for any torsion-free group.

The Embedding Conjecture is proved for all *ordered* groups, which include finitely generated free groups and torsion free 1-relator groups (see e.g., Theorem 8.5) thus Lemma 3.36 holds for ordered groups. All noncommutative clusters we construct in this paper are built over such groups.

Definition 3.38. We say that a seed $\mathbf{S}$ is *quasi-Laurent* if the canonical homomorphism $\mathbf{j}:\mathbb{Z}G\to\mathcal{A}_G$ given by (3.7) extends to an injective homomorphism of algebras:

$$\hat{\mathbf{j}}:\mathcal{A}(\mathbf{S})\hookrightarrow\mathcal{A}_G$$

We say that $\mathbf{S}$ *Laurent* if the image $\hat{\mathbf{j}}(\mathbf{X})$ of each cluster $\mathbf{X}'$ under belongs to $\mathbb{Z}G$.???

3.7. **Strong mutations.** ???? We say that $\mu_i(\mathbf{S})$ is a *strong* i-th (monomial) mutation of $\mathbf{S}$ if

$$\underline{\mu_i}(\gamma'_j)\in\Gamma^{ij}$$

for each $j\in[n]\setminus\{i\}$, where Γ^{ij} is the normal subgroup of G generated by γ_i and γ_j.

For a given cluster group $\mathbf{S}$ denote by $H=H(\mathbf{S})$ the quotient of G by the normal subgroup generated by $\alpha_i(\mathbf{S})$, $i\in[n]$.

Proposition 3.39. *For any noncommutative seeds* $\mathbf{S}$ *and* $\mathbf{S}'$ *related by strong* i*-th mutation, the isomorphisms* $\mu_i^{(s)}:G\to G'$ *restrict to a unique (thus canonical) isomorphism*

$$\mu_i:H(\mathbf{S})\to H(\mathbf{S}')\ .$$

We say that $\langle\mathcal{S}\rangle$ is a *(strong) cluster group structure* (of rank n over C) if for each $\mathbf{S}'\in\langle\mathbf{S}\rangle$ and each $i\in[n]$ there exists (strong) i-th mutation $\mathbf{S}'\in\langle\mathbf{S}\rangle$ of $\mathbf{S}$.

Corollary 3.40. *For a given strong cluster group structure* $\langle\mathcal{S}\rangle$ *there exists a unique group* H *such that* H *is canonically isomorphic to* $H(\mathbf{S}')$, $\mathbf{S}'\in\langle\mathbf{S}\rangle$.

Remark 3.41. Based on this and results of Section ??, it makes sense to refer to $H(\mathcal{S})$ as a *noncommutative homology* of the strong mutation-equivalence class $\langle\mathbf{S}\rangle$.

3.8. **Folding of noncommutative clusters.** The following result is our main tool for constructing new noncommutative (pre-)clusters.

Proposition 3.42. *Let* H *be a subgroup of* G, H' *be another group,* $\pi_0:H\twoheadrightarrow H'$ *be a surjective group homomorphism, and let* G' *be another group and* $\pi:G\to G'$ *be a surjective group homomorphism determined by the pushout diagram of groups:*

$$\begin{tikzcd} H \arrow[r, "\varphi"] \arrow[d, "\pi_0"'] & G \arrow[d, "\pi"] \\ H' \arrow[r] & G' \end{tikzcd} \tag{3.8}$$

Let $\varphi\in\mathbf{pClust}_G$ *be such that*

$$\langle\varphi(Ker\ \tilde{\pi}_0)\rangle=\langle\mathbf{j}(Ker\ \tilde{\pi}_0))\rangle\ , \tag{3.9}$$

where $\tilde{\pi}_0:\Bbbk H\twoheadrightarrow\Bbbk H'$ *is the canonical extension of* π_0, $\mathbf{j}:\Bbbk H\subset\Bbbk G\to\mathcal{A}_G$ *is the canonical homomorphism, and* $\langle X\rangle$ *denotes the ideal in* $\mathcal{A}_G$ *generated by* X.

Then there is a unique $\varphi'\in\mathbf{pClust}_{G'}$ *such that*

$$\begin{tikzcd} \Bbbk G \arrow[r, "\varphi"] \arrow[d, "\tilde{\pi}"'] & \mathcal{A}_G \arrow[d, "\hat{\pi}"] \\ \Bbbk G' \arrow[r, "\varphi'"] & \mathcal{A}_{G'} \end{tikzcd} \tag{3.10}$$

is a pushout diagram, where $\hat{\pi}:\mathcal{A}_G\to\mathcal{A}_{G'}$ *is given by* (3.11)*.???*

Proof. We need the following fact.

Lemma 3.43. *For any group homomorphisms* $\pi:G'\to G$ *there is a unique homomorphism* $\hat{\pi}:\mathcal{A}_G\to\mathcal{A}_{G'}$ *such that the following is a pushout diagram:*

$$\begin{tikzcd} \Bbbk G \arrow[r, "\mathbf{j}"] \arrow[d, "\tilde{\pi}"'] & \mathcal{A}_G \arrow[d, "\hat{\pi}"] \\ \Bbbk G' \arrow[r, "\mathbf{j}'"] & \mathcal{A}_{G'} \end{tikzcd} \tag{3.11}$$

where $\tilde{\pi}:\Bbbk G'\to\Bbbk G$ *is the canonical extension of* π*, and the horizontal arrows stand for the canonical homomorphisms* $\Bbbk G\to\mathcal{A}_G$ *and* $\Bbbk G'\to\mathcal{A}_{G'}$*.*

Proof. Let us abbreviate $R=\Bbbk G$, $R=\Bbbk G$, $S=\mathbb{Q}_{>0}G$, $S'=\mathbb{Q}_{>0}G'$. Then $\mathcal{A}_G=R[S^{-1}]$, $\mathcal{A}_{G'}=R[{S'}^{-1}]$ and $\tilde{\pi}:R\to R'$ is a homomorphism of algebras such that $\tilde{\pi}(S)=S'$. Then Lemma 8.2 verifies that (3.11) is the pushout diagram. □

Furthermore, the linearization of (3.8) is the pushout diagram:

$$\begin{tikzcd}
\Bbbk H \arrow[r, "\varphi"] \arrow[d, "\tilde{\pi}_0"'] & \Bbbk G \arrow[d, "\tilde{\pi}"] \\
\Bbbk H' \arrow[r, "\varphi''"] & \Bbbk G'
\end{tikzcd} \tag{3.12}$$

Composing it "horizontally" with (3.11), gives the pushout diagram

$$\begin{tikzcd}
\Bbbk H \arrow[r, "\mathbf{j}"] \arrow[d, "\tilde{\pi}_0"'] & \mathcal{A}_G \arrow[d, "\hat{\pi}"] \\
\Bbbk H' \arrow[r] & \mathcal{A}_{G'}
\end{tikzcd} \tag{3.13}$$

In particular, $\mathcal{A}_{G'}=\mathcal{A}_G/\langle\mathbf{j}(Ker\ \tilde{\pi}_0)\rangle$.

Furthermore, $\tilde{\pi}$ and φ determine a pushout diagram

$$\begin{tikzcd}
\Bbbk G \arrow[r, "\varphi"] \arrow[d, "\tilde{\pi}"'] & \mathcal{A}_G \arrow[d, "\psi'"] \\
\Bbbk G' \arrow[r, "\varphi'"] & \mathcal{A}'
\end{tikzcd} \tag{3.14}$$

which, after composing with (3.13) gives the pushout diagram:

$$\begin{tikzcd}
\Bbbk H \arrow[r, "\varphi\circ\mathbf{j}"] \arrow[d, "\tilde{\pi}_0"'] & \mathcal{A}_G \arrow[d, "\psi'"] \\
\Bbbk H' \arrow[r] & \mathcal{A}'
\end{tikzcd}$$

In particular, the kernel of the (surjective) homomorphism $\psi':\mathcal{A}_G\to\mathcal{A}'$ is $\langle\varphi(Ker\ \tilde{\pi}_0)\rangle$. The condition (3.9) guarantees that

$$\mathcal{A}'=\mathcal{A}_G/Ker\ \psi'=\mathcal{A}_G/\langle\varphi(Ker\ \tilde{\pi}_0)\rangle=\mathcal{A}_G/\langle\mathbf{j}(Ker\ \tilde{\pi}_0)\rangle=\mathcal{A}_{G'}$$

Thus, $\psi'=\psi$ in the pushout diagram (3.14) which identifies the latter diagram with (3.10). The proposition is proved.

□

Remark 3.44. Most of noncommutative clusters from Section 6 are constructed via such a pushout in Proposition 3.42, when φ commutes with the natural inclusions $H\subset\Bbbk G$ and $\mathbf{j}:H\to\mathcal{A}_G$. Another important case is when $G=H$ and H' is obtained by the "folding" of G along an automorphism σ of G commuting with φ.

4. NONCOMMUTATIVE RECURSIONS

Let $\mathcal{F}_2$ be a $\mathbb{Q}$-algebra freely generated by $a_{12}^{\pm1}, a_{21}^{\pm1}, a_{23}^{\pm1}, a_{32}^{\pm1}$. Denote by $\mathcal{F}_2(Y_1,Y_2)$ the algebra generated by $\mathcal{F}_2$ and $Y_1^{\pm1}$, $Y_2^{\pm1}$ subject to the relations

$$Y_1a_{12}Y_2a_{23}=a_{32}Y_2a_{21}Y_1\ . \tag{4.1}$$

Clearly, $\mathcal{F}_2(Y_1,Y_2)$ is isomorphic to a group algebra of a free group in 6 generators.

Define the elements $a_{k,k\pm1}$, $k\in\mathbb{Z}$ in $\mathcal{F}_2$ recursively by:

$$a_{k+2,k+1}=a_{k-1,k}^{-1},\ a_{k+1,k+2}=a_{k,k-1}^{-1}$$

for $k\in\mathbb{Z}$.

Then define elements $Y_k \in Frac(\mathcal{F}_2(Y_1,Y_2))$, $k \in \mathbb{Z}\setminus\{1,2\}$ by the formula:

$$Y_{k+1}Y_{k-1} = h_k(a_{k-1,k}Y_k a_{k,k+1}), \tag{4.2}$$

for all $k \in \mathbb{Z}$, where all $h_k(x) \in \mathbb{Q}[x]$ and $h_k(x) = h_{k-2}(x)$ for $k \in \mathbb{Z}$.

Proposition 4.1. *The elements Y_k satisfy:*

$$Y_{k-1}Y_{k+1} = h_k(a_{k+1,k}Y_k a_{k,k-1}) \tag{4.3}$$

$$Y_k a_{k,k+1}Y_{k+1}a_{k+1,k+2} = a_{k+2,k+1}Y_{k+1}a_{k+1,k}Y_k \tag{4.4}$$

for $k \in \mathbb{Z}$,

Proof. For each $k \in \mathbb{Z}$ denote

$$Y_k^- := a_{k-1,k}Y_k a_{k,k+1},\ \ Y_k^+ := a_{k+1,k}Y_k a_{k,k-1} \tag{4.5}$$

Then, the relations (4.2), (4.3) and (4.4) simplify and become respectively:

$$Y_{k+1}Y_{k-1} = h_k(Y_k^-) \tag{4.6}$$

$$Y_{k-1}Y_{k+1} = h_k(Y_k^+), \tag{4.7}$$

$$Y_k Y_{k+1}^- = Y_{k+1}^+ Y_k \tag{4.8}$$

for $k \in \mathbb{Z}$. We prove (4.7) and (4.4) by induction in k. For simplicity, consider the case $k \ge 2$, and suppose that (4.8) holds for $k-1$, i.e.,

$$Y_{k-1}Y_k^- = Y_k^+ Y_{k-1} \tag{4.9}$$

Then, conjugating (4.2) with Y_{k-1} on the left and using (4.9), we obtain:

$$Y_{k-1}Y_{k+1} = Y_{k-1}h_k(Y_k^-)Y_{k-1}^{-1} = h_k(Y_k^+)\ ,$$

which gives (4.7).

Thus, it remains to prove (4.4). Indeed, since ??$a_{k,k+1} = b_{k,k-1}$ and $b_{k,k+1} = a_{k-1,k}^{-1}$, (4.4) is equivalent to:

$$Y_k^- Y_{k+1} = Y_{k+1}Y_k^+$$

Multiplying (4.7) by $Y_k^- Y_{k-1}^{-1}$ on the left and using (4.9), we obtain:

$$Y_k^- Y_{k+1} = Y_k^- Y_{k-1}^{-1}h_k(Y_k^+) = Y_k^- Y_k^+ h_k(Y_k^+) = Y_{k-1}^{-1}h_k(Y_k^+)Y_k^+ = Y_{k-1}^{-1}(Y_{k-1}Y_{k+1})Y_k^+ = Y_{k+1}Y_k^+$$

This proves (4.4).

The case $k \le 0$ is proved by the same argument. Proposition 4.1 is proved. □

Definition 4.2. Let now $h_k(x) = 1 + x^{r_k}$, where $r_k = \begin{cases} r_1 & \text{if } k \text{ is odd} \\ r_2 & \text{if } k \text{ is even}\end{cases}$. Denote by $\mathcal{A}_{r_1,r_2}$ the subalgebra of $\mathcal{F}_2$ generated by $a_i, \overline{a}_i$, $i = 1,2$ and Y_0, Y_1, Y_2, Y_3. We refer to $\mathcal{A}_{r_1,r_2}$ as the *purely noncommutative* rank 2 cluster algebra.

Theorem 4.3. *For each $k \in \mathbb{Z}$ the subalgebra of $\mathcal{F}_2(Y_1,Y_2)$ generated by $\mathcal{F}_2$ and $Y_k, Y_{k+1}, Y_{k+2}, Y_{k+3}$ equals to $\mathcal{A}_{r_1,r_2}$.*

Proof. We proceed similarly to the proof of [5, Theorem 5]. Denote by $\mathcal{A}_k$ the subalgebra of $Frac(\mathcal{F}_2(Y_1,Y_2))$ generated by $\mathcal{F}_2$ and $Y_k, Y_{k+1}, Y_{k+2}, Y_{k+3}$ (so that $\mathcal{A}_{r_1,r_2} = \mathcal{A}_0$). It suffices to prove that

$$\mathcal{A}_{k+1} = \mathcal{A}_k \tag{4.10}$$

For simplicity (and without loss of generality) we assume that $k = 0$. We need the following result.

Proposition 4.4. *For each $k \in \mathbb{Z}$ one has:*

$$Y_4 = Y_0(Y_3^-)^{r_1} - \sum_{s=0}^{r_1-1}(Y_1^+)^s a_{21}(Y_2^+)^{r_2-1}a_{32}(Y_3^+)^s\ .$$

Proof. We start with the following technical result.

Lemma 4.5. *For all $m \geq 0$ one has and $k \in \mathbb{Z}$:*

$$(Y_{k-1}^-)^m (Y_{k+1}^+)^m = 1 + \sum_{s=0}^{m-1} (Y_{k-1}^-)^s a_{k+1,k}^{-1} (Y_k^+)^{r_k} a_{k+1,k} (Y_{k+1}^+)^s \ .$$

Proof. We proceed by induction on m. For $m = 0$ the assertion is clear. Assume that $m > 0$ and it holds for $m - 1$. Let us prove it for m. Note that

$$Y_{k-1}^- Y_{k+1}^+ = a_{k-2,k-1} Y_{k-1} a_{k-1,k} a_{k+2,k+1} Y_k a_{k+1,k} = a_{k+1,k}^{-1} (1 + (Y_k^+)^{r_k}) a_{k+1,k} = 1 + a_{k+1,k}^{-1} (Y_k^+)^{r_k} a_{k+1,k}$$

because $a_{k-1,k} a_{k+2,k+1} = 1$ and $a_{k-2,k-1} = a_{k+1,k}^{-1}$. Using this, we obtain:

$$(Y_{k-1}^-)^m (Y_{k+1}^+)^m = (Y_{k-1}^-)(Y_{k-1}^- Y_{k+1}^+)(Y_{k+1}^+)^{m-1} = (Y_{k-1}^-)^{m-1} (Y_{k+1}^+)^{m-1} + (Y_{k-1}^-)^{m-1} a_{k+1,k}^{-1} (Y_k^+)^{r_k} a_{k+1,k} (Y_{k+1}^+)^{m-1}$$

$$= 1 + \sum_{s=0}^{m-2} (Y_{k-1}^-)^s a_{k+1,k}^{-1} (Y_k^+)^{r_k} a_{k+1,k} (Y_{k+1}^+)^s + (Y_{k-1}^-)^{m-1} a_{k+1,k}^{-1} (Y_k^+)^{r_k} a_{k+1,k} (Y_{k+1}^+)^{m-1} \ .$$

The lemma is proved. □

Furthermore, combining (4.7) with $k = 3$ and (4.6) with $k = 1$, we obtain:

$$Y_4 = Y_2^{-1}(1 + (Y_3^+)^{r_1}) = Y_2^{-1}(Y_3^+)^{r_1} + Y_2^{-1} = (Y_0 - Y_2^{-1}(Y_1^-)^{r_1})(Y_3^+)^{r_1} + Y_2^{-1}$$

$$= Y_0 (Y_3^-)^{r_1} - Y_2^{-1}((Y_1^+)^{r_1}(Y_3^-)^{r_1} - 1)$$

Furthermore, using Lemma 4.5 with $m = r_1$, $k = 2$, we obtain:

$$Y_4 = Y_0 (Y_3^-)^{r_1} - Y_2^{-1} \sum_{s=0}^{r_1-1} (Y_1^-)^s a_{32}^{-1} (Y_2^+)^{r_2} a_{32} (Y_3^+)^s = Y_0 (Y_3^-)^{r_1} - Y_2^{-1} \sum_{s=0}^{r_1-1} (Y_1^-)^s a_{32}^{-1} (Y_2^+)^{r_2} a_{32} (Y_3^+)^s$$

$$= Y_0 (Y_3^-)^{r_1} - \sum_{s=0}^{r_1-1} (Y_1^+)^s a_{21} (Y_2^+)^{r_2-1} a_{32} (Y_3^+)^s$$

by (4.9) with $k = 2$ and the fact that $Y_2^{-1} a_{32}^{-1} Y_2^+ = a_{21}$. This finishes the proof of Proposition 4.4. □

Therefore, the theorem is proved. □

Corollary 4.6. *(noncommutative Laurent phenomenon) $Y_k \in \mathcal{A}_{r_1,r_2}$ for all $k \in \mathbb{Z}$. In particular, all Y_k are noncommutative Laurent polynomials in Y_1, Y_2 with coefficients in $\mathcal{F}_2$.*

Define an involutive anti-automorphism $x \mapsto \overline{x}$ of $\mathcal{F}_2(Y_1, Y_2)$ by $\overline{a}_{k,k\pm1} = a_{k\pm1,k}$, $\overline{c}_{k,k+1} = c_{k+1,k}$, $\overline{Y}_1 = Y_1$, $\overline{Y}_2 = Y_2$.

Denote by τ the automorphism of $Frac(\mathcal{F}_2(Y_1, Y_2))$ given by

$$a_{k,k\pm1} \mapsto a_{k+1,k+1\pm1},\ Y_1 \mapsto Y_2,\ Y_2 \mapsto Y_3 = Y_1^{-1} c_{12} (1 + (Y_2^+)^{r_2})$$

Theorem 4.7. *(symmetries)*

(a) One has $\overline{Y}_k = Y_k$ for all $k \in \mathbb{Z}$. In particular, the algebra $\mathcal{A}_{r_1,r_2}$ is invariant under the anti-involution $x \mapsto \overline{x}$ of $\mathcal{F}_2(Y_1, Y_2)$.

(b) ??? The restriction of τ to $\mathcal{A}_{r_1,r_2}$ is an automorphism

$$\mathcal{A}_{r_1,r_2} \widetilde{\to} \mathcal{A}_{r_1,r_2} \ .$$

Let $\hat{G}(r_1, r_2)$ be the group generated by x_i, τ_i, $i = 1, 2$, $a, \overline{a}$ subject to the relations:

1. x_i, τ_i, $i = 1, 2$ form a principal rank 2 torus $G(B)$, where $B = \begin{pmatrix} 0 & -r_1 \\ r_2 & 0 \end{pmatrix}$.

2. x_1 commutes with a and $\overline{a}$.

Denote by $\mathbb{T}_2^{nc}$ group generated by $Y_i, a_{i,i+1}, a_{i+1,i}$, $i = 1, 2$ subject to the relations

$$Y_1 a_{12} Y_2 a_{23} = a_{32} Y_2 a_{21} Y_1 \ . \tag{4.11}$$

Lemma 4.8. *The assignment $Y_i \mapsto x_i \tau_i^{-1}$, $i = 1, 2$ and $a_{12} \mapsto a$, $a_{21} \mapsto \overline{a}$,*

$$a_{23} \mapsto a_{21} \tau_1^{-1},\ a_{32} \mapsto \overline{\tau}_1^{-1} a_{12}$$

defines a homomorphism $f : \mathbb{T}_2^{nc} \to \hat{G}(r_1, r_2)$.

Theorem 4.9. *Under the above homomorphism, one has*

$$f(Y_k) = x_k \tau_k^{-1}$$

for $k \in \mathbb{Z}$ *where* τ_k *is determined by* $\tau_k = \overline{\tau}_{k-2}^{-1}$ *for* $k \in \mathbb{Z}$.

5. Noncommutative polygons

5.1. **Definition and main results.** In this section we define the *noncommutative polygon* algebra $\mathcal{A}_n$ (over $\mathbb{Q}$) and relate it to the noncommutative cluster structure on polygons from Section 3.4, from which we also borrow some notation.

Definition 5.1. Denote by $\mathcal{A}_n$ the $\mathbb{Q}$-algebra generated by x_{ij} and x_{ij}^{-1}, $i, j \in [n]$, $i \neq j$ subject to the relations:

(i) (Triangle relations) For any distinct indices $i, j, k \in [n]$:

$$x_{ij} x_{kj}^{-1} x_{ki} = x_{ik} x_{jk}^{-1} x_{ji} \ . \tag{5.1}$$

(ii) (Exchange relations) For any cyclic (i, j, k, ℓ) in $[n]$:

$$x_{j\ell} = x_{jk} x_{ik}^{-1} x_{i\ell} + x_{ji} x_{ki}^{-1} x_{k\ell} \ . \tag{5.2}$$

Remark 5.2. It is easy to see that the exchange relations (5.37) are equivalent to noncommutative Ptolemy relations (2.6) provided the triangular relations (5.1).

At the first glance the number of relations of $\mathcal{A}_n$ greatly exceeds the number of generators. However, we will demonstrate below that the algebra $\mathcal{A}_n$ is "rationally" generated only by $3n-4$ free generators.

Following A. A Malcev and P. M. Cohn (see also Section 8), we say that an algebra $\mathcal{A}$ is of class $\mathcal{E}$ if it embeds into a skew field. We say that algebra $\mathcal{A} \in \mathcal{E}$ is of class $\mathcal{E}_0$ if the category of epic $\mathcal{A}$-fields admits a (unique) initial object (see [10], Section 7.2). Such object is called in [10] the universal $\mathcal{A}$-field and we denote it by $Frac(\mathcal{A})$.

Essentially $Frac(\mathcal{A})$ is the largest or the "freemost" skew field such that $\mathcal{A} \subset Frac(\mathcal{A})$ and $\mathcal{A}$ generates $Frac(\mathcal{A})$ (we explicitly construct such skew fields for algebras considered in this paper.) For example, it is well-known (see e.g., Theorem 8.5) that the free algebra $F_m := \mathbb{Q} < x_1, \ldots, x_m >$ is of class $\mathcal{E}_0$. The free skew field $Frac(F_m)$ was consructed by S. Amitsur and P.M. Cohn (see [10]). In what follows, we abbreviate $\mathcal{F}_m := Frac(F_m)$ and refer to it as the *free skew field* in m generators.

Theorem 5.3. *The algebra* $\mathcal{A}_n$ *is of class* $\mathcal{E}_0$ *and* $Frac(\mathcal{A}_n)$ *is isomorphic to* $\mathcal{F}_{3n-4}$.

We prove the theorem in Section 5.4. In fact, it will follow from a more precise assertion (Theorem 5.5).

Remark 5.4. By contrast, we expect that the subalgebra of $\mathcal{A}_n$ generated by all x_{ij} is isomorphic to F_{n^2-n}.

Now we explore the cluster structure of $\mathcal{A}_n$. For each triangulation Δ of $[n]$ define the subalgebra $\mathcal{A}_\Delta$ of $\mathcal{A}_n$ generated by x_{ij}, $i, j \in [n]$ and x_{ij}^{-1}, $(i,j) \in \Delta$.

Clearly, the assignment $t_{ij} \mapsto x_{ij}$, $(i,j) \in \Delta$ defines a homomorphism of algebras:

$$\mathbf{i}_\Delta : \mathbb{Q}\mathbb{T}_\Delta \to \mathcal{A}_\Delta \ , \tag{5.3}$$

where $\mathbb{Q}\mathbb{T}_\Delta$ is the group algebra of $\mathbb{T}_\Delta$ (defined in Section 3.4).

Recall (see, e.g., (8.1)) that for a given algebra $\mathcal{A}$ with no zero divisors and a subset $S \subset \mathcal{A} \setminus \{0\}$ one has a freemost localization $\mathcal{A}[S^{-1}]$ of $\mathcal{A}$ by S.

Theorem 5.5. *For each triangulation* Δ *of* $[n]$ *one has:*

(a) The homomorphism $\mathbf{i}_\Delta$ *given by* (5.3) *is an isomorphism.*

(b) There is a multiplicative sub-monoid $S = S_\Delta$ *of* $\mathcal{A}_\Delta \setminus \{0\}$ *such that* $\mathcal{A}_n = \mathcal{A}_\Delta[S^{-1}]$.

We will prove Theorem 5.5 in Section 5.4. In fact, Theorem 5.5(a) establishes a *noncommutative cluster structure* on $\mathcal{A}_n$ and Theorem 5.5(b) – a *noncommutative Laurent Phenomenon* (see also Section 5.2).

Now we illustrate Theorem 5.5 for each *starlike* triangulation

$$\Delta_i = \{(i,j), (j,i) | j \in [n] \setminus \{i\}, (k, k^{\pm}), k \in [n]\} \ ,$$

$i \in [n]$.

Proposition 5.6. *Fix $i \in [n]$. Then for each $k, \ell \in [n] \setminus \{i\}$ such that $(i, k, \ell)$is cyclic, the following relation holds in $\mathcal{A}_n$:*

$$x_{k\ell} = \sum_s x_{ki} x_{si}^{-1} x_{s,s^+} x_{i,s^+}^{-1} x_{i\ell}$$

where summation is over all $s = k, k^+, \ldots, \ell^-$ in cyclic order. Hence $x_{k\ell} = \mathbf{i}_{\Delta_i}(\sum_s t_{ki} t_{si}^{-1} t_{s,s^+} t_{i,s^+}^{-1} t_{i\ell})$.

In fact, this result is a direct corollary of Theorem ?? below.

Let $\overline{\mathcal{A}}_n$ be the subalgebra of $\mathcal{A}_n$ generated by x_{ij}, $i, j \in [n]$, $i \neq j$ and $x_{i,i^\pm}^{-1}$, $i \in [n]$. By definition, $\overline{\mathcal{A}}_n \subset \mathcal{A}_\Delta$ for each triangulation Δ of $[n]$.

Conjecture 5.7. *For each $n \geq 2$ one has:*

$$\overline{\mathcal{A}}_n = \bigcap_\Delta \mathcal{A}_\Delta \ , \tag{5.4}$$

where the intersection is over all triangulations Δ of n.

Remark 5.8. This conjecture means that $\overline{\mathcal{A}}_n$ is a totally noncommutative analogue of the upper cluster algebra (of type A_{n-3}).

Conjecture 5.9. *$\overline{\mathcal{A}}_n$ is generated by all x_{ij}, $i \neq j$ and $x_{i,i^\pm}^{-1}$ subject to:*

(a) the reduced triangle relations:

$$x_{ij} x_{j^+,j}^{-1} x_{j^+,i} = x_{i,j^+} x_{j,j^+}^{-1} x_{ji} \ . \tag{5.5}$$

for all $i, j \in [n]$, $i \notin \{j, j^+\}$;

(b) the reduced exchange relations:

$$x_{ji} x_{i^-,i}^{-1} x_{i^-,i^+} = x_{j,i^+} + x_{j,i^-} x_{i,i^-}^{-1} x_{i,i^+}, \ x_{i^+,i^-} x_{i,i^-}^{-1} x_{ij} = x_{i^+,i} + x_{i^+,i} x_{i^-,i}^{-1} x_{i^-,j} \tag{5.6}$$

for all distinct $i, j \in [n]$ such that $j \notin \{i^-, i^+\}$.

This allows for finding a more "economical" presentation of $\mathcal{A}_n$.

Corollary 5.10. *???? The algebra $\mathcal{A}_n$ is generated by all $x_{ij}^{\pm 1}$, $i \neq j$ subject to the relations* (5.5) *and* (5.6).

5.2. **Noncommutative Laurent Phenomenon.** For each even sequence $\mathbf{i} = (i_1, \ldots, i_{2m}) \in [n]^{2m}$ such that adjacent indices are distinct define the monomial $x_\mathbf{i} \in \mathcal{A}_n$ by:

$$x_\mathbf{i} := x_{i_1,i_2} x_{i_3,i_2}^{-1} x_{i_3,i_4} \cdots x_{i_{2m-1},i_{2m-2}}^{-1} x_{i_{2m-1},i_{2m}} \ .$$

For a directed chord (i, j), we say that a sequence $\mathbf{i} = (i_1, \ldots, i_{2m}) \in [n]^{2m}$ of indices is (i, j)-*admissible* if:
(i) $i_1 = i$, $i_{2m} = j$ and $i_s \neq i_{s+1}$ for $s = 1, \ldots, 2m-1$ and all chords (i_s, i_{s+1}) are distinct;
(ii) each chord (i_{2s}, i_{2s+1}), $s = 2, \ldots, m-1$ intersects (i, j);
(iii) for each $s < t$ the intersection point $(i_{2s}, i_{2s+1}) \cap (i, j)$ is closer to i than $(i_{2t}, i_{2t+1}) \cap (i, j)$.

Theorem 5.11. *(Refined Noncommutative Laurent Phenomenon) Let Δ be a triangulation of $[n]$. Then for any $i \neq j$ each element x_{ij} of $\mathcal{A}_n$ belongs to $\mathcal{A}_\Delta$, more precisely,*

$$x_{ij} = \sum x_\mathbf{i} \ , \tag{5.7}$$

where the summation is over all (i, j)-admissible sequences $\mathbf{i} = (i_1, \ldots, i_{2m})$ such that $(i_s, i_{s+1}) \in \Delta$ for $s = 1, \ldots, 2m-1$.

Remark 5.12. This is a noncommutative version of Schiffler's formula ([24]).

Example 5.13. (a) If $n = 5$ and $\Delta = \{(1,3), (1,4); (12), (23), (34), (45), (51)\}$, then

$$x_{25} = x_{21} x_{41}^{-1} x_{45} + x_{23} x_{13}^{-1} x_{15} + x_{21} x_{31}^{-1} x_{34} x_{14}^{-1} x_{15}.$$

(b) If $n = 6$ and $\Delta = \{(13), (36), (46); (12), (23), (34), (45), (56), (61)\}$, then

$$x_{25} = x_{23} x_{63}^{-1} x_{65} + x_{21} x_{31}^{-1} x_{36} x_{46}^{-1} x_{45} + x_{21} x_{31}^{-1} x_{34} x_{64}^{-1} x_{65} + x_{23} x_{13}^{-1} x_{16} x_{46}^{-1} x_{45} x_{23} x_{13}^{-1} x_{16} x_{36}^{-1} x_{34} x_{64}^{-1} x_{65}.$$

In fact, we can streamline the formula for x_{ij} (and even prove Theorem 5.11) by introducing new coordinates associated with Δ. For each triple of distinct indices $i,j,k\in[n]$ define elements y_{ij}^k of $\mathcal{A}_n$ by:

$$y_{ij}^k := x_{ki}^{-1}x_{kj}\ .$$

We refer to y_{ij}^k as *noncommutative sectors* and denote by $\mathcal{Q}_n$ the subalgebra of $\mathcal{A}_n$ generated by all y_{ij}^k (with the convention $y_{ii}^k=1$).

Theorem 5.14. *The algebra $\mathcal{Q}_n$ is generated by all y_{ij}^k subject to the relations:*

(i) Triangle relations:

$$y_{ij}^k y_{ji}^k = 1,\ y_{ij}^k y_{jk}^i y_{ki}^j = 1 \tag{5.8}$$

for distinct $i,j,k\in[n]$ and

$$y_{ij}^\ell y_{jk}^\ell y_{ki}^\ell = 1 \tag{5.9}$$

for distinct $i,j,k,\ell\in[n]$.

(ii) Exchange relations:

$$y_{i\ell}^j = y_{ij}^k y_{j\ell}^i + y_{i\ell}^k \tag{5.10}$$

for all cyclic (i,j,k,ℓ) in $[n]$.

For each odd sequence $\mathbf{j}=(i_0,i_1,\ldots,i_{2m})\in[n]^{2m+1}$ such that adjacent indices are distinct define the monomial $y_{\mathbf{i}}\in\mathcal{Q}_n$ by:

$$y_{\mathbf{j}} := y_{i_0i_2}^{i_1}y_{i_2i_4}^{i_3}\cdots y_{i_{2m-2}i_{2m}}^{i_{2m-1}}\ .$$

The following result is a "polynomial equivalent" in $\mathcal{Q}_n$ of Theorem 5.11.

Theorem 5.15. *(Noncommutative polynomial phenomenon) Let Δ be a triangulation of $[n]$. Then for any triple (i,j,k) of distinct indices such that $(i,k)\in\Delta$ one has:*

$$y_{kj}^i = \sum y_{(k,\mathbf{i})}\ ,$$

where $\mathbf{i}$ runs of all (i,j)-admissible sequences $\mathbf{i}=(i_1,\ldots,i_{2m})$ such that $(i_s,i_{s+1})\in\Delta$ for $s=1,\ldots,2m-1$.

Example 5.16. (a) If $n=5$ and $\Delta=\{(1,3),(3,1),(1,4),(4,1);(i,i^\pm)|i\in[5]\}$, then

$$y_{15}^2 = y_{15}^4 + y_{13}^2y_{35}^1 + y_{14}^3y_{45}^1.$$

(b) If $n=6$ and $\Delta=\{(13),(36),(46);(12),(23),(34),(45),(56),(61)\}$, then

$$y_{15}^2 = y_{16}^3y_{65}^4 + y_{13}^2y_{35}^6 + y_{14}^3y_{46}^5 + y_{13}^2y_{36}^1y_{65}^4 + y_{13}^2y_{36}^1y_{64}^3y_{45}^6.$$

Similarly to Section 5.1, for each triangulation Δ of $[n]$ define:

- The subalgebra $\mathcal{Q}_\Delta$ of $\mathcal{Q}_n$ generated by all y_{ij}^k, $i,j,k\in[n]$ such that $(i,k),(k,j)\in\Delta$.
- the subgroup $\mathbb{U}_\Delta$ of $\mathbb{T}_\Delta$ generated by

$$u_{ij}^k := t_{ki}^{-1}t_{kj}\ ,$$

for $i,j,k\in[n]$ such that $(i,k),(kj)\in\Delta$.

Clearly, the restriction of the homomorphism $\mathbf{i}_\Delta$ given by (5.3) to $\mathbb{QU}_\Delta\subset\mathbb{QT}_\Delta$ is a homomorphism of algebras:

$$\mathbf{i}'_\Delta : \mathbb{QU}_\Delta \to \mathcal{Q}_\Delta\ . \tag{5.11}$$

The following is an analogue of Theorem 5.5.

Theorem 5.17. *For each triangulation Δ one has:*

(a) The homomorphism $\mathbf{i}'_\Delta$ given by (5.11) is an isomorphism.

(b) There exists a multiplicative sub-monoid $S'=S'_\Delta\subset\mathcal{Q}_\Delta\setminus\{0\}$ such that $\mathcal{Q}_n=\mathcal{Q}_\Delta[S'^{-1}]$.

We prove Theorem 5.17 in Section 5.4.

The following is immediate.

Corollary 5.18. *For each $n\ge 2$ the algebra $\mathcal{Q}_n$ is of class $\mathcal{E}_0$ and $Frac(\mathcal{Q}_n)\cong\mathcal{F}_{2n-4}$.*

5.3. **Free factorizations of $\mathcal{A}_n$ and proof of Theorem 5.14.** For any $\mathbb{Q}$-algebras $\mathcal{A}$ and $\mathcal{B}$ denote by $\mathcal{A} * \mathcal{B}$ their free product, i.e., the universal algebra generated by $\mathcal{A}$ and $\mathcal{B}$ as subalgebras (with no relations between them). The most fundamental property of the free product is that any algebra homomorphisms $f_1 : \mathcal{A} \to \mathcal{C}$, $f_2 : \mathcal{B} \to \mathcal{C}$ canonically lift to an algebra homomorphism $f_1 * f_2 : \mathcal{A} * \mathcal{B} \to \mathcal{C}$.

Denote by $\hat{F}_m$ the free Laurent polynomial algebra $\mathbb{Q} < c_1^{\pm 1}, \ldots, c_m^{\pm 1} >$. By definition, $\hat{F}_m = \mathbb{Q}[c_1^{\pm 1}] * \cdots * \mathbb{Q}[c_m^{\pm 1}]$ is the group algebra of the free group generated by $c_i^{\pm 1}$, $i = 1, \ldots, m$.

Proposition 5.19. *For each $n \geq 2$ the assignment $x_{ij} \mapsto c_i * y^i_{i^-,j}$, $i, j \in [n]$, $i \neq j$ defines an isomorphism of algebras*

$$f : \mathcal{A}_n \widetilde{\to} \hat{F}_n * \mathcal{Q}_n \ . \tag{5.12}$$

Proof. Let us prove that the homomorphism (5.12) is well-defined. We need the following obvious fact.

Lemma 5.20. *Let $\mathcal{B}$ be a $\mathbb{Q}$-algebra and let $c_1, \ldots, c_n$ be invertible elements of $\mathcal{B}$. Then the assignment*

$$x_{ij} \mapsto c_i * x_{ij} \tag{5.13}$$

for $i, j \in [n]$, $i \neq j$ defines a homomorphism of algebras:

$$\mathcal{A}_n \to \mathcal{B} * \mathcal{A}_n$$

By the above Lemma $\mathcal{B} = \hat{F}_n$ generated by $c_i^{\pm 1}$, $i \in [n]$, the assignment (5.13) defines a homomorphism of algebras

$$\mathcal{A}_n \to \hat{F}_n * \mathcal{A}_n \ . \tag{5.14}$$

Furthermore, the assignment $c_i \mapsto c_i * x^{-1}_{i,i^-}$, $i \in [n]$ defines an algebra homomorphism $f_1 : \hat{F}_n \to \hat{F}_n * \mathcal{A}_n$ and the identity map $\mathcal{A}_n \to \mathcal{A}_n$ defines a homomorphism of algebras $f_2 : \mathcal{A}_n \to \hat{F}_n * \mathcal{A}_n$. This gives an algebra homomorphism $f_1 * f_2 : \hat{F}_n * \mathcal{A}_n \to \hat{F}_n * \mathcal{A}_n$ determined by $c_i \mapsto c_i * x^{-1}_{i,i^-}$, $x_{ij} \mapsto x_{ij}$. Then the composition of the homomorphism (5.14) with $f_1 * f_2 : \hat{F}_n * \mathcal{A}_n$ is a homomorphism of algebras

$$\mathcal{A}_n \to \hat{F}_n * \mathcal{A}_n$$

given by

$$x_{ij} \mapsto c_i * x_{ij} \mapsto c_i * x^{-1}_{i,i^-} x_{ij} = c_i * y^i_{i^-,j}$$

for all $i, j \in [n]$, $i \neq j$. Since the image of the latter homomorphism belongs to $\hat{F}_n * \mathcal{Q}_n$, we see that the algebra homomorphism $f : \mathcal{A}_n \to \hat{F}_n * \mathcal{Q}_n$ given by (5.12) is well-defined.

It remains to show that f is invertible. Indeed, denote by $f_1' : \hat{F}_n \to \mathcal{A}_n$ the homomorphism of algebras given by $c_i \mapsto x_{i,i^-}$, $i \in [n]$ and denote by f_2' the natural inclusion $\mathcal{Q}_n \hookrightarrow \mathcal{A}_n$. This defines a homomorphism of algebras $g = f_1' * f_2' : \hat{F}_n * \mathcal{Q}_n \to \mathcal{A}_n$ which is determined by $c_i \mapsto x_{i,i^-}$, $y_{ij} \mapsto y_{ij}$. This immediately implies that

$$(g \circ f)(x_{ij}) = g(c_i * y^i_{i^-,j}) = x_{i,i^-} y^i_{i^-,j} = x_{ij}$$

for all $i \neq j$. Therefore, $g \circ f = Id$. Similarly,

$$(f \circ g)(c_i) = f(x_{i,i^-}) = c_i * y^i_{i^-,i^-} = c_i * 1 = c_i, \ (f \circ g)(y^i_{i^-,j}) = f(y^i_{i^-,j})$$

$$= f(x^{-1}_{i,i^-} x_{ij}) = f(x_{i,i^-})^{-1} f(x_{ij}) = (c_i * x_{i,i^-})^{-1} c_i * x_{ij} = x^{-1}_{i,i^-} x_{ij} = y_{ij}$$

Therefore, $f \circ g = Id$ as well.

The proposition is proved. □

Remark 5.21. Proposition 5.19 is a noncommutative algebraic analogue of the following assertion: if a group G acts freely on a set X, then there a bijection $X \widetilde{\to} G \times X/G$.

For any groups G and H denote by $G * H$ their free product. It is well-known (see, e.g., [10]) that $\mathbb{Q}(G * H) = (\mathbb{Q}G) * (\mathbb{Q}H)$.

Proposition 5.22. *For each triangulation* Δ *of* $[n]$ *the assignment*

$$t_{ij} \to c_i * u^i_{i^-,j}$$

for all $(i,j) \in \Delta$ *(in the notation of* (5.11)*) defines an isomorphism of groups*

$$\mathbb{T}_\Delta \widetilde{\to} G_n * \mathbb{U}_\Delta \ , \tag{5.15}$$

where G_n *is the free group generated by* $c_1, \ldots, c_n$*.*

Proof. We essentially copy the proof of Proposition 5.19. Indeed, the following fact is obvious.

Lemma 5.23. *Let* G *be any any group and let* $c_1, \ldots, c_n \in G$*. Then for any triangulation* Δ *of* $[n]$ *the assignment*

$$t_{ij} \mapsto c_i * t_{ij} \tag{5.16}$$

for $i,j \in \Delta$*, defines a homomorphism of groups:*

$$\mathbb{T}_\Delta \to G * \mathbb{U}_\Delta$$

Now we take $G = G_n$, the free group generated by $c_1, \ldots, c_n$. Clearly, the assignment

$$c_i \mapsto c_i * t^{-1}_{i,i^-}$$

for $i \in [n]$ defines a group homomorphism $G_n \to G_n * \mathbb{T}_\Delta$. Composing this with (5.16), we obtain a group homomorphism:

$$\mathbb{T}_\Delta \to G_n * \mathbb{T}_\Delta$$

given by

$$t_{ij} \mapsto c_i * u_{i^-,j}$$

for all $i,j \in \Delta$. Clearly, the image of this homomorphism contains all c_i and u^k_{ij}, $(i,j),(jk) \in \Delta$, hence this gives a group homomorphism (5.15). Clearly, the homomorphism $G_n * \mathbb{U}_\Delta \to \mathbb{T}_\Delta$ given by

$$c_i \mapsto t_{i,i^-}, \ u^k_{ij} \mapsto u^k_{ij}$$

is inverse of (5.15).

The proposition is proved. □

Taking into account that $G_n * G_m \cong G_{m+n}$, we obtain an obvious corollary from Theorem 3.29.

Corollary 5.24. *For each triangulation* Δ *of* n *the group* $\mathbb{U}_\Delta$ *is isomorphic to* G_{2n-4}*, the free group in* $2n-4$ *generators.*

Now we are ready to prove Theorem 5.14.

Proof of Theorem 5.14. First, we verify that the relations (5.8), (5.9), and (5.18) hold. The left hand side of the first relation (5.8) is:

$$y^k_{ij} y^k_{ji} = (x^{-1}_{ki} x_{kj})(x^{-1}_{kj} x_{ki}) = 1 \ .$$

Furthermore, the left hand side of the second relation (5.8) is:

$$y^k_{ij} y^i_{jk} y^j_{ki} = (x^{-1}_{ki} x_{kj})(x^{-1}_{ij} x_{ik})(x^{-1}_{jk} x_{ji}) = (x^{-1}_{ki} x_{kj} x^{-1}_{ij})(x_{ik} x^{-1}_{jk} x_{ji}) = 1 \ .$$

for all distinct $i,j,k \in [n]$ by the triangle relations (5.1). Similarly, the left hand side of (5.9) is:

$$y^\ell_{ij} y^\ell_{jk} y^\ell_{ki} = (x^{-1}_{\ell i} x_{\ell j})(x^{-1}_{\ell j} x_{\ell k})(x^{-1}_{\ell k} x_{\ell i}) = 1 \ .$$

for all distinct quadruples (i,j,k,ℓ).

Finally, the difference between the right and left hand sides of (5.18) is:

$$y^k_{ij} y^i_{j\ell} + y^k_{i\ell} - y^j_{i\ell} = (x^{-1}_{ki} x_{kj})(x^{-1}_{ij} x_{i\ell}) + x^{-1}_{ki} x_{k\ell} - x^{-1}_{ji} x_{j\ell} = (x^{-1}_{ji} x_{jk} x^{-1}_{kj}) x_{i\ell} + x^{-1}_{ki} x_{k\ell} - x^{-1}_{ji} x_{j\ell}$$

$$x^{-1}_{ji}(x_{jk} x^{-1}_{kj} x_{i\ell} + x_{ji} x^{-1}_{ki} x_{k\ell} - x_{j\ell}) = 0$$

for all cyclic (i,j,k,ℓ) by the exchange relations (5.37).

Now let us show that the relations (5.8), (5.9), (5.18) are defining. Indeed, Proposition 5.19 implies that there is a surjective homomorphism of algebras $\mathcal{A}_n \twoheadrightarrow \mathcal{Q}_n$ given by

$$x_{ij} \mapsto y^i_{i^-,j} \ .$$

Therefore, we obtain the following obvious result.

Lemma 5.25. *The algebra* $\mathcal{Q}_n$ *is generated by all* $y_{ij} := y^i_{i^-,j}$ *and* y_{ij}^{-1}, $i, j \in [n]$, $i \neq j$, *subject to* $y_{i,i^-} = 1$, $i \in [i]$ *and the relations* (5.1), (5.37), *i.e.,*

$$y_{ij} y_{kj}^{-1} y_{ki} = y_{ik} y_{jk}^{-1} y_{ji} \ . \tag{5.17}$$

for any distinct indices $i, j, k \in [n]$*;*

$$y_{j\ell} = y_{jk} y_{ik}^{-1} y_{i\ell} + y_{ji} y_{ki}^{-1} y_{k\ell} \ . \tag{5.18}$$

for all cyclic (l, k, j, i) *in* $[n]$.

Since $y^k_{ij} = y_{ki}^{-1} y_{kj}$, the relations (5.17) directly follow from (5.8) and the relations (5.18) directly follow from (5.18) (this is obvious if we “reverse engineer” the fist part of the proof and replace all x_{ij} by y_{ij} there).

Therefore, Theorem 5.14 is proved. $\square$

The following obvious corollary from the proof of Theorem 5.14 will be instrumental in Section 6.

Corollary 5.26. *For each triangulation* Δ *of* $[n]$ *the* $\mathbb{U}_\Delta$ *is generated by* u^k_{ij}, $(i,k), (jk) \in \Delta$ *subject to the relations* (5.8) *and* (5.9), *i.e., for all distinct* $i, j, k \in [n]$ *such that* $(i,j), (jk) \in \Delta$ *one has:*

- $u^k_{ii} = 1$ *and* $u^k_{ij} u^k_{ji} = 1$
- $u^k_{ij} u^i_{jk} u^j_{ki} = 1$
- $u^\ell_{ij} u^\ell_{jk} u^\ell_{ki} = 1$ *for any* $\ell \notin \{i, j, k\}$.

5.4. **Proof of Theorems 5.3, 5.5, 5.17 and representation by noncommutative** $2 \times n$ **matrices.** In fact, Theorem 5.5 implies Theorem 5.3 because for a given triangulation Δ of $[n]$ one has:

- The group $\mathbb{T}_\Delta$ is free by Theorem 3.29 hence $\mathbb{Q}\mathbb{T}_\Delta$ is of class $\mathcal{E}_0$ by Theorem 8.5) $Frac(\mathbb{Q}\mathbb{T}_\Delta) \cong \mathcal{F}_{3n-4}$.
- $\mathcal{A}_n = \mathcal{A}_\Delta(S^{-1}) \subset Frac(\mathcal{A}_\Delta)$.
- This implies that $\mathcal{A}_n$ is also of class $\mathcal{E}_0$ and its image in $Frac(\mathcal{A}_\Delta)$ generates $Frac(\mathcal{A}_\Delta)$.
- On the other hand, $Frac(\mathcal{A}_\Delta) \cong Frac(\mathbb{Q}\mathbb{T}_\Delta) \cong \mathcal{F}_{3n-4}$.

Therefore, $\mathcal{A}_n$ embeds into $\mathcal{F}_{3n-4}$ and generates it, i.e., $Frac(\mathcal{A}_n) \cong \mathcal{F}_{3n-4}$.

This finishes proof of the implication Theorem 5.5 => Theorem 5.3. $\square$

In fact, Theorem Theorem 5.5 follows from Theorem 5.17 because for a given triangulation Δ of $[n]$ one has:

- The restriction of the isomorphism (5.12) to $\mathcal{A}_\Delta$ gives an isomorphism $f_\Delta : \mathcal{A}_\Delta \widetilde{\to} \hat{F}_n * \mathcal{Q}_\Delta$.
- The isomorphism (5.15) of groups extends to the isomorphism of algebras

$$\mathbb{Q}\mathbb{T}_\Delta \widetilde{\to} \mathbb{Q}(G_n * \mathbb{U}_\Delta) = (\mathbb{Q}G_n) * (\mathbb{Q}\mathbb{U}_\Delta) = \hat{F}_n * \mathbb{Q}\mathbb{U}_\Delta \ .$$

- And the following diagram commutes

$$\begin{tikzcd} \mathbb{Q}\mathbb{T}_\Delta \arrow[r] \arrow[d, "\mathbf{i}_\Delta"'] & \hat{F}_n * \mathbb{Q}\mathbb{U}_\Delta \arrow[d, "Id_{\hat{F}_n} * \mathbf{i}'_\Delta"] \\ \mathcal{A}_\Delta \arrow[r, "f_\Delta"] & \hat{F}_n * \mathcal{Q}_\Delta \end{tikzcd} \tag{5.19}$$

Thus, Theorem 5.17(a) implies Theorem 5.5(a) because if $\mathbf{i}'_\Delta$ is an isomorphism, then so is $\mathbf{i}_\Delta$ in the commutative diagram (5.19). Also Theorem 5.17(b) implies Theorem 5.5(b) due to the following obvious ?? identity:

$$(\mathcal{B}_1 * \mathcal{B}_2)[(S_1 * S_2)^{-1}] = \mathcal{B}_1[S_1^{-1}] * \mathcal{B}_2[S_2^{-1}] \tag{5.20}$$

for any algebra $\mathcal{B}_i$, $i = 1, 2$ with no zero divisors where S_i is a multiplicative sub-monoid of $\mathcal{B}_i \setminus \{0\}$ (free product of monoids $S_1 * S_2$ is defined in the most obvious way).

This gives Theorem 5.5(b) by taking $\mathcal{B}_1 = F_n$, $S_1 = G_n$ (so that $\mathcal{B}_1[S_1^{-1}] = \mathbb{Q}G_n = \hat{F}_n$) and $\mathcal{B}_2 = \mathcal{Q}_\Delta$, $S_2 = S'$ as in Theorem 5.17(b), so that $\mathcal{B}_2[S_2^{-1}] = \mathcal{Q}_n$. Then (5.20) gives

$$(\hat{F}_n * \mathcal{Q}_\Delta)[(G_n * S')^{-1}] = \hat{F}_n * \mathcal{Q}_n \ .$$

Then, applying the isomorphism f^{-1} given by (5.12) and taking taking $S = f^{-1}(G_n * S')$, we obtain Theorem 5.5(b).

This finishes the implication Theorem 5.17 => Theorem 5.5. $\square$

It remains to prove Theorem 5.17.

Proof of Theorem 5.17. In what follows, we identify the free skew field generated by by all a_{1i}, a_{2i}, $i \in [n]$ with $\mathcal{F}_{2n}$ and view it as the set of totally noncommutative rational functions on the space $Mat_{2\times n}$ of $2 \times n$ matrices.

Following [20] define 2×2-*quasiminors* by

$$\begin{vmatrix} a_{1i} & \boxed{a_{1j}} \\ a_{2i} & a_{2j} \end{vmatrix} = a_{1j} - a_{1i}a_{2i}^{-1}a_{2j}, \ \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & \boxed{a_{2j}} \end{vmatrix} = a_{2j} - a_{2i}a_{1i}^{-1}a_{1j} \ .$$

for $i, j \in [n]$ and the quasi-Plücker coordinates q_{ij}^k for distinct $i, j, k \in [n]$ by:

$$q_{ij}^k = \begin{vmatrix} a_{1k} & \boxed{a_{1i}} \\ a_{2k} & a_{2i} \end{vmatrix}^{-1} \cdot \begin{vmatrix} a_{1k} & \boxed{a_{1j}} \\ a_{2k} & a_{2j} \end{vmatrix} = \begin{vmatrix} a_{1k} & a_{1j} \\ a_{2k} & \boxed{a_{2j}} \end{vmatrix}^{-1} \cdot \begin{vmatrix} a_{1k} & a_{1j} \\ a_{2k} & \boxed{a_{2j}} \end{vmatrix} \tag{5.21}$$

(the last identity is proved in Proposition 4.2.1 and Section 4.3 of [20]).

Proposition 5.27. *For each $n \geq 2$ the assignment*

$$y_{ij}^k \mapsto \operatorname{sgn}(i-k)\operatorname{sgn}(j-k)q_{ij}^k$$

defines a homomorphism of algebras

$$\varphi : \mathcal{Q}_n \to \mathcal{F}_{2n} \ . \tag{5.22}$$

Proof. First, we establish a new presentation of $\mathcal{A}_n$ (and $\mathcal{Q}_n$) by using generators $\tilde{x}_{ij}^{\pm 1} := \operatorname{sgn}(j-i)x_{ij}^{\pm 1}$, $i \neq j$ and the elements $\tilde{T}_i^{jk} \in \mathcal{A}$ given by:

$$\tilde{T}_i^{jk} = \tilde{x}_{ji}^{-1}\tilde{x}_{jk}\tilde{x}_{ik}^{-1} = \operatorname{sgn}(i-j)\operatorname{sgn}(k-j)\operatorname{sgn}(k-i)x_{ji}^{-1}\tilde{x}_{jk}\tilde{x}_{ik}^{-1} \tag{5.23}$$

(see also Section 5.6). Similarly, we define

$$\tilde{y}_{ij}^k = \tilde{x}_{ki}^{-1}\tilde{x}_{kj} = \operatorname{sgn}(i-k)\operatorname{sgn}(j-k)y_{ij}^k \tag{5.24}$$

for distinct $i, j, j \in [n]$

We need the following useful fact.

Lemma 5.28. *For each $n \geq 2$ one has:*

(a) The algebra $\mathcal{A}_n$ is generated by $\tilde{x}_{ij}$ for distinct $i, j \in [n]$ subject to the relations:

$$\tilde{T}_i^{jk} = -\tilde{T}_i^{kj} \ . \tag{5.25}$$

for any distinct $i, j, k \in [n]$:

$$\tilde{T}_i^{jk} + \tilde{T}_i^{k\ell} + \tilde{T}_i^{\ell j} = 0 \ . \tag{5.26}$$

for any distinct $i, j, k, \ell \in [n]$.

(b) The algebra $\mathcal{Q}_n$ is generated by all $\tilde{y}_{ij}^k$ subject to the relations:

$$\tilde{y}_{ij}^k\tilde{y}_{ji}^k = 1, \ \tilde{y}_{ij}^k\tilde{y}_{jk}^i\tilde{y}_{ki}^j = -1 \tag{5.27}$$

for distinct $i, j, k \in [n]$,

$$\tilde{y}_{ij}^\ell\tilde{y}_{jk}^\ell\tilde{y}_{ki}^\ell = 1, \ \tilde{y}_{ik}^j\tilde{y}_{ki}^\ell + \tilde{y}_{i\ell}^j\tilde{y}_{\ell i}^k = 1 \tag{5.28}$$

for distinct $i, j, k, \ell \in [n]$.

Proof. Prove (a). Denote by $\mathcal{A}_n''$ the algebra freely generated by all $\tilde{x}_{ij}^{\pm 1}$, $i \neq j$. That is, $\mathcal{A}_n''$ is the group algebra of a free group in $n^2 - n$ generators. And define

$$\tilde{r}_{ijk} = \tilde{T}_i^{kj}(\tilde{T}_i^{jk})^{-1}$$

for all distinct $i, j, k \in [n]$. Clearly,

$$\tilde{r}_{ijk} = \tilde{x}_{ki}^{-1}\tilde{x}_{kj}\tilde{x}_{ij}^{-1}\tilde{x}_{ik}\tilde{x}_{jk}^{-1}\tilde{x}_{ji} = -x_{ki}^{-1}x_{kj}x_{ij}^{-1}x_{ik}x_{jk}^{-1}x_{ji} = \tilde{y}_{ij}^k\tilde{y}_{jk}^i\tilde{y}_{ki}^j = -y_{ij}^k y_{jk}^i y_{ki}^j$$

for all distinct $i, j, k \in [n]$. Denote by $\mathcal{I}'$ the ideal in $\mathcal{A}''$ generated by all $\tilde{r}_{ijk} + 1$. Then the quotient $\mathcal{A}_n' := \mathcal{A}_n''/\mathcal{I}'$ is an algebra generated by x_{ij}, $i, j \in [n]$, $i \neq j$ subject to the triangle relations (5.1).

Furthermore, for any distinct $i, j, k, \ell \in [n]$ define $\tilde{r}_{i;j,k,\ell} \in \mathcal{A}'_n$ by

$$\tilde{r}_{i;j,k,\ell} = \tilde{T}_i^{jk} + \tilde{T}_i^{k\ell} + \tilde{T}_i^{\ell j} \ .$$

Clearly, $\tilde{r}_{i;j,k,\ell} = -r_{i;k,j,\ell} = -r_{i;j,\ell,k}$ for all i, j, k, ℓ, i.e., $r_{i;j,k,\ell}$ is skew-symmetric in j, k, ℓ because of (5.25). Note also that

$$\tilde{r}_{i;j,k,\ell} = \tilde{x}_{ji}^{-1}(\tilde{x}_{jk}\tilde{x}_{ik}^{-1}\tilde{x}_{i\ell} + \tilde{x}_{ji}\tilde{x}_{ki}^{-1}\tilde{x}_{k\ell} - \tilde{x}_{j\ell})\tilde{x}_{i\ell}^{-1} = (\tilde{y}_{ik}^{j}\tilde{y}_{k\ell}^{i} + \tilde{y}_{i\ell}^{k} - \tilde{y}_{i\ell}^{j})\tilde{x}_{i\ell}^{-1} = (-\tilde{y}_{ik}^{j}\tilde{y}_{ki}^{\ell} + 1 - \tilde{y}_{i\ell}^{j}\tilde{y}_{\ell i}^{k})\tilde{x}_{i\ell}^{-1}$$

for all distinct i, j, k, ℓ. Moreover, if (i, j, k, ℓ) is cyclic, i.e., (ik) crosses (j, ℓ), this gives:

$$\tilde{r}_{i;j,k,\ell} = \pm x_{ji}^{-1}(x_{jk}x_{ik}^{-1}x_{i\ell} + x_{ji}x_{ki}^{-1}x_{k\ell} - x_{j\ell})x_{i\ell}^{-1} \ .$$

Therefore, if we denote by $\mathcal{I}$ the ideal in $\mathcal{A}'_n$ generated by all $\tilde{r}_{i;j,k,\ell}$, then, obviously, $\mathcal{A}'_n/\mathcal{I} \cong \mathcal{A}_n$.

This proves (a).

Part (b) also follows because the relations (5.27) and (5.28) are equivalent to (5.8), (5.9), and (5.17). The lemma is proved. □

Finally, note that quasi-Plücker coordinates also satisfy (5.27) and (5.28) by the results of [20, Section 4.4]. This proves that the assignment

$$\tilde{y}_{ij}^{k} \mapsto q_{ij}^{k}$$

is a homomorphism of algebras. Taking into account (5.24), this finishes the proof of Proposition 5.27. □

The following is an immediate corollary of Propositions 5.19 and 5.27.

Corollary 5.29. *For each $n \geq 2$ the assignments*

$$x_{ij} \mapsto \mathrm{sgn}(j-i)\begin{vmatrix} a_{1i} & \boxed{a_{1j}} \\ a_{2i} & a_{2j} \end{vmatrix}, \ x_{ij} \mapsto \mathrm{sgn}(j-i)\begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & \boxed{a_{2j}} \end{vmatrix}$$

for all $i \neq j$ define homomorphisms of algebras

$$\varphi_+ : \mathcal{A}_n \to \mathcal{F}_{2n}, \ \varphi_- : \mathcal{A}_n \to \mathcal{F}_{2n} \ . \tag{5.29}$$

Furthermore, denote by $\mathcal{F}'_{2n-4}$ the skew sub-field of $\mathcal{F}_{2n}$ generated by $\varphi(\mathcal{Q}_n)$, i.e., by all q_{ij}^k.

Proposition 5.30. *$\mathcal{F}'_{2n-4}$ is isomorphic to $\mathcal{F}_{2n-4}$.*

Proof. Denote:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \end{pmatrix}, B = \begin{pmatrix} a_{13} & \cdots & a_{1n} \\ a_{23} & \cdots & a_{2n} \end{pmatrix}, C = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{5.30}$$

so that $A = [C \,|\, B]$.

Lemma 5.31. ([20, Theorem 4.4.4]) *The matrix $C^{-1}B$ equals:*

$$C^{-1}B = \begin{pmatrix} q_{13}^2 & \cdots & q_{1n}^2 \\ q_{23}^1 & \cdots & q_{2n}^1 \end{pmatrix},$$

where $q_{ij}^k = q_{ij}^k(A)$ are quasi-Plücker coordinates on A given by (5.21).

It was proved in [20, Section 4] that $q_{ij}^k(A) = q_{ij}^k(DA)$ for all distinct $i, j, j \in [n]$ and any invertible 2×2 matrix D over $\mathcal{F}_{2n}$. In particular, taking $D = C^{-1}$, we see that $q_{ij}^k = q_{ij}^k([C \,|\, B]) = q_{ij}^k([I_2 \,|\, C^{-1}B])$, therefore, each q_{ij}^k belongs to the sub-field of $\mathcal{F}_{2n}$ generated by the matrix coefficients of C (here I_2 is the 2×2 identity matrix). This proves that $\mathcal{F}'_{2n-4}$ is a sub-field of $\mathcal{F}_{2n}$ generated by the entries of C, i.e., by all q_{1j}^2, q_{2j}^1, $j = 3, \ldots, n$.

It remains to show that matrix coefficients of $C^{-1}B$ (freely) generate a free subfield of $\mathcal{F}_{2n}$. We need the following obvious fact.

Lemma 5.32. *Let $\mathcal{F}$ be a skew field, $C \in GL_m(\mathcal{F})$ and $B \in Mat_{m,n-m}(\mathcal{F})$ such that matrix coefficients of the partitioned matrix $A = [C \,|\, B]$ generate $\mathcal{F}$. Then the matrix coefficients of $[C \,|\, C^{-1}B]$ also generate $\mathcal{F}$.*

Now we take $m = 2$ and B, C as in (5.30), $\mathcal{F} = \mathcal{F}_{2n}$, the free field freely generated by matrix coefficients of $A = [C\,|\,B]$. Then, clearly, $C \in GL_2(\mathcal{F}_{2n})$ and $B \in Mat_{2,n-2}(\mathcal{F})$. Then, by Lemma 5.32, the matrix coefficients $A' = [C\,|\,C^{-1}B]$ also generate $\mathcal{F}_{2n}$. Since A' is $2 \times n$, then Proposition 8.6 implies that the matrix coefficients of A' are free generators of $\mathcal{F}_{2n}$. In particular, the matrix coefficients of the $2 \times (n-2)$ matrix $C^{-1}B$ are free generators of the free skew sub-field of $\mathcal{F}_{2n}$. That is, $\mathcal{F}'_{2n-2}$ is freely generated by the matrix coefficients q^2_{1j}, q^1_{2j}, $j = 3, \ldots, n$ of $C^{-1}B$.

The proposition is proved. □

Remark 5.33. Proposition 5.30 and its proof generalize verbatim to $m \times n$ matrices.

Now we are ready to finish the proof of Theorem 5.17(a). It follows from the following key fact.

Theorem 5.34. *For each triangulation Δ of $[n]$ the homomorphism*

$$\varphi \circ \mathbf{i}'_\Delta : \mathbb{Q}\mathbb{U}_\Delta \to \mathcal{F}'_{2n-4} \tag{5.31}$$

is injective.

Proof. We need the following result, which is a particular case of [25, Theorem 10.10].

Proposition 5.35. *Let $m \geq 1$ and assume that m elements $t_1, \ldots, t_m$ of $\mathcal{F}_m$ generate $\mathcal{F}_m$. Then $t_1, \ldots, t_m$ are free generators, in particular, the assignment $c_i \mapsto t_i$, $i = 1, \ldots, m$ defines an injective homomorphism of algebras*

$$\mathbb{Q}G_m \hookrightarrow \mathcal{F}_m\ .$$

Taking $m = 2n-4$ and any free generating set $u_1, \ldots, u_{2n-4}$ of the free group $\mathbb{U}_\Delta \cong G_{2n-4}$, we see that $t_i := \varphi(\mathbf{i}'_\Delta(u_i))$, $i = 1, \ldots, 2n-4$ generate $\mathcal{F}'_{2n-4}$ due to the following fact.

Lemma 5.36. *For each triangulation Δ of $[n]$ the image $\varphi(\mathcal{Q}_\Delta)$ generates the skew field $\mathcal{F}'_{2n-4}$.*

Proof. Denote by $\mathcal{F}''_{2n-4}$ the skew subfield of $\mathcal{F}_{2n}$ generated by image $\varphi(\mathcal{Q}_\Delta)$. Since image $\mathcal{Q}_\Delta \subset \mathbb{Q}_n$, we have an obvious inclusion $\mathcal{F}''_{2n-4} \subseteq \mathcal{F}'_{2n-4}$. On the other hand, Theorem 5.15 implies that all elements $y_{ij} := y^i_{i^-,i}$, $i \neq j$ belong to $\mathcal{Q}_\Delta$. Taking into account that $y^k_{ij} = y^{-1}_{ki} y_{kj}$ for all distinct $i, j, k \in [n]$, we see that $\mathcal{F}''_{2n-4}$ contains all quotients $\varphi(y^k_{ij}) = \varphi(y_{ki})^{-1}\varphi(y_{kj})$, thus $\mathcal{F}''_{2n-4}$ contains the image $\varphi(\mathcal{Q}_n)$. This implies the opposite inclusion $\mathcal{F}''_{2n-4} \supseteq \mathcal{F}'_{2n-4}$ and hence $\mathcal{F}''_{2n-4} = \mathcal{F}'_{2n-4}$. The lemma is proved. □

Therefore using Proposition 8.6 with $\ell = 2n-4$, we see that $t_1, \ldots, t_{2n-4}$ are free generators of $\mathcal{F}'_{2n-4}$ and hence the homomorphism (5.31) is injective.

Theorem 5.34 is proved. □

Finally, the injectivity of (5.31) implies injectivity of $\mathbf{i}'_\Delta$. This finishes the proof of Theorem 5.17(a).

Now prove Theorem 5.17(b). Indeed, by the above arguments the restriction $\varphi_\Delta = \varphi|_{\mathcal{Q}_\Delta}$ is an injective homomorphism of algebras:

$$\varphi_\Delta : \mathcal{Q}_\Delta \hookrightarrow \mathcal{F}'_{2n-4} \tag{5.32}$$

Let S' be the sub-monoid of $\mathcal{Q}_\Delta$ generated by $y_{ij} = y^i_{i^-,i}$ for all distinct $i, j \in [n]$. By Lemma 8.3, the homomorphism (5.32) uniquely extends to an injective homomorphism of algebras

$$\hat{\varphi}_\Delta : \mathcal{Q}_\Delta[{S'}^{-1}] \hookrightarrow \mathcal{F}'_{2n-4} \tag{5.33}$$

By the construction, the image of $\hat{\varphi}_\Delta$ equals $\varphi(\mathcal{Q}_n$ hence we obtain an isomorphism:

$$\varphi(\mathcal{Q}_n) \widetilde{\to} \mathcal{Q}_\Delta[{S'}^{-1}]$$

which splits the surjective homomorphism of algebras:

$$\varphi : \mathcal{Q}_n \twoheadrightarrow \varphi(\mathcal{Q}_n)\ .$$

This implies that the latter homomorphism is also an isomorphism, and thus proves Theorem 5.17(b).

Theorem 5.17 is proved. □

5.5. **Some symmetries of noncommutative polygons.** In the notation of Lemma 5.28 define the action of the symmetric group S_n on the set $\tilde{X} = \{\tilde{x}_{ij} | i, j \in [n], i \neq j\}$ by the formula

$$w(\tilde{x}_{ij}) = \tilde{x}_{w(i),w(j)}$$

for all $w \in S_n$, $i, j \in [n]$, $i \neq j$.

Proposition 5.37. *For each $n \geq 2$ one has:*

(a) The above action uniquely extends to an action of S_n on $\mathcal{A}_n$ by algebra automorphisms.

(b) The action commutes with homomorphisms $\varphi_+, \varphi_- : \mathcal{A}_n \to \mathcal{F}_{2n}$ given by (5.29)*, where the action of S_n on $\mathcal{F}_{2n}$ is given by*

$$w(a_{s,i}) = a_{s,w(i)}$$

for $s = 1, 2$, $i \in [n]$, $w \in S_n$.

(c) The subalgebra $\mathcal{Q}_n$ is invariant under the S_n-action, more precisely,

$$w(\tilde{y}^k_{ij}) = \tilde{y}^{w(k)}_{w(i),w(j)}$$

for all $i, j, k \in [n]$, $w \in S_n$.

Proof. Prove (a). In what follows, we borrow all notation from the proof of Proposition 5.27. The following fact is obvious.

Lemma 5.38. *The S_n action on $\tilde{X}$ uniquely extends to the S_n-action on $\mathcal{A}''_n = \mathbb{Q} < \tilde{X} >$ by algebra automorphisms.*

Thus, it suffices to prove that the S_n-action on $\mathcal{A}''_n$ preserves the ideal of triangle relations (5.25) and exchange relations (5.26).

Let us prove that the ideal $\mathcal{I}'$ of $\mathcal{A}''$ generated by all r_{ijk} is invariant under the S_n-action. Indeed, for distinct $i, j, k \in [n]$ and $w \in S_n$ one has

$$w(\tilde{r}_{ijk}) = w(\tilde{x}_{ij})w(\tilde{x}_{kj})^{-1}w(\tilde{x}_{ki})w(\tilde{x}_{ji})^{-1}w(\tilde{x}_{jk})w(\tilde{x}_{ik})^{-1} = \tilde{r}_{w(i),w(j),w(k)} \ .$$

This proves that $S_n(\mathcal{I}') = \mathcal{I}'$ hence S_n acts on $\mathcal{A}'_n$ by algebra automorphisms.

It remains to prove that the ideal of exchange relations (5.26) in $\mathcal{A}'_n$ is invariant under the S_n-action. Now we show that the ideal $\mathcal{I}$ of $\mathcal{A}'_n = \mathcal{A}''_n/\mathcal{I}'_n$ generated by all $\tilde{r}_{i;j,k,\ell}$ is invariant under the S_n-action. Indeed,

$$w(\tilde{r}_{i;j,k,\ell}) = w(\tilde{T}^{jk}_i) + w(\tilde{T}^{k\ell}_i) + w(\tilde{T}^{\ell j}_i) = \tilde{T}^{w(j),w(k)}_{w(i)} + \tilde{T}^{w(k),w(\ell)}_{w(i)} + \tilde{T}^{w(\ell),w(j)}_{w(i)} = \tilde{r}_{w(i);w(j),w(k),w(\ell)}$$

for all distinct $i, j, k, \ell \in [n]$ (where $\tilde{T}^{jk}_i$ are defined in (5.23)). This proves that $S_n(\mathcal{I}) = \mathcal{I}$.

Part (a) is proved.

Part (b) follows from the fact that the homomorphisms $\varphi_+, \varphi_- : \mathcal{A}_n \to \mathcal{F}_{2n}$ are determined respectively by the assignments:

$$\tilde{x}_{ij} \mapsto \begin{vmatrix} a_{1i} & \boxed{a_{1j}} \\ a_{2i} & a_{2j} \end{vmatrix}, \ \tilde{x}_{ij} \mapsto \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & \boxed{a_{2j}} \end{vmatrix}$$

which, clearly commute with the S_n-action.

Part (c) is obvious.

The proposition is proved. □

The Lie algebra $gl_n(\mathbb{Q})$ (viewed as $Mat_{n\times n}(\mathbb{Q})$) naturally acts on the space $Mat_{2\times n}(\mathbb{Q})$ by right multiplications, i.e.,

$$E_{ji}(a_{s,t}) = \delta_{t,j} a_{s,i}$$

for $s \in \{1, 2\}$, $i, j, t \in [n]$), where $E_{ij} \in gl_N(\mathbb{Q})$ are the matrix units.

This action uniquely extends to $\mathcal{F}_{2n}$ by the Leibniz rule:

$$E(fg) = E(f)g + fE(g), \ E(h^{-1}) = -h^{-1}E(h)h^{-1}$$

for any $E \in gl_n(\mathbb{Q})$, $f, g \in \mathcal{F}_{2n}$, $h \in \mathcal{F}_{2n} \setminus \{0\}$.

Proposition 5.39. *For each $n \geq 2$ there exists a unique action of $gl_n(\mathbb{Q})$ on $\mathcal{Q}_n$ by derivations such that the homomorphism $\varphi : \mathcal{Q}_n \to \mathcal{F}_{2n}$ given by (5.22) is $gl_n(\mathbb{Q})$-equivariant. The action is given by:*

$$E_{j',i'}(\tilde{y}^k_{i,j}) = \begin{cases} 0 & \textit{if } j' \notin \{i,j,k\} \\ -\tilde{y}^k_{i,i'}\tilde{y}^k_{ij} & \textit{if } j' = i \\ \tilde{y}^k_{i,i'}\tilde{y}^i_{k,j} & \textit{if } j' = k \\ \tilde{y}^k_{i,i'} & \textit{if } j' = j \end{cases} \tag{5.34}$$

for any distinct indices $i, j, k \in [n]$.

Proof. Indeed, since the homomorphism φ is injective, it suffices to prove (5.34) for $q^k_{ij} = \varphi(\tilde{y}^k_{ij})$. Indeed, if we abbreviate $\underline{x}_{ij} = \begin{vmatrix} a_{1i} & \boxed{a_{1j}} \\ a_{2i} & a_{2j} \end{vmatrix}$ for distinct $i, j \in [n]$, then

$$E_{j'i'}(\underline{x}_{ij}) = E_{j',i'}(a_{1j} - a_{1i}a_{2i}^{-1}a_{2j}) = \begin{cases} 0 & \text{if } j' \notin \{i,j\} \\ a_{1,i'} - a_{1i}a_{2i}^{-1}a_{2,i'} & \text{if } j' = j \\ -E_{i,i'}(a_{1i}a_{2i}^{-1})a_{2j} & \text{if } j' = i \end{cases} = \begin{cases} 0 & \text{if } j' \notin \{i,j\} \\ \underline{x}_{i,i'} & \text{if } j' = j \\ \underline{x}_{i,i'}\underline{x}_{ji}^{-1}\underline{x}_{ij} & \text{if } j' = i \end{cases}$$

because

$$-E_{i,i'}(a_{1i}a_{2i}^{-1}) = -a_{1,i'}a_{2i}^{-1} + a_{1i}a_{2i}^{-1}a_{2,i'}a_{2i}^{-1} = -\underline{x}_{i,i'}a_{2i}^{-1}$$

and

$$a_{2i}^{-1}a_{2j} = -\underline{x}_{ji}^{-1}\underline{x}_{ij}$$

for $i \neq j$. Therefore,

$$E_{j'i'}(q^k_{ij}) = E_{j'i'}(\underline{x}_{ki}^{-1}\underline{x}_{kj}) = E_{j'i'}(\underline{x}_{ki}^{-1})\underline{x}_{kj} + \underline{E}_{j'i'}(x_{ki}^{-1}\underline{x}_{kj}) = \begin{cases} 0 & \text{if } j' \notin \{i,j,k\} \\ \underline{x}_{ki}^{-1}E_{j,i'}(\underline{x}_{kj}) & \text{if } j' = j \\ E_{ii'}(\underline{x}_{ki}^{-1})\underline{x}_{kj} & \text{if } j' = i \\ E_{ki'}(\underline{x}_{ki}^{-1})\underline{x}_{kj} + \underline{x}_{ki}^{-1}\underline{E}_{ki'}(x_{kj}) & \text{if } j' = i \end{cases}$$

Note that

$$E_{ki'}(\underline{x}_{ki}^{-1})\underline{x}_{kj} + \underline{x}_{ki}^{-1}E_{i'k}(\underline{x}_{kj}) = -\underline{x}_{ki}^{-1}(\underline{x}_{k,i'}\underline{x}_{ik}^{-1}\underline{x}_{ki})\underline{x}_{ki}^{-1}\underline{x}_{kj} + \underline{x}_{ki}^{-1}(\underline{x}_{k,i'}\underline{x}_{jk}^{-1}\underline{x}_{kj}) = \underline{x}_{ki}^{-1}\underline{x}_{k,i'}(-\underline{x}_{ik}^{-1} + \underline{x}_{jk}^{-1})\underline{x}_{kj}$$

$$= \underline{x}_{ki}^{-1}\underline{x}_{k,i'}\underline{x}_{ik}^{-1}(\underline{x}_{ik} - \underline{x}_{jk})\underline{x}_{jk}^{-1}\underline{x}_{kj} = \underline{x}_{ki}^{-1}\underline{x}_{k,i'}\underline{x}_{ik}^{-1}\underline{x}_{ij}$$

because

$$(\underline{x}_{ik} - \underline{x}_{jk})\underline{x}_{jk}^{-1}\underline{x}_{kj} = ((a_{1k} - a_{1i}a_{2i}^{-1}a_{2k}) - (a_{1k} - a_{1j}a_{2j}^{-1}a_{2k}))a_{2k}^{-1}a_{2j} = -a_{1i}a_{2i}^{-1}a_{2j} + a_{1j} = \underline{x}_{ij}$$

Therefore,

$$E_{j'i'}(q^k_{ij}) = \begin{cases} 0 & \text{if } j' \notin \{i,j,k\} \\ \underline{x}_{ki}^{-1}\underline{x}_{k,i'} & \text{if } j' = j \\ -\underline{x}_{ki}^{-1}(\underline{x}_{k,i'})\underline{x}_{ki}^{-1}\underline{x}_{kj} & \text{if } j' = i \\ \underline{x}_{ki}^{-1}\underline{x}_{k,i'}\underline{x}_{ik}^{-1}\underline{x}_{ij} & \text{if } j' = i \end{cases} = \begin{cases} 0 & \text{if } j' \notin \{i,j,k\} \\ q^k_{i,i'} & \text{if } j' = j \\ -q^k_{i,i'}q^k_{ij} & \text{if } j' = i \\ q^k_{i,i'}q^i_{kj} & \text{if } j' = i \end{cases}.$$

The proposition is proved. □

For $i, j \in [n]$ define the elements $y_{ij} \in \mathcal{F}_n$ by:

$$\tilde{y}_{ij} = \tilde{y}^i_{i^-,j} = \tilde{x}_{i,i^-}^{-1}\tilde{x}_{ij}$$

(with the convention that $y_{ii} = 0$). Clearly, $\tilde{y}_{i,i^-} = 1$ and $\tilde{y}_{i,i^+} = \tilde{x}_{i,i^-}^{-1}\tilde{x}_{i,i^+}$.

Denote by $\overline{\mathcal{A}}'_n$ the subalgebra of $\mathcal{Q}_n$ generated by all $\tilde{y}_{ij}$ and $\tilde{y}_{i,i^+}^{-1}$. The following is an immediate corollary of Proposition 5.39.

Corollary 5.40. *For each $i, j, i', j' \in [n]$ one has:*

$$E_{i',j'}(\tilde{y}_{ij}) = \begin{cases} 0 & \text{if } j' \notin \{i^-, i, k\} \\ -\tilde{y}_{i,i'}\tilde{y}_{ij} & \text{if } j' = i^- \\ -\tilde{y}_{i,i'}\tilde{y}_{i^-,i}^{-1}\,\tilde{y}_{i^-,j} & \text{if } j' = i \\ \tilde{y}_{i,i'} & \text{if } j' = j \end{cases}$$

In particular, $\overline{\mathcal{A}}_n'$ is invariant under the $gl_n(\mathbb{Q})$-action.

Remark 5.41. Note, however, that the subalgebra $\overline{\mathcal{A}}_n$ of $\mathcal{A}_n$ generated by x_{ij}, $i, j \in [n]$, $i \neq j$ and x_{i,i^+}^{-1}, $i \in [n]$ is not $gl_n(\mathbb{Q})$-invariant.

5.6. **Noncommutative angles.** Now we take advantage of the "invariant" algebra $\mathcal{Q}_n$ and will view the ambient algebra $\mathcal{A}_n$ as some "Galois extension" of $\mathcal{Q}_n$: in fact, Proposition 5.19 guarantees that $\mathcal{A}_n$ is (freely) generated by x_{i,i^-}, $i \in [n]$ and $\mathcal{Q}_n$.

However, we want a more symmetric and "geometric" presentation of $\mathcal{A}_n$ over $\mathcal{Q}_n$. Recall that $y_{ij}^k = x_{ki}^{-1}x_{jk}$ and set

$$T_i^{jk} = x_{ji}^{-1}x_{jk}x_{ik}^{-1}.$$

The following result provides such a presentation of $\mathcal{A}_n$.

Proposition 5.42. *The algebra $\mathcal{A}_n$ is generated by y_{ij}^k, $(T_i^{jk})^{\pm 1}$ for all distinct triples $i, j, k \in [n]$ subject to the relations:*

(i) triangle relations for all distinct (i, j, k):

$$T_i^{jk} = T_i^{kj}$$

(ii) modified exchange relations for any cyclic (i, j, k, ℓ) in $[n]$:

$$T_i^{j\ell} = T_i^{jk} + T_i^{k\ell}$$

(iii) consistency relations:

$$T_i^{jk} = y_{ij}^k y_{j\ell}^i y_{\ell i}^m T_i^{\ell m} \tag{5.35}$$

for all $i, j, k, \ell, m \in [n]$ such that the triples (i, j, k) and (i, ℓ, m) are distinct (with the convention $y_{ii}^j = 1$ for all i, j.

Proof. Denote by $\mathcal{A}_n'$ the algebra whose presentation is given in the proposition. We need to construct an isomorphism $\mathcal{A}_n \to \mathcal{A}_n'$. Indeed, we define $x_{ij} \in \mathcal{A}_n'$ for distinct $i, j \in [n]$ by:

$$x_{ij} := (T_i^{jk})^{-1}y_{ij}^k$$

for any $k \notin \{i, j\}$. This definition is correct because (5.35) guarantees that one has in $\mathcal{A}_n'$:

- $T_i^{jk} = y_{ij}^k y_{jj}^i y_{ji}^k T_i^{jk}$ hence $y_{ij}^k y_{ji}^k = 1$ for distinct i, j, k.
- $T_i^{jk} = y_{ij}^k y_{jj}^i y_{ji}^m T_i^{jm}$ hence $(y_{ij}^k)^{-1}T_i^{jk} = y_{ji}^m T_i^{jm} = (y_{ij}^m)^{-1}T_i^{jm}$ for distinct i, j, k, m. □

We view T_i^{jk} as *noncommutative angles* first, because of the triangle relations (i) (so that we can attach T_i^{jk} to the angle in the triangle (i, j, k) at the vertex i) and, second, because of the modified exchange relations (ii) of Proposition 5.42 can be viewed as an "addition law" of angles in a quadrilateral. In fact, such an addition law holds in more general situation.

Corollary 5.43. *For any cyclic $(i_0, i_1, i_2, \ldots, i_\ell)$ one has:*

$$T_{i_0}^{i_1,i_k} = T_{i_0}^{i_1,i_2} + T_{i_0}^{i_2,i_3} + \cdots + T_{i_0}^{i_{\ell-1},i_\ell} ,$$

in particular,

$$T_1^{2,n} = T_1^{23} + T_1^{34} + \cdots + T_1^{n-1,n}$$

Moreover, this view is supported by the following observation. For each triangulation Δ of n and each $i \in [n]$ define the *total angle* T_i^Δ around the vertex i to be the sum of all noncommutative angles in Δ at the vertex i. For instance, we have in Example 5.16:

$$T_1^\Delta = T_1^{23} + T_1^{34} + T_1^{45},\ T_2^\Delta = T_2^{13},\ T_3^\Delta = T_3^{12} + T_3^{14},\ T_4^\Delta = T_4^{13} + T_4^{15},\ T_5^\Delta = T_5^{14}\ .$$

Corollary 5.44. *For any triangulation Δ of $[n]$ and any $i\in[n]$, we have*

$$T_i^\Delta = T_i^{i^-,i^+} \ .$$

In particular, T_i^Δ does not depend on a choice of Δ.

Remark 5.45. Based on Corollary 5.44, we can view $T_i := T_i^{i^-,i^+}$ as the *total angle* of the noncommutative n-gon at the vertex i. The sum of all total angles $T := T_1 + T_2 + \cdots + T_n$ also does not depend on a choice of triangulations and, in particular, can be specialized to any constant value (e.g., to $\pi\cdot(n-2)$).

Remark 5.46. The independence of T_i of a choice of Δ means that T_i is invariant of noncommutative mutations. We will encounter the noncommutative angles again in Section 6.

5.7. **Extended noncommutative n-gons.**

In this section we define a larger algebra $\mathcal{A}_n^\pm$ which is an extension of $\mathcal{A}_n$ and can be viewed as a carrier of *double* noncommutative triangulations of the n-gon.

Definition 5.47. Let $\mathcal{A}_n^\pm$ be the algebra generated by x_{ij}^ε and $(x_{ij}^\varepsilon)^{-1}$, $i,j\in[n]$, $i\ne j$, $\varepsilon\in\{-,+\}$ subject to the relations:

(i) (Triangle relations) For any triple (i,j,k) of distinct indices in $[n]$:

$$x_{ij}^-(x_{kj}^+)^{-1}x_{ki}^- = x_{ik}^+(x_{jk}^-)^{-1}x_{ji}^+ \ . \tag{5.36}$$

(ii) (Exchange relations) For all cyclic (i,j,k,ℓ) in $[n]$:

$$x_{j\ell}^+ = x_{jk}^-(x_{ik}^-)^{-1}x_{i\ell}^+ + x_{ji}^+(x_{ki}^-)^{-1}x_{k\ell}^-,\ x_{j\ell}^- = x_{jk}^-(x_{ik}^+)^{-1}x_{i\ell}^- + x_{ji}^+(x_{ki}^+)^{-1}x_{k\ell}^- \ . \tag{5.37}$$

The following result is obvious.

Proposition 5.48. *The assignment $x_{ij}^\pm\mapsto x_{ij}$ defines a surjective homomorphism of algebras $\pi_n:\mathcal{A}_n^\pm\to\mathcal{A}_n$.*

In what follows, we adopt a convention for all distinct i,j,k:
$x_{ij}^k := x_{ij}^+$ if the triangle (i,j,k) is to the **left** of the interval (i,j) when one goes from i to j;
$x_{ij}^k := x_{ij}^-$ if the triangle (i,j,k) is to the **right** of the interval (i,j) when one goes from i to j.
In particular, we have

$$x_{ij}^k = x_{ij}^\ell$$

whenever (ik) crosses $(j\ell)$.

The following result is a generalization of Proposition 5.42. Let

$$\tilde y_{ij}^k = (x_{ki}^j)^{-1}x_{jk}^i,\ \tilde T_i^{jk} = (x_{ji}^k)^{-1}x_{jk}^i(x_{ik}^j)^{-1}.$$

Theorem 5.49. *The algebra $\mathcal{A}_n^\pm$ is generated by y_{ij}^k, $(\tilde T_i^{jk})^{\pm1}$ for all distinct triples (i,j,k) subject to the relations*

(i) triangular relations:

$$\tilde T_i^{jk} = \tilde T_i^{kj}$$

for all distinct (i,j,k);

(ii) modified exchange relations:

$$\tilde T_i^{j\ell} = \tilde T_i^{jk} + \tilde T_i^{k\ell}$$

whenever (i,k) crosses (j,ℓ);

(iii) consistency relations:

$$(\tilde T_i^{jk})^{-1}\tilde y_{ij}^k = (\tilde T_i^{j\ell})^{-1}\tilde y_{ij}^\ell$$

for all distinct quadruples (i,j,k,ℓ) such that (i,k) crosses (j,ℓ).

The generators x_{ij}^k can be recovered by:

$$x_{ij}^k = (\tilde T_i^{jk})^{-1}\tilde y_{ij}^k$$

for any $k\notin\{i,j\}$.

Theorem 5.50. *Fix a triangulation Δ of $[n]$. Then for each i,j,k one has:*

$$x_{ij}^+ = ???$$

$$x_{ij}^- = ???$$

5.8. **Further generalizations and specializations.**

Definition 5.51. Let $\widehat{\mathcal{A}}_n$ be the algebra generated by all $x_{ij}^k, (x_{ij}^k)^{-1}$, where i, j, k are distinct indices in $[1, n]$ subject to the relations:

(i) (triangular relations) $\hat{T}_i^{jk} = \hat{T}_i^{kj}$ for all distinct i, j, k, where

$$\hat{T}_i^{jk} = (x_{ji}^k)^{-1} x_{jk}^i (x_{ik}^j)^{-1} \ .$$

(ii) (exchange relations) $\hat{T}_i^{j\ell} = \hat{T}_i^{jk} + \hat{T}_i^{k\ell}$ whenever (i, k) crosses (j, ℓ);

The following result is obvious.

Lemma 5.52. *(a) The assignment* $x_{ij}^k \mapsto x_{ij}$ *defines a surjective homomorphism of algebras* $\widehat{\mathcal{A}}_n \to \mathcal{A}_n$.

(b) The assignment $x_{ij}^k \mapsto x_{ij}^k$ *defines a surjective homomorphism of algebras* $\widehat{\mathcal{A}}_n \to \widetilde{\mathcal{A}}_n$.

We refer to each $\hat{T}_i^{jk}$ as the generalized noncommutative angle and view it as a certain measure of the angle at the vertex i in the triangle (ijk).

For any triangulation Δ of the n-gon and $i \in [n]$, define the *total angle* $\hat{T}_i^{\Delta}$ to be the sum of all noncommutative angles of all triangles of Δ at the vertex i.

Theorem 5.53. *For any triangulations* Δ *and* Δ' *of the* n*-gon, we have*

$$\hat{T}_{\Delta} = \hat{T}_{\Delta'} \ .$$

Furthermore, let $\mathcal{A}'_n$ be the algebra generated by x_{ij}, $c_i^{jk} = c_i^{kj}$, $d_i^{jk} = d_i^{kj}$ and their inverses subject to the relations:

(i) (triangular relations) $T_i^{jk} = T_i^{kj}$ for all distinct i, j, k, where

$$T_i^{jk} = x_{ji}^{-1} x_{jk} x_{ik}^{-1} \ ;$$

(ii) (exchange relations) $(d_i^{j\ell})^{-1} T_i^{j\ell} (c_i^{j\ell})^{-1} = (d_i^{jk})^{-1} T_i^{jk} (c_i^{jk})^{-1} + (d_i^{k\ell})^{-1} T_i^{k\ell} (c_i^{k\ell})^{-1}$ whenever (i, k) crosses (j, ℓ).

Proposition 5.54. *The assignment* $x_{ij}^k \mapsto c_i^{jk} x_{ij} d_j^{ik}$ *defines a homomorphism of algebras:*

$$\hat{\varphi} : \hat{\mathcal{A}}_n \hookrightarrow \mathcal{A}'_n \tag{5.38}$$

Proof. Denote by $\hat{\mathcal{A}}'_n$ the algebra freely generated by all x_{ij}^k. Then, clearly, the assignment $x_{ij}^k \mapsto c_i^{jk} x_{ij} d_j^{ik}$ defines an algebra homomorphism

$$\hat{\mathcal{A}}'_n \to \mathcal{A}'_n \ .$$

Denote $\hat{T}_i^{'jk} := (x_{ji}^k)^{-1} x_{jk}^i (x_{ik}^j)^{-1}$. We need the following fact.

Lemma 5.55.

$$\hat{\varphi}'(\hat{T}_i^{'jk}) = (d_i^{jk})^{-1} T_i^{jk} (c_i^{jk})^{-1} \ .$$

Proof. Indeed,

$$\begin{aligned} \hat{\varphi}'(\hat{T}_i^{'jk}) &= \hat{\varphi}'((x_{ji}^k)^{-1} x_{jk}^i (x_{ik}^j)^{-1}) = (c_j^{ik} x_{ji} d_i^{jk})^{-1} c_j^{ik} x_{jk} d_k^{ij} (c_i^{jk} x_{ik} d_k^{ij})^{-1} \\ &= (d_i^{jk})^{-1} x_{ji} x_{jk} x_{ik} (c_i^{jk})^{-1} = (d_i^{jk})^{-1} T_i^{jk} (c_i^{jk})^{-1} \ . \end{aligned}$$

The lemma is proved. □

The lemma implies that $\hat{\varphi}'(\hat{T}_i^{'jk}) = \hat{\varphi}'(\hat{T}_i^{'kj})$ and:

$$\hat{\varphi}'(\hat{T}_i^{'j\ell} - \hat{T}_i^{'jk} - \hat{T}_i^{'k\ell}) = (d_i^{j\ell})^{-1} T_i^{j\ell} (c_i^{j\ell})^{-1} - (d_i^{jk})^{-1} T_i^{jk} (c_i^{jk})^{-1} - (d_i^{k\ell})^{-1} T_i^{k\ell} (c_i^{k\ell})^{-1} = 0 \ .$$

This proves the proposition. □

Corollary 5.56. *For each collection of integers* $a = \{a_i^{jk} = a_i^{kj} | i, j, k \in [n]$ *are distinct*$\}$, *the assignment*

$$x_{ij}^k \mapsto (T_i^{jk})^{a_i^{jk}} x_{ij} (T_j^{ik})^{-a_i^{jk}}$$

defines an algebra homomorphism

$$\varphi_a : \hat{\mathcal{A}}_n \to \mathcal{A}_n$$

(the latter algebra is defined in Definition 5.1).

Proof. Clearly, $\varphi=\psi\circ\hat{\varphi}$, where $\hat{\varphi}$ is given by (5.38) and $\psi:\mathcal{A}'_n\to\mathcal{A}_n$ is a surjective algebra homomorphism given by

$$x_{ij}\mapsto x_{ij},\ c_i^{jk}\mapsto (T_i^{jk})^a,\ d_i^{jk}\mapsto (T_i^{jk})^{-a}\ .$$

□

Remark 5.57. Note that if $a_i^{jk}=1$, then $\varphi_a(x_{ij}^k)=x_{ki}^{-1}x_{kj}x_{jk}x_{ik}^{-1}x_{ij}$.

6. Noncommutative triangulated surfaces

6.1. **Definitions and main results.** In this section we extend all the constructions and results of Section 2 to each orientable surface Σ possibly with boundary and with some finite set I_0 marked boundary points and a finite set I_1 of punctures (see e.g., Goncharov, Fock, Fomin, Thurston, Shapiro, Muziker, Schiffler, Williams, etc).

Denote by $\Gamma=\Gamma_\Sigma$ the set of isotopy classes of directed paths on Σ between any points of $I=I_0\sqcup I_1$ and for each $\gamma\in\Gamma$ we denote by $s(\gamma),t(\gamma)\in I$ respectively the source and the tail of γ. Also denote by $\gamma\mapsto\overline{\gamma}$ the change of direction in the path $\gamma\in\Gamma$.

In what follows an *n-gon* in Σ a sequence $P=(\gamma_1,\ldots,\gamma_n)$ in Γ such that $t(\gamma_i)=s(\gamma_{i^+})$, $i\in[n]$ and such that the interior of P contains no punctures. Note that for each n-gon in Σ and each $i\ne j$ there exists a unique $\gamma_{ij}\in\Gamma$ strictly in the interior of P such that $s(\gamma_{ij})=s(\gamma_i)$ and $t(\gamma_{ij})=s(\gamma_j)$. By definition, $\overline{\gamma}_{ij}=\gamma_{ji}$ and $\gamma_{i,i^+}=\gamma_i$. By definition, for each n-gon P in Σ one has (in the notation of Section 5) a continuous map f_P from the regular n-gon D_n to Σ such that $f_P(i,j)=\gamma_{ij}$ for all $i,j\in[n]$, $i\ne j$.

Note that each pair γ,γ' that intersect at one point defines a unique quadrilateral Q in Γ.

Definition 6.1. Define the algebra $\mathcal{A}_\Sigma$ to be generated by all $x_\gamma,\gamma\in\Gamma$ subject to:

(1) (triangle relations) (i) For any triangle $T=(\gamma_1,\gamma_2,\gamma_3)$ in Σ one has

$$x_{\gamma_{12}}x_{\gamma_{32}}^{-1}x_{\gamma_{31}}=x_{\gamma_{13}}x_{\gamma_{23}}^{-1}x_{\gamma_{21}}\ . \tag{6.1}$$

(ii) (Exchange relations) For any quadrilateral $Q=(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$:

$$x_{\gamma_{24}}=x_{\gamma_{21}}x_{\gamma_{31}}^{-1}x_{\gamma_{34}}+x_{\gamma_{23}}x_{\gamma_{13}}^{-1}x_{\gamma_{14}}\ . \tag{6.2}$$

A *triangulation* Δ of Σ is a maximal non-crossing subset of Γ (we require that $\overline{\Delta}=\Delta$).

For each triangulation Δ of Σ we define the group $\mathbb{T}_\Delta=\mathbb{T}_\Delta(\Sigma)$ to be generated by all $t_\gamma^{\pm1}$, $\gamma\in\Delta$ subject to the triangle relations:

$$t_{\gamma_{12}}t_{\gamma_{32}}^{-1}t_{\gamma_{31}}=t_{\gamma_{13}}t_{\gamma_{23}}^{-1}t_{\gamma_{21}}\ . \tag{6.3}$$

for any triangle ??? $T=(\gamma_1,\gamma_2,\gamma_3)$ in Σ.

Also for each triangulation Δ of Σ denote by $\mathbb{U}_\Delta$ the subgroup of $\mathbb{T}_\Delta$ generated by all

$$y_{\gamma,\gamma'}:=x_{\overline{\gamma}}^{-1}x_{\gamma'}$$

for all $\gamma,\gamma'\in\Delta$ such that the composition $\gamma\circ\gamma'$ is defined and also belongs to Δ.

Ultimately, for each Δ define the group $\mathbb{H}_\Delta$ to be the quotient of $\mathbb{T}_\Delta$ by the *quadrilateral relations*:

$$t_{\gamma_{12}}t_{\gamma_{32}}^{-1}t_{\gamma_{34}}t_{\gamma_{41}}^{-1}=1$$

for any triangle $T=(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ in Σ.

Theorem 6.2. *For any two triangulations Δ and Δ' of Σ there exists a group isomorphism:*

$$f_{\Delta,\Delta'}:\mathbb{T}_\Delta\cong\mathbb{T}_{\Delta'}$$

such that

$$f_{\Delta,\Delta'}(\mathbb{U}_\Delta)=\mathbb{U}_{\Delta'}\ .$$

Proof. It suffices to prove the assertion only for neighboring triangulations Δ and Δ', i.e., $\Delta\backslash\Delta'=\{\gamma_{13},\gamma_{31}\}$ and $\Delta\setminus\Delta'=\{\gamma_{24},\gamma_{42}\}$ for some intersecting paths $\gamma_{13}\in\Gamma_{i_1,i_3}$, $\gamma_{24}\in\Gamma_{i_2,i_4}$, where $\gamma_{31}=\overline{\gamma}_{13}$, $\gamma_{42}=\overline{\gamma}_{24}$.

The following result is obvious.

Lemma 6.3. *In the notation as above, the assignment*

$$x_{\gamma_{ij}} \mapsto \begin{cases} x_{\gamma_{12}} x_{\gamma_{42}}^{-1} x_{\gamma_{43}} & \text{if } (i,j) = (1,3) \\ x_{\gamma_{34}} x_{\gamma_{24}}^{-1} x_{\gamma_{21}} & \text{if } (i,j) = (3,1) \\ x_{\gamma_{ij}} & \text{otherwise} \end{cases}$$

for $i, j \in [1,4]$, $i \neq j$, defines an isomorphism

$$\varphi_{\Delta,\Delta'} : \mathbb{T}_\Delta \widetilde{\to} \mathbb{T}_{\Delta'}$$

This proves the theorem. ??? □

This allows to define a group $\mathbb{T}(\Sigma)$ (resp. $\mathbb{U}(\Sigma) \subset \mathbb{T}(\Sigma)$) such that $\mathbb{T}(\Sigma) \cong \mathbb{T}_\Delta$ (resp. $\mathbb{U}(\Sigma) \cong \mathbb{U}_\Delta$) for each triangulation Δ of Σ. We sometimes refer to the group as $\mathbb{T}_\Sigma$ as the *triangular group* of the surface Σ.

In fact, the groups $\mathbb{T}_\Sigma$, $\mathbb{U}_\Sigma$, and $\mathbb{H}_\Sigma$ are topological (and even differential-geometric invariants).

Theorem 6.4. *??? The assignment $\Sigma \mapsto \mathbb{T}(\Sigma)$ and the assignment $\Sigma \mapsto \mathbb{T}(\Sigma)$ each extends to a functor from the category of surfaces (with continuous maps as morphisms) to the category of groups.*

Proof. It suffices prove that any continuous map $f : \Sigma \to \Sigma'$ defines a group homomorphism $f_\star : T_\Sigma \to T_{\Sigma'}$, which is functorial. Indeed, let us chose triangulations Δ and Δ' of respectively Σ and Σ' such that $f(\Delta) \subset \Delta'$.

Then, by definition of the triangular group, the assignment $x_\gamma \mapsto x'_{f(\gamma)}$ for $\gamma \in \Delta$ extends to a homomorphism of groups $\mathbb{T}_\Delta \to \mathbb{T}_{\Delta'}$.

□

Our next result provides an initial classification of triangular groups of surfaces.

Theorem 6.5. *For any orientable surface Σ as above we have:*

(a) If Σ is a surface with boundary, then $\mathbb{T}_\Sigma$ is a free group in $4(2g+b-2)+3|I|+p$ generators, where g is the genus, b is the number of boundary components, and p is the number of punctures of Σ.

(b) If Σ is the sphere with 3 punctures, then $\mathbb{T}_\Sigma$ is a free group in 5 generators.

(c) If Σ is a closed surface not homeomorphic to sphere with 3 punctures, then the group $\mathbb{T}_\Sigma$ is (non-free) 1-relator torsion free (in the sense of Definition 8.4) in $8(g-1)+4|I|+1$ generators.

Proof. ???We need the following results.

Lemma 6.6. *For each triangulation Δ of Σ there exists $n > 0$, a triangulation $\tilde{\Delta}$ of $[n]$ and an equivalence relation $\equiv$ on $[n]$ such that*

- *Each equivalence class consists of at most 2 elements*
- *$\mathbb{T}_\Delta$ is the quotient of $\mathbb{T}_{\tilde{\Delta}}$ by the relations $x_{i,i^+} = x_{j,j^-}$ whenever $i \equiv j$.*

In fact, in what follows we choose a star-like triangulation of Σ so that all paths in Δ originate at a single vertex p so that $\tilde{\Delta} = \Delta_1$ as in ???.

We need the following result.

Lemma 6.7. *Let Δ_1 be the star-like triangulation of $[n]$ (as in ??). Then the (free) group $\mathbb{T}_{\Delta_1}$ of the n-gon is generated by $t_j = T_j^{1,j^+}$, $j = 3, \ldots, n-1$, $c_k = x_{k,k^+}$, $\overline{c}_k = x_{k^+,k}$, $k \in [n]$ (so that $c_n = x_{n,1}$, $\overline{c}_n = x_{1,n}$, $t_1 = t_n = 1$) subject to the relation:*

$$c_2 t_3 \cdots t_{n-1} c_{n-1} \overline{c}_n^{-1} c_1 = \overline{c}_1 c_n^{-1} \overline{c}_{n-1} t_{n-1} \overline{c}_{n-2} \cdots t_3 \overline{c}_2 \tag{6.4}$$

Proof. It is easy to see that

$$x_{1j} = c_1 t_2 c_2 \cdots t_{j-1} c_{j-1},\ x_{j1} = \overline{c}_{j-1} t_{j-1} \cdots \overline{c}_2 t_2 \overline{c}_1$$

for $j = 1, \ldots, n$. Thus, $\mathbb{T}_{\Delta_1}$ is generated by $t_2, \ldots, t_{n-1}$, c_k, $\overline{c}_k$, $k = 1, \ldots, n$ subject to the relations:

$$\overline{c}_n = c_1 t_2 c_2 \cdots c_{n-2} t_{n-1} c_{n-1}, c_n = \overline{c}_{n-1} t_{n-1} \cdots \overline{c}_2 t_2 \overline{c}_1$$

By eliminating t_2, we see that $\mathbb{T}_{\Delta_1}$ is subject to the relation (6.4).

The lemma is proved. □

By combining Lemmas 6.6 and (6.4), we see that $\mathbb{T}(\Sigma)$ is generated by t_j, $j = 3, \ldots, n-1$, c_k, $\overline{c}_k$, $k = 1, \ldots, n$ subject to the relations: $c_i = \overline{c}_{j^-}$ whenever $i \equiv j$ and:

$$c_2 t_3 \cdots t_{n-1} c_{n-1} \overline{c}_n^{-1} c_1 = \overline{c}_1 c_n^{-1} \overline{c}_{n-1} t_{n-1} \overline{c}_{n-2} \cdots t_3 \overline{c}_2 \tag{6.5}$$

This implies that $\mathbb{T}(\Sigma)$ is free if $\equiv$ has an equivalence class consisting of a single element, i.e., when Σ has boundary.

Assume now that Σ has no boundary. Then clearly, $\mathbb{T}(\Sigma)$ is a 1-relator group. More precisely, $\mathbb{T}(\Sigma)$ is free only if Σ is a sphere with $n = 3$ punctures. Otherwise, it is not free.

This finishes the proof of Theorem 6.5. □

Example 6.8. If Σ is a torus with one puncture, then $\mathbb{T}_\Sigma$ is isomorphic to the group generated by a, b, c, d, e subject to the relation $abcde = edcba$.

Remark 6.9. The classification of 1-relator torsion free groups is a rather nontrivial task (see ???).

Clearly, for each triangulation Δ of Σ the assignment $t_\gamma \mapsto x_\gamma$, $\gamma \in \Delta$ defines a homomorphisms of algebras:

$$\mathbf{i}_\Delta : \mathbb{Q}\mathbb{T}_\Delta \to \mathcal{A}_\Sigma \ . \tag{6.6}$$

Theorem 6.10. *For each triangulation Δ of Σ the homomorphism* (5.3) *is injective.*

Proof. Follows from Theorems 6.5 and 8.5.

□

Similarly to Section 5.1, for each triangulation Δ of Σ denote by $\mathcal{A}_\Delta$ the image (6.6), i.e., the subalgebra of $\mathcal{A}_\Sigma$ generated by x_{ij}, x_{ij}^{-1}, $(i,j) \in \Delta$. Then define

$$\overline{\mathcal{A}}_\Sigma := \bigcap_\Delta \mathcal{A}_\Delta$$

The following is an analogue of Conjecture 5.7.

Conjecture 6.11. *If Σ has bounary, then $\overline{\mathcal{A}}_\Sigma$ is finitely generated.*

Now we consider surfaces Σ with possible "black holes", i.e, boundary components with no marked points.

For such a surface Σ with "black holes" denote by $\Gamma_{(i,h)} \subset \Gamma$ the set of all isotopy classes of minimal ??? loops at the vertex i which become trivial after gluing in the black hole h by a disc. Clearly, $|\Gamma_{(i,h)}| = 2$.

Definition 6.12. Let $\mathcal{A}_\Sigma$ be the algebra generated by all x_γ, $\gamma \in \Gamma$ subject to the relations:

(1) All triangle and exchange relations from Definition 6.1;

(2) $x_{\gamma_i} = x_{\overline{\gamma_i}}$ for all $\gamma \in \Gamma_{(i,h)}$;

(3) $x_{\gamma_i} = x_{\gamma_{ij}} x_{\gamma_j}^{-1} x_{\overline{\gamma}_{ij}} + x_{\gamma'_{ij}} x_{\gamma_j}^{-1} x_{\overline{\gamma'_{ij}}}$ for all $\gamma_i \in \Gamma_{(i,h)}$, $\gamma_j \in \Gamma_{(j,h)}$, $i \neq j$, where $\{\gamma_{ij}, \gamma'_{ji}\}$ is the set of two paths in Γ_{ij} such that the bi-gon spanned by γ_{ij} and γ'_{ji} contains the black hole h, but does not contain other punctures or boundary components.

Theorem 6.13. *If Σ is the annulus with no punctures and n marked points on one boundary component (so that the remaining one is the black hole), then $\mathcal{C}_n := \mathcal{A}_\Sigma$ is generated by x_{ij}, $i \neq j$ and y_{ij}, $i, j \in [n]$ subject to the relations: ?????????*

(i) x_{ij}, $i \neq j$ satisfy (6.1) *and* (6.2).

(ii) For each $i \neq j$ one has:

$$y_{j\ell} = y_{jk} y_{ik}^{-1} y_{i\ell} + x_{ji} y_{ki}^{-1} x_{k\ell}$$

(iii)(mutual triangle relations) For any triple (i,j,k) of distinct indices in $[n]$:

$$y_{ij} x_{kj}^{-1} y_{ki} = y_{ik} x_{jk}^{-1} y_{ji}, \ \ x_{ij} y_{kj}^{-1} x_{ki} = x_{ik} y_{jk}^{-1} x_{ji} \ . \tag{6.7}$$

(iv) (special mutual exchange relations) For any distinct indices $i, j, k \in [n]$ such that j is between i and k in the linear order on $[n]$:

$$???x_{ik} = x_i y_{ji}^{-1} x_{jk} + x_{ij} y_{ij}^{-1} y_{ik}, \ \ y_{ij} = x_i x_{ki}^{-1} y_{kj} + y_{ik} x_{ik}^{-1} x_{ij} \ . \tag{6.8}$$

Corollary 6.14. *????Commutative limit of $\mathcal{C}_n$ is the cluster algebra of type C_{n-1}.*

????????????????????????????

???? Let now $I = \{1, 2\}$ and $\Gamma = \{\gamma_{ij} | i, j \in \mathbb{Z}\}$ be generated by $\gamma_{ij}, \overline{\gamma}_{ji} \in \Gamma_{ij}$, $i, j \in I$ subject to the relations

$$\gamma_{ij}\overline{\gamma}_{ij} = \overline{\gamma}_{ji}\gamma_{ji} = id_i$$

for all $i, j \in I$. Clearly, the inertia group of Γ is $\mathbb{Z}$ and the $\mathbb{Z}$-action on $\gamma \in \Gamma_{ij}$ is given by $[n](\gamma_{ii}^n \gamma \gamma_{jj}^n)$.

To each $[\gamma] \in \Gamma/\mathbb{Z}$ we associate a variable $x_{[\gamma]}$.

Notations: $x_{ab} = \bar{x}_{ba}$, $x_{12} = x_{23} = ... = u$, $x_{1'2'} = x_{2'3'} = ... = v$, $x_{11'} = x_{22'} = ... = w$, $x_{12'} = x_{23'} = ... = x$.

Then

$$x_{1'2} = vx^{-1}u + \bar{w}\bar{x}^{-1}\bar{w},$$

$$x_{1'3'} = vx^{-1}x + \bar{w}\bar{x}^{-1}v + vx^{-1}u\bar{w}^{-1}v,$$

$$x_{1'3} = vw^{-1}u + \bar{w}\bar{x}^{-1}vx^{-1}u +$$
$$+vx^{-1}u\bar{w}^{-1}vx^{-1}u + vx^{-1}u\bar{x}^{-1}\bar{w} + (\bar{w}\bar{x}^{-1})\bar{w},$$

$$x_{1'4'} = vw^{-1}u\bar{w}^{-1}v + (barw\bar{x}^{-1})^2 u\bar{w}^{-1}v + (vx^{-1}u\bar{w}^{-1})^2 v +$$
$$+vx^{-1}u\bar{x}^{-1}v + (\bar{w}\bar{x}^{-1})v + v(w^{-1}x)2 + \bar{w}\bar{x}^{-1}vw^{-1}x + vx^{-1}u\bar{w}^{-1}vw^{-1}.x$$

6.2. **Kontsevich discrete integrable system and its generalizations.** We illustrate the above structures and results by the description of $\mathcal{A}_{\Sigma_{1,r}}$ where $\Sigma_{1,r}$ is a (vertical) cylinder with no punctures, one marked point p on the top boundary circle and r marked points $p_1, \ldots, p_r$ on the bottom boundary circle. It is easy to see that classes of the arcs from p to $\{p_1, \ldots, p_r\}$ in $\Sigma_{1,r}$ are in a natural bijection with $\mathbb{Z}$: the n-th arc γ_n goes from p to p_s where $s \equiv n \mod r$ and γ_n has the winding number q such that $n = rq + s$.

Thus, we obtain the following result.

Theorem 6.15. *The algebra $\mathcal{A}_{\Sigma_{1,r}}$ contains a subalgebra $\mathcal{A}'$ generated by $d^{\pm 1}, \overline{d}^{\pm 1}$ and $c_n^{\pm 1}, \overline{c}_n^{\pm 1}$, $x_n, \overline{x}_n$, $n \in \mathbb{Z}$ such that $c_{n+r} = c_r$, $\overline{c}_{n+r} = \overline{c}_n$ for all $n \in \mathbb{Z}$ and:*

(i) (Triangle relations)

$$x_{n-1}\overline{c}_n^{-1}\overline{x}_n = x_n c_n^{-1}\overline{x}_{n-1},\ \overline{x}_n\overline{d}^{-1}x_{n-r} = \overline{x}_{n-r}d^{-1}x_n \tag{6.9}$$

(ii) (Exchange relations) For each $n \in \mathbb{Z}$:

$$\overline{x}_{n-r-1}d^{-1}x_n = c_n + \overline{x}_{n-1}\overline{d}^{-1}x_{n-r},\ \overline{x}_n\overline{d}^{-1}x_{n-r-1} = \overline{c}_n + \overline{x}_{n-r}d^{-1}x_{n-1} \tag{6.10}$$

for all $n \in \mathbb{Z}$.

In the theorem, each x_n corresponds to γ_n, c_n corresponds to the short (counterclockwise) boundary arc in the bottom circle, and d – to the short boundary arc in the top circle.

Note that for each $m \in \mathbb{Z}$ the cylinder $\Sigma_{1,r}$ has a triangulation Δ_m such that the set $x_{\Delta_m} = \{x_\gamma, \gamma \in \Delta_m\}$ is given by:

$$x_{\Delta_0} = \{d, \overline{d}, c_s, \overline{c}_s, s = 1, \ldots, r,\ x_{m+t}, \overline{x}_{m+t}, t = 1, \ldots, r+1\}\ .$$

Hence the triangle group $\mathbb{T}_{\Delta_0}$ is generated by $x_n, \overline{x}_n$, $n \in \mathbb{Z}$, $c_s, \overline{c}_s$, $s = 1, \ldots, r$, $d, \overline{d}$ subject to the triangle relations

$$\overline{x}_{r+1}\overline{d}^{-1}x_1 = \overline{x}_1 d^{-1}x_{r+1},\ x_{s-1}\overline{c}_s^{-1}\overline{x}_s = x_s c_s^{-1}\overline{x}_{s-1}, \tag{6.11}$$

$s = 2, \ldots, r+1$ (with the convention $c_{r+1} = c_1$, $\overline{c}_{r+1} = \overline{c}_1$).

Theorem 6.16. *For each $r > 0$ we have*

(a) (Noncommutative Laurent Phenomenon) Each $x_n, \overline{x}_n$, $n \in \mathbb{Z}$ is sum of elements of $\mathbb{T}_{\Delta_0}$ in $\mathbb{Z}\mathbb{T}_{\Delta_0}$.

(b) The total noncommutative angle $T_{\Sigma_{1,r}} \in \mathcal{A}_{\Sigma_{1,r}}$ is given by

$$T_{\Sigma_{1,r}} = x_{n+r}^{-1} d\,\overline{x}_n^{-1} + d^{-1}x_{n+r}x_n^{-1} + \overline{d}^{-1}x_n x_{n+r}^{-1} + \sum_{m=n+1}^{n+r} (\overline{x}_{m-1}^{-1}c_m x_m^{-1} + c_m^{-1}\overline{x}_{m-1}x_m^{-1} + x_{m-1}^{-1}x_m\overline{c}_m^{-1})$$

(and does not depend on n).

If r is even, we can refine this observations and thus recover noncommutative integrable Kontsevich recursion from the recent paper [21]. Indeed, by setting $U_n := \begin{cases} x_n & \text{if } n \text{ is even} \\ \overline{x}_n & \text{if } n \text{ is odd} \end{cases}$, $C_n := \begin{cases} c_n & \text{if } n \text{ is even} \\ \overline{c}_n & \text{if } n \text{ is odd} \end{cases}$, and $D := d^{-1}, \overline{D} := \overline{d}^{-1}$ we see that the subgroup of the triangle group $\mathbb{T}_{\Delta_0}$ generated by D, $\overline{D}$ and C_s, $s = 1, \ldots, r$, U_t, $t = 1, \ldots, r+1$ is free.

Furthermore, denote by $\mathbb{T}_r$ the free group generated by $D, \overline{D}, C_1, \ldots, C_r, U_1, \ldots, U_{r+1}$ and by $\mathcal{F}_r$ the (free) skew field of fractions of $\mathbb{T}_r$. Define the elements $U_n \in \mathcal{F}_r$, $n \in \mathbb{Z}$ by

$$\begin{cases} U_{n-r-1}DU_n = C_n + U_{n-1}\overline{D}U_{n-r} & \text{if } n \text{ is even} \\ U_n\overline{D}U_{n-r-1} = C_n + U_{n-r}DU_{n-1} & \text{if } n \text{ is odd} \end{cases}. \tag{6.12}$$

(with the convention $C_{n+r} = C_r$).

The following corollary of Theorem 6.16, in particular, proves Kontsevich conjecture from [21, Section 4].

Theorem 6.17. *Let $r > 0$ be even. Then each U_n belongs to the group algebra $\mathbb{Z}\mathbb{T}_r$ of the free group $\mathbb{T}_r$, more precisely, U_n is a sum of elements of $\mathbb{T}_r$.*

7. NONCOMMUTATIVE RANK 2 SURFACES

7.1. **Type $A_1^{(1)}$.** Let $\mathcal{B}_1^{(1)}$ be the algebra generated by $X_k^{\pm 1}$, $\overline{X}_k^{\pm 1}$, $k \in \mathbb{Z}$, $c_i^{\pm 1}$, $\overline{c}_i^{\pm 1}$, $i = 1, 2$ subject to the relations.

$$X_{k+1} = X_kX_{k-1}^{-1}X_k + \overline{c}_1\overline{X}_{k-1}^{-1}\overline{c}_2,\ \overline{X}_{k+1} = \overline{X}_k\overline{X}_{k-1}^{-1}\overline{X}_k + c_2X_{k-1}^{-1}c_1\ , \tag{7.1}$$

$$X_kX_{k-1}^{-1}c_1 = \overline{c}_1\overline{X}_{k-1}^{-1}\overline{X}_k,\ c_2X_{k-1}^{-1}X_k = \overline{X}_k\overline{X}_{k-1}^{-1}\overline{c}_2\ . \tag{7.2}$$

for all $k \in \mathbb{Z}$.

Denote by $\mathcal{A}_1^{(1)}$ the subalgebra of $\mathcal{B}_1^{(1)}$ generated by X_k, $\overline{X}_k$, $k = 0, 1, 2, 3$, $c_i^{\pm 1}$, $\overline{c}_i^{\pm 1}$, $i = 1, 2$ and refer to $\mathcal{A}_1^{(1)}$ as the (upper) cluster algebra of *noncommutative type* $A_1^{(1)}$.

The following result is obvious.

Lemma 7.1. *(a) The assignment $X_k \mapsto \overline{X}_k$, $c_i \mapsto \overline{c}_i$, extends to an involutive anti-automorphism $\overline{\cdot}$ of $\mathcal{B}_1^{(1)}$ preserving $\mathcal{A}_1^{(1)}$.*

(a) The assignment $X_k \mapsto X_{k+1}$, $\overline{X}_k \mapsto \overline{X}_{k+1}$, $c_i \mapsto c_i$, $\overline{c}_i \mapsto \overline{c}_i$, extends to an automorphism of $\mathcal{B}_1^{(1)}$.

(b) The assignment $X_k \mapsto \overline{X}_k$, $c_i \mapsto \overline{c}_i$ extends to an involutive anti-automorphism $\overline{\cdot}$ of $\mathcal{B}_1^{(1)}$ preserving $\mathcal{A}_1^{(1)}$.

The following result shows a number of new relations in $\mathcal{B}_1^{(1)}$ and $\mathcal{A}_1^{(1)}$.

Lemma 7.2. *The following relations hold in $\mathcal{B}_1^{(1)}$ for all $k \in \mathbb{Z}$:*

$$X_{k-1}c_2^{-1}\overline{X}_{k+1} = c_1 + X_k\overline{c}_2^{-1}\overline{X}_k, X_{k+1}\overline{c}_2^{-1}\overline{X}_{k-1} = \overline{c}_1 + X_kc_2^{-1}\overline{X}_k\ , \tag{7.3}$$

$$\overline{X}_{k-1}\overline{c}_1^{-1}X_{k+1} = \overline{c}_2 + \overline{X}_kc_1^{-1}X_k, \overline{X}_{k+1}c_1^{-1}X_{k-1} = c_2 + \overline{X}_k\overline{c}_1^{-1}X_k\ , \tag{7.4}$$

$$X_{k-1} = X_kX_{k+1}^{-1}X_k + c_1\overline{X}_{k+1}^{-1}c_2,\ \overline{X}_{k-1} = \overline{X}_k\overline{X}_{k+1}^{-1}\overline{X}_k + \overline{c}_2X_{k+1}^{-1}\overline{c}_1\ . \tag{7.5}$$

Proof. Prove the first relation. Indeed, the defining relations (7.1) and (7.2) imply:

$$\overline{X}_{k+1} = \overline{X}_k\overline{X}_{k-1}^{-1}\overline{X}_k + c_2X_{k-1}^{-1}c_1 = c_2X_{k-1}^{-1}(c_1 + X_k\overline{c}_2^{-1}\overline{X}_k)$$

This verifies the first relation (7.3). In turn, this implies that

$$X_{k-1} = c_1\overline{X}_{k+1}^{-1}c_2 + X_k\overline{c}_2^{-1}\overline{X}_k\overline{X}_{k+1}^{-1}c_2 = c_1\overline{X}_{k+1}^{-1}c_2 + X_kX_{k+1}^{-1}X_k$$

by the second relation (7.2). Furthermore, this implies that

$$\overline{X}_{k+1}c_1^{-1}X_{k-1} = c_2 + \overline{X}_{k+1}c_1^{-1}X_kX_{k+1}^{-1}X_k = c_2 + \overline{X}_k\overline{c}_1^{-1}X_k$$

by (7.2), which verifies the second relation (7.4).

The remaining relations follow by applying Lemma 7.1(b). □

The following result is an obvious corollary from relations (7.5).

Lemma 7.3. *(b) The assignment $X_k \mapsto X_{-k}$, $\overline{X}_k \mapsto \overline{X}_{-k}$, $c_i \mapsto \overline{c}_i$, $\overline{c}_i \mapsto c_i$, extends to an automorphism of $\mathcal{B}_1^{(1)}$.*

Theorem 7.4. *(Noncommutative Laurent phenomenon) Each X_k and $\overline{X}_k$ belongs to $\mathcal{A}_1^{(1)}$.*

Proof. For each $k \in \mathbb{Z}$ define the element $Z_k \in \mathcal{B}_1^{(1)}$ by:

$$Z_k = \overline{c}_1^{-1} X_{k+1} X_k^{-1} + c_1^{-1} X_{k-1} X_k^{-1} \ . \tag{7.6}$$

Clearly, $\overline{Z}_k = Z_k$ and

$$Z_k X_k = \overline{c}_1^{-1} X_{k+1} + c_1^{-1} X_{k-1} \ . \tag{7.7}$$

Lemma 7.5. *The elements Z_k satisfy for each $k \in \mathbb{Z}$:*
(a) $Z_k = c_1^{-1}(X_{k-2} c_2^{-1} \overline{X}_{k+1} - X_{k-1} \overline{c}_2^{-1} \overline{X}_k) c_1^{-1}$.
(b) $Z_k = Z_2$.

Proof. Prove (a). Using (7.3) along with the first relation (7.2), we obtain:

$$c_1^{-1} X_{k-2} c_2^{-1} \overline{X}_{k+1} c_1^{-1} = c_1^{-1}(c_1 + X_{k-1} \overline{c}_2^{-1} \overline{X}_{k-1}) \overline{X}_k^{-1} \overline{X}_{k+1} c_1^{-1} = (1 + c_1^{-1} X_{k-1} \overline{c}_2^{-1} \overline{X}_{k-1}) \overline{c}_1^{-1} X_{k+1} X_k^{-1}$$
$$= \overline{c}_1^{-1} X_{k+1} X_k^{-1} + c_1^{-1} X_{k-1} \overline{c}_2^{-1} \overline{X}_{k-1} \overline{c}_1^{-1} X_{k+1} X_k^{-1} = \overline{c}_1^{-1} X_{k+1} X_k^{-1} + c_1^{-1} X_{k-1} \overline{c}_2^{-1} (\overline{c}_2 + \overline{X}_k c_1^{-1} X_k) X_k^{-1}$$

by the first relation (7.4). Therefore,

$$c_1^{-1} X_{k-2} c_2^{-1} \overline{X}_{k+1} c_1^{-1} = \overline{c}_1^{-1} X_{k+1} X_k^{-1} + c_1^{-1} X_{k-1} X_k^{-1} + c_1^{-1} X_{k-1} \overline{c}_2^{-1} \overline{X}_k c_1^{-1} = Z_k + c_1^{-1} X_{k-1} \overline{c}_2^{-1} \overline{X}_k c_1^{-1} \ .$$

This proves (a).

Prove (b). It suffices to prove that $Z_k = Z_{k+1}$. Indeed, using (7.5), we obtain:

$$Z_{k+1} X_k = \overline{c}_1^{-1} X_{k+2} X_{k+1}^{-1} X_k + c_1^{-1} X_k X_{k+1}^{-1} X_k = \overline{c}_1^{-1} X_{k+2} X_{k+1}^{-1} X_k - \overline{X}_{k+1}^{-1} c_2 + c_1^{-1} X_{k-1} \ .$$

Furthermore, using (7.5) we have:

$$\overline{c}_1^{-1} X_{k+2} X_{k+1}^{-1} X_k = \overline{c}_1^{-1} (X_{k+1} X_k^{-1} X_{k+1} + \overline{c}_1 \overline{X}_k^{-1} \overline{c}_2) X_{k+1}^{-1} X_k$$
$$= \overline{c}_1^{-1} X_{k+1} + \overline{X}_k^{-1} \overline{c}_2 X_{k+1}^{-1} X_k = \overline{c}_1^{-1} X_{k+1} + \overline{X}_{k+1}^{-1} c_2$$

by (7.1) taken with $k+1$. Therefore,

$$Z_{k+1} X_k = \overline{c}_1^{-1} X_{k+1} + c_1^{-1} X_{k-1} = Z_k X_k$$

and $Z_{k+1} = Z_k$. This proves (b).

The lemma is proved. □

We will finish the proof of the theorem now. First, note that $Z_k = Z_2 \in \mathcal{A}_1^{(1)}$ by Lemma 7.5. Furthermore, we proceed by induction in k. The assertion is obvious for $k = 0$. Assume that for some $k > 0$ we have $X_\ell \in \mathcal{A}_1^{(1)}$ for all $0 \le \ell \le k$. Then Lemma 7.5 implies that $X_{k+1} \in \mathcal{A}_1^{(1)}$ as well. This finishes the induction. Since $\overline{\mathcal{A}_1^{(1)}} = \mathcal{A}_1^{(1)}$, this also proves that $\overline{X}_k \in \mathcal{A}_1^{(1)}$ for all k.

The case $k < 0$ can be treated identically.

The theorem is proved. □

Lemma 7.6. *For $k \ge 1$ one has*

$$X_{k+1} = X_2 X_1^{-1} X_k + \overline{c}_1 \overline{X}_1^{-1} \overline{c}_2 X_2^{-1} \sum_{s=0}^{k-2} (X_1 c_2^{-1} \overline{c}_2 X_2^{-1})^s X_{k-s} \ . \tag{7.8}$$

Proof. We proceed by induction in all $k \ge 1$. For $k = 1, 2$ we have nothing to prove because $X_2 = X_2 X_1^{-1} X_1$ and $X_3 = X_2 X_1^{-1} X_2 + \overline{c}_1 \overline{X}_1^{-1} \overline{c}_2 X_2^{-1} X_2$.

Assume that $k \ge 3$ and the assertion holds for all $\ell < k$.

Taking into account that

$$Z_k = Z_2 = \overline{c}_1^{-1} X_3 X_2^{-1} + c_1^{-1} X_1 X_2^{-1} = \overline{c}_1^{-1} X_2 X_1^{-1} + \overline{X}_1^{-1} \overline{c}_2 X_2^{-1} + c_1^{-1} X_1 X_2^{-1}$$

by Lemma 7.5(b) and (7.6), we obtain using (7.7):

$$\overline{c}_1^{-1} X_{k+1} = Z_2 X_k - c_1^{-1} X_{k-1} = (\overline{c}_1^{-1} X_2 X_1^{-1} + \overline{X}_1^{-1} \overline{c}_2 X_2^{-1} + c_1^{-1} X_1 X_2^{-1}) X_k - c_1^{-1} X_{k-1} =$$

$$\overline{c}_1^{-1}X_2X_1^{-1}X_k+\overline{X}_1^{-1}\overline{c}_2X_2^{-1}X_k+c_1^{-1}(X_1X_2^{-1}X_k-X_{k-1})\ .$$

Therefore, in order to obtain (7.8), it suffices to prove that

$$c_1^{-1}(X_1X_2^{-1}X_k-X_{k-1})=\overline{X}_1^{-1}\overline{c}_2X_2^{-1}\sum_{s=1}^{k-2}(X_1c_2^{-1}\overline{c}_2X_2^{-1})^sX_{k-s}\ . \tag{7.9}$$

Using the inductive hypothesis for $\ell=k-1$ and the first relation (7.2), we obtain:

$$X_1X_2^{-1}X_k-X_{k-1}=X_1X_2^{-1}\overline{c}_1\overline{X}_1^{-1}\overline{c}_2X_2^{-1}\sum_{s=0}^{k-3}(X_1c_2^{-1}\overline{c}_2X_2^{-1})^sX_{k-1-s}$$

$$=c_1\overline{X}_2^{-1}\overline{c}_2X_2^{-1}\sum_{s=0}^{k-3}(X_1c_2^{-1}\overline{c}_2X_2^{-1})^sX_{k-1-s}=c_1\overline{X}_2^{-1}\overline{c}_2X_2^{-1}\sum_{s=1}^{k-2}(X_1c_2^{-1}\overline{c}_2X_2^{-1})^{s-1}X_{k-s}$$

$$=c_1\overline{X}_2^{-1}c_2X_1^{-1}\sum_{s=1}^{k-2}(X_1c_2^{-1}\overline{c}_2X_2^{-1})^sX_{k-s}$$

and we obtain (7.9) (using second relation (7.2)). This finishes the inductive proof of (7.8). The lemma is proved.

□

Example 7.7. For small k we have:

$X_3=\overline{c}_1\overline{X}_1^{-1}(\overline{X}_2c_1^{-1}X_2+\overline{c}_2)$

$X_4=\overline{c}_1\overline{X}_1^{-1}(\overline{X}_2c_1^{-1}+\overline{c}_2X_2^{-1})X_3+\overline{c}_1\overline{X}_2^{-1}\overline{c}_2=\overline{c}_1\overline{X}_1^{-1}(\overline{X}_2c_1^{-1}+\overline{c}_2X_2^{-1})(\overline{c}_1+X_2c_2^{-1}\overline{X}_2)\overline{X}_1^{-1}\overline{c}_2+\overline{c}_1\overline{X}_2^{-1}\overline{c}_2$

$X_5=\overline{c}_1\overline{X}_1^{-1}(\overline{X}_2c_1^{-1}+\overline{c}_2X_2^{-1})X_4+\overline{c}_1\overline{X}_2^{-1}\overline{c}_2X_2^{-1}(X_3+X_1c_2^{-1}\overline{c}_2)$

$=(\overline{c}_1\overline{X}_1^{-1}(\overline{X}_2c_1^{-1}+\overline{c}_2X_2^{-1}))^2X_3+\overline{c}_1\overline{X}_1^{-1}(c_2+\overline{c}_2X_2^{-1}\overline{c}_1\overline{X}_2^{-1}\overline{c}_2)+\overline{c}_1\overline{X}_2^{-1}\overline{c}_2X_2^{-1}(X_3+X_1c_2^{-1}\overline{c}_2)$

$=[\bar{c}_1\bar{X}_1^{-1}(\bar{X}_2c_1^{-1}+\bar{c}_2X_2^{-1})]^2(\bar{c}_1+X_2c_2^{-1}\bar{X}_2)\bar{X}_1^{-1}c_2+\bar{c}_1[\bar{X}_1^{-1}(\bar{X}_2c_1^{-1}+\bar{c}_2X_2^{-1})\bar{c}_1+\bar{X}_2^{-1}\bar{X}_1]\bar{X}_2^{-1}\bar{c}_2+\bar{c}_1\bar{X}_2^{-1}\bar{c}_2(c_2^{-1}\bar{X}_2+X_2^{-1}\bar{c}_1)\bar{X}_1^{-1}\bar{c}_2$.

Lemma 7.8. *The specialization homomorphism* $\overline{X}_k\mapsto qX_k$, $c_1\mapsto 1$, $\overline{c}_1\mapsto q$, $c_2\mapsto q$, $\overline{c}_2\mapsto 1$ *results in quantum cluster algebra of type* $A_1^{(1)}$ *(as defined in* [8]*):*

$$X_kX_{k+1}=qX_{k+1}X_k,\ X_{k-1}X_{k+1}=qX_k^2+1,\ X_{k+1}X_{k-1}=q^{-1}X_k^2+1\ .$$

7.2. Type $A_1^{(2)}$.

Let $\mathcal{B}_1^{(2)}$ be the algebra generated by $X_k^{\pm1}$, $\overline{X}_k^{\pm1}$ $k\in\mathbb{Z}$ subject to the relations.

$$X_{2k-1}X_{2k+1}=1+X_{2k},\ X_{2k+1}X_{2k-1}=1+\overline{X}_{2k},\overline{X}_{2k-2}X_{2k}=\overline{X}_{2k}X_{2k-2}=1+X_{2k-1}^4,$$

$$\overline{X}_{2k+1}=X_{2k+1},\ X_{k\pm1}X_k^{-1}=\overline{X}_k^{-1}\overline{X}_{k\pm1}\ .$$

for all $k\in\mathbb{Z}$.

Denote by $\mathcal{A}_1^{(2)}$ the subalgebra of $\mathcal{B}_1^{(2)}$ generated by X_k, $\overline{X}_k$, $k=0,1,2,3$, and refer to $\mathcal{A}_1^{(2)}$ as the (upper) cluster algebra of *noncommutative type* $A_1^{(2)}$.

The following result is obvious.

Lemma 7.9. *(a) The assignment* $X_k\mapsto\overline{X}_k$ *extends to an involutive anti-automorphism* $\bar{\ }$ *of* $\mathcal{B}_1^{(2)}$ *preserving* $\mathcal{A}_1^{(2)}$.

(b) The assignment $X_k\mapsto X_{k+2}$, $\overline{X}_k\mapsto\overline{X}_{k+2}$, $k\in\mathbb{Z}$ *extends to an automorphism of* $\mathcal{B}_1^{(2)}$.

(c) The assignment $X_k\mapsto\overline{X}_{-k}$, $k\in\mathbb{Z}$ *extends to an involutive automorphism* φ *of* $\mathcal{B}_1^{(2)}$.

Theorem 7.10. *(Noncommutative Laurent phenomenon) Each* X_k *and* $\overline{X}_k$ *belongs to* $\mathcal{A}_1^{(2)}$*. In particular, each* X_k *and* $\overline{X}_k$ *is a noncommutative Laurent polynomial in* $X_1,X_2,\overline{X}_2$.

Proof. We need is the following result.

Lemma 7.11. *We have for each* $k\in\mathbb{Z}$*:*

$$X_{k+4}=\begin{cases}X_kX_{k+3}-X_{k+2}^3 & \text{if } k \text{ is odd}\\ X_kX_{k+3}^4+\overline{X}_{k+2}^{-1}(1-((1+X_{k+2})(1+\overline{X}_{k+2}))^2) & \text{if } k \text{ is even}\end{cases}, \tag{7.10}$$

Proof. In view of Lemma 7.9(b), in order to prove (7.10), it suffices to do so only for $k = 0, 1$. Indeed,

$$X_0X_3^4 = \overline{X}_2^{-1}(X_1^4+1)X_3^4 = \overline{X}_2^{-1}X_1^4X_3^4 + \overline{X}_2^{-1}X_3^4 = \overline{X}_2^{-1}(((1+X_{k+2})(1+\overline{X}_{k+2}))^2-1)+X_4$$

because

$$X_1^2X_3^2 = X_1(1+X_2)X_3 = (1+\overline{X}_2)(1+X_2), X_1^4X_3^4 = X_1^2(1+\overline{X}_2)(1+X_2)X_3^2 = ((1+X_{k+2})(1+\overline{X}_{k+2}))^2 \ .$$

Finally,

$$X_1X_4 = X_3^{-1}(1+X_2)X_4 = X_3^{-1}X_4 + X_3^{-1}X_2X_4 = X_3^{-1}X_4 + X_3^{-1}(1+X_3^4) = X_5 + X_3^3 \ .$$

The lemma is proved. □

To finish the proof of the Theorem, let us denote by $\mathcal{A}_{1,k}^{(2)}$, $k \in \mathbb{Z}$ the subalgebra of $\mathcal{B}_1^{(2)}$ generated by X_{k+i}, $\overline{X}_{k+i}$, $i = 0, 1, 2, 3$, which we refer to as the *upper bound*. Clearly, $\mathcal{A}_1^{(2)} = \mathcal{A}_{1,0}^{(2)}$.

Lemma 7.12. *$\mathcal{A}_{1,k}^{(2)} = \mathcal{A}_1^{(2)}$ for all $k \in \mathbb{Z}$.*

Proof. It suffices to prove that $\mathcal{A}_{1,k}^{(2)} = \mathcal{A}_{1,k+1}^{(2)}$ for all $k \in \mathbb{Z}$. It follows from (7.10) that $X_{k+4} \in \mathcal{A}_{1,k}^{(2)}$ for all $k \in \mathbb{Z}$. Applying Lemma 7.9(a), we see that $\overline{X}_{k+4} \in \mathcal{A}_{1,k}^{(2)}$ for all $k \in \mathbb{Z}$ hence $\mathcal{A}_{1,k}^{(2)} \supseteq \mathcal{A}_{1,k+1}^{(2)}$. By applying the homomorphism φ from Lemma 7.9(c) we obtain the opposite inclusion because $\varphi(\mathcal{A}_{1,k}^{(2)}) = \mathcal{A}_{1,-k-3}^{(2)}$ for all k.
The lemma is proved. □

Theorem 7.10 is proved. □

Example 7.13. For small k we have:
$X_3 = X_1^{-1}X_2 + X_1^{-1}$
$X_4 = X_4 = X_2^{-1} + X_1^{-4}X_2^{-1}((1+\bar{X}_2)(1+X_2))^2$
$X_5 = X_1\bar{X}_2^{-1} + X_1^{-3}\bar{X}_2^{-1}(1+X_2)(1+\bar{X}_2)(1+X_2)$.

8. APPENDIX: NONCOMMUTATIVE LOCALIZATIONS

Given a ring R and a monoid S we define the *free localization* of R by S to be the free product $R*\mathbb{Z}S$, where $\mathbb{Z}S$ is the *linearlization* of S, i.e., $\mathbb{Z}S = \bigoplus_{s\in S}\mathbb{Z}\cdot[s]$ is the ring with the natural extension of multiplication on S.

If S is a multiplicative sub-monoid a unital ring of R, following Ore (see e.g., [11, Section 0.7]), we define the *universal localization* $R[S^{-1}]$ of R by S to be quotient of the free localization $R*(\mathbb{Z}S^{op})$ by the ideal generated by all elements of the form $s*[s]-1$, $[s]*s-1$ for any $s \in S$. By definition, one has a canonical ring homomorphism

$$\mathbf{j} : R \to R[S^{-1}] \tag{8.1}$$

In other words, $R[S^{-1}]$ is the unital ring R' with the universal property that for any ring homomorphism $\pi : R \to R'$ such that each element of $\pi(S)$ in invertible there exists a unique homomorphism $\hat{\pi} : R[S^{-1}] \to R'$ such that π factors as a composition of $\hat{\pi}$ and $\mathbf{j} : R \to R[S^{-1}]$.

For any subset $T \subset R$ containing no zero divisors, we sometimes abbreviate $R[T^{-1}] := R[S^{-1}]$, where S is the multiplicative sub-monoid of R generated by X.
???

Proposition 8.1. *Let R and R' be $\Bbbk$-algebras and let $f : R \to R'$ be a homomorphism of algebras. Then for any $S \subset R$ and any homomorphism $\varphi : R \to R[S^{-1}]$ one has a pushout diagram:*

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\pi"'] & R[S^{-1}] \arrow[d, "\psi"] \\ R' \arrow[r, "\varphi'"] & R''[S''^{-1}] \end{tikzcd} \tag{8.2}$$

where R'' is determined by the pushout diagram:

$$\begin{tikzcd}
R \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \varphi(R) \arrow[d, "\pi'"] \\
R' \arrow[r] & R''
\end{tikzcd} \tag{8.3}$$

and $S'' = \varphi(\pi'(S))$.

Proof. We need the following fact.

Lemma 8.2. *Let R and R' be rings and $\pi : R \to R'$ be a homomorphism. Then for any $S \subset R$ there exists a ring homomorphism $\hat{\pi} : R[S^{-1}] \to R'[\pi(S)^{-1}]$ such that following diagram is a pushout.*

$$\begin{tikzcd}
R \arrow[r, "\mathbf{j}"] \arrow[d, "\pi"'] & R[S^{-1}] \arrow[d, "\hat{\pi}"] \\
R' \arrow[r, "\mathbf{j}'"] & R'[\pi(S)^{-1}]
\end{tikzcd} \tag{8.4}$$

Proof. First, we construct $\hat{\pi}$ and verify the commutativity of (8.4). Denote $\pi' := \mathbf{j}' \circ \pi$. Clearly, each element in $\pi'(S) = \pi(S)$ is invertible in $R'[\pi(S)^{-1}]$. This implies that π' factors as $\pi' = \hat{\pi} \circ \mathbf{j}$, where $\hat{\pi} : R[S^{-1}] \to R'[\pi(S)^{-1}]$ is the canonical lifting of π'.

Thus, it suffices to verify the universality of (8.4). Indeed, consider a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\mathbf{j}"] \arrow[d, "\pi"'] & R[S^{-1}] \arrow[d, "\hat{\pi}''"] \\
R' \arrow[r, "\mathbf{j}''"] & R_1
\end{tikzcd}$$

where R_1 is any ring generated by the images of $\mathbf{j}''$ and π''. Since $\mathbf{j}''$ is an $\pi(S)$-inverting homomorphism, there exists a homomorphism $\psi : R'[\pi(S)^{-1}]$ such that $\mathbf{j}'' = \mathbf{j}' \circ \psi$. Furthermore, it is easy to see that R_1 is generated by $\mathbf{j}''(R') = \psi(\mathbf{j}'(R')$ and by $\mathbf{j}''(S)^{-1} = \psi(\mathbf{j}'(S)^{-1}$. This verifies surjectivity of ψ. Finally, let us show that

$$\pi'' = \psi \circ \hat{\pi} \tag{8.5}$$

Indeed, let $\underline{r} \in \mathbf{j}(R)$, i.e., there is $r \in R$ such that $\underline{r} = \mathbf{j}(r)$. Then

$$\pi''(\underline{r}) = \mathbf{j}''(\pi(r)) = \psi(\mathbf{j}'(\pi(r))) = \psi(\hat{\pi}(\mathbf{j}(r)) = \psi(\hat{\pi}(\underline{r})) \ .$$

This verifies (8.5).

Thus, ψ makes the total "pentagonal" diagram fully commutative and is surjective.

Therefore (8.2) is the pushout diagram.

The lemma is proved. □

Using Lemma 8.2, we obtain a pushout diagram:

$$\begin{tikzcd}
\varphi(R) \arrow[r] \arrow[d, "\pi'"'] & R[S^{-1}] \arrow[d, "\psi"] \\
R'' \arrow[r] & R''[S^{-1}]
\end{tikzcd}$$

Composing it "horizontally" with (8.3) gives the pushout diagram (8.2).

The proposition is proved. □

Note that the canonical S-inverting homomorphism (8.1) in not always injective. Below, following A.I. Malcev and P.M. Cohn, we establish a sufficient conditions on injectivity of (8.1).

Following A. Malcev and P. M. Cohn, we say that a unital ring is of class $\mathcal{E}$ if it can be embedded into a skew-field.

The following result is obvious.

Lemma 8.3. *Let R be any ring of class $\mathcal{E}$. Then for any $S \subset R \setminus \{0\}$ the canonical homomorphism* (8.1) *is injective.*

Below we provide a sufficient criterion for a group algebra $\Bbbk G$ to belong to class $\mathcal{E}$.

Definition 8.4. A group G is said to be 1-relator torsion-free if G is isomorphic to $\mathbb{F}/\langle x\rangle$ where $\mathbb{F}$ is a finitely generated free group, $x \in \mathbb{F} \setminus \{1\}$ is not a proper power in $\mathbb{F}$, and $\langle x\rangle$ denotes a normal subgroup of $\mathbb{F}$ generated by x.

The following result is proved by Malcev, Newman, J. Lewin and T. Lewin (see e.g., [10, Section 8.7], [23]).

Theorem 8.5. *Let G be any finitely generated free group or any 1-relator torsion free group. Then the group algebra $\Bbbk G$ is of class $\mathcal{E}$. In particular, for any submonoid $S \subset \Bbbk G \setminus \{0\}$ the canonical homomorphism* (8.1)

$$\Bbbk G \to \Bbbk G[S^{-1}]$$

is injective.

Denote by G_ℓ the free group in ℓ generators $c_1, \ldots, c_\ell$. We will need the following result, which is a particular case of [25, Theorem 10.10].

Proposition 8.6. *Let $\ell \geq 1$ and assume that ℓ elements $t_1, \ldots, t_\ell$ of $\mathcal{F}_\ell$ generate $\mathcal{F}_\ell$. Then $t_1, \ldots, t_\ell$ are free generators. In particular, the assignment $c_i \mapsto t_i$ for $i = 1, \ldots, \ell$ defines an injective homomorphism of algebras*

$$\mathbb{Q}G_\ell \hookrightarrow \mathcal{F}_\ell \ .$$

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